

Improved Maximin Share Guarantee for Additive Valuations

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Abstract

The maximin share (MMS) is the most prominent share-based fairness notion in the fair allocation of indivisible goods. Recent years have seen significant efforts to improve the approximation guarantees for MMS for different valuation classes, particularly for additive valuations. For the additive setting, it has been shown that for some instances, no allocation can guarantee a factor better than $1 - \frac{1}{n^4}$ of maximin share value to all agents. However, the best currently known algorithm achieves an approximation guarantee of $\frac{3}{4} + \frac{3}{3836}$ for MMS. In this work, we narrow this gap and improve the best-known approximation guarantee for MMS to $\frac{10}{13}$.

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1 Introduction

Fair allocation is a fundamental problem that spans multiple disciplines, including mathematics, social sciences, economics, and computer science. Given m goods and n agents, each agent has a valuation function v_i that assigns a non-negative value to every subset of goods. The goal is to allocate the goods fairly. In this paper, we focus on the setting where the valuations are *additive*.

What does it mean for an allocation to be fair? How can fairness be measured and ensured? These questions have been extensively studied for over eight decades. The foundation of modern fair division theory dates back to Hugo Steinhaus’ seminal work [Ste49] in 1949, where he provided a mathematically rigorous definition of the *cake-cutting problem*—a fair allocation problem involving a continuous, heterogeneous resource. Since then, numerous fairness criteria have been proposed, which can be broadly classified into two main categories:

- **Envy-based:** Agents evaluate fairness by comparing their own bundle to either the entire bundle or a subset of another agent’s bundle. Examples include *envy-freeness* [Fol66], *envy-freeness up to one good* [Bud11], and *envy-freeness up to any good* [CKM⁺19].
- **Share-based:** An agent evaluates fairness based on the value they receive, independently of others’ allocations. Examples include *maximin share* [Bud11] and *proportionality* [Ste49].

In this paper, we focus on one of the most well-studied share-based fairness notions in recent years: the *maximin share* (MMS) [Bud11]. Suppose we aim to define a share-based notion of fairness by setting a threshold τ_i for each agent a_i to determine whether their share is fair. A reasonable expectation is $\tau_i \leq \frac{v_i(M)}{n}$, since if all agents have similar valuation functions, guaranteeing a larger value to every agent would be impossible. This quantity, $v_i(M)/n$, is called the *proportional share* and has been extensively studied in the literature of fair allocation [Ste49]. When goods are *divisible*, proportionality can always be guaranteed [DS61]. However, with *indivisible* goods, this is no longer the case. Consider a simple example: if there are two agents and one indivisible good, one agent will receive the good while the other gets nothing—far below their proportional share.

A natural alternative is the *maximin share* (MMS), which provides a more flexible fairness benchmark. To define MMS, consider a different way to set an upper bound for τ_i . We ask agent a_i to divide the goods into n bundles in a way that maximizes the value of the least valuable bundle. The value of this least valuable bundle is called the *maximin share* (MMS) value of agent a_i .

By definition, an agent’s maximin share value is always at most their proportional share. They coincide when an agent can partition the goods into n bundles of equal value. Moreover, MMS value serves as an upper bound for τ_i ; if all agents have similar valuations, at least one agent receives a bundle worth at most their maximin share value. This naturally leads to a question: *Can we guarantee that every agent receives a bundle which she values as much as her maximin share value?*

Unfortunately, the answer to this question is negative; there exist instances where no allocation can ensure that every agent receives a bundle with value at least as their maximin share value [PW14]. However, unlike proportionality, there always exist allocations that guarantee each agent a constant fraction of their maximin share value.

Over the past decade, significant efforts have been made to improve approximation guarantees for the maximin share problem in the additive setting [PW14, BK20, GHS⁺18, AMNS17, GT20, AGST23, AG24]. A $(\frac{1}{2})$ -approximation guarantee is easy to achieve, but the first nontrivial bound of $\frac{2}{3}$ was introduced by Procaccia and Wang [PW14]. This was later improved to $\frac{3}{4}$ by Ghodsi *et al.* [GHS⁺18]. Subsequent work slightly improved this bound to $(\frac{3}{4} + \frac{1}{12n})$ -MMS and $(\frac{3}{4} + \min(\frac{1}{36}, \frac{3}{16n-4}))$ -MMS [GT20, AGST23], but no breakthrough occurred until the recent work of Akrami and Garg [AG24], who improved the approximation guarantee to $\frac{3}{4} + \frac{3}{3836}$.

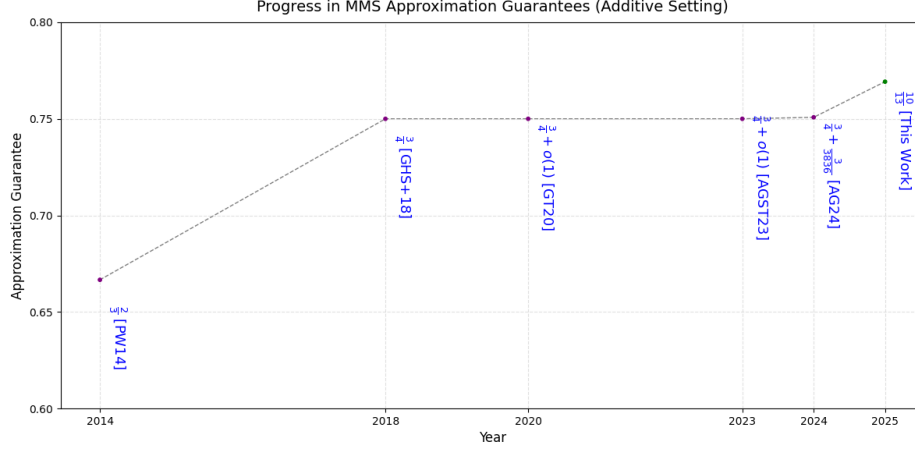


Figure 1: Recent progress on approximation guarantees for MMS in the additive setting.

Although the result of Akrami and Garg [AG24] breaks the $(\frac{3}{4})$ -approximation barrier, the improvement remains small. In this paper, we advance the frontier by proving that a $(\frac{10}{13})$ -approximation of MMS can always be guaranteed for all agents (See Figure 1).

1.1 Further Related Work

Much of the work on MMS approximations for *additive valuations* has been discussed in the introduction. Here, we briefly mention additional results and focus on broader settings.

On the impossibility side, MMS allocations do not always exist [PW14]. Kurokawa et al. [KPW16] showed this even when $m \leq 3n + 4$, and Feige et al. [FST21] proved an upper bound of $1 - \frac{1}{n^4}$ and also established a bound of $\frac{39}{40}$ for three agents.

Beyond additive valuations, MMS has been studied under *submodular*, *fractionally subadditive*, and *subadditive valuations*. For submodular valuations, Barman and Krishnamurthy [BK20] initiated this direction with a 0.21-approximation. Ghodsi et al. [GHS⁺18] later improved the bound to $\frac{1}{3}$, and more recently, Uziah and Feige [UF23] achieved a $\frac{10}{27}$ -approximation. The best-known upper bound remains $\frac{3}{4}$ [GHS⁺18]. For fractionally subadditive valuations, Ghodsi et al. [GHS⁺18] provided an initial $\frac{1}{5}$ approximation with an upper bound of $\frac{1}{2}$. Seddighin and Seddighin [SS24] improved this to $\frac{1}{4.6}$, and Akrami et al. [AMSS23] further improved the bound to $\frac{3}{13}$. For subadditive valuations, Ghodsi et al. [GHS⁺18] proved an $\Omega(\frac{1}{\log m})$ approximation. Seddighin and Seddighin [SS24] improved the lower bound to $\Omega(\frac{1}{\log n \log \log n})$. Subsequently, Feige and Huang [FH25] improved the approximation to $\Omega(\frac{1}{\log n})$, which was further improved to $\Omega(\frac{1}{\log \log n^2})$ by Seddighin and Seddighin [SS25], and to $\Omega(\frac{1}{\log \log n})$ by Feige [Fei25].

For a *small number of agents or goods*, MMS allocations exist for two agents. For three agents, successive improvements have raised the best-known approximation guarantee from $\frac{7}{8}$ [AMNS17] to $\frac{8}{9}$ [GM19] and later to $\frac{11}{12}$ [FN22]. Ghodsi et al. [GHS⁺18] established a $\frac{4}{5}$ guarantee for four agents. Further existence results hold when $m \leq n + 3$ [AMNS17] or $m \leq n + 5$ [FST21]. Recently, Garg and Shahkar [GS25] provided better guarantees for two and three types of agents.

For *chores* (undesirable goods), Aziz et al. [ARSW17] extended the definition of MMS and provided a 2-approximation, which was improved to $\frac{4}{3}$ by Barman and Krishnamurthy [BK20] and further to $\frac{11}{9}$ by Huang and Lu [HL21], and Huang et al. [HSH23] subsequently provided a $\frac{13}{11}$ -approximation for chores. MMS guarantees have also been studied in *ordinal settings* [HSSH22,

AGST24, GHS25] and in *weighted* MMS [ACL19, FGH⁺19], for which Wang, Li, and Lu [WLL24] achieved an $O(\log n)$ -approximation.

2 Basic Notations

We denote the input instance of our MMS allocation Algorithm by $\mathcal{I} = (N, M)$, where N is the set of agents and M is the set of goods. Also, we have $|N| = n$ and $|M| = m$. For each agent a_i , we denote their valuation function by $v_i : 2^M \rightarrow \mathbb{R}^{\geq 0}$, which assigns a non-negative value to every subset of goods. We assume valuations are additive, meaning that for any disjoint subsets $S, T \subseteq M$, $v_i(S \cup T) = v_i(S) + v_i(T)$. Thus, the valuation of a subset S simplifies to $v_i(S) = \sum_{g \in S} v_i(\{g\})$.

A key assumption we make is that the input instance is **ordered**, meaning all agents rank the goods in the same order according to their values. Barman and Krishnamurthy [BK20] showed that any MMS allocation instance with additive valuations can be reduced to an ordered instance.

Theorem 2.1 (Theorem 3.2 of [BK20](Restated)). *For every instance $\hat{\mathcal{I}}$, there exists an ordered instance $\tilde{\mathcal{I}}$ such that any α -MMS allocation for $\tilde{\mathcal{I}}$ can be converted into an α -MMS allocation for $\hat{\mathcal{I}}$.*

By Theorem 2.1, we assume all agents rank the goods in a common order. Hence, we denote $M = \langle g_1, g_2, \dots, g_m \rangle$, where the goods are sorted in non-increasing order of their values for all agents, i.e., for every agent $a_i \in N$ and every $j < k \leq m$, we have $v_i(\{g_j\}) \geq v_i(\{g_k\})$.

Given a constant d , a valuation function v , and a set S of goods, the maximin share value of v with respect to d and S is defined as

$$\Psi_v^d(S) = \max_{\langle \pi_1, \pi_2, \dots, \pi_d \rangle \in \Pi_d(S)} \min_{j=1}^d v(\pi_j),$$

where $\Pi_d(S)$ is the set of all partitionings of S into d bundles. For an agent $a_i \in N$, we refer to $\Psi_{v_i}^n(M)$ as her maximin share value. Our goal is to compute an allocation that guarantees each agent a constant-factor approximation of her maximin share value. In Definition 1, we formally define approximate maximin share allocations.

Definition 1. *For a constant α , we say an allocation that allocates a distinct bundle A_i to each agent a_i is α -MMS, if for every agent a_i , $v_i(A_i) \geq \alpha \Psi_{v_i}^n(M)$.*

Our goal in this paper is to prove the existence of a $(\frac{10}{13})$ -MMS allocation. For this purpose, we set $\alpha = \frac{10}{13}$. A key property of the maximin share is that it is scale-free. That is, an agent's maximin share depends only on her valuations, so multiplying or dividing all values by a constant factor does not affect the approximation guarantee of an allocation. Hence, we suppose without loss of generality that for every agent a_i , their maximin share is scaled such that $\Psi_{v_i}^n(M) = 1$. The goal is then to allocate each agent a bundle with value at least α according to their valuation.

2.1 Algorithm Structure and Notation

In Figure 2, we present a flowchart of our algorithm. As mentioned, the input instance is denoted by $\mathcal{I} = (N, M)$. Our algorithm proceeds as follows:

- **Primary Reductions:** We apply a set of primary reductions to the input instance $\mathcal{I}$. The output of this step is denoted by $\tilde{\mathcal{I}} = (\tilde{N}, \tilde{M})$, where $\tilde{N} \subseteq N$ is the remaining set of agents, and $\tilde{M} \subseteq M$ is the remaining set of goods.

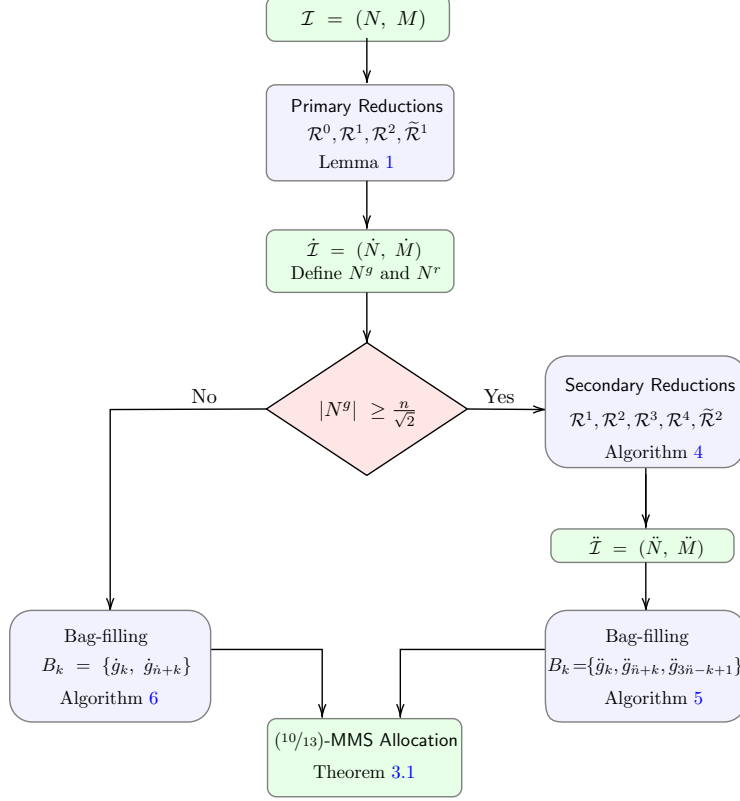


Figure 2: A flowchart of our algorithm.

- **Secondary Reductions for Case 1:** After primary reductions, the algorithm branches into two cases. For the first case, we apply a set of secondary reductions. The output of these reductions is denoted by $\tilde{\mathcal{I}} = (\tilde{N}, \tilde{M})$, where $\tilde{N} \subseteq \hat{N}$ is the set of agents after the reductions, and $\tilde{M} \subseteq \hat{M}$ is the set of goods after the reductions.

Afterwards, the algorithm executes a Bag-filling process tailored to each case. Further details on these reductions and Bag-filling procedures are provided in Section 3. For convenience, we assume the following notations for the instances: $|N| = n$, $|\hat{N}| = \hat{n}$, and $|\tilde{N}| = \tilde{n}$ for the agents, $|M| = m$, $|\hat{M}| = \hat{m}$, and $|\tilde{M}| = \tilde{m}$ for the goods.

All three instances $(\mathcal{I}, \hat{\mathcal{I}}, \text{ and } \tilde{\mathcal{I}})$ are assumed to be ordered. Specifically $\tilde{M} = \langle \tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_{\tilde{m}} \rangle$, $\hat{M} = \langle \hat{g}_1, \hat{g}_2, \dots, \hat{g}_{\hat{m}} \rangle$, and $M = \langle g_1, g_2, \dots, g_m \rangle$, where the goods are sorted in non-increasing order of their values for all agents. Recall that for every agent a_i , we have $\Psi_{v_i}^n(M) = 1$. For convenience, for a function v , we denote $\Psi_v^{\hat{n}}(\hat{M})$ by $\hat{\Psi}_v$, and $\Psi_v^{\tilde{n}}(\tilde{M})$ by $\tilde{\Psi}_v$.

To state and prove some of our lemmas and theorems in a general setting, we occasionally consider arbitrary instances, which we denote by $\hat{\mathcal{I}} = (\hat{N}, \hat{M})$ and $\tilde{\mathcal{I}} = (\tilde{N}, \tilde{M})$. Following our notational convention, we let $|\hat{N}| = \hat{n}$, $|\hat{M}| = \hat{m}$, $|\tilde{N}| = \tilde{n}$, and define $\hat{\Psi}_{\hat{v}} = \Psi_{\hat{v}}^{\hat{n}}(\hat{M})$ and $\tilde{\Psi}_{\tilde{v}} = \Psi_{\tilde{v}}^{\tilde{n}}(\tilde{M})$.

Finally, to simplify the analysis and presentation, we make a few standard assumptions about the input. First, to ensure that the indices of goods used in reductions or during the Bag-filling process do not exceed the total number of goods, we assume that the number of goods is at least $5n$. This can always be ensured by adding dummy goods that have value 0 for all agents. Second, we assume that no good has a value greater than 1 to any agent. This assumption is common in

prior work and has been shown to be without loss of generality [GHS⁺18]. In fact, it is immediately justified by the first reduction we introduce.

Remark 1. *For the reader’s convenience, we provide Table 4, which summarizes notations that are frequently used throughout the paper.*

3 Highlights of Techniques

3.1 Algorithmic Overview

Our approach follows the classical framework adopted in much of the prior work. This framework consists of two main components: **a set of reductions** and **a Bag-filling process**. The reduction phase focuses on allocating *large* goods. Roughly speaking, a reduction identifies a small subset of goods (typically of size fewer than four) each valued at least α to an agent^{*} and (almost) preserving the maximin share value of the remaining agents for the remaining goods. If such a subset exists, the algorithm allocates it to that agent and reduces the problem to a smaller instance with fewer goods and agents.

As a simple example, the most basic form of a reduction checks whether there exists a good valued at least α by some agent. If so, we allocate it to that agent, remove both the good and the agent, and recurse on the rest. This works because removing one good and one agent does not decrease the maximin share values of the remaining agents for the remaining goods. Moreover, this reduction ensures that in the residual instance—where this reduction is no longer applicable—each good has a value of at most α to every agent. A similar—though more complex—principle applies to other reductions. These reductions impose a rich collection of upper bounds and structural constraints on the values that agents assign to various goods.

When no further reductions apply, the instance consists of goods that hold relatively low value for the agents. At this stage, the algorithm invokes a Bag-filling process to allocate the remaining goods. While the Bag-filling procedure can be intricate, its core idea draws inspiration from the classic moving-knife method in cake-cutting: starting with an empty bag, goods are added one by one until an agent among those still participating calls “STOP!”—indicating that the current bundle meets the targeted approximation guarantee of the agent’s maximin share. The bundle is then given to that agent, and they are removed from the process.

The core idea of the Bag-filling process is as follows: when a bundle is allocated to an agent who shouts “STOP!”, this bundle must have value less than α for any agent who has not yet shouted. Our goal is to upper bound the value that each allocated bundle holds for the remaining agents.

However, complications arise when multiple agents shout “STOP!” at the same time. If we give the bundle to one of them, the value of that bundle to the others may exceed α (or sometimes even 1). Note that since no one shouted before the last good was added, this excess value is bounded by the value of a single remaining good.

To handle this, several key strategies can be employed. First, bags may be initialized with higher-value remaining goods to ensure these goods are evenly distributed and do not end up among the last goods added. Second, a priority order among agents can be used to break ties when multiple agents shout simultaneously. This priority typically favors agents who are more likely to face difficulties later in the Bag-filling process.

Together, these strategies—along with the bounds established through the reductions and a careful analysis of bundle values—yield Theorem 3.1, which is the main result of the paper. We

^{*}We present this section under the assumption that the goal is to find an α -MMS allocation. While this is not the objective in prior studies, the assumption does not affect the description of the underlying ideas.

note that deriving some of these bounds is highly nontrivial and relies heavily on the structure of the agents’ maximin share partitions.

Theorem 3.1. *The allocation returned by Algorithm 1 is $(\frac{10}{13})$ -MMS.*

3.2 Techniques

While the overall approach shares similarities with that of Akrami and Garg [AG24], our method introduces several key improvements and novel insights, which can be summarized as follows. We note that the analysis of Akrami and Garg [AG24] is tight for their algorithm.

- **Dynamic reductions:** In previous approaches, a reduction is typically defined by fixing the indices of the goods considered in the ordered list of goods. These indices remain static for each reduction. In contrast, we introduce more flexible reductions: we allow the index of the smallest allocated good (i.e., the good with the largest index) to be determined dynamically based on the input—specifically, we let it be as large as possible. Although this flexibility might seem minor, it plays a crucial role in uncovering useful patterns in the valuations of agents whose MMS values decrease during the reduction process, which in turn allow us to partially offset this decrease in later steps. Further details are provided in Section 3.3.
- **Deferred matching:** We introduce a more flexible bundle allocation strategy in the reduction phase. When multiple agents are eligible for a bundle, we initially assign it to one agent temporarily, but keep the option to reassign it later, after the reduction phase is complete. This deferred matching approach allows us to make more informed allocation decisions, based on agents’ valuations over the remaining goods. We elaborate on this in Section 3.3.
- **Calibration functions:** A key conceptual innovation in our approach is the introduction of *calibration functions*. These functions streamline analysis and enable a more precise approach that improves the approximation guarantee. Unlike prior work, which often treated reductions as a black box—focusing only on the maximin share values after reduction—our method preserves additional structural details through these calibration functions. They capture how good values change during a reduction, allowing us to analyze complex cases with greater accuracy that would otherwise be challenging to handle. More detail is given in Section 3.5.
- **Bundle initialization:** Another key difference from the approach of Akrami and Garg [AG24] lies in how we initialize the bundles in the Bag-filling process. This modification leads to a stronger approximation guarantee in both cases we consider, especially in the second case. More detail is given in Section 3.4.

Below, we discuss our techniques in more detail and highlight how they compare to previous approaches. We emphasize that, for clarity and ease of presentation, the notation and arguments in this section have been simplified and are not fully rigorous.

3.3 Reductions

As discussed earlier, a reduction simplifies the problem by allocating large goods. Previous studies have introduced several useful types of reductions, which here we denote by R^0 to R^3 .

Let us first review these reductions. Consider an ordered instance $\hat{\mathcal{I}} = (\hat{N}, \hat{M})$ where $\hat{M} = \langle \hat{g}_1, \hat{g}_2, \dots, \hat{g}_{\hat{m}} \rangle$. Reduction R^0 checks whether $\hat{g}_1$ is valued at least α by some agent. If so, we allocate $\hat{g}_1$ to that agent and recursively solve the problem for the remaining agents and goods. As

mentioned earlier, it has been shown that allocating in this manner does not reduce the maximin share values of the remaining agents for the remaining goods. Consequently, any approximation guarantee achieved for the new instance also applies to the original instance.

Reductions R^1 , R^2 , and R^3 follow a similar approach for specific subsets of goods. R^1 considers subset $\{\hat{g}_{\hat{n}}, \hat{g}_{\hat{n}+1}\}$, R^2 considers $\{\hat{g}_{2\hat{n}-1}, \hat{g}_{2\hat{n}}, \hat{g}_{2\hat{n}+1}\}$, and R^3 considers $\{\hat{g}_{3\hat{n}-2}, \hat{g}_{3\hat{n}-1}, \hat{g}_{3\hat{n}}, \hat{g}_{3\hat{n}+1}\}$. Each rule checks whether these goods together have value at least α for some agent and, if so, allocates them accordingly and solves the problem recursively for the remaining goods and agents.

There are also two special reductions $\tilde{R}^1$ and $\tilde{R}^2$ introduced respectively by Garg and Taki [GT20], and Akrami and Garg [AG24]. These reductions check whether bundles $\{\hat{g}_1, \hat{g}_{2\hat{n}+1}\}$ and $\{\hat{g}_1, \hat{g}_2\}$, respectively, are worth more than α to some agent. If so, the bundle is allocated to that agent, and the agent is removed from the instance. What sets these reductions apart from the previous ones (which is why they are denoted by tilde) is that they may slightly decrease the maximin share value for some agents. However, they show that this decrease is limited and remains within a tolerable range.

Our reductions. In this paper, we introduce a new reduction and refine the existing ones— $R^0, R^1, R^2, R^3, \tilde{R}^1$, and $\tilde{R}^2$. We refer to our set of reductions as $\mathcal{R}^0, \mathcal{R}^1, \mathcal{R}^2, \mathcal{R}^3, \mathcal{R}^4, \tilde{\mathcal{R}}^1$, and $\tilde{\mathcal{R}}^2$. The modification we make to obtain $\mathcal{R}^0, \mathcal{R}^1, \mathcal{R}^2, \mathcal{R}^3, \tilde{\mathcal{R}}^1$, and $\tilde{\mathcal{R}}^2$ is simple yet effective: rather than fixing in advance which goods are used in each reduction, we determine the rightmost index dynamically, based on the instance: we allow its index to be shifted as far to the right as possible. For example, rule $\mathcal{R}^1$ identifies the largest index $x > \hat{n}$ (if it exists) such that set $\{\hat{g}_{\hat{n}}, \hat{g}_x\}$ has value α for some agent. Or reduction $\tilde{\mathcal{R}}^2$ identifies the largest index x for which the bundle $\{\hat{g}_1, \hat{g}_x\}$ is worth at least α to some agent. If such an index exists, the instance is updated accordingly. Note that $\tilde{\mathcal{R}}^1, \tilde{\mathcal{R}}^2$ may slightly decrease the maximin share value for the remaining agents.

Also, reduction $\mathcal{R}^4$ follows the same pattern as the previous rules. It identifies the largest index $x > 4\hat{n}$ (if it exists) such that set $\{\hat{g}_{4\hat{n}-3}, \hat{g}_{4\hat{n}-2}, \hat{g}_{4\hat{n}-1}, \hat{g}_{4\hat{n}}, \hat{g}_x\}$ values at least α to some agent.

Shifting the last index to the right as far as possible serves two main purposes. For reductions $\mathcal{R}^0$ to $\mathcal{R}^4$, this modification helps establish a tighter bound on the value of the allocated bundle during the reduction process. For $\tilde{\mathcal{R}}^1$ and $\tilde{\mathcal{R}}^2$, this adjustment reveals a key structural property: If applying one of these reductions causes an agent's maximin share value to drop below 1, it must be because the agent values the selected bundle significantly—that is, $\hat{v}_i(\{\hat{g}_1\}) + \hat{v}_i(\{\hat{g}_x\}) > 1$. At the same time, since $\hat{g}_x$ is the rightmost good that could be included in the reduction, replacing it with $\hat{g}_{x+1}$ would not satisfy the reduction condition; thus, we also have $\hat{v}_i(\{\hat{g}_1\}) + \hat{v}_i(\{\hat{g}_{x+1}\}) < \alpha$. This implies a value gap of at least $1 - \alpha$ between $\hat{g}_x$ and $\hat{g}_{x+1}$ from the agent's perspective.

This value gap leads to an important consequence: If applying $\tilde{\mathcal{R}}^1$ causes an agent's maximin share value to drop below 1, and considering that $\hat{v}_i(\{\hat{g}_1\}) < \alpha$ (since $\mathcal{R}^0$ is not applicable) and $\hat{v}_i(\{\hat{g}_x\}) < \alpha/3$ (since $\mathcal{R}^2$ is not applicable), we can deduce: $\hat{v}(\{\hat{g}_x\}) > 1 - \alpha$ and $\hat{v}(\{\hat{g}_{x+1}\}) < \frac{4\alpha}{3} - 1$. Therefore, the agent has no good valued in the interval $[\frac{4\alpha}{3} - 1, 1 - \alpha]$. A similar argument holds for reduction $\tilde{\mathcal{R}}^2$. These insights plays critical role in our analysis.

Deferred matching. In the primary reductions, we apply the reductions with priority $\mathcal{R}^0 \succ \mathcal{R}^1 \succ \mathcal{R}^2 \succ \tilde{\mathcal{R}}^1$ until none is applicable. The obtained irreducible instance has $\hat{n}$ agents. We color the agents in N (the original set of agents before reductions) into two categories: green or red. An agent is green if their value for $\hat{g}_{2\hat{n}+1}$ (the good ranked $2\hat{n} + 1$ in the reduced instance) is at least $1 - \alpha$; otherwise, they are red. We then prioritize either green or red agents based on their sizes: if the number of green agents in N is at least $\frac{n}{\sqrt{2}}$, we prioritize red agents; otherwise, we prioritize green agents. This prioritization guides the Bag-filling process.

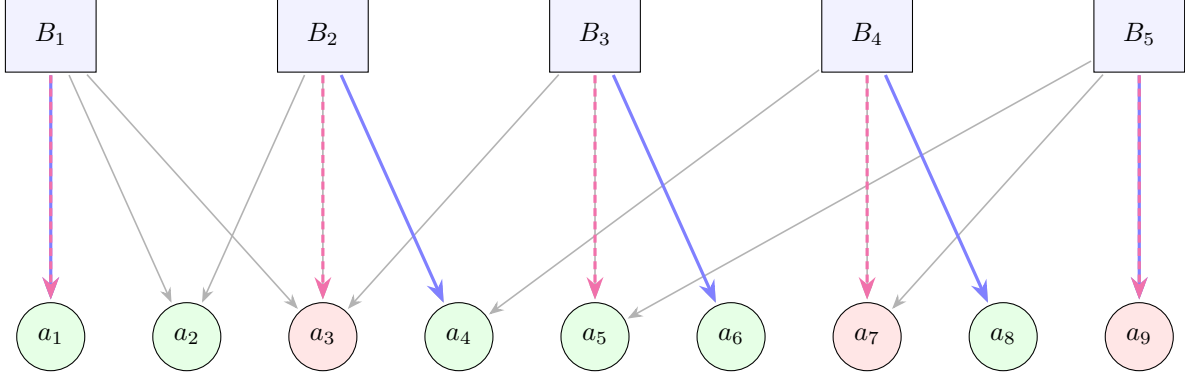


Figure 3: Bipartite graph demonstrating: (1) Reduction matching (solid blue) covering all bundles $\{B_i\}_{i=1}^5$, and (2) A matching (dashed magenta) prioritizing red over green agents. In both matchings, B_1 is paired with agent a_1 and B_5 is paired with agent a_9 .

The intuition behind deferred matching is as follows: To achieve a better approximation guarantee, the prioritization must also be considered during the reduction process itself. At first glance, this may seem paradoxical since prioritization depends on the outcome of reduction sequence. However, we show that it is indeed achievable. To address this, the allocations in the reduction sequence remain temporary—we may reallocate the bundle to another agent once the sequence of reductions is complete. Suppose we reach an irreducible instance after $n - \bar{n}$ reductions. At this point, we can determine whether each agent in N is green or red based on their value for $\dot{g}_{2\bar{n}+1}$. Now, we finalize the reduction by constructing a bipartite graph G as follows:

- For each bundle allocated during the reduction phase, we add a vertex to the first part of G .
- For each agent in N , we add a vertex to the second part of G .
- We draw an edge between the corresponding vertices of an agent and a bundle if the agent values that bundle at least α .

Note that, by construction, the reduction sequence corresponds to a matching that saturates all bundle vertices. On the other hand we show that, by selecting a specific sequence of reductions, called perfect sequence of reductions, each maximum matching in G corresponds to a valid reduction sequence. Among all possible maximum matchings, we select one that maximizes the number of agents from the prioritized color (either red or green). We prove that such a matching incorporates the prioritization into the reduction process. See Figure 3.

3.4 Bag-filling

As mentioned earlier, there are two key components we can adjust in a Bag-filling process: how we initialize the bundles and how we prioritize agents when multiple agents simultaneously shout “STOP.” Based on the number of green and red agents, we run two different versions of Bag-filling:

- **Case (i): The number of green agents is at least $\frac{n}{\sqrt{2}}$.** After applying the primary reductions, we obtain the instance $\dot{I} = (\dot{N}, \dot{M})$. We then apply the secondary reductions with priority $\mathcal{R}^1 \succ \mathcal{R}^2 \succ \mathcal{R}^3 \succ \mathcal{R}^4 \succ \tilde{\mathcal{R}}^2$ until we reach an irreducible instance $\tilde{I} = (\tilde{N}, \tilde{M})$. Next, we create $\bar{n}$ bags, where the k -th bag initially contains the goods $\{\dot{g}_k, \ddot{g}_{\bar{n}+k}, \ddot{g}_{3\bar{n}-k+1}\}$. We then perform a Bag-filling process, giving priority to red agents.

- **Case (ii): Otherwise.** We create $\hat{n}$ bags, where the k -th bag initially contains goods $\{\dot{g}_k, \dot{g}_{\hat{n}+k}\}$. We then perform a Bag-filling process, giving priority to green agents.

Note that some bags may already exceed the α threshold for some agents before any additional goods are added.

A key innovation of our algorithm lies in how we initialize the bags in both cases. This distinction is especially crucial in the second case: while previous studies including [AG24] pair goods as $\{g_k, g_{2\hat{n}+1-k}\}$, we pair them as $\{g_k, g_{\hat{n}+k}\}$. Our pairing ensures a consistent ordinal ranking of bundles for all agents—for instance, $\{g_1, g_{\hat{n}+1}\}$ is more valuable than $\{g_2, g_{\hat{n}+2}\}$ for every agent. Though subtle, this modification plays an important role in the guarantee of our algorithm.

3.5 Calibration Functions

To facilitate and advance our analysis, we introduce a set of functions, which we call calibration functions, designed to systematically modify agents valuations. These functions do not affect the actual allocation process but serve as theoretical tools to simplify our arguments and also handle more complex cases.

One key challenge in our framework is that reductions $\tilde{\mathcal{R}}^1$ and $\tilde{\mathcal{R}}^2$ may reduce an agent’s maximin share, making it harder to guarantee that they receive a sufficiently valuable bundle during the Bag-filling steps. Calibration functions help address this issue by carefully modifying valuations so that the agent’s maximin share, when computed under the calibrated valuation, does not decrease. Moreover, these functions are designed to ensure that the maximin share value under the calibrated valuation stays sufficiently close to its original counterpart. This allows us to establish a meaningful lower bound on the value each agent receives in the final allocation.

These functions play an important role in providing more in-depth analysis of what happens in reduction steps. Since calibration functions comprise a family of functions parameterized differently, we can define a suitable calibrated value for each agent. This allows us to perform more accurate analysis suitable for an agents valuations. Moreover, because the definition of calibration functions is deterministic, we always have access to both the original and the calibrated values, allowing us to employ both simultaneously for a more refined analysis.

For a formal definition and detailed properties of these functions, see Section 5. Also, see Figure 4 and Figure 5 for a visual representation of these functions.

3.6 Organization of the Paper

The remainder of the paper is organized as follows. In Section 4, we describe our reduction steps. Section 5 introduces the calibration functions, along with key properties that are used throughout the paper. The formal proofs of the MMS bounds for calibrated valuations are deferred to the appendix. Section 6 outlines the overall structure of our algorithm, including the initial reductions and introduces the two main cases. The first case is presented in Section 7, while Section 8 addresses the second case.

To keep the presentation focused and accessible, we defer full proofs of Lemmas 8, 9, 10, 11, 14, and 15 to the appendix, and include only brief proof sketches in the main text. Furthermore, to aid intuition and readability, we include several supporting tables, figures, and examples. Table 3 summarizes which lemmas establish MMS guarantees for different subsets of agents. Detailed examples illustrating key steps of the algorithm and its analysis can be found in Appendix B. We also provide a table of notations in Appendix A, listing essential definitions used throughout the algorithm.

4 Reductions

Similar to most previous studies on maximin share, especially in the additive setting, our algorithm begins with a series of reductions. However, in the previous studies on maximin share in the additive setting, the reduction process was often treated as an unstructured procedure, where a set of reduction rules were applied sequentially with some priority over the reductions to the input until no further reduction is possible. In contrast, our approach introduces a more clever way to apply these reductions. Hence, here we need a more rigorous definition of a reduction.

In this section, we consider a fixed ordered instance $\hat{\mathcal{I}} = (\hat{N}, \hat{M})$ where the set of agents is $\hat{N} = \{\hat{a}_1, \hat{a}_2, \dots, \hat{a}_{\hat{n}}\}$ and the set of goods is $\hat{M} = \{\hat{g}_1, \hat{g}_2, \dots, \hat{g}_{\hat{m}}\}$. For each agent $\hat{a}_i$, let $\hat{v}_i$ denote the valuation function of $\hat{a}_i$. In what follows, we will provide our definitions based on this setup.

Reduction patterns are typically defined as subsets of indices in the sorted sequence of goods. Previous studies have introduced several fixed reduction patterns, such as:

$$\begin{aligned} &\{1\}, \\ &\{\hat{n}, \hat{n} + 1\}, \\ &\{2\hat{n} - 1, 2\hat{n}, 2\hat{n} + 1\}, \\ &\{3\hat{n} - 2, 3\hat{n} - 1, 3\hat{n}, 3\hat{n} + 1\}, \\ &\{4\hat{n} - 3, 4\hat{n} - 2, 4\hat{n} - 1, 4\hat{n}, 4\hat{n} + 1\}. \end{aligned}$$

For instance, the reduction pattern $\{\hat{n}, \hat{n} + 1\}$ refers to the goods at positions $\hat{n}$ and $\hat{n} + 1$ in the sorted list. As seen above, these patterns are fully determined by the number of agents. In contrast, our approach introduces a more flexible notion of reduction patterns by allowing the last index to extend further to the right, depending on the values of the goods. Specifically, we define a reduction pattern as $\mathcal{R}^k$ where it has **static part**

$$S(\mathcal{R}^k) = \{k(\hat{n} - 1) + 1, k(\hat{n} - 1) + 2, \dots, k\hat{n}\}$$

and **dynamic index** $x \geq k\hat{n} + 1$, which is the largest index satisfying the condition that some agent values the set of goods $\{\hat{g}_{k(\hat{n}-1)+1}, \hat{g}_{k(\hat{n}-1)+2}, \dots, \hat{g}_{k\hat{n}}, \hat{g}_x\}$ at least α . Note that, it might be the case that no such index exists. In that case, we say that $\mathcal{R}^k$ is **not applicable**.

In this paper, we introduce two additional reduction patterns, denoted by $\tilde{\mathcal{R}}^1$ and $\tilde{\mathcal{R}}^2$. These reductions extend the following fixed reduction patterns originally proposed by Garg and Taki [GT20], and Akrami and Garg [AG24]: $\{1, 2\hat{n} + 1\}$, and $\{1, 2\}$.

As with earlier reduction patterns, we allow the last index in each set to shift dynamically based on the values of the goods. Specifically, we define $\tilde{\mathcal{R}}^1, \tilde{\mathcal{R}}^2$ as follows:

- For $\tilde{\mathcal{R}}^1$, we set **static part** $S(\tilde{\mathcal{R}}^1) = \{1\}$, and **dynamic index** $x \geq 2\hat{n} + 1$, which is the largest index such that some agent values the set $\{\hat{g}_1, \hat{g}_x\}$ at least α .
- For $\tilde{\mathcal{R}}^2$, we set **static part** $S(\tilde{\mathcal{R}}^2) = \{1\}$, and **dynamic index** $x \geq 2$, which is the largest index such that some agent values the set $\{\hat{g}_1, \hat{g}_x\}$ at least α .

Note that each reduction comes with a lower bound on its dynamic index. These reductions play a central role in improving the approximation guarantee. The notion of *inapplicability* naturally extends to $\tilde{\mathcal{R}}^1$ and $\tilde{\mathcal{R}}^2$ as well.

Now, we define a reduction in Definition 2.

Definition 2. A reduction is denoted by $\rho = (\hat{\mathcal{I}}, \mathcal{R}, x, \hat{a}_i, \tilde{\mathcal{I}})$, where $\mathcal{R} \in \{\mathcal{R}^0, \mathcal{R}^1, \dots, \mathcal{R}^4, \tilde{\mathcal{R}}^1, \tilde{\mathcal{R}}^2\}$ is a reduction pattern and x is dynamic index of $\mathcal{R}$ and $\hat{a}_i \in \hat{N}$ is the agent to whom the reduction is applied. The resulting instance $\tilde{\mathcal{I}} = (\tilde{N}, \tilde{M})$ is defined as: $\tilde{N} = \hat{N} \setminus \{\hat{a}_i\}$, and $\tilde{M} = \hat{M} \setminus \{\hat{g}_k \mid k \in S(\mathcal{R}) \cup \{x\}\}$. Reduction ρ is **valid** if the following conditions hold:

- x satisfies the lower bound determined for the dynamic index of $\mathcal{R}$,
- $\hat{v}_i(\{\hat{g}_k \mid k \in S(\mathcal{R}) \cup \{x\}\}) \geq \alpha$,
- $x = \hat{m}$ or there is no agent $\hat{a}_j \in \hat{N}$ such that $\hat{v}_j(\{\hat{g}_k \mid k \in S(\mathcal{R}) \cup \{x+1\}\}) \geq \alpha$.

Observation 1 follow from standard arguments used in previous studies [AG24, AGST23, GMT19, GT20], which show that $\mathcal{R}^0, \mathcal{R}^1, \mathcal{R}^2, \mathcal{R}^3$ and $\mathcal{R}^4$ do not decrease the maximin share value of the remaining agents over the remaining goods.

Observation 1. Let $\rho = (\hat{\mathcal{I}}, \mathcal{R}, x, \hat{a}_i, \tilde{\mathcal{I}})$ be a valid reduction, such that $\mathcal{R} \in \{\mathcal{R}^0, \mathcal{R}^1, \dots, \mathcal{R}^4\}$. Then, for every agent $\hat{a}_j \in \hat{N}$, we have $\tilde{\Psi}_{\hat{v}_j} \geq \hat{\Psi}_{\hat{v}_j}$.

In contrast to rules $\mathcal{R}^0$ to $\mathcal{R}^4$, reductions $\tilde{\mathcal{R}}^1, \tilde{\mathcal{R}}^2$ may decrease an agent's maximin share value. However, under certain conditions, these reductions also preserve the agents' maximin share values. Observation 2 introduces one of these conditions.

Observation 2. Let $\hat{M}$ be a set of goods, let $\hat{v}$ be a valuation function on $\hat{M}$, and let $\hat{g}_x, \hat{g}_y$ be two distinct goods in $\hat{M}$ such that $\hat{v}(\{\hat{g}_x, \hat{g}_y\}) \leq \Psi_{\hat{v}}^d(\hat{M})$. Then $\Psi_{\hat{v}}^{d-1}(\hat{M} \setminus \{\hat{g}_x, \hat{g}_y\}) \geq \Psi_{\hat{v}}^d(\hat{M})$.

We define a total order $\succ$ over reduction patterns based on their static and dynamic indices as follows. We have $\mathcal{R}^0 \succ \mathcal{R}^1 \succ \mathcal{R}^2 \succ \mathcal{R}^3 \succ \mathcal{R}^4 \succ \tilde{\mathcal{R}}^1 \succ \tilde{\mathcal{R}}^2$, meaning that $\mathcal{R}^0$ has the highest priority and $\tilde{\mathcal{R}}^2$ the lowest.

4.1 Reduction Sequence

Typically, a reduction is viewed as an independent process, where the order and choice of reductions do not matter—only that the instance eventually becomes irreducible, meaning no further reductions can be applied. However, in this paper, we take a different approach by carefully considering the sequence of reductions in our analysis. Among the various ways to reduce the problem to an irreducible instance, we select a specific sequence that follows a structured pattern. This allows us to improve the performance of our algorithm.

Definition 3. Let $\hat{\mathcal{I}} = (\hat{N}, \hat{M})$ be an ordered instance, and let $\mathfrak{R} \subseteq \{\mathcal{R}^0, \mathcal{R}^1, \dots, \mathcal{R}^4, \tilde{\mathcal{R}}^1, \tilde{\mathcal{R}}^2\}$. A sequence of valid reductions $\boldsymbol{\rho} = (\rho_1, \dots, \rho_r)$ on $\hat{\mathcal{I}}$, where each reduction uses a pattern from $\mathfrak{R}$, is called a **perfect sequence of reductions** (with respect to $\mathfrak{R}$), if the corresponding sequence $\langle \mathcal{R}_1, x_1, \mathcal{R}_2, x_2, \dots, \mathcal{R}_r, x_r \rangle$ is lexicographically maximum (over all such sequence of valid reductions).[†] Here, $\mathcal{R}_\ell$ denotes the reduction pattern of ρ_ℓ , and x_ℓ denotes the dynamic index of ρ_ℓ .

Observation 3. Let $\hat{\mathcal{I}} = (\hat{N}, \hat{M})$ be an ordered instance, and let $\mathfrak{R} \subseteq \{\mathcal{R}^0, \mathcal{R}^1, \dots, \mathcal{R}^4, \tilde{\mathcal{R}}^1, \tilde{\mathcal{R}}^2\}$. Suppose $\boldsymbol{\rho} = (\rho_1, \dots, \rho_r)$ is a perfect sequence of reductions (with respect to $\mathfrak{R}$). Then the following hold:

1. For every $1 \leq \ell \leq r$, there does not exist a reduction ρ' such that $(\rho_1, \rho_2, \dots, \rho_{\ell-1}, \rho')$ is a sequence of valid reductions on $\hat{\mathcal{I}}$ and the reduction type of ρ' has strictly higher priority than that of ρ_ℓ .

[†]Here lexicographically maximum means comparing each reduction pattern $\mathcal{R}_\ell$ according to the order $\succ$, and comparing each index x_ℓ by its numerical value.

2. If $\boldsymbol{\rho}' = (\rho'_1, \dots, \rho'_{r'})$ is another perfect sequence of reductions (with respect to $\mathfrak{R}$), then $r = r'$, and for every $1 \leq \ell \leq r$, the reduction ρ_ℓ allocates exactly the same bundle as ρ'_ℓ .

Proof. By Definition 3, the sequence $\boldsymbol{\rho}$ corresponds to a sequence $\langle \mathcal{R}_1, x_1, \mathcal{R}_2, x_2, \dots, \mathcal{R}_r, x_r \rangle$, that is lexicographically maximum among all sequences of valid reductions.

For the first part, if a reduction ρ' as described existed, then replacing ρ_ℓ with ρ' would yield a sequence whose corresponding sequence is lexicographically larger, contradicting maximality.

For the second part, suppose $\boldsymbol{\rho}'$ is another perfect sequence. Since both sequences correspond to lexicographically maximum sequences, their lengths must coincide ($r = r'$). Moreover, by the uniqueness of the lexicographically maximum sequence, the tuples coincide entry by entry. Hence, for every position ℓ , the reduction types and dynamic indices of ρ_ℓ and ρ'_ℓ must be the same, and therefore ρ_ℓ and ρ'_ℓ allocate the same bundle. $\square$

Intuitively, after applying a sequence of reductions, we obtain a reduced instance of the problem. At this point, we analyze how the reductions have affected the agents' values for the good in rank $2\hat{n} + 1$ and classify them into two groups. We then choose between two algorithms based on the sizes of these two groups. Each of these algorithms prioritizes one of the groups. The main challenge, however, is that for our approximation guarantee to hold, this prioritization must be considered not only in the second stage but also during the reduction process itself. In other words, when multiple reduction choices are available, we must select agents based on this prioritization. At first glance, this may seem paradoxical, as the prioritization depends on the reduction sequence. Surprisingly, we demonstrate that a carefully designed selection strategy allows us to achieve this goal. This idea is built upon Lemma 1, which we state below and prove in this section.

Lemma 1. Let $\hat{\mathcal{I}} = (\hat{N}, \hat{M})$ be an ordered instance and $\mathfrak{R} \subseteq \{\mathcal{R}^0, \mathcal{R}^1, \dots, \mathcal{R}^4, \tilde{\mathcal{R}}^1, \tilde{\mathcal{R}}^2\}$. Let $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_r)$ be a perfect sequence of reductions (with respect to $\mathfrak{R}$) on $\hat{\mathcal{I}}$. Additionally, let $\hat{N}^1$ and $\hat{N}^2$ be a partition of $\hat{N}$ into two subsets. Then, there exists another perfect sequence of reductions (with respect to $\mathfrak{R}$), $\boldsymbol{\rho}' = (\rho'_1, \rho'_2, \dots, \rho'_r)$, such that, after applying $\boldsymbol{\rho}'$, for any $\hat{a}_x \in \hat{N}^1$ who has not received a bundle and any $\hat{a}_y \in \hat{N}^2$ who has, we have: $\hat{v}_x(B^y) < \alpha$, where B^y is the bundle allocated to agent $\hat{a}_y$ during the reduction process.

Proof. For each $1 \leq j \leq r$, let $\mathcal{R}_j$ be the reduction pattern, x_j the dynamic index, and B_j the bundle associated with reduction ρ_j . We construct a bipartite graph where the first set of nodes represents these bundles, and the second set of nodes represents the agents in $\hat{N}$. An edge exists between a bundle node B_j and an agent node $\hat{a}_i$ if $\hat{v}_i(B_j) \geq \alpha$.

A perfect sequence of reductions $\boldsymbol{\rho}$ corresponds to a matching that saturate all nodes in the first part of the graph. We now show that any matching that saturate all nodes in the first part of the graph, yields a perfect sequence of reductions. Let $\boldsymbol{\rho}' = (\rho'_1, \dots, \rho'_r)$ be the sequence obtained from this matching. By Definition 3, it is enough to show that each reduction ρ'_j is valid. The remaining conditions for a perfect sequence depend only on the reduction patterns and dynamic indices, and not on the specific agents involved.

Note that for each ℓ , the reduction pattern and the dynamic index of ρ' remain the same as those of ρ , since we only reallocate bundles while keeping the goods in the bundles unchanged. Suppose, for contradiction, that some reduction in the sequence is not valid. Let ρ'_ℓ be the first such reduction (i.e., the one with the smallest index ℓ).

The invalidity of ρ'_ℓ implies that there exists another valid reduction ρ'' with the same reduction pattern as ρ'_ℓ but with a strictly larger dynamic index $x'' > x_\ell$, applied to the instance obtained after executing $\rho'_1, \dots, \rho'_{\ell-1}$. Since ρ'_ℓ is the first invalid reduction in the sequence, all reductions in the updated sequence $(\rho'_1, \dots, \rho'_{\ell-1}, \rho'')$ are valid. However the corresponding sequence

$\langle \mathcal{R}_1, x_1, \mathcal{R}_2, x_2, \dots, \mathcal{R}_{\ell-1}, x_{\ell-1}, \mathcal{R}_\ell, x'' \rangle$ is lexicographically larger than $\langle \mathcal{R}_1, x_1, \mathcal{R}_2, x_2, \dots, \mathcal{R}_r, x_r \rangle$ contradicting the definition of a perfect sequence.

Among all possible matchings that saturate all nodes in the first part of the graph, we choose one that maximizes the number of matched agents in $\hat{N}^1$. We argue that this matching satisfies the desired properties of Lemma 1.

Suppose for contradiction that there exists an agent $\hat{a}_x \in \hat{N}^1$ who has not received a bundle while there exists $\hat{a}_y \in \hat{N}^2$ who has received one, with $\hat{v}_x(B^y) \geq \alpha$. Replacing $\hat{a}_y$ by $\hat{a}_x$ yields another matching that saturate all nodes in the first part of the graph, covering more agents in $\hat{N}^1$, contradicting our construction. $\square$

5 Calibration

To simplify the analysis, we define *calibration functions* for certain agents. These functions make slight adjustments to the agents' valuation functions, while ensuring that the change in their maximin share remains small and within a known bound. These modifications are purely analytical and do not affect the actual execution of the algorithm.

Definition 4. Let $\hat{v}$ be a valuation over goods $\hat{M}$ such that for all $\hat{g} \in \hat{M}$, we have $\hat{v}(\{\hat{g}\}) \leq 1$, and let $\hat{f} : [0, 1] \rightarrow [0, 1]$ be a non-decreasing function with $\hat{f}(x) \leq x$ for all x . The **calibration** of $\hat{v}$ by $\hat{f}$, denoted by $(\hat{f} \star \hat{v})$, is the **additive** valuation defined by $(\hat{f} \star \hat{v})(\{\hat{g}\}) = \hat{f}(\hat{v}(\{\hat{g}\}))$ for all $\hat{g} \in \hat{M}$.

Throughout our analysis, different agents may use different calibration functions. Here we define the specific calibration functions used in this paper.

Definition 5. Define the functions:

$$\begin{aligned}
 (\text{For } 0 \leq \lambda \leq \frac{4\alpha}{3} - 1) \quad f_\lambda(x) &= \begin{cases} x, & x \in [0, \frac{\alpha}{3} - \lambda) \\ \max(\frac{\alpha}{3} - \lambda, x - \lambda), & x \in [\frac{\alpha}{3} - \lambda, 1 - \frac{2\alpha}{3}) \\ \max(1 - \frac{2\alpha}{3} - \lambda, x - \frac{3\lambda}{2}), & x \in [1 - \frac{2\alpha}{3}, 1 - \frac{\alpha}{3} - \frac{\lambda}{2}) \\ \max(1 - \frac{\alpha}{3} - 2\lambda, x - 3\lambda), & x \in [1 - \frac{\alpha}{3} - \frac{\lambda}{2}, 1] \end{cases} \\
 h(x) &= \begin{cases} x, & x \in [0, 2 - \frac{7\alpha}{3}) \\ \max(2 - \frac{7\alpha}{3}, x - \frac{4\alpha}{3} + 1), & x \in [2 - \frac{7\alpha}{3}, 2 - \frac{13\alpha}{6}) \\ \max(3 - \frac{7\alpha}{2}, x - \frac{8\alpha}{3} + 2), & x \in [2 - \frac{13\alpha}{6}, 1] \end{cases} \\
 (\text{For } 0 \leq \lambda \leq \frac{1}{2}) \quad w_\lambda(x) &= \begin{cases} x, & x \in [0, \frac{1}{2} - \lambda) \\ \max(\frac{1}{2} - \lambda, x - 2\lambda), & x \in [\frac{1}{2} - \lambda, 1] \end{cases} \\
 (\text{For } 0 \leq \lambda \leq 2(1 - \alpha)) \quad z_\lambda(x) &= \begin{cases} x, & x \in [0, 2 - 2\alpha - \lambda) \\ \max(2 - 2\alpha - \lambda, x - 2\lambda), & x \in [2 - 2\alpha - \lambda, 1] \end{cases}
 \end{aligned}$$

For convenience, we denote $f_{\frac{4\alpha}{3}-1}$ by $\hat{f}$ throughout the remainder of the paper.

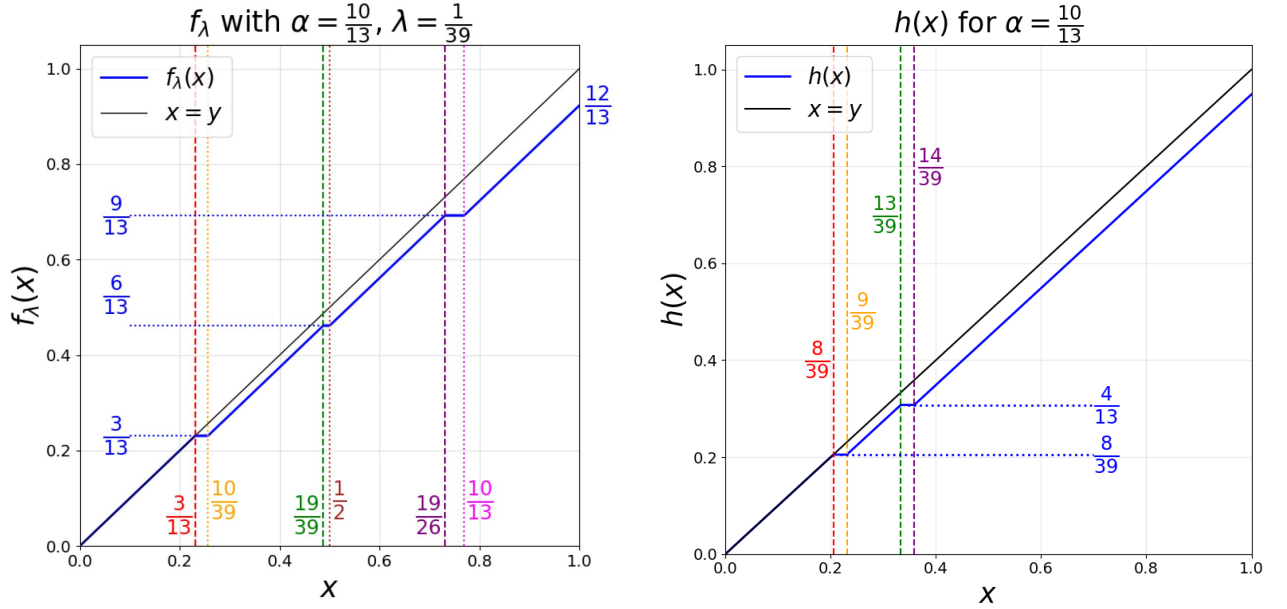


Figure 4: Plot of the calibration functions f_λ and h for $\alpha = \frac{10}{13}$.

Figures 4 and 5 illustrates the calibration functions. Note that all functions satisfy $\hat{f}(x) \leq x$ for every $x \in [0, 1]$. In Appendix C, we present lemmas establishing lower bounds on the maximin share values under these calibration functions. These bounds are summarized in Tables 1 and 2.

Lemma	Function	Precondition	MMS Bound
Lemma 17	f_λ	$\Psi_{\hat{v}}^d(\hat{M}) \geq 1$	$\Psi_{(f_\lambda \star \hat{v})}^{\hat{M}}(d) \geq 1 - 3\lambda$
Lemma 18	h	$\Psi_{\hat{v}}^d(\hat{M}) \geq 4(1 - \alpha)$	$\Psi_{(h \star \hat{v})}^d(\hat{M}) \geq 4(2 - \frac{7\alpha}{3})$
Lemma 19	w_λ	$\Psi_{\hat{v}}^d(\hat{M}) \geq 1$	$\Psi_{(w_\lambda \star \hat{v})}^{\hat{M}}(d) \geq 1 - 2\lambda$
Lemma 20	z_λ	$\Psi_{\hat{v}}^d(\hat{M}) \geq 4(1 - \alpha)$	$\Psi_{(z_\lambda \star \hat{v})}^d(\hat{M}) \geq 4(1 - \alpha) - 2\lambda$

Table 1: Bounds on the maximin share (MMS) under calibration functions. For any instance satisfying the given preconditions, the corresponding lemma establishes the stated lower bound.

Lemma 2. Let $\hat{\mathcal{I}} = (\hat{N}, \hat{M})$ be an ordered instance, and let $\mathfrak{R}_1 = [\mathcal{R}^0 \succ \mathcal{R}^1 \succ \mathcal{R}^2 \succ \tilde{\mathcal{R}}^1]$ and $\mathfrak{R}_2 = [\mathcal{R}^1 \succ \mathcal{R}^2 \succ \mathcal{R}^3 \succ \mathcal{R}^4 \succ \tilde{\mathcal{R}}^2]$. Assume that $\tilde{\mathcal{I}} = (\tilde{N}, \tilde{M})$ is the result of applying a sequence of valid reductions with respect to either $\mathfrak{R}_1$ or $\mathfrak{R}_2$. Then, the conditions shown in Table 2 satisfy.

Finally, we introduce a calibration function with a structure that differs from the earlier ones. This function will be used in Section 8 to simplify the analysis.

Definition 6. Let v be a valuation over $\hat{M}$ such that $\Psi_{\hat{v}}^d(\hat{M}) \geq \lambda$, and let $P = (P_1, \dots, P_d)$ be a maximin partition of $\hat{M}$ under v . Define the multiset

$$\mathcal{M} = \left\{ \frac{v(\{\hat{g}\}) \cdot \lambda}{v(P_j)} \mid 1 \leq j \leq d, \hat{g} \in P_j \right\}.$$

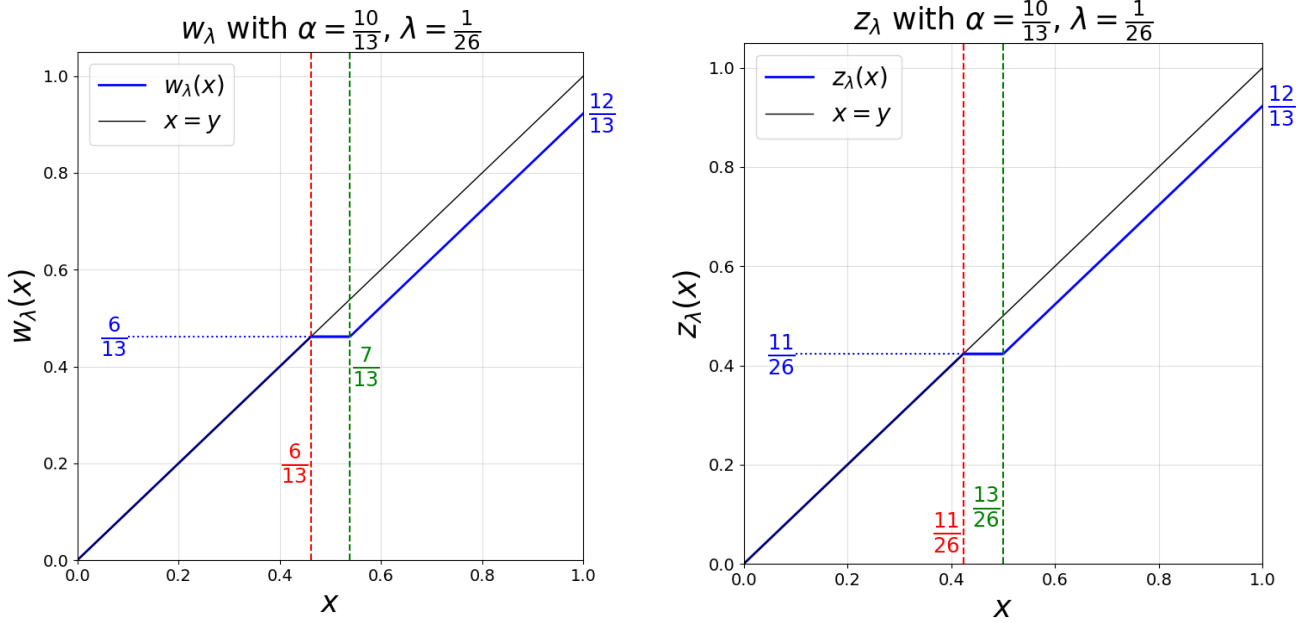


Figure 5: Plot of the calibration functions w_λ and z_λ for $\alpha = \frac{10}{13}$.

Func	Red	Prec 1	Prec 2	Prec 3	MMS Guarantee
f_λ	$\mathfrak{R}_1$	$\lambda \leq \frac{4\alpha}{3} - 1$	$\hat{\Psi}_{\hat{v}_i} \geq 1$	$\hat{v}_i(\{\hat{g}_1\}) \leq 1 - \frac{\alpha}{3} + \lambda$	$\check{\Psi}_{(f_\lambda \star \hat{v}_i)} \geq 1 - 3\lambda$
w_λ	$\mathfrak{R}_2$	$\lambda \leq \frac{1}{2}$	$\hat{\Psi}_{\hat{v}_i} \geq 1$	$\hat{v}_i(\{\hat{g}_1\}) \leq \frac{1}{2} + \lambda$	$\check{\Psi}_{(w_\lambda \star \hat{v}_i)} \geq 1 - 2\lambda$
z_λ	$\mathfrak{R}_2$	$\lambda \leq 2(1 - \alpha)$	$\hat{\Psi}_{\hat{v}_i} \geq 4(1 - \alpha)$	$\hat{v}_i(\{\hat{g}_1\}) \leq 2(1 - \alpha) + \lambda$	$\check{\Psi}_{(z_\lambda \star \hat{v}_i)} \geq 4(1 - \alpha) - 2\lambda$

Table 2: Calibrated MMS bounds under various reduction sequences. If an instance satisfies both preconditions, the stated guarantee holds for the calibrated MMS after applying the reductions.

Now define $\text{normalized}_\lambda^d(v, \hat{M})$ as the valuation obtained by assigning the values in $\mathcal{M}$ to the goods in $\hat{M}$, preserving their original order under v . That is, if $\hat{g}_i$ is the i -th highest-valued good under v , then value of $\{\hat{g}_i\}$ in $\text{normalized}_\lambda^d(v, \hat{M})$ is the i -th highest value in $\mathcal{M}$.

Observation 4. Let v be a valuation over $\hat{M}$ such that $\Psi_v^d(\hat{M}) \geq \lambda$. Then $\nu = \text{normalized}_\lambda^d(v, \hat{M})$ satisfies the following conditions:

1. There exists a partition $(P_1, \dots, P_d)$ of $\hat{M}$ such that $\nu(P_j) = \lambda$ for every $1 \leq j \leq d$.
2. $\nu(\{\hat{g}\}) \leq v(\{\hat{g}\})$ for every $\hat{g} \in \hat{M}$.
3. v and ν rank the goods in the same order.

6 A $(\frac{10}{13})$ -MMS Allocation Algorithm

Algorithm 1 provides the pseudocode for our allocation algorithm. By the scaling assumption, the maximin share of all the agents in the initial instance is equal to 1. After applying the reductions, we denote the resulting instance by $\dot{\mathcal{I}} = (\dot{N}, \dot{M})$, which is irreducible with respect to reduction rules $\mathcal{R}^0, \mathcal{R}^1, \mathcal{R}^2$ and $\tilde{\mathcal{R}}^1$. Along with the reductions, we also divide the agents into two subsets: green agents (N^g) and red agents (N^r). Depending on the size of these subsets, we consider two cases: whether $|N^g| \geq \frac{n}{\sqrt{2}}$ or not. For each case, we design a separate algorithm and prove that it guarantees an α -MMS allocation.

Algorithm 1 $(\frac{10}{13})$ -MMS Allocation

Input: $\mathcal{I} = (N, M)$

Output: Allocation satisfying $(\frac{10}{13})$ -MMS

- 1: $\dot{\mathcal{I}}, N^g, N^r \leftarrow \text{PRIMARY-REDUCTIONS}(\mathcal{I})$ ▷ See Algorithm 2
 - 2: **if** $|N^g| \geq \frac{n}{\sqrt{2}}$ **then**
 - 3: Run $\text{ALGORITHM-CASE1}(\dot{\mathcal{I}}, N^g, N^r)$ ▷ See Algorithm 3
 - 4: **else**
 - 5: Run $\text{ALGORITHM-CASE2}(\dot{\mathcal{I}}, N^g, N^r)$ ▷ See Algorithm 6
 - 6: **end if**
-

6.1 Primary Reductions

In the primary reductions, we first find a perfect sequence of reductions with respect to $\mathcal{R}^0 \succ \mathcal{R}^1 \succ \mathcal{R}^2 \succ \tilde{\mathcal{R}}^1$. Let $\dot{M}$ denote the set of goods obtained after applying all reductions, and let $\dot{N}$ be the resulting set of agents. We partition the agents in N into two subsets, N^g and N^r , as follows:

$$N^g = \{a_i \in N \mid v_i(\{\dot{g}_{2\dot{n}+1}\}) \geq 1 - \alpha\} \quad \text{and} \quad N^r = \{a_i \in N \mid v_i(\{\dot{g}_{2\dot{n}+1}\}) < 1 - \alpha\}.$$

Note that we partition agents in N , not $\dot{N}$. Using Lemma 1, we can modify the reduction sequence so that the primary reductions give priority to one of N^g or N^r , based on the size of N^g as follows: (i) If $|N^g| \geq \frac{n}{\sqrt{2}}$ we follow a perfect reduction sequence that prioritizes agents in N^r . (ii) If $|N^g| < \frac{n}{\sqrt{2}}$, we choose a perfect reduction sequence that prioritizes agents in N^g . Algorithm 2 presents a pseudo code of our algorithm for the primary reductions.

Recall that, after the reductions, in instance $\dot{\mathcal{I}}$ the agents' maximin share values may no longer be at least 1, since $\tilde{\mathcal{R}}^1$ can decrease these values. We prove Lemma 3 for the agents whose maximin share values decrease due to the primary reductions.

Lemma 3. *Let a_i be an agent in $\dot{N}$ whose $\dot{\Psi}_{v_i} < 1$ after the primary reductions. Then there exists a real number $s_i > 0$ such that the following conditions hold:*

$$v_i(\{\dot{g}_1\}) - (1 - \frac{\alpha}{3}) \leq s_i < \frac{4\alpha}{3} - 1, \tag{1}$$

$$\forall \dot{g} \in \dot{M}, \quad v_i(\{\dot{g}\}) \notin \left[\frac{4\alpha}{3} - 1 - s_i, \frac{\alpha}{3} - s_i \right], \tag{2}$$

$$\forall \lambda \in [s_i, \frac{4\alpha}{3} - 1], \quad \dot{\Psi}_{(f_\lambda \star v_i)} \geq 1 - 3\lambda. \tag{3}$$

Proof. Consider the first reduction $(\hat{\mathcal{I}}, \mathcal{R}, x, a_j, \check{\mathcal{I}})$ such that a_i 's maximin share drops below 1. That is, before the reduction, $\hat{\Psi}_{v_i} \geq 1$, but after the reduction, $\dot{\Psi}_{v_i} < 1$. By Observation 1, among

Algorithm 2 PRIMARY-REDUCTIONS

Input: $\mathcal{I} = (N, M)$
Output: $\dot{\mathcal{I}}, N^g, N^r$

- 1: $\mathfrak{R} \leftarrow [\mathcal{R}^0 \succ \mathcal{R}^1 \succ \mathcal{R}^2 \succ \tilde{\mathcal{R}}^1]$ ▷ Reduction order
 - 2: $\boldsymbol{\rho} \leftarrow$ Perfect sequence of reductions on $\mathcal{I}$ with respect to $\mathfrak{R}$ ▷ See Definition 3
 - 3: $\dot{M} \leftarrow$ Set of goods after applying $\boldsymbol{\rho}$ on $\mathcal{I}$
 - 4: $\dot{n} \leftarrow n - |\boldsymbol{\rho}|$ ▷ Number of remaining agents
 - 5: $N^g \leftarrow \{a_i \in N \mid v_i(\{\dot{g}_{2\dot{n}+1}\}) \geq 1 - \alpha\}$
 - 6: $N^r \leftarrow \{a_i \in N \mid v_i(\{\dot{g}_{2\dot{n}+1}\}) < 1 - \alpha\}$
 - 7: **if** $|N^g| \geq \frac{n}{\sqrt{2}}$ **then**
 - 8: $\boldsymbol{\rho}' \leftarrow$ Reallocate $\boldsymbol{\rho}$ with maximum number of red agents
 - 9: **else**
 - 10: $\boldsymbol{\rho}' \leftarrow$ Reallocate $\boldsymbol{\rho}$ with maximum number of green agents
 - 11: **end if** ▷ See Lemma 1
 - 12: $\dot{\mathcal{I}} \leftarrow$ Output of $\boldsymbol{\rho}'$ on $\mathcal{I}$.
 - 13: **return** $\dot{\mathcal{I}}, N^g, N^r$
-

all reduction patterns, only $\tilde{\mathcal{R}}^1$ can decrease the maximin share of an agent. So we assume this reduction allocated two goods, $\hat{g}_1$ and $\hat{g}_x$, and define $s_i = v_i(\{\hat{g}_1\}) - (1 - \frac{\alpha}{3})$.

We first show that $0 < s_i < \frac{4\alpha}{3} - 1$. The drop in the maximin share implies that $v_i(\{\hat{g}_1, \hat{g}_x\}) > 1$, by Observation 2. Meanwhile, because the reduction $\mathcal{R}^2$ does not apply here, we have $v_i(\{\hat{g}_x\}) < \frac{\alpha}{3}$. Putting these together, we conclude that $v_i(\{\hat{g}_1\}) > 1 - \frac{\alpha}{3}$, so $s_i > 0$. On the other hand, the fact that $\mathcal{R}^0$ does not apply means that $v_i(\{\hat{g}_1\}) < \alpha$, which gives $s_i < \frac{4\alpha}{3} - 1$. Therefore, s_i is positive and lies in the desired range. Since all goods in $\dot{M}$ are drawn from $\hat{M}$, we have

$$\begin{aligned} v_i(\{\dot{g}_1\}) &\leq v_i(\{\hat{g}_1\}) \\ &= (1 - \frac{\alpha}{3}) + s_i. \end{aligned}$$

This confirms that Inequality (1) holds.

To show Equation (2), we use two properties of $\tilde{\mathcal{R}}^1$: the pair $\hat{g}_1$ and $\hat{g}_x$ has total value above 1, while replacing $\hat{g}_x$ with $\hat{g}_{x+1}$ [‡] gives a total value below α . Plugging the expression for $v_i(\{\hat{g}_1\})$ into these inequalities, we get $v_i(\{\hat{g}_x\}) > \frac{\alpha}{3} - s_i$, and $v_i(\{\hat{g}_{x+1}\}) < \frac{4\alpha}{3} - 1 - s_i$. This implies that no good in $\hat{M}$ has a value (according to agent a_i) that falls within the interval

$$\left[\frac{4\alpha}{3} - 1 - s_i, \frac{\alpha}{3} - s_i \right].$$

Since $\dot{M} \subseteq \hat{M}$, the same holds for all goods in $\dot{M}$, implying Equation (2).

To prove Inequality (3), recall that $\hat{\Psi}_{v_i} \geq 1$. Then for any $s_i \leq \lambda \leq \frac{4\alpha}{3} - 1$, we have $v_i(\{\hat{g}_1\}) \leq (1 - \frac{\alpha}{3}) + \lambda$, therefore applying Lemma 2 implies:

$$\dot{\Psi}_{(f_\lambda \star v_i)} \geq 1 - 3\lambda.$$

□

[‡] In the boundary case $x = \hat{m}$, we define $\hat{g}_{x+1}$ to be an auxiliary good that is assigned value 0 by every agent.

7 Algorithm 3: Frequent Green Agents

In this section, we present our algorithm for the case that $|N^g| \geq \frac{n}{\sqrt{2}}$. The pseudocode for this case is given in Algorithm 3. This algorithm takes as input an instance that is irreducible with respect to $\mathcal{R}^0, \mathcal{R}^1, \mathcal{R}^2, \tilde{\mathcal{R}}^1$. First we run a set of further reductions on the input instance. Next, we run a Bag-filling on the set of remaining agents. Throughout this section, whenever we need to choose between multiple agents, we prioritize agents in N^r .

Algorithm 3 ALGORITHM-CASE1

Input: $\tilde{\mathcal{I}}, N^g, N^r$

▷ Assumption: $|N^g| \geq \frac{n}{\sqrt{2}}$

Output: Allocation satisfying $(\frac{10}{13})$ -MMS

1: $\tilde{\mathcal{I}} = \text{SECONDARY-REDUCTIONS}(\tilde{\mathcal{I}})$

▷ See Algorithm 4

2: Run BAG-FILLING1($\tilde{\mathcal{I}}, N^g, N^r$)

▷ See Algorithm 5

We organize this section in two parts. In Section 7.1, we present the additional reductions used by our algorithm. Then, in Section 7.2, we describe the Bag-filling process and prove the approximation guarantee of the resulting allocation.

7.1 Secondary Reductions

In this case, we further apply a sequence of the secondary reductions, following the priority order

$$\mathcal{R}^1 \succ \mathcal{R}^2 \succ \mathcal{R}^3 \succ \mathcal{R}^4 \succ \tilde{\mathcal{R}}^2.$$

The pseudocode for this step is provided in Algorithm 4. We denote by $\tilde{\mathcal{I}} = (\ddot{N}, \ddot{M})$ the instance obtained after applying these reductions.

Algorithm 4 SECONDARY-REDUCTIONS

Input: $\tilde{\mathcal{I}} = (\ddot{N}, \ddot{M})$

Output: $\tilde{\mathcal{I}} = (\ddot{N}, \ddot{M})$

1: $\mathfrak{R} \leftarrow [\mathcal{R}^1 \succ \mathcal{R}^2 \succ \mathcal{R}^3 \succ \mathcal{R}^4 \succ \tilde{\mathcal{R}}^2]$

▷ Reduction order

2: $\tilde{\mathcal{I}} \leftarrow \tilde{\mathcal{I}}$

3: **while** there exists a valid reduction from $\mathfrak{R}$ on $\tilde{\mathcal{I}}$ **do**

4: $\mathcal{R} \leftarrow$ the highest-priority valid reduction from $\mathfrak{R}$ on $\tilde{\mathcal{I}}$

5: Apply a valid reduction $\rho = (\tilde{\mathcal{I}}, \mathcal{R}, x, a_i, \tilde{\mathcal{I}}')$ on $\tilde{\mathcal{I}}$

▷ Priority is given to agents in N^r

6: **end while**

7: **return** $\tilde{\mathcal{I}} = (\ddot{N}, \ddot{M})$

We now establish some useful bounds. Specifically, in Observation 5, we show that the values of goods $\dot{g}_1, \dot{g}_2$, and $\dot{g}_{2\ddot{n}+1}$ are bounded above under the functions $v_i, (\dot{f} \star v_i)$, and $(h \star \dot{f} \star v_i)$ for green agents. These bounds are later used to prove our claims.

Observation 5. Let a_i be a green agent in $\ddot{N}$. Then, for goods $\dot{g}_1$, $\ddot{g}_2$, and $\ddot{g}_{2\ddot{n}+1}$, the following upper bounds hold:

Goods	$v_i(\cdot)$	$(\mathring{f} \star v_i)(\cdot)$	$(h \star \mathring{f} \star v_i)(\cdot)$
$\{\dot{g}_1\}$	$< 2\alpha - 1$	$\leq \frac{1}{2}$	$\leq \frac{5}{2} - \frac{8\alpha}{3}$
$\{\ddot{g}_2\}$	$< \frac{\alpha}{2}$	$\leq 1 - \frac{5\alpha}{6}$	$\leq 3 - \frac{7\alpha}{2}$
$\{\ddot{g}_{2\ddot{n}+1}\}$	$< \frac{\alpha}{3}$	$\leq 1 - \alpha$	$\leq 2 - \frac{7\alpha}{3}$

Proof. For $\dot{g}_1$, since a_i is a green agent, she values $\dot{g}_{2\ddot{n}+1}$ at least $1 - \alpha$. Moreover, since $\tilde{\mathcal{R}}^1$ is not applicable, $v_i(\{\dot{g}_1, \dot{g}_{2\ddot{n}+1}\}) \leq \alpha$. Together, these imply the first inequality. Noting that $\frac{3}{4} < \alpha < \frac{5}{6}$, we have

$$\begin{aligned}
 (\mathring{f} \star v_i)(\{\dot{g}_1\}) &\leq \mathring{f}(2\alpha - 1) && \mathring{f} \text{ is non-decreasing,} \\
 &= \max\left(2(1 - \alpha), \frac{1}{2}\right) && 2\alpha - 1 \in [1 - \frac{2\alpha}{3}, 1 - \frac{\alpha}{3} - (\frac{2\alpha}{3} - \frac{1}{2})], \\
 &= \frac{1}{2}.
 \end{aligned}$$

By the definition of h and noting that $\alpha \geq \frac{9}{13}$ we have

$$\begin{aligned}
 (h \star \mathring{f} \star v_i)(\{\dot{g}_1\}) &\leq h\left(\frac{1}{2}\right) && h \text{ is non-decreasing,} \\
 &= \max\left(3 - \frac{7\alpha}{2}, \frac{5}{2} - \frac{8\alpha}{3}\right) && \frac{1}{2} \in [2 - \frac{13\alpha}{6}, 1], \\
 &= \frac{5}{2} - \frac{8\alpha}{3}.
 \end{aligned}$$

For $\ddot{g}_2$, since reduction $\tilde{\mathcal{R}}^2$ is not applicable we have $v_i(\{\ddot{g}_2\}) < \frac{\alpha}{2}$. Since $\frac{2}{3} < \alpha < \frac{6}{7}$, we have

$$\begin{aligned}
 (\mathring{f} \star v_i)(\{\ddot{g}_2\}) &\leq \mathring{f}\left(\frac{\alpha}{2}\right) && \mathring{f} \text{ is non-decreasing,} \\
 &= \max\left(1 - \alpha, 1 - \frac{5\alpha}{6}\right) && \frac{\alpha}{2} \in [\frac{\alpha}{3} - (\frac{4\alpha}{3} - 1), 1 - \frac{2\alpha}{3}], \\
 &= 1 - \frac{5\alpha}{6}.
 \end{aligned}$$

By the definition of h and noting that $\alpha \geq \frac{3}{4}$ we have

$$\begin{aligned}
 (h \star \mathring{f} \star v_i)(\{\ddot{g}_2\}) &\leq h\left(1 - \frac{5\alpha}{6}\right) && h \text{ is non-decreasing,} \\
 &= \max\left(3 - \frac{7\alpha}{2}, 1 - \frac{5\alpha}{6} - \frac{8\alpha}{3} + 2\right) && 1 - \frac{5\alpha}{6} \in [2 - \frac{13\alpha}{6}, 1], \\
 &= 3 - \frac{7\alpha}{2}.
 \end{aligned}$$

For $\ddot{g}_{2\ddot{n}+1}$, since reduction $\mathcal{R}^2$ is not applicable, we have $v_i(\{\ddot{g}_{2\ddot{n}+1}\}) < \frac{\alpha}{3}$. Since $\alpha > \frac{3}{4}$, we have

$$\begin{aligned} (f \star v_i)(\{\ddot{g}_{2\ddot{n}+1}\}) &\leq f\left(\frac{\alpha}{3}\right) && f \text{ is non-decreasing,} \\ &= \max\left(\frac{\alpha}{3} - \left(\frac{4\alpha}{3} - 1\right), \frac{\alpha}{3} - \left(\frac{4\alpha}{3} - 1\right)\right) && \frac{\alpha}{3} \in \left[\frac{\alpha}{3} - \left(\frac{4\alpha}{3} - 1\right), 1 - \frac{2\alpha}{3}\right), \\ &= 1 - \alpha. \end{aligned}$$

Finally, by the definition of h , and noting that $\frac{3}{4} \leq \alpha \leq \frac{6}{7}$, we have

$$\begin{aligned} (h \star f \star v_i)(\{\ddot{g}_{2\ddot{n}+1}\}) &\leq h(1 - \alpha) && h \text{ is non-decreasing,} \\ &= \max\left(2 - \frac{7\alpha}{3}, (1 - \alpha) - \frac{4\alpha}{3} + 1\right) && 1 - \alpha \in \left[2 - \frac{7\alpha}{3}, 2 - \frac{13\alpha}{6}\right), \\ &= 2 - \frac{7\alpha}{3}. \end{aligned}$$

□

Also, for $\ddot{g}_{3\ddot{n}+1}$, we establish a strong upper bound in Observation 6.

Observation 6. *Let $a_i \in \ddot{N}$ be an agent with $\dot{\Psi}_{v_i} < 1$ after the primary reductions. Then $v_i(\{\ddot{g}_{3\ddot{n}+1}\}) < \frac{4\alpha}{3} - 1$.*

Proof. By Lemma 3, for every $\ddot{g} \in \ddot{M}$ we have $v_i(\{\ddot{g}\}) \notin [\frac{4\alpha}{3} - 1 - s_i, \frac{\alpha}{3} - s_i]$. From Inequality (1), $0 < s_i < \frac{4\alpha}{3} - 1$, hence $[\frac{4\alpha}{3} - 1, 1 - \alpha] \subseteq [\frac{4\alpha}{3} - 1 - s_i, \frac{\alpha}{3} - s_i]$. Since $\mathcal{R}^3$ is not applicable, $v_i(\{\ddot{g}_{3\ddot{n}+1}\}) < \frac{\alpha}{4}$, and as $\frac{\alpha}{4} \in [\frac{4\alpha}{3} - 1, 1 - \alpha]$, the claim follows. □

As shown in Observation 1, applying $\mathcal{R}^1, \mathcal{R}^2, \mathcal{R}^3, \mathcal{R}^4$ does not reduce the MMS value of any agent. However, this is not necessarily true for $\tilde{\mathcal{R}}^2$. In Lemmas 4 and 5, we provide a set of bounds on the valuation of agents after the secondary reductions.

Lemma 4. *Let a_i be a green agent in $\ddot{N}$ such that $\dot{\Psi}_{v_i} \geq 1$ after the primary reductions, and $\ddot{\Psi}_{v_i} < 1$ after the secondary reductions. Then, there exists a positive real number t_i satisfying the following conditions:*

$$v_i(\{\ddot{g}_1\}) - \frac{1}{2} \leq t_i < 2\alpha - \frac{3}{2}, \quad (4)$$

$$\forall \ddot{g} \in \ddot{M}, \quad v_i(\{\ddot{g}\}) \notin \left[\alpha - \frac{1}{2} - t_i, \frac{1}{2} - t_i\right], \quad (5)$$

$$\ddot{\Psi}_{(w_{t_i} \star v_i)} \geq 1 - 2t_i. \quad (6)$$

Proof. Consider the first reduction $\rho = (\hat{\mathcal{I}}, \mathcal{R}, x, a_j, \hat{\mathcal{I}})$ such that a_i 's maximin share drops below 1, that is, before the reduction, $\hat{\Psi}_{v_i} \geq 1$, but after the reduction, $\ddot{\Psi}_{v_i} < 1$. By Observation 1, among all reduction patterns, only $\tilde{\mathcal{R}}^2$ can decrease the maximin share of an agent. So we assume this reduction allocated goods $\{\hat{g}_1, \hat{g}_x\}$, and define $t_i = v_i(\{\hat{g}_1\}) - \frac{1}{2}$. The drop in the maximin share after this reduction implies $v_i(\{\hat{g}_1, \hat{g}_x\}) > 1$, by Observation 2. Meanwhile, since $v_i(\{\hat{g}_1\}) \geq v_i(\{\hat{g}_x\})$, we have $v_i(\{\hat{g}_1\}) > \frac{1}{2}$, which means $t_i > 0$. By Observation 5, we have $v_i(\{\hat{g}_1\}) < 2\alpha - 1$, which gives $t_i < 2\alpha - \frac{3}{2}$. Therefore, t_i is positive and lies in the desired range. Since all goods in $\ddot{M}$ are drawn from $\hat{M}$, we have $v_i(\{\ddot{g}_1\}) \leq v_i(\{\hat{g}_1\})$ and hence $v_i(\{\ddot{g}_1\}) \leq \frac{1}{2} + t_i$. This proves Inequality (4).

To show Equation (5), note that by the construction of $\tilde{\mathcal{R}}^2$, we have

$$v_i(\{\hat{g}_1, \hat{g}_x\}) > 1 \quad \text{and} \quad v_i(\{\hat{g}_1, \hat{g}_{x+1}\}) < \alpha.^\dagger$$

Plugging the bound for $v_i(\{\hat{g}_1\})$ into these inequalities, we get

$$v_i(\{\hat{g}_x\}) > \frac{1}{2} - t_i \quad \text{and} \quad v_i(\{\hat{g}_{x+1}\}) < \alpha - \frac{1}{2} - t_i.$$

This implies that no good in $\hat{M}$ has a value to agent a_i that falls within the interval $[\alpha - \frac{1}{2} - t_i, \frac{1}{2} - t_i]$. Since $\ddot{M} \subseteq \hat{M}$, the same holds for all goods in $\ddot{M}$, implying Equation (5).

To prove Inequality (6), recall that $\hat{\Psi}_{v_i} \geq 1$. Since $v_i(\{\hat{g}_1\}) = \frac{1}{2} + t_i$, applying Lemma 2 implies $\ddot{\Psi}_{(w_{t_i} \star v_i)} \geq 1 - 2t_i$. $\square$

Lemma 5. *Let a_i be a green agent in $\ddot{N}$ such that $\dot{\Psi}_{v_i} < 1$ after the primary reductions, and $\ddot{\Psi}_{(f \star v_i)} < 4(1 - \alpha)$ after the secondary reductions. Then, there exists a positive real number t_i satisfying the following conditions:*

$$(f \star v_i)(\{\ddot{g}_1\}) - 2(1 - \alpha) \leq t_i \leq 2\alpha - \frac{3}{2}, \quad (7)$$

$$\forall \ddot{g} \in \ddot{M}, \quad (f \star v_i)(\{\ddot{g}\}) \notin \left[\frac{5\alpha}{3} - 1 - t_i, 2(1 - \alpha) - t_i \right], \quad (8)$$

$$\ddot{\Psi}_{(z_{t_i} \star f \star v_i)} \geq 4(1 - \alpha) - 2t_i. \quad (9)$$

Proof. By setting $\lambda = \frac{4\alpha}{3} - 1$ in Inequality (3), we obtain

$$\dot{\Psi}_{(f \star v_i)} \geq 4(1 - \alpha).$$

Hence, consider the first reduction $\rho = (\hat{\mathcal{I}}, \mathcal{R}, x, a_j, \check{\mathcal{I}})$ such that a_i 's maximin share under $(f \star v_i)$ drops below $4(1 - \alpha)$. By Observation 1, among all reduction patterns, only $\tilde{\mathcal{R}}^2$ can decrease the maximin share value of an agent. Thus, we assume this reduction allocated two goods, $\hat{g}_1$ and $\hat{g}_x$, and define $t_i = (f \star v_i)(\{\hat{g}_1\}) - 2(1 - \alpha)$.

We first show that t_i is positive and less than $2\alpha - \frac{3}{2}$. The drop in the maximin share by ρ implies that $(f \star v_i)(\{\hat{g}_1, \hat{g}_x\}) > 4(1 - \alpha)$, by Observation 2. Since $\hat{g}_1$ is more valuable than $\hat{g}_x$, we conclude that $(f \star v_i)(\{\hat{g}_1\}) > 2(1 - \alpha)$, which means $t_i > 0$. On the other hand, by Observation 5, we have $(f \star v_i)(\{\hat{g}_1\}) \leq \frac{1}{2}$, which gives $t_i \leq 2\alpha - \frac{3}{2}$. Therefore, t_i is positive and lies in the desired range. Since all goods in $\ddot{M}$ are drawn from $\hat{M}$, we have

$$\begin{aligned} (f \star v_i)(\{\ddot{g}_1\}) &\leq (f \star v_i)(\{\hat{g}_1\}) \\ &= 2(1 - \alpha) + t_i. \end{aligned}$$

This proves Inequality (7).

To show Equation (8), we use two properties of $\tilde{\mathcal{R}}^2$:

$$(f \star v_i)(\{\hat{g}_1, \hat{g}_x\}) > 4(1 - \alpha) \quad \text{and} \quad v_i(\{\hat{g}_1, \hat{g}_{x+1}\}) < \alpha.^\dagger$$

Since $0 < t_i \leq 2\alpha - \frac{3}{2}$ and $\frac{3}{4} < \alpha < \frac{6}{7}$ we conclude $1 - \frac{2\alpha}{3} + t_i \in [1 - \frac{2\alpha}{3}, 1 - \frac{\alpha}{3} - (\frac{2\alpha}{3} - \frac{1}{2})]$. Hence,

$$\begin{aligned} f(1 - \frac{2\alpha}{3} + t_i) &= \max\left(2(1 - \alpha), \frac{5}{2} - \frac{8\alpha}{3} + t_i\right) && \text{Definition 5,} \\ &< 2(1 - \alpha) + t_i && \alpha > \frac{3}{4}, \\ &= (f \star v_i)(\{\hat{g}_1\}). \end{aligned}$$

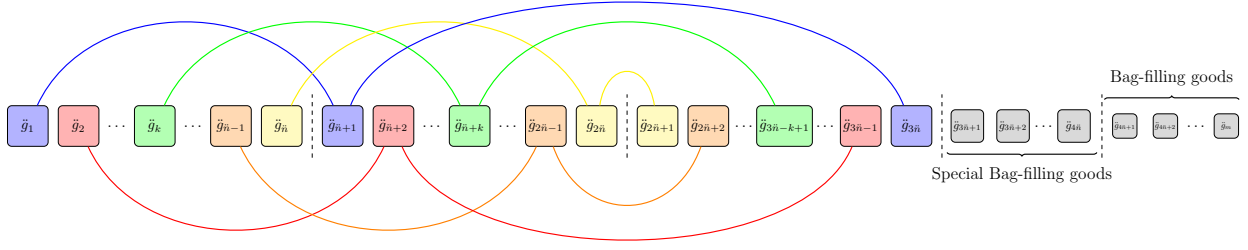


Figure 6: Structure of the bags in Algorithm 5.

Using monotonicity of $\mathring{f}$, we obtain $v_i(\{\hat{g}_1\}) \geq 1 - \frac{2\alpha}{3} + t_i$. Plugging $(\mathring{f} \star v_i)(\{\hat{g}_1\}) = 2(1 - \alpha) + t_i$ and $v_i(\{\hat{g}_1\}) \geq 1 - \frac{2\alpha}{3} + t_i$ into these inequalities, we get

$$(\mathring{f} \star v_i)(\{\hat{g}_x\}) > 2(1 - \alpha) - t_i \quad \text{and} \quad (\mathring{f} \star v_i)(\{\hat{g}_{x+1}\}) \leq v_i(\{\hat{g}_{x+1}\}) < \frac{5\alpha}{3} - 1 - t_i.$$

This implies that no good in $\hat{M}$ has a value under $(\mathring{f} \star v_i)$ that falls within the interval $[\frac{5\alpha}{3} - 1 - t_i, 2(1 - \alpha) - t_i]$. Since $\tilde{M} \subseteq \hat{M}$, the same holds for all goods in $\tilde{M}$, implying Equation (8).

To prove Inequality (9), recall that $\hat{\Psi}_{(\mathring{f} \star v_i)} \geq 4(1 - \alpha)$, and since $(\mathring{f} \star v_i)(\{\hat{g}_1\}) = 2(1 - \alpha) + t_i$, applying Lemma 2 implies $\ddot{\Psi}_{(z_{t_i} \star \mathring{f} \star v_i)} \geq 4(1 - \alpha) - 2t_i$. $\square$

7.2 Bag-filling

After the secondary reductions, we apply the Bag-filling method shown in Algorithm 5. The algorithm begins by constructing $\ddot{n}$ bundles $B_1, B_2, \dots, B_{\ddot{n}}$, where

$$B_k = \{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}\}.$$

Then, starting from $k = \ddot{n}$ and proceeding downwards to $k = 1$, the algorithm attempts to allocate bag B_k to an agent. At each step, if no remaining agent values B_k at least α , one additional remaining good is added to the bag until some agent finds its value at least α . **We select the next good according to the following priority:**

- (i) $\ddot{g}_{3\ddot{n}+k}$,
- (ii) any good from the set $\{\ddot{g}_{4\ddot{n}+1}, \ddot{g}_{4\ddot{n}+2}, \dots, \ddot{g}_{\ddot{n}}\}$,
- (iii) the remaining good with the smallest index.

To analyze our algorithm, we categorize green agents into four distinct groups based on their maximin share properties after the reduction phases:

1. Agents whose maximin share satisfies $\dot{\Psi}_{v_i} \geq 1$ after the primary reductions and $\ddot{\Psi}_{v_i} \geq 1$ after the secondary reductions.
2. Agents for whom $\dot{\Psi}_{v_i} < 1$ after the primary reductions, but the calibrated share satisfies $\ddot{\Psi}_{(\mathring{f} \star v_i)} \geq 4(1 - \alpha)$ after the secondary reductions.
3. Agents with $\dot{\Psi}_{v_i} \geq 1$ after the primary reductions, but $\ddot{\Psi}_{v_i} < 1$ after the secondary reductions.

Algorithm 5 BAG-FILLING1

Input: $\vec{I} = (\vec{N}, \vec{M})$ **Output:** Allocation satisfying $(\frac{10}{13})$ -MMS

```
1: for  $k : 1 \rightarrow \vec{n}$  do
2:    $B_k \leftarrow \{\vec{g}_k, \vec{g}_{\vec{n}+k}, \vec{g}_{3\vec{n}-k+1}\}$ 
3: end for
4: for  $k : \vec{n} \rightarrow 1$  do
5:   while There does not exist a remaining agent  $a_i$  s.t.  $v_i(B_k) \geq \alpha$  do
6:     if  $\vec{g}_{3\vec{n}+k}$  is remaining then
7:       Add  $\vec{g}_{3\vec{n}+k}$  to  $B_k$ 
8:     else if  $\exists x \geq 4\vec{n} + 1$  s.t.  $\vec{g}_x$  is remaining then
9:       Add  $\vec{g}_x$  to  $B_k$ 
10:    else
11:      Add the remaining good with the smallest index to  $B_k$ 
12:    end if
13:  end while
14:  Allocate  $B_k$  to  $a_i$  with  $v_i(B_k) \geq \alpha$  ▷ Priority is given to agents in  $N^r$ 
15: end for
```

4. Agents for whom $\dot{\Psi}_{v_i} < 1$ after the primary reductions and $\ddot{\Psi}_{(f^\star v_i)} < 4(1 - \alpha)$ after the secondary reductions.

Lemma 6 provides general tools for analyzing the first two groups of agents, while Lemma 7 helps with the analysis of the last two groups. In Lemma 8, we show that the agents in the first group receive a bundle. Lemma 9 establishes the same for the second group, Lemma 10 for the third group, and Lemma 11 for the fourth group.

Lemma 6. *Let $a_i \in \vec{N}$ be an agent, and let $\hat{v}$ be a valuation function that ranks the goods in the same order as v_i . Assume the following conditions hold:*

$$\forall_{\vec{g} \in \vec{M}} \quad \hat{v}(\{\vec{g}\}) \leq v_i(\{\vec{g}\}) \quad (10)$$

$$\alpha + \hat{v}(\{\vec{g}_{3\vec{n}+1}\}) \leq \ddot{\Psi}_{\hat{v}}, \quad (11)$$

$$\forall_{2 \leq k \leq \vec{n}}, \quad \hat{v}(\{\vec{g}_k, \vec{g}_{\vec{n}+k}, \vec{g}_{3\vec{n}+1-k}\}) \leq \ddot{\Psi}_{\hat{v}}, \quad (12)$$

$$\hat{v}(\{\vec{g}_1, \vec{g}_{\vec{n}+1}, \vec{g}_{3\vec{n}}\}) + \alpha \leq 2 \ddot{\Psi}_{\hat{v}}. \quad (13)$$

Then, a_i receives a bundle of value at least α in Algorithm 5.

Proof. Suppose, for the sake of contradiction, that agent a_i receives no bundle, and let γ be the index at which the algorithm halts while filling bag B_γ .

We claim that for every $2 \leq k \leq \vec{n}$, the value of bag B_k satisfies $\hat{v}(B_k) \leq \ddot{\Psi}_{\hat{v}}$. If no good is added to bag k during the Bag-filling process—either because it already holds value at least α for some remaining agent or its turn has not yet come—then this inequality follows directly from Inequality (12). Otherwise, if goods are added to bag k , we know that just before the last good was added, the bundle had value less than α to agent a_i . Furthermore, by the construction of the Bag-filling process, the last added good has an index at least $3\vec{n} + 1$. Therefore, its value under $\hat{v}$ is at most $\hat{v}(\{\vec{g}_{3\vec{n}+1}\})$. By the additivity of $\hat{v}$ and using Inequalities (10) and (11), it follows that the total value of B_k does not exceed $\ddot{\Psi}_{\hat{v}}$, as claimed. Now, assume $\gamma \neq 1$. Since the bags

$B_1, B_2, \dots, B_{\ddot{n}}$ form a partition of $\ddot{M}$, we have:

$$\begin{aligned}
\hat{v}(\ddot{M}) &= \sum_{k=1}^{\ddot{n}} \hat{v}(B_k) \\
&= \hat{v}(B_1) + \hat{v}(B_\gamma) + \sum_{k=2}^{\gamma-1} \hat{v}(B_k) + \sum_{k=\gamma+1}^{\ddot{n}} \hat{v}(B_k) \\
&= \hat{v}(B_1) + \hat{v}(B_\gamma) + (\ddot{n} - 2) \cdot \ddot{\Psi}_{\hat{v}} \\
&< \hat{v}(B_1) + \alpha + (\ddot{n} - 2) \cdot \ddot{\Psi}_{\hat{v}} \\
&= \hat{v}(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) + \alpha + (\ddot{n} - 2) \cdot \ddot{\Psi}_{\hat{v}} && \gamma \neq 1, \\
&\leq 2 \cdot \ddot{\Psi}_{\hat{v}} + (\ddot{n} - 2) \cdot \ddot{\Psi}_{\hat{v}} && \text{Inequality (13),} \\
&= \ddot{n} \cdot \ddot{\Psi}_{\hat{v}}.
\end{aligned}$$

which is a contradiction. If $\gamma = 1$, since for every $2 \leq k \leq \ddot{n}$ we have $\hat{v}(B_k) \leq \ddot{\Psi}_{\hat{v}}$, and $\hat{v}(B_1) < \alpha \leq \ddot{\Psi}_{\hat{v}}$, it follows that $\hat{v}(\ddot{M}) < \ddot{n} \cdot \ddot{\Psi}_{\hat{v}}$, which is a contradiction. $\square$

Lemma 7. *Let $a_i \in \ddot{N}$ be an agent, and let $\hat{v}$ be a valuation function that ranks the goods in the same order as v_i . Assume the following conditions hold:*

$$\forall_{\ddot{g} \in \ddot{M}} \quad \hat{v}(\{\ddot{g}\}) \leq v_i(\{\ddot{g}\}), \quad (14)$$

$$\alpha + \hat{v}(\{\ddot{g}_{4\ddot{n}+1}\}) \leq \ddot{\Psi}_{\hat{v}}, \quad (15)$$

$$\alpha + \hat{v}(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) \leq 2 \ddot{\Psi}_{\hat{v}}, \quad (16)$$

$$\forall_{2 \leq k \leq \ddot{n}} \quad \hat{v}(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}+1-k}, \ddot{g}_{3\ddot{n}+k}\}) \leq \ddot{\Psi}_{\hat{v}}, \quad (17)$$

$$2\alpha + \hat{v}(\{\ddot{g}_{3\ddot{n}+1}\}) \leq 2 \ddot{\Psi}_{\hat{v}}, \quad (18)$$

$$2\alpha + \hat{v}(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}, \ddot{g}_{3\ddot{n}+1}\}) \leq 3 \ddot{\Psi}_{\hat{v}}. \quad (19)$$

Then, a_i receives a bundle of value at least α in Algorithm 5.

Proof. Assume, to reach a contradiction, that agent $a_i \in \ddot{N}$ receives no bundle. Then the algorithm must enter its third priority at least once. Let the first such moment occur while filling bag ℓ . For each $1 \leq k \leq \ddot{n}$, denote by C_k the contents of bag k at that time. Observe that during the first priority, the algorithm added $\ddot{g}_{3\ddot{n}+k}$ into bag k , and during the second priority it added $\ddot{g}_x$ to bag k for some $x \geq 4\ddot{n} + 1$. Moreover, since the algorithm only enters the third priority after exhausting all higher-priority goods, none of

$$\ddot{g}_{4\ddot{n}+1}, \dots, \ddot{g}_{\ddot{m}}$$

is remaining. And since this is the first time we reach the third priority, for each $1 \leq k \leq \ddot{n}$, the good $\ddot{g}_{3\ddot{n}+k}$ has either been placed into bag k or is still remaining; in particular, for bag ℓ we have $\ddot{g}_{3\ddot{n}+\ell} \in C_\ell$. Therefore, the sets

$$C_k \cup \{\ddot{g}_{3\ddot{n}+k}\}, \quad k = 1, \dots, \ddot{n},$$

indeed form a partition of all goods in $\ddot{M}$. We will show that

$$\sum_{k=1}^{\ddot{n}} \hat{v}(C_k \cup \{\ddot{g}_{3\ddot{n}+k}\}) < \ddot{n} \ddot{\Psi}_{\hat{v}},$$

contradicting the definition of the maximin share.

First, we claim that for every $k \geq 2$, $\hat{v}(C_k \cup \{\check{g}_{3\check{n}+k}\}) \leq \check{\Psi}_{\hat{v}}$. Indeed, if $C_k \cup \{\check{g}_{3\check{n}+k}\} = \{\check{g}_k, \check{g}_{\check{n}+k}, \check{g}_{3\check{n}+1-k}, \check{g}_{3\check{n}+k}\}$, then the desired bound follows immediately from Inequality (17). Otherwise, the algorithm must have reached the second priority: let $\check{g}_x$ be the last good added to C_k . By Inequality (14) we have $\hat{v}(C_k \setminus \{\check{g}_x\}) < \alpha$, and since $x \geq 4\check{n} + 1$, it follows that

$$\begin{aligned} \hat{v}(C_k \cup \{\check{g}_{3\check{n}+k}\}) &= \hat{v}(C_k) \\ &= \hat{v}(C_k \setminus \{\check{g}_x\}) + \hat{v}(\{\check{g}_x\}) \\ &< \alpha + \hat{v}(\{\check{g}_{4\check{n}+1}\}) \\ &\leq \check{\Psi}_{\hat{v}} \end{aligned} \quad \text{Inequality (15).}$$

Next, if $\ell = 1$, then since for every $k \geq 2$ we have $\hat{v}(C_k \cup \{\check{g}_{3\check{n}+k}\}) \leq \check{\Psi}_{\hat{v}}$, we deduce $\hat{v}(C_1 \cup \{\check{g}_{3\check{n}+1}\}) \geq \check{\Psi}_{\hat{v}}$. By Inequality (15), $\check{\Psi}_{\hat{v}} \geq \alpha$, and since $\check{g}_{3\check{n}+1}$ remains available, agent a_i would accept $C_1 \cup \{\check{g}_{3\check{n}+1}\}$. This contradicts our assumption, so $\ell \geq 2$. We now show that for every $j \notin \{1, \ell\}$,

$$\hat{v}(C_j \cup \{\check{g}_{3\check{n}+j}\}) \geq \alpha.$$

Suppose, towards a contradiction, that for some such j we have $\hat{v}(C_j \cup \{\check{g}_{3\check{n}+j}\}) < \alpha$. Noting that $\check{g}_{3\check{n}+\ell} \in C_\ell$, we have

$$\begin{aligned} &\hat{v}(C_1 \cup \{\check{g}_{3\check{n}+1}\}) + \hat{v}(C_j \cup \{\check{g}_{3\check{n}+j}\}) + \hat{v}(C_\ell \cup \{\check{g}_{3\check{n}+\ell}\}) \\ &< \hat{v}(C_1 \cup \{\check{g}_{3\check{n}+1}\}) + 2\alpha \\ &= \hat{v}(\{\check{g}_1, \check{g}_{\check{n}+1}, \check{g}_{3\check{n}}, \check{g}_{3\check{n}+1}\}) + 2\alpha \leq 3\check{\Psi}_{\hat{v}} \end{aligned} \quad \text{Inequality (19).}$$

Since each of the other bundles also has value at most $\check{\Psi}_{\hat{v}}$, summing yields $\hat{v}(\check{M}) < \check{n} \check{\Psi}_{\hat{v}}$, again a contradiction. Hence for all $j \notin \{1, \ell\}$ we have $\hat{v}(C_j \cup \{\check{g}_{3\check{n}+j}\}) \geq \alpha$.

Note that this is the first time at which the algorithm enters the third priority. Hence goods $\check{g}_{3\check{n}+1}, \check{g}_{3\check{n}+2}, \dots, \check{g}_{3\check{n}+\ell-1}$ are remaining. In the third priority the algorithm selects the remaining good with the smallest index, so it adds $\check{g}_{3\check{n}+1}$ to bag ℓ . We show that $\hat{v}(C_\ell \cup \{\check{g}_{3\check{n}+1}\}) \geq \alpha$. Suppose, towards a contradiction, that $\hat{v}(C_\ell \cup \{\check{g}_{3\check{n}+1}\}) < \alpha$. Then:

$$\begin{aligned} \hat{v}(C_1) + \hat{v}(C_\ell \cup \{\check{g}_{3\check{n}+1}\}) &< \hat{v}(C_1) + \alpha \\ &= \hat{v}(\{\check{g}_1, \check{g}_{2\check{n}+1}, \check{g}_{3\check{n}}\}) + \alpha & \ell \neq 1, \\ &\leq \check{\Psi}_{\hat{v}} \end{aligned} \quad \text{Inequality (16).}$$

Since each of the other bundles also has value at most $\check{\Psi}_{\hat{v}}$, summing yields $\hat{v}(\check{M}) < \check{n} \check{\Psi}_{\hat{v}}$, again a contradiction. Therefore, agent a_i would accept bag ℓ upon adding $\check{g}_{3\check{n}+1}$. The algorithm then fills the remaining bags in descending order from $\ell - 1$ down to 1. As shown above, for each $k \notin \{1, \ell\}$, agent a_i would accept bag k after adding $\check{g}_{3\check{n}+k}$. Hence, if the algorithm reaches bag 1 with $\hat{v}(C_1) < \alpha$, then we have:

$$\begin{aligned} \hat{v}(C_1) + \hat{v}(C_\ell \cup \{\check{g}_{3\check{n}+1}\}) &< \alpha + \hat{v}(C_\ell \cup \{\check{g}_{3\check{n}+1}\}) \\ &\leq 2\alpha + \hat{v}(\{\check{g}_{3\check{n}+1}\}) \\ &\leq 2\check{\Psi}_{\hat{v}} \end{aligned} \quad \text{Inequality (18).}$$

Since each of the other bundles also has value at most $\check{\Psi}_{\hat{v}}$, summing yields $\hat{v}(\check{M}) < \check{n} \check{\Psi}_{\hat{v}}$, again a contradiction. $\square$

Using Lemma 6, we analyze two groups of green agents. The first group consists of those who satisfy $\dot{\Psi}_{v_i} \geq 1$ after the primary reductions and $\ddot{\Psi}_{v_i} \geq 1$ after the secondary reductions (see Lemma 8). The second group includes agents with $\dot{\Psi}_{v_i} < 1$ after the primary reductions and $\ddot{\Psi}_{(\mathring{f} \star v_i)} \geq 4(1 - \alpha)$ after the secondary reductions (see Lemma 9).

Lemma 8. *Every green agent $a_i \in \ddot{N}$ with $\dot{\Psi}_{v_i} \geq 1$ after the primary reductions, and $\ddot{\Psi}_{v_i} \geq 1$ after the secondary reductions, receives a bundle in Algorithm 5.*

Proof sketch. We analyze two cases based on the value agent a_i assigns to good $\check{g}_{3\ddot{n}}$. If this value is at most $\frac{4\alpha}{3} - 1$, we show that $(h \star \mathring{f} \star v_i)$ satisfies all three conditions of Lemma 6. Otherwise, we assume $v_i(\{\check{g}_{3\ddot{n}}\}) > \frac{4\alpha}{3} - 1$, and consider whether for any $2 \leq k \leq \ddot{n}$, the bundle $\{\check{g}_k, \check{g}_{\ddot{n}+k}, \check{g}_{3\ddot{n}-k+1}\}$ has value at least 1. If none do, we again verify all conditions of Lemma 6 for v_i . If for some $2 \leq k \leq \ddot{n}$, the bundle $\{\check{g}_k, \check{g}_{\ddot{n}+k}, \check{g}_{3\ddot{n}-k+1}\}$ has value at least 1, we prove that all such bundles have value at least α , ensuring the agent receives a bundle. $\square$

Lemma 9. *Every green agent $a_i \in \ddot{N}$ with $\dot{\Psi}_{v_i} < 1$ after the primary reductions, and $\ddot{\Psi}_{(\mathring{f} \star v_i)} \geq 4(1 - \alpha)$ after the secondary reductions, receives a bundle in Algorithm 5.*

Proof sketch. We analyze two cases based on the value agent a_i assigns to good $\check{g}_{3\ddot{n}}$. If this value is at most $\frac{4\alpha}{3} - 1$, we show that $(h \star \mathring{f} \star v_i)$ satisfies all conditions of Lemma 6. Otherwise, we assume $v_i(\{\check{g}_{3\ddot{n}}\}) > \frac{4\alpha}{3} - 1$, and consider whether for any $2 \leq k \leq \ddot{n}$, the bundle $\{\check{g}_k, \check{g}_{\ddot{n}+k}, \check{g}_{3\ddot{n}-k+1}\}$ has value at least $4(1 - \alpha)$, under the function $(\mathring{f} \star v_i)$. If none do, we again verify all conditions of Lemma 6 for $(\mathring{f} \star v_i)$. If for some $2 \leq k \leq \ddot{n}$,

$$(\mathring{f} \star v_i)(\{\check{g}_k, \check{g}_{\ddot{n}+k}, \check{g}_{3\ddot{n}-k+1}\}) \geq 4(1 - \alpha),$$

we prove that all such bundles have value at least α , ensuring the agent receives a bundle. $\square$

We handle the remaining two groups using similar arguments: agents with $\dot{\Psi}_{v_i} \geq 1$ after the primary reductions but $\ddot{\Psi}_{v_i} < 1$ after the secondary reductions (Lemma 10), and those with $\dot{\Psi}_{v_i} < 1$ after the primary reductions and $\ddot{\Psi}_{(\mathring{f} \star v_i)} < 4(1 - \alpha)$ after the secondary reductions (Lemma 11).

Lemma 10. *Every green agent a_i in $\ddot{N}$ such that $\dot{\Psi}_{v_i} \geq 1$ after the primary reductions and $\ddot{\Psi}_{v_i} < 1$ after the secondary reductions, receives a bundle in Algorithm 5.*

Proof sketch. Noting that by Lemma 4, there exists a number t_i satisfying Inequality (4), Equation (5), and Inequality (6), we verify that valuation function v_i satisfies all conditions required by Lemma 7. $\square$

Lemma 11. *Every green agent a_i in $\ddot{N}$ such that $\dot{\Psi}_{v_i} < 1$ after the primary reductions and $\ddot{\Psi}_{(\mathring{f} \star v_i)} < 4(1 - \alpha)$ after the secondary reductions, receives a bundle in Algorithm 5.*

Proof sketch. Noting that by Lemma 5, there exists a number t_i satisfying Inequality (7), Equation (8), and Inequality (9), we verify that valuation function $(z_{t_i} \star \mathring{f} \star v)$ satisfies all conditions required by Lemma 7. $\square$

Lemma 12. *Every red agent $a_i \in \ddot{N}$ receives a bundle in Algorithm 5.*

Proof. We argue by contradiction. Suppose there exists a red agent $a_i \in \ddot{N}$ who does not receive any bundle in Algorithm 5. Let the algorithm terminate while filling bag B_γ . This means $n - \gamma$ agents have received a bundle of value at least α , and bags $B_1, B_2, \dots, B_\gamma$ are unallocated. Note

that by Lemma 8, Lemma 9, Lemma 10 and Lemma 11, all green agents have received a bundle, and only red agents remain. Our goal is to show $v_i(M) < n$, contradicting $\Psi_{v_i}^n(M) = 1$.

For every agent a_j such that a_j received a bundle during the primary reductions, secondary reductions, or Bag-filling, we denote A_j by the bundles allocated to her. By the prioritization of red agents in Algorithms 2, 4 and 5, every bundle allocated to a green agent satisfies $v_i(A_j) < \alpha$. We now show that every bundle allocated to a red agent satisfies $v_i(A_j) \leq 2\alpha$.

Primary Reductions. Consider a red agent a_j such that we allocated a bundle to her, during the primary reductions, and let $\hat{\mathcal{I}} = (\hat{N}, \hat{M})$ be the instance before applying the reduction. If a_j receives a bundle via reduction $\mathcal{R}^0$, then $v_i(A_j) \leq 1 < 2\alpha$. If the bundle is assigned via $\mathcal{R}^1$, note that $\mathcal{R}^0$ was not applicable at that point, meaning $v_i(\{\hat{g}_1\}) < \alpha$. Since $\mathcal{R}^1$ allocates two goods, it follows that $v_i(A_j) < 2\alpha$. The same reasoning applies to $\tilde{\mathcal{R}}^1$. Finally, if a_j receives a bundle via $\mathcal{R}^2$, the inapplicability of $\mathcal{R}^1$ at that time implies $v_i(\{\hat{g}_{\hat{n}+1}\}) < \frac{\alpha}{2}$. Since $\mathcal{R}^2$ allocates three goods, we get $v_i(A_j) < \frac{3\alpha}{2} < 2\alpha$.

Secondary Reductions. Consider a red agent a_j such that we allocated a bundle to her, during the secondary reductions, and let $\hat{\mathcal{I}} = (\hat{N}, \hat{M})$ be the instance before applying the reduction. The bounds for $\mathcal{R}^1$ and $\mathcal{R}^2$ were already established in the primary reductions. For $\tilde{\mathcal{R}}^2$, the same reasoning as in $\mathcal{R}^1$ applies, since the bundle size is also 2, leading to $v_i(A_j) < 2\alpha$. For $\mathcal{R}^3$, since $\mathcal{R}^2$ was not applicable at the time of allocation, we have $v_i(\{\hat{g}_{2\hat{n}+1}\}) < \frac{\alpha}{3}$, and hence, $v_i(A_j) < \frac{4\alpha}{3} < 2\alpha$. Similarly, for $\mathcal{R}^4$, the inapplicability of $\mathcal{R}^3$ implies $v_i(\{\hat{g}_{3\hat{n}+1}\}) < \frac{\alpha}{4}$, which leads to $v_i(A_j) < \frac{5\alpha}{4} < 2\alpha$.

Bag-filling. For each $1 \leq k \leq \ddot{n}$, B_k is either $\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}\}$ or at least one good is added to it. In the first case, since $\tilde{\mathcal{R}}^2$ is not applicable, $v_i(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}\}) < \alpha$, so $v_i(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}\}) < 2\alpha$. In the second case, let $\ddot{g}_x$ be the last good added to it. Since both $v_i(B_k \setminus \{\ddot{g}_x\}) < \alpha$ and $v_i(\{\ddot{g}_x\}) < \alpha$, we have $v_i(B_k) < 2\alpha$.

Now we calculate sum of all bundles $B_1, B_2, \dots, B_\gamma$ and A_j for all satisfied agents a_j . Since all green agents are satisfied, we have:

$$v_i(M) < \alpha |N^g| + 2\alpha (|N^r| - \gamma) + 2\alpha \gamma.$$

Using $|N^g| \geq \frac{n}{\sqrt{2}}$ we obtain

$$\begin{aligned} v_i(M) &< \left(\alpha \frac{1}{\sqrt{2}} + 2\alpha \left(1 - \frac{1}{\sqrt{2}} \right) \right) n \\ &< n \end{aligned} \qquad \alpha < \frac{4 + \sqrt{2}}{7} \approx 0.7735.$$

This contradicts $\Psi_{v_i}^n(M) = 1$. Therefore, every red agent in $\ddot{N}$ receives a bundle. $\square$

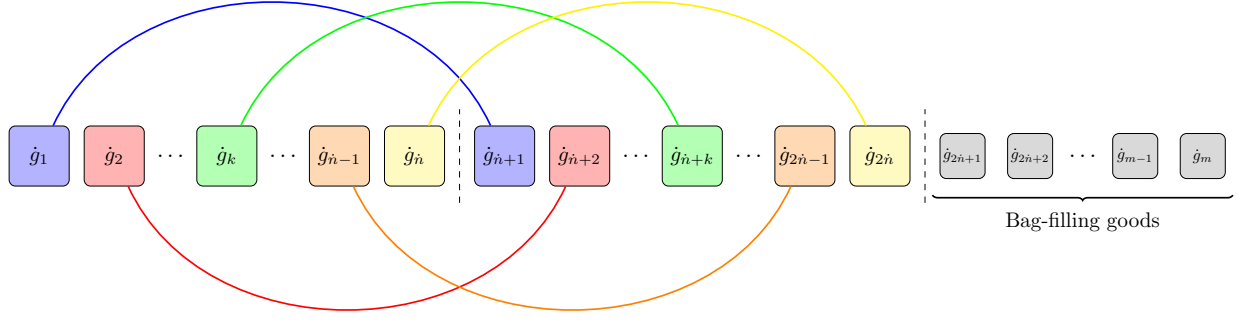


Figure 7: Structure of the bags in Algorithm 6.

8 Algorithm 6: Less Frequent Green Agents

In this section, we consider the case where $|N^g| < \frac{n}{\sqrt{2}}$. To handle this scenario, we use Algorithm 6, which gives priority to green agents. We first show that every red agent receives a bundle. Then, we prove that, due to the prioritization of green agents, each green agent also receives a bundle.

Algorithm 6 ALGORITHM-CASE2

Input: $\dot{\mathcal{I}}, N^g, N^r$

Output: Allocation satisfying $(\frac{10}{13})$ -MMS

- 1: **for** $k : 1 \rightarrow \dot{n}$ **do**
 - 2: $B_k \leftarrow \{\dot{g}_k, \dot{g}_{\dot{n}+k}\}$
 - 3: **end for**
 - 4: **for** $k : 1 \rightarrow \dot{n}$ **do**
 - 5: **while** There does not exist a remaining agent a_i s.t. $v_i(B_k) \geq \alpha$ **do**
 - 6: Add an arbitrary remaining good to B_k
 - 7: **end while**
 - 8: Allocate B_k to a_i with $v_i(B_k) \geq \alpha$ $\triangleright$ Priority is given to agents in N^g
 - 9: **end for**
-

In Algorithm 6, we use a simple Bag-filling algorithm. We begin by forming initial bags of the form

$$\{\dot{g}_k, \dot{g}_{\dot{n}+k}\} \quad \text{for } k = 1, \dots, \dot{n}.$$

Next, we sequentially add the remaining goods $\dot{g}_{2\dot{n}+1}, \dot{g}_{2\dot{n}+2}, \dots, \dot{g}_{\dot{m}}$ to the bags, one good at a time. As soon as the total value of a bag reaches at least α for some agent a_i , we allocate that bag to $a_i \in \dot{N}$. If multiple agents are eligible at the same time, priority is given to those in N^g .

To analyze our algorithm, we categorize red agents into two groups based on their maximin share after the primary reductions:

1. Agents with $\dot{\Psi}_{v_i} \geq 1$ after the primary reductions.
2. Agents with $\dot{\Psi}_{v_i} < 1$ after the primary reductions.

Lemma 13 provides general tools for analyzing these agents. In Lemma 14, we show that the agents in the first group receive a bundle. Lemma 15 establishes the same for the second group.

Lemma 13. Let a_i be an agent in $\dot{N}$, consider integers $0 \leq x \leq y \leq \dot{n}$ and define

$$M' = \dot{M} \setminus \bigcup_{k=x+1}^y \{\dot{g}_k, \dot{g}_{\dot{n}+k}\} \quad \text{and} \quad n' = \dot{n} - (y - x).$$

Let $\hat{v}$ be a valuation function on M' that ranks the goods in the same order as v_i . Assume the following conditions hold:

$$\forall \dot{g} \in M' \quad \hat{v}(\{\dot{g}\}) \leq v_i(\{\dot{g}\}), \quad (20)$$

$$\forall 1 \leq k \leq y \quad v_i(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \geq \alpha, \quad (21)$$

$$\forall y < k \leq \dot{n} \quad \hat{v}(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) < \alpha, \quad (22)$$

$$x\hat{v}(\{\dot{g}_1\}) + \sum_{k=1}^x \hat{v}(\{\dot{g}_{\dot{n}+k}\}) + (n' - x)(\alpha + \hat{v}(\{\dot{g}_{2\dot{n}+1}\})) < n' \Psi_{\hat{v}}^{n'}(M'). \quad (23)$$

Then a_i receives a bundle in Algorithm 6.

Proof. Assume, for contradiction, that agent a_i does not receive any bundle and Algorithm 6 terminates while filling bag B_γ . Note that the bags $B_1, B_2, \dots, B_{\dot{n}}$ form a partition of $\dot{M}$. For $1 \leq k \leq y$, we know from Inequality (21) that $v_i(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \geq \alpha$, so the algorithm does not add any additional goods to these bags. Thus, $B_k = \{\dot{g}_k, \dot{g}_{\dot{n}+k}\}$ for all $k \leq y$. It follows that the following collection of bundles forms a partition of M' :

$$\{\dot{g}_1, \dot{g}_{\dot{n}+1}\}, \{\dot{g}_2, \dot{g}_{\dot{n}+2}\}, \dots, \{\dot{g}_x, \dot{g}_{\dot{n}+x}\}, B_{y+1}, B_{y+2}, \dots, B_{\dot{n}}.$$

Therefore,

$$\hat{v}(M') = \sum_{k=1}^x \hat{v}(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) + \sum_{k=y+1}^{\dot{n}} \hat{v}(B_k).$$

We now show that

$$\sum_{k=1}^x \hat{v}(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) + \sum_{k=y+1}^{\dot{n}} \hat{v}(B_k) < n' \cdot \Psi_{\hat{v}}^{n'}(M'),$$

which contradicts the assumption that $\hat{v}(M') \geq n' \cdot \Psi_{\hat{v}}^{n'}(M')$.

For each $y < k \leq n'$, the bag B_k is either left as $\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}$, or at least one additional good is added to it during the Bag-filling process. In the first case, by Inequality (22), we have $\hat{v}(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) < \alpha$. In the second case, let $\dot{g}_x$ denote the last good added to the bag (with $x \geq 2\dot{n}+1$). Then by Inequality (20), $\hat{v}(B_k \setminus \{\dot{g}_x\}) < \alpha$. Thus, in both cases,

$$\hat{v}(B_k) < \alpha + \hat{v}(\{\dot{g}_{2\dot{n}+1}\}).$$

We can now upper bound the total value of M' as follows:

$$\begin{aligned} \hat{v}(M') &= \sum_{k=1}^x \hat{v}(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) + \sum_{k=y+1}^{\dot{n}} \hat{v}(B_k) \\ &\leq \sum_{k=1}^x \hat{v}(\{\dot{g}_k\}) + \sum_{k=1}^x \hat{v}(\{\dot{g}_{\dot{n}+k}\}) + (n' - x)(\alpha + \hat{v}(\{\dot{g}_{2\dot{n}+1}\})) \\ &\leq x \cdot \hat{v}(\{\dot{g}_1\}) + \sum_{k=1}^x \hat{v}(\{\dot{g}_{\dot{n}+k}\}) + (n' - x)(\alpha + \hat{v}(\{\dot{g}_{2\dot{n}+1}\})) \\ &< n' \cdot \Psi_{\hat{v}}^{n'}(M') \end{aligned} \quad \text{Inequality (23).}$$

This contradicts the assumption that $\hat{v}(M') \geq n' \cdot \Psi_{\hat{v}}^{n'}(M')$, completing the proof. $\square$

We now prove Lemmas 14 and 15, which together establish that the red agents receive a bundle in Algorithm 6. To maintain the flow of the paper, we present only a brief proof sketch here and defer the full proofs to the appendix.

Lemma 14. *Every red agent a_i in $\dot{N}$ with $\dot{\Psi}_{v_i} \geq 1$ after the primary reductions, receives a bundle in Algorithm 6.*

Proof sketch. We begin by defining an index $0 \leq y \leq \dot{n}$ such that:

- $v_i(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \geq \alpha$ for all $1 \leq k \leq y$, and
- $v_i(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) < \alpha$ for all $y < k \leq \dot{n}$.

Next, we identify the smallest index $x \leq y$ such that the maximin share with respect to the remaining $\dot{n} - (y - x)$ bundles is at least 1:

$$\Psi_{v_i}^{\dot{n}-(y-x)} \left(\dot{M} \setminus \bigcup_{k=x+1}^y \{\dot{g}_k, \dot{g}_{\dot{n}+k}\} \right) \geq 1.$$

Such an x must exist because $\dot{\Psi}_{v_i} \geq 1$ by assumption. Now, define:

$$n' = \dot{n} - (y - x), \quad M' = \dot{M} \setminus \bigcup_{k=x+1}^y \{\dot{g}_k, \dot{g}_{\dot{n}+k}\},$$

and let by Definition 6 $v_i^{\text{norm}} = \text{normalized}_1^{n'}(v_i, M')$, the normalized valuation of agent a_i over M' for n' bundles.

We observe that $0 \leq x \leq y \leq \dot{n}$ and verify that v_i^{norm} satisfies all the conditions required in Lemma 13. Inequalities (21) and (22) follow directly from the definition of y . To verify Inequality (23), we estimate the sum

$$\sum_{k=1}^x v_i^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}),$$

and provide separate bounds depending on the ratio $\frac{x}{n'}$. $\square$

By Lemma 3, for every red agent $a_i \in \dot{N}$ with $\dot{\Psi}_{v_i} < 1$, there exists a value s_i such that conditions Inequality (1), Equation (2), and Inequality (3) are satisfied. Fixing these values s_i , we now proceed to prove Observation 7.

Observation 7. *For every red agent a_i in $\dot{N}$ with $\dot{\Psi}_{v_i} < 1$, we have*

$$(f_{s_i} \star v_i)(\{\dot{g}_1\}) \leq 1 - \frac{\alpha}{3} - 2s_i, \tag{24}$$

$$(f_{s_i} \star v_i)(\{\dot{g}_{\dot{n}+1}\}) \leq \frac{\alpha}{2} - s_i, \tag{25}$$

$$(f_{s_i} \star v_i)(\{\dot{g}_{2\dot{n}+1}\}) < \frac{4\alpha}{3} - 1 - s_i. \tag{26}$$

Proof. We prove it one by one.

By Lemma 3, $v_i(\{\dot{g}_1\}) \leq 1 - \frac{\alpha}{3} + s_i$, and by the definition of f_{s_i} , we have

$$\begin{aligned} (f_{s_i} \star v_i)(\{\dot{g}_1\}) &\leq (f_{s_i} \star v_i)(1 - \frac{\alpha}{3} + s_i) \\ &= 1 - \frac{\alpha}{3} - 2s_i. \end{aligned}$$

Since $\mathcal{R}^1$ is not applicable, it follows that $v_i(\{\dot{g}_{\dot{n}+1}\}) < \frac{\alpha}{2}$. Now, by the definition of f_{s_i} , since $\frac{\alpha}{2} \geq \frac{\alpha}{3}$ it follows that

$$\begin{aligned} (f_{s_i} \star v_i)(\{\dot{g}_{\dot{n}+1}\}) &\leq (f_{s_i} \star v_i)\left(\frac{\alpha}{2}\right) \\ &\leq \frac{\alpha}{2} - s_i. \end{aligned}$$

By definition of N^r , we have $v_i(\{\dot{g}_{2\dot{n}+1}\}) \leq 1 - \alpha$ and by Lemma 3 there is no good in $\dot{M}$ with value in

$$\left[\frac{4\alpha}{3} - 1 - s_i, 1 - \alpha\right]$$

. Therefore $(f_{s_i} \star v_i)(\{\dot{g}_{2\dot{n}+1}\}) < \frac{4\alpha}{3} - 1 - s_i$. $\square$

Lemma 15. *Every red agent a_i in $\dot{N}$ with $\dot{\Psi}_{v_i} < 1$ after the primary reductions, receives a bundle in Algorithm 6.*

Proof sketch. We begin by defining an index $0 \leq y \leq \dot{n}$ such that:

- $(f_{s_i} \star v_i)(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \geq \alpha$ for all $1 \leq k \leq y$, and
- $(f_{s_i} \star v_i)(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) < \alpha$ for all $y < k \leq \dot{n}$.

Next, we identify the smallest index $x \leq y$ such that the maximin share with respect to the remaining $\dot{n} - (y - x)$ bundles is at least $1 - 3s_i$:

$$\Psi_{(f_{s_i} \star v_i)}^{\dot{n}-(y-x)}\left(\dot{M} \setminus \bigcup_{k=x+1}^y \{\dot{g}_k, \dot{g}_{\dot{n}+k}\}\right) \geq 1 - 3s_i.$$

Such an x must exist because $\dot{\Psi}_{(f_{s_i} \star v_i)} \geq 1 - 3s_i$ by assumption. Now, define:

$$n' = \dot{n} - (y - x), \quad M' = \dot{M} \setminus \bigcup_{k=x+1}^y \{\dot{g}_k, \dot{g}_{\dot{n}+k}\},$$

and let by Definition 6 $(f_{s_i} \star v_i)^{\text{norm}} = \text{normalized}_{1-3s_i}^{n'}((f_{s_i} \star v_i), M')$, the normalized valuation of agent a_i over M' for n' bundles. We observe that $0 \leq x \leq y \leq \dot{n}$ and verify that $(f_{s_i} \star v_i)^{\text{norm}}$ satisfies all the conditions required in Lemma 13. Inequalities (21) and (22) follow directly from the definition of y . To verify Inequality (23), we estimate the sum

$$\sum_{k=1}^x (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}),$$

and provide separate bounds depending on the ratio $\frac{x}{n'}$. $\square$

Finally, in Lemma 16, we show that every green agent is allocated a bundle during the Bag-Filling process.

Lemma 16. *Every green agent $a_i \in \dot{N}$ receives a bundle in Algorithm 6.*

Proof. We argue by contradiction. Suppose there exists a green agent $a_i \in \dot{N}$ who does not receive any bundle in Algorithm 6. Let the algorithm terminate while filling bag B_γ . This means $n - (\dot{n} - \gamma + 1)$ agents have received a bundle of value at least α , and bags $B_\gamma, B_{\gamma+1}, \dots, B_{\dot{n}}$ are unallocated. Note that by Lemma 14 and Lemma 15, all red agents have received a bundle, and only green agents remain. Our goal is to show $v_i(M) < n$, contradicting $\Psi_{v_i}^n(M) = 1$. For every agent a_j such that a_j received a bundle during the primary reductions or Bag-filling, we denote A_j by the bundles allocated to her. By the prioritization of green agents in Algorithms 2 and 6, every bundle allocated to a red agent satisfies $v_i(A_j) < \alpha$. We now show that every bundle allocated to a green agent satisfies $v_i(A_j) \leq 4\alpha - 2$.

Primary Reductions. Consider a green agent $a_j \in N^g$, and let A_j be the bundle allocated to a_j through a reduction on the instance $\hat{\mathcal{I}} = (\hat{N}, \hat{M})$. Let y be the index such that $\hat{g}_y = \hat{g}_{2\hat{n}+1}$ (i.e., $\hat{g}_{2\hat{n}+1}$ is the y -th good in $\hat{M}$). By the definition of green agents, the value of $\hat{g}_y$ for a_j satisfies

$$1 - \alpha \leq v_i(\{\hat{g}_y\}) \leq \frac{\alpha}{3}.$$

We now consider several cases based on the pattern of reduction applied in this step.

If the reduction is $\rho = (\hat{\mathcal{I}}, \mathcal{R}^0, x, a_j, \hat{\mathcal{I}})$, then the allocated bundle satisfies $v_i(A_j) \leq 1 < 4\alpha - 2$, since $\alpha > \frac{3}{4}$. Now, suppose the reduction is $\rho = (\hat{\mathcal{I}}, \mathcal{R}^1, x, a_j, \hat{\mathcal{I}})$, so the allocated bundle is $A_j = \{\hat{g}_{\hat{n}}, \hat{g}_x\}$. If the second good in the bundle has value at most $\frac{\alpha}{3}$, then the total value is at most $\alpha + \frac{\alpha}{3} \leq 4\alpha - 2$.

Otherwise, $v_i(\{\hat{g}_x\}) > \frac{\alpha}{3}$, which implies $x < y$. By definition of $\mathcal{R}^1$, we have $v_i(\{\hat{g}_{\hat{n}}, \hat{g}_y\}) < \alpha$, and since $\hat{g}_y$ has value at least $1 - \alpha$, we get $v_i(\{\hat{g}_{\hat{n}}\}) < 2\alpha - 1$. Therefore, the value of the bundle is at most

$$v_i(A_j) \leq 2v_i(\{\hat{g}_{\hat{n}}\}) < 4\alpha - 2.$$

Now consider the case where the reduction is $\rho = (\hat{\mathcal{I}}, \mathcal{R}^2, x, a_j, \hat{\mathcal{I}})$, which assigns the bundle $A_j = \{\hat{g}_{2\hat{n}-1}, \hat{g}_{2\hat{n}}, \hat{g}_x\}$. Since $\mathcal{R}^1$ is not applicable, we have

$$v_i(\{\hat{g}_{2\hat{n}-1}\}) + v_i(\{\hat{g}_{2\hat{n}}\}) < \alpha.$$

If the value of the third good is small, say $v_i(\{\hat{g}_x\}) \leq \frac{\alpha}{3}$, then the total value is bounded by $\alpha + \frac{\alpha}{3} \leq 4\alpha - 2$. Otherwise, $v_i(\{\hat{g}_x\}) > \frac{\alpha}{3}$, and must have $x < y$, by definition of $\mathcal{R}^2$ we can conclude

$$v_i(\{\hat{g}_{2\hat{n}-1}\}) + v_i(\{\hat{g}_{2\hat{n}}\}) + v_i(\{\hat{g}_y\}) < \alpha, \quad \text{and} \quad v_i(\{\hat{g}_y\}) \geq 1 - \alpha.$$

This implies

$$v_i(\{\hat{g}_{2\hat{n}-1}\}) + v_i(\{\hat{g}_{2\hat{n}}\}) < 2\alpha - 1.$$

Since $x > 2\hat{n}$, we get

$$v_i(A_j) \leq \frac{3}{2}(v_i(\{\hat{g}_{2\hat{n}-1}\}) + v_i(\{\hat{g}_{2\hat{n}}\})) < 3\alpha - \frac{3}{2} < 4\alpha - 2.$$

Finally, if the reduction is $\rho = (\hat{\mathcal{I}}, \tilde{\mathcal{R}}^1, x, a_j, \hat{\mathcal{I}})$, then $\mathcal{R}^0$ and $\mathcal{R}^2$ are not applicable, which means

$$v_i(\{\hat{g}_1\}) < \alpha \quad \text{and} \quad v_i(\{\hat{g}_{2\hat{n}+1}\}) < \frac{\alpha}{3}.$$

Hence, the total value of the bundle is at most

$$v_i(A_j) \leq \alpha + \frac{\alpha}{3} \leq 4\alpha - 2, \quad \text{since } \alpha > \frac{3}{4}.$$

Bag-filling For each $1 \leq k \leq \dot{n}$, B_k is either $\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}$ or at least one good is added to it. In the first case, by definition of green agents, $v_i(\{\dot{g}_{2\dot{n}+1}\}) \geq 1 - \alpha$, and since $\mathcal{R}^1$ is not applicable, we have $v_i(\{\dot{g}_1\}) < 2\alpha - 1$. Therefore

$$v_i(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) < (2\alpha - 1) + \frac{\alpha}{2} < 4\alpha - 2 \quad \alpha \geq \frac{2}{3}.$$

For the second case, we have

$$\begin{aligned} v_i(B_k) &= v_i(B_k \setminus \{\dot{g}_x\}) + v_i(\{\dot{g}_x\}) \\ &< \alpha + \frac{\alpha}{3} \\ &< 4\alpha - 2 \end{aligned} \quad \begin{array}{l} \mathcal{R}^2 \text{ is not applicable,} \\ \alpha > \frac{3}{4}. \end{array}$$

In both cases $v_i(B_k) < 4\alpha - 2$ holds.

Now we calculate sum of all bundles $B_\gamma, \dots, B_{\dot{n}}$ and A_j for all satisfied agents a_j . Since all red agents are satisfied, we have: $n < \alpha |N^r| + (4\alpha - 2) (|N^g| - (\dot{n} - \gamma + 1)) + (4\alpha - 2) (\dot{n} - \gamma + 1)$. Using $|N^g| < \frac{n}{\sqrt{2}}$ we obtain

$$\begin{aligned} n &< \alpha |N^r| + (4\alpha - 2) |N^g| \\ &\leq (4\alpha - 2) \frac{1}{\sqrt{2}} n + \alpha \left(1 - \frac{1}{\sqrt{2}}\right) n \\ &\leq n \end{aligned} \quad \alpha \leq \frac{4 + \sqrt{2}}{7} \approx 0.7735.$$

a contradiction. Therefore, every green agent in $\dot{N}$ receives a bundle. □

Case	Agent	Condition	Lemma
Case 1: $ N^g \geq \frac{n}{\sqrt{2}}$	green agents	$\dot{\Psi}_{v_i} \geq 1, \ddot{\Psi}_{v_i} \geq 1$	Lemma 8
		$\ddot{\Psi}_{v_i} < 1, \ddot{\Psi}_{(\hat{f} \star v_i)} \geq 4(1 - \alpha)$	Lemma 9
		$\dot{\Psi}_{v_i} \geq 1, \ddot{\Psi}_{v_i} < 1$	Lemma 10
		$\ddot{\Psi}_{v_i} < 1, \ddot{\Psi}_{(\hat{f} \star v_i)} < 4(1 - \alpha)$	Lemma 11
	red agents	—	Lemma 12
Case 2: $ N^g < \frac{n}{\sqrt{2}}$	red agents	$\dot{\Psi}_{v_i} \geq 1$	Lemma 14
		$\dot{\Psi}_{v_i} < 1$	Lemma 15
	green agents	—	Lemma 16

Table 3: Categorization for Theorem 3.1.

9 Putting the Pieces Together

Finally, in this section, we bring all the components together to show that our algorithm guarantees a $(\frac{10}{13})$ -MMS allocation.

Theorem 3.1. *The allocation returned by Algorithm 1 is $(\frac{10}{13})$ -MMS.*

Proof. First, by Theorem 2.1, without loss of generality, we can transform any instance into an ordered and normalized instance. After running the primary reductions, the algorithm branches into two cases. We analyze each case separately.

Case 1. When $|N^g| \geq n/\sqrt{2}$, we first apply the secondary reductions from Algorithm 4, and then run the Bag-filling process from Algorithm 5. By Lemma 8, Lemma 9, Lemma 10, and Lemma 11, every green agent receives a bundle worth at least α . In addition, Lemma 12 ensures that the red agents also receive a bundle in this case. Since these groups together include all agents, it follows that every agent receives a bundle of value at least α .

Case 2. When $|N^g| < n/\sqrt{2}$, we perform the Bag-filling procedure in Algorithm 6. By Lemma 14 and Lemma 15, we have that all red receive a bundles of value at least α . Moreover, in Lemma 16, we show that the green agents also receive a bundle of value at least α in this case.

In both cases, the lemmas collectively ensure that every agent is allocated a bundle of value at least α . Moreover, all the constraints and assumptions imposed on α throughout the analysis are indeed satisfied when $\alpha = \frac{10}{13}$, and no larger value of α satisfies all these conditions simultaneously. Table 3 summarizes the lemmas that cover all agent categories and conditions. Therefore, the algorithm guarantees a $(\frac{10}{13})$ -MMS allocation. $\square$

Finally, as a consequence of Theorem 3.1, we show that for some constant $\varepsilon > 0$, a $(\frac{10}{13} - \varepsilon)$ -MMS allocation can be computed in polynomial time.

Theorem 9.1. *For every constant $\varepsilon > 0$, we can find a $(\frac{10}{13} - \varepsilon)$ -MMS allocation in polynomial time.*

Proof. All steps of our algorithm run in polynomial time, except for the normalization step, which requires computing the exact MMS of each agent which is NP-hard. However, a PTAS due to [Woe97] provides a $(1 - \varepsilon)$ -approximation for MMS for a constant ε in polynomial time. Using this approximation, we can estimate each agent's MMS value closely enough to ensure an overall $(\frac{10}{13} - \varepsilon)$ -approximation guarantee in polynomial time for constant ε .

The polynomial time implementation of the rest of the algorithm is mostly straightforward, except for identifying a perfect sequence of reductions. In this step, we iteratively select reductions one by one, always choosing the highest-priority reduction that still allows for a perfect matching between the resulting bundles and agents. The existence of such a matching can be verified in polynomial time using standard bipartite matching algorithms. Moreover, we can enforce agent priorities by treating the problem as a weighted matching: assign weight n to ordinary edges and $n + 1$ to prioritized ones. A maximum-weight matching under this scheme maximizes the number of matched prioritized agents and is computable in polynomial time.

Therefore, for any constant $\varepsilon > 0$, we can find a $(\frac{10}{13} - \varepsilon)$ -MMS allocation in polynomial time. $\square$

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A Table of Frequently Used Notation

Label	Instance	Agents	Goods	Good	MMS
Initial	$\mathcal{I}$	N	M	g_k	$\Psi_v^n(M)$
Primary	$\dot{\mathcal{I}}$	$\dot{N}$	$\dot{M}$	$\dot{g}_k$	$\dot{\Psi}_v$
Secondary	$\ddot{\mathcal{I}}$	$\ddot{N}$	$\ddot{M}$	$\ddot{g}_k$	$\ddot{\Psi}_v$
N^g	Green agents. $\{a_i \in N \mid v_i(\{\dot{g}_{2\hat{n}+1}\}) \geq 1 - \alpha\}$				
N^r	Red agents. $\{a_i \in N \mid v_i(\{\dot{g}_{2\hat{n}+1}\}) < 1 - \alpha\}$				
B_k	Bags. Initialized with $\{\ddot{g}_k, \ddot{g}_{\hat{n}+k}, \ddot{g}_{3\hat{n}+1-k}\}$ in Case 1, and $\{\dot{g}_k, \dot{g}_{\hat{n}+k}\}$ in Case 2.				
$\overset{\circ}{f}$	$f_{\frac{4}{3}\alpha-1}$				
$\mathcal{R}^1$	$\{\hat{n}, x\}$		$\mathcal{R}^2$	$\{2\hat{n} - 1, 2\hat{n}, x\}$	
$\mathcal{R}^3$	$\{3\hat{n} - 2, 3\hat{n} - 1, 3\hat{n}, x\}$		$\mathcal{R}^4$	$\{4\hat{n} - 3, 4\hat{n} - 2, 4\hat{n} - 1, 4\hat{n}, x\}$	
$\tilde{\mathcal{R}}^1$	$\{1, x\} \quad x \geq 2\hat{n} + 1$		$\tilde{\mathcal{R}}^2$	$\{1, x\} \quad x \geq 2$	

Table 4: Notation Table

B Examples

Example 1 (Reductions). We visualize three of our modified reduction rules $\mathcal{R}^1$, $\mathcal{R}^2$, and $\tilde{\mathcal{R}}^1$ and illustrate how their outcomes differ from the corresponding classical reductions. Consider an instance with 2 agents and 7 goods, whose valuations are given in the table below:

Good	g_1	g_2	g_3	g_4	g_5	g_6	g_7
Agent 1	$7/13$	$7/13$	$4/13$	$3/13$	$3/13$	$1/13$	$1/13$
Agent 2	$8/13$	$5/13$	$5/13$	$3/13$	$2/13$	$2/13$	$1/13$

For $\alpha = 10/13$:

- The classical rule $\mathcal{R}^1$ assigns the bundle $\{g_2, g_3\}$, while our modified rule selects $\{g_2, g_5\}$.
- The classical rule $\mathcal{R}^2$ assigns $\{g_3, g_4, g_5\}$, whereas our version picks $\{g_3, g_4, g_6\}$.
- The classical rule $\tilde{\mathcal{R}}^1$ assigns $\{g_1, g_5\}$, while our modified rule selects $\{g_1, g_6\}$.

Note that unlike the classical reductions, we do not allocate these bundles immediately and defer the matching process.

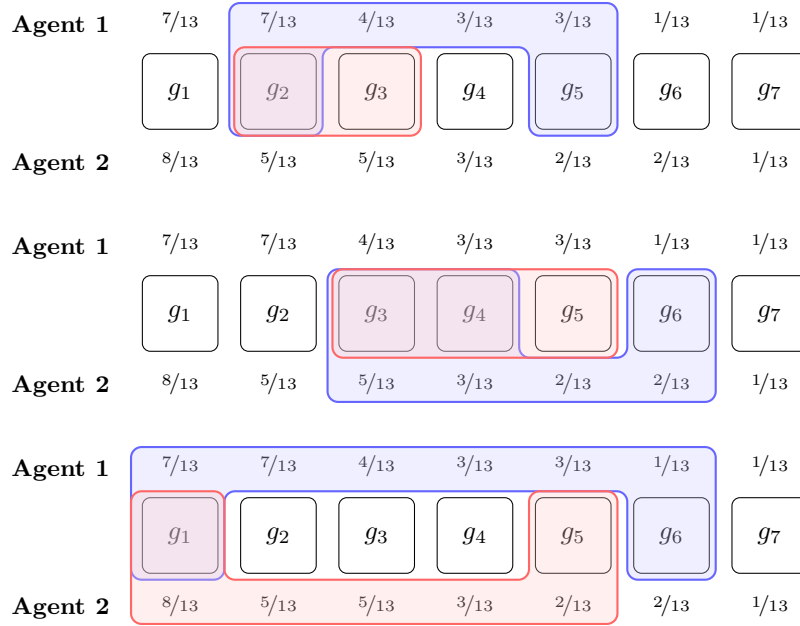
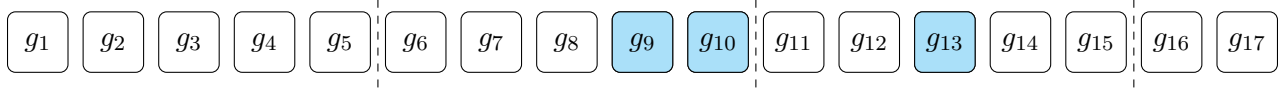


Figure 8: Comparison of classical and our modified reductions. Red boxes indicate the bundle chosen by the classical reductions, while blue boxes indicate the bundle chosen by our reductions.

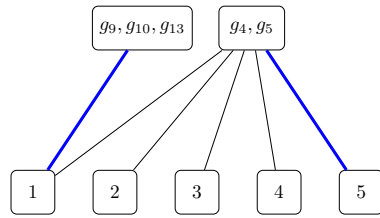
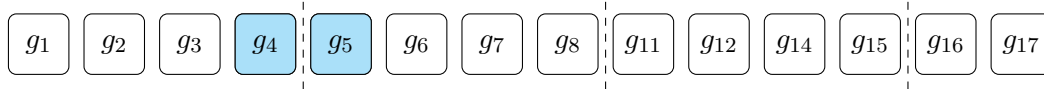
Example 2 (Primary Reductions). Consider an instance with 5 agents and 17 goods. We illustrate the primary reductions on this instance step by step. In this example, we consider $\alpha = 10/13$. The valuation functions are given in the table below: [§]

Good	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}	g_{17}
Agent 1	9/13	8/13	7/13	6/13	5/13	4/13	4/13	4/13	4/13	4/13	2/13	2/13	2/13	1/13	1/13	1/13	1/13
Agent 2	9/17	9/17	8/17	8/17	8/17	5/17	4/17	4/17	4/17	4/17	4/17	3/17	3/17	3/17	3/17	3/17	3/17
Agent 3	10/19	10/19	9/19	9/19	9/19	5/19	5/19	4/19	4/19	4/19	4/19	4/19	4/19	4/19	4/19	3/19	3/19
Agent 4	11/21	11/21	11/21	11/21	11/21	5/21	5/21	5/21	5/21	5/21	5/21	4/21	4/21	4/21	3/21	3/21	2/21
Agent 5	7/13	7/13	5/13	5/13	5/13	4/13	3/13	3/13	3/13	3/13	3/13	3/13	3/13	3/13	3/13	3/13	2/13

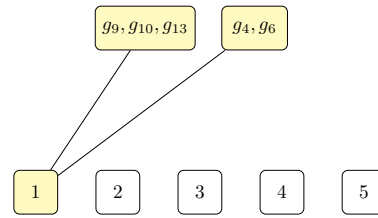
Initially, rule $\mathcal{R}^1$ is not applicable since no one values the bundle $\{g_5, g_6\}$ at a value of at least α . However, rule $\mathcal{R}^2$ is applicable, and the modified version of this rule selects the goods $\{g_9, g_{10}, g_{13}\}$. Note that the bundle is not allocated and is only considered in the matching.



In the second step, we consider the following reductions: $\mathcal{R}^1$ with the bundle $\{g_4, g_5\}$, $\mathcal{R}^2$ with $\{g_7, g_8, g_{11}\}$, and $\tilde{\mathcal{R}}^1$ with $\{g_1, g_{11}\}$. Rule $\mathcal{R}^1$ is applicable, and adding the bundle g_4, g_5 results in a matching of size two. However, we cannot select the bundle g_4, g_6 since, although Agent 1 still values this bundle, no matching of size two exists for it. [¶]

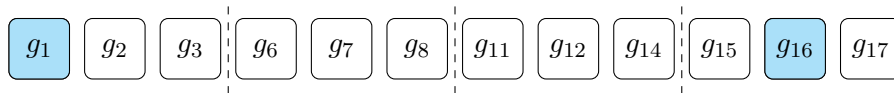


(a) Rule $\mathcal{R}^1$ is applicable.



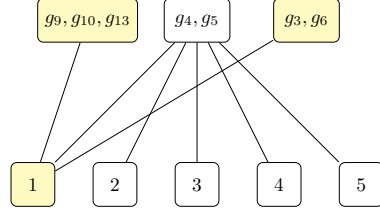
(b) Bundle $\{g_4, g_6\}$ cannot be selected.

In the third step, we consider the following reductions: $\mathcal{R}_1$ with the bundle $\{g_3, g_6\}$, $\mathcal{R}_2$ with $\{g_7, g_8, g_{11}\}$, and $\tilde{\mathcal{R}}^1$ with $\{g_1, g_{11}\}$. The rules $\mathcal{R}^1$ and $\mathcal{R}^2$ are not applicable, and the matching does not exist for these rules. However, $\tilde{\mathcal{R}}^1$ is applicable and can select the goods g_1 and g_{16} . For the bundle $\{g_1, g_{17}\}$, no matching exists.

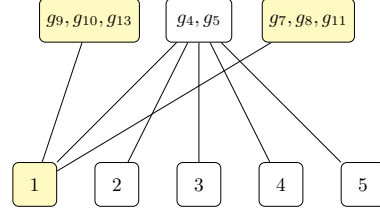


[§] Given $\alpha = 10/13$ (i) $13/17 < \alpha < 14/17$, (ii) $14/19 < \alpha < 15/19$, (iii) $16/21 < \alpha < 17/21$.

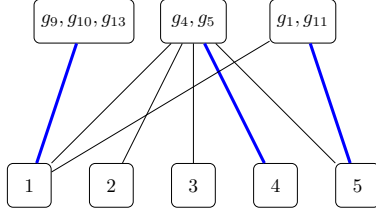
[¶] A saturating matching is shown with blue edges; otherwise, the Hall-violating set is marked in yellow.



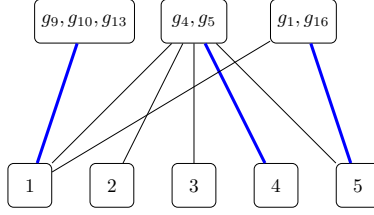
(a) Rule $\mathcal{R}^1$ is not applicable.



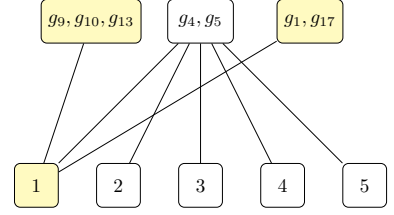
(b) Rule $\mathcal{R}^2$ is not applicable.



(a) Rule $\tilde{\mathcal{R}}^1$ is applicable.

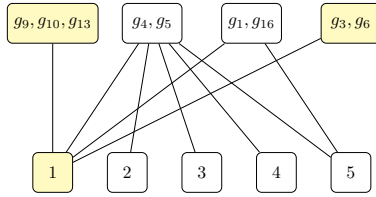
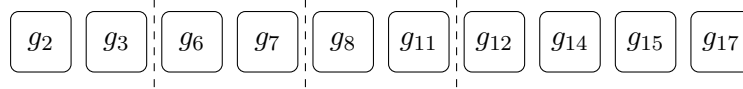


(b) Bundle $\{g_1, g_{16}\}$ is valid.

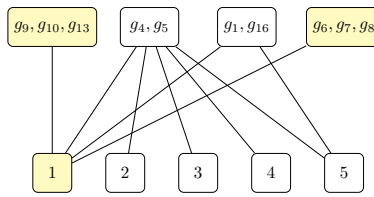


(c) Bundle $\{g_1, g_{17}\}$ cannot be selected.

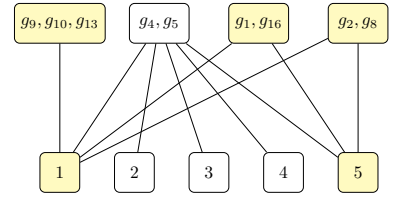
In the final step, none of the remaining reduction rules is applicable, so we complete the primary reduction phase. At this stage, we divide all agents into two groups, N^g and N^r , based on their valuation of the good g_8 , where N^g consists of agents who value g_8 at least $3/13$, and N^r includes the remaining agents. In this case, $N^g = \{1, 2, 4, 5\}$ and $N^r = \{3\}$ and $|N^g| \geq n/\sqrt{2}$. We then select a matching that maximizes the number of matched agents from N^r , and prioritize them in both primary reductions and the subsequent steps of the algorithm.



(a) Rule $\mathcal{R}^1$ is not applicable.



(b) Rule $\mathcal{R}^2$ is not applicable.



(c) Rule $\tilde{\mathcal{R}}^1$ is not applicable.

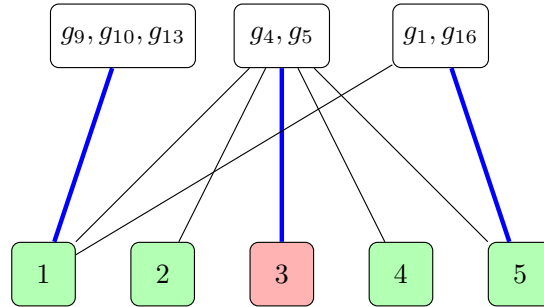


Figure 13: The final result of primary reductions.

C Bounds on MMS Values for Calibrated Valuations

Lemma 17. *Let $\hat{M}$ be a set of goods, d be a constant, and let $\hat{v}$ be a valuation function such that $\Psi_{\hat{v}}^d(\hat{M}) \geq 1$, and for all $\hat{g} \in \hat{M}$ we have $\hat{v}(\{\hat{g}\}) \leq 1$. Then for every $0 \leq \lambda \leq \frac{4\alpha}{3} - 1$ we have $\Psi_{(f_\lambda \star \hat{v})}^d(\hat{M}) \geq 1 - 3\lambda$.*

Proof. Since $\Psi_{\hat{v}}^d(\hat{M}) \geq 1$, we can partition $\hat{M}$ into $(P_1, \dots, P_d)$ such that each subset P_j satisfies $\hat{v}(P_j) \geq 1$. It suffices to show that for every $1 \leq j \leq d$, $(f_\lambda \star \hat{v})(P_j) \geq 1 - 3\lambda$, which directly implies $\Psi_{(f_\lambda \star \hat{v})}^d(\hat{M}) \geq 1 - 3\lambda$. Let $S = \{\hat{g} \in P_j \mid \hat{v}(\{\hat{g}\}) \geq \frac{\alpha}{3} - \lambda\}$. If $|S| \geq 4$, we have

$$\begin{aligned} (f_\lambda \star \hat{v})(P_j) &\geq 4 \cdot \left(\frac{\alpha}{3} - \lambda \right) \\ &= \frac{4\alpha}{3} - 4\lambda \\ &\geq 1 - 3\lambda \end{aligned} \quad \lambda \leq \frac{4\alpha}{3} - 1.$$

Therefore, assume $|S| \leq 3$. Note that for every good $\hat{g} \in P_j \setminus S$, we have $\hat{v}(\{\hat{g}\}) < \frac{\alpha}{3} - \lambda$, and thus $f_\lambda(\hat{v}(\{\hat{g}\})) = \hat{v}(\{\hat{g}\})$. We consider two cases.

- **At least one good $\hat{g}$ in S has value at least $1 - \frac{\alpha}{3} - \frac{\lambda}{2}$:** In particular, for this good, we have $f_\lambda(\hat{v}(\{\hat{g}\})) = \max(1 - \frac{\alpha}{3} - 2\lambda, \hat{v}(\{\hat{g}\}) - 3\lambda)$, therefore, the transformation f_λ reduces the original value by at most 3λ . Now If $\hat{g}$ is the only good in S , we get $(f_\lambda \star \hat{v})(P_j) \geq 1 - 3\lambda$. Otherwise, if there is another good in P_j with value at least $\frac{\alpha}{3} - \lambda$, then combining both goods ensures:

$$\begin{aligned} (f_\lambda \star \hat{v})(P_j) &\geq \left(1 - \frac{\alpha}{3} - 2\lambda\right) + \left(\frac{\alpha}{3} - \lambda\right) \\ &= 1 - 3\lambda. \end{aligned}$$

Thus, in both cases, we have $(f_\lambda \star \hat{v})(P_j) \geq 1 - 3\lambda$.

- **All goods in S have values below $1 - \frac{\alpha}{3} - \frac{\lambda}{2}$:** If there are at most two such goods in S , the transformation f_λ reduces their original values by at most $\frac{3\lambda}{2}$. Therefore the calibrated value satisfies $(f_\lambda \star \hat{v})(P_j) \geq 1 - 3\lambda$. Now, suppose there are exactly three such goods with value at least $\frac{\alpha}{3} - \lambda$. If at least one of them has a value of at least $1 - \frac{2\alpha}{3}$, then grouping it with the other two ensures:

$$\begin{aligned} (f_\lambda \star \hat{v})(P_j) &\geq \left(1 - \frac{2\alpha}{3} - \lambda\right) + 2\left(\frac{\alpha}{3} - \lambda\right) \\ &= 1 - 3\lambda. \end{aligned}$$

Otherwise, if all three goods have values below $1 - \frac{2\alpha}{3}$, the transformation f_λ reduces their original values by at most λ , leading to a total loss of at most 3λ , which again guarantees $(f_\lambda \star \hat{v})(P_j) \geq 1 - 3\lambda$.

Thus, in all cases, the bound holds. $\square$

Lemma 18. *Let $\hat{v}$ be a valuation function on $\hat{M}$ with $\Psi_{\hat{v}}^d(\hat{M}) \geq 4(1 - \alpha)$ and for all $\hat{g} \in \hat{M}$ we have $\hat{v}(\{\hat{g}\}) \leq 1$. Then $\Psi_{(h \star \hat{v})}^d(\hat{M}) \geq 4(2 - \frac{7\alpha}{3})$.*

Proof. Since $\Psi_{\hat{v}}^d(\hat{M}) \geq 4(1-\alpha)$, there exists a partition $(P_1, \dots, P_d)$ of $\hat{M}$ such that $\hat{v}(P_j) \geq 4(1-\alpha)$ for each j . We aim to show that for every P_j , $(h \star \hat{v})(P_j) \geq 4(2 - \frac{7\alpha}{3})$ which directly implies $\Psi_{(h \star \hat{v})}^d(\hat{M}) \geq 4(2 - \frac{7\alpha}{3})$. Let $S = \{\hat{g} \in P_j \mid \hat{v}(\{\hat{g}\}) \geq 2 - \frac{7\alpha}{3}\}$.

Note that if $|S| \geq 4$, we have: $(h \star \hat{v})(P_j) \geq 4(2 - \frac{7\alpha}{3})$. For the case that $|S| \leq 3$, by definition of S , there are at most three goods in P_j with $\hat{v}(\{\hat{g}\}) \geq 2 - \frac{7\alpha}{3}$, all other goods in P_j have $\hat{v}(\{\hat{g}\}) < 2 - \frac{7\alpha}{3}$, and thus $h(\hat{v}(\{\hat{g}\})) = \hat{v}(\{\hat{g}\})$. If $|S| \leq 2$, then the transformation h reduces their original values by at most $\frac{8\alpha}{3} - 2$, therefore

$$\begin{aligned} \sum_{\hat{g} \in P_j} (h \star \hat{v})(\{\hat{g}\}) &\geq \sum_{\hat{g} \in P_j} \hat{v}(\{\hat{g}\}) - 2\left(\frac{8\alpha}{3} - 2\right) \\ &\geq 4(1-\alpha) - 4\left(\frac{4\alpha}{3} - 1\right) \\ &= 4\left(2 - \frac{7\alpha}{3}\right). \end{aligned}$$

For $|S| = 3$ we consider two cases.

- **At most one good in S has value at least $2 - \frac{13\alpha}{6}$:** Then by definition of h , the transformation h reduces the original value of one good by at most $\frac{8\alpha}{3} - 2$, and two goods by at most $\frac{4\alpha}{3} - 1$. Hence:

$$\begin{aligned} \sum_{\hat{g} \in P_j} (h \star \hat{v})(\{\hat{g}\}) &\geq \sum_{\hat{g} \in P_j} \hat{v}(\{\hat{g}\}) - \left(\frac{8\alpha}{3} - 2\right) - 2\left(\frac{4\alpha}{3} - 1\right) \\ &\geq 4(1-\alpha) - 4\left(\frac{4\alpha}{3} - 1\right) \\ &= 4\left(2 - \frac{7\alpha}{3}\right). \end{aligned}$$

- **At least two goods in S have value at least $2 - \frac{13\alpha}{6}$:** Therefore there are two goods with value at least $2 - \frac{13\alpha}{6}$ and one good with value at least $2 - \frac{7\alpha}{3}$, ensures:

$$\begin{aligned} \sum_{\hat{g} \in P_j} (h \star \hat{v})(\{\hat{g}\}) &\geq 2h\left(2 - \frac{13\alpha}{6}\right) + h\left(2 - \frac{7\alpha}{3}\right) \\ &\geq 2\left(3 - \frac{7\alpha}{2}\right) + \left(2 - \frac{7\alpha}{3}\right) \\ &= 4\left(2 - \frac{7\alpha}{3}\right). \end{aligned}$$

Thus $\Psi_{(h \star \hat{v})}^d(\hat{M}) \geq 4(2 - \frac{7\alpha}{3})$, as desired. $\square$

Lemma 19. Let $\hat{M}$ be a set of goods, d be a constant, and let $\hat{v}$ be a valuation function such that $\Psi_{\hat{v}}^d(\hat{M}) \geq 1$ and for all $\hat{g} \in \hat{M}$ we have $\hat{v}(\{\hat{g}\}) \leq 1$. Then we have $\Psi_{(w_\lambda \star \hat{v})}^d(\hat{M}) \geq 1 - 2\lambda$.

Proof. Since $\Psi_{\hat{v}}^d(\hat{M}) \geq 1$, there exists a partition $(P_1, \dots, P_d)$ of $\hat{M}$ such that $\hat{v}(P_j) \geq 1$ for each $1 \leq j \leq d$. We want to show that for every $1 \leq j \leq d$, $(w_\lambda \star \hat{v})(P_j) \geq 1 - 2\lambda$, which directly implies $\Psi_{(w_\lambda \star \hat{v})}^d(\hat{M}) \geq 1 - 2\lambda$. Let $S = \{\hat{g} \in P_j \mid \hat{v}(\{\hat{g}\}) \geq \frac{1}{2} - \lambda\}$.

We consider two cases:

- $|S| < 2$: By definition of w_λ , the transformation w_λ reduces the original value of one good by at most 2λ . Hence:

$$\begin{aligned} \sum_{\hat{g} \in P_j} (w_\lambda \star \hat{v})(\{\hat{g}\}) &\geq \sum_{\hat{g} \in P_j} v(\{\hat{g}\}) - 2\lambda \\ &\geq 1 - 2\lambda. \end{aligned}$$

- $|S| \geq 2$: In this case for each $\hat{g} \in S$ we have $w_\lambda(\hat{v}(\{\hat{g}\})) \geq \frac{1}{2} - \lambda$. Therefore:

$$\begin{aligned} \sum_{\hat{g} \in P_j} (w_\lambda \star \hat{v})(\{\hat{g}\}) &\geq 2\left(\frac{1}{2} - \lambda\right) \\ &= 1 - 2\lambda. \end{aligned}$$

□

Lemma 20. *Let $\hat{M}$ be a set of goods, d be a constant, and let $\hat{v}$ be a valuation function such that $\Psi_{\hat{v}}^d(\hat{M}) \geq 4(1-\alpha)$ and for all $\hat{g} \in \hat{M}$ we have $\hat{v}(\{\hat{g}\}) \leq 1$. Then we have $\Psi_{(z_\lambda \star \hat{v})}^d(\hat{M}) \geq 4(1-\alpha) - 2\lambda$.*

Proof. Since $\Psi_{\hat{v}}^d(\hat{M}) \geq 4(1-\alpha)$, there exists a partition $(P_1, \dots, P_d)$ of $\hat{M}$ such that $\hat{v}(P_j) \geq 4(1-\alpha)$ for each $1 \leq j \leq d$. We want to show that for every $1 \leq j \leq d$, $(z_\lambda \star \hat{v})(P_j) \geq 4(1-\alpha) - 2\lambda$, which directly implies $\Psi_{(z_\lambda \star \hat{v})}^d(\hat{M}) \geq 4(1-\alpha) - 2\lambda$. Let $S = \{\hat{g} \in P_j \mid \hat{v}(\{\hat{g}\}) \geq 2(1-\alpha) - \lambda\}$. We consider two cases:

- $|S| < 2$: By definition of z_λ , the transformation z_λ reduces the original value of one good by at most 2λ . Hence:

$$\begin{aligned} \sum_{\hat{g} \in P_j} (z_\lambda \star \hat{v})(\{\hat{g}\}) &\geq \sum_{\hat{g} \in P_j} v(\{\hat{g}\}) - 2\lambda \\ &\geq 4(1-\alpha) - 2\lambda. \end{aligned}$$

- $|S| \geq 2$: In this case for each $\hat{g} \in S$ we have $z_\lambda(\hat{v}(\{\hat{g}\})) \geq 2(1-\alpha) - \lambda$. Therefore:

$$\begin{aligned} \sum_{\hat{g} \in P_j} (z_\lambda \star \hat{v})(\{\hat{g}\}) &\geq 2(2(1-\alpha) - \lambda) \\ &= 4(1-\alpha) - 2\lambda. \end{aligned}$$

□

D Proofs for Section 5 (Calibration)

Lemma 2. *Let $\hat{I} = (\hat{N}, \hat{M})$ be an ordered instance, and let $\mathfrak{R}_1 = [\mathcal{R}^0 \succ \mathcal{R}^1 \succ \mathcal{R}^2 \succ \tilde{\mathcal{R}}^1]$ and $\mathfrak{R}_2 = [\mathcal{R}^1 \succ \mathcal{R}^2 \succ \mathcal{R}^3 \succ \mathcal{R}^4 \succ \tilde{\mathcal{R}}^2]$. Assume that $\check{I} = (\check{N}, \check{M})$ is the result of applying a sequence of valid reductions with respect to either $\mathfrak{R}_1$ or $\mathfrak{R}_2$. Then, the conditions shown in Table 2 satisfy.*

Proof. Case 1: Since $\hat{\Psi}_{\hat{v}_i} \geq 1$, by Lemma 17, we have $\hat{\Psi}_{(f_\lambda \star \hat{v}_i)} \geq 1 - 3\lambda$. Therefore it suffices to show that no reduction makes the MMS under $(f_\lambda \star \hat{v}_i)$ less than $1 - 3\lambda$. By Observation 1, the only reduction that can decrease the MMS value of an agent is $\tilde{\mathcal{R}}^1$. Suppose that the rule allocates goods $\hat{g}_1$ and $\hat{g}_x$, where $x \geq 2\hat{n} + 1$. As Precondition 2 holds, we have $\hat{v}_i(\hat{g}_1) \leq 1 - \frac{\alpha}{3} + \lambda$, and

Func	Red	Prec 1	Prec 2	Prec 3	MMS Guarantee
f_λ	$\mathfrak{R}_1$	$\lambda \leq \frac{4\alpha}{3} - 1$	$\hat{\Psi}_{\hat{v}_i} \geq 1$	$\hat{v}_i(\{\hat{g}_1\}) \leq 1 - \frac{\alpha}{3} + \lambda$	$\check{\Psi}_{(f_\lambda \star \hat{v}_i)} \geq 1 - 3\lambda$
w_λ	$\mathfrak{R}_2$	$\lambda \leq \frac{1}{2}$	$\hat{\Psi}_{\hat{v}_i} \geq 1$	$\hat{v}_i(\{\hat{g}_1\}) \leq \frac{1}{2} + \lambda$	$\check{\Psi}_{(w_\lambda \star \hat{v}_i)} \geq 1 - 2\lambda$
z_λ	$\mathfrak{R}_2$	$\lambda \leq 2(1 - \alpha)$	$\hat{\Psi}_{\hat{v}_i} \geq 4(1 - \alpha)$	$\hat{v}_i(\{\hat{g}_1\}) \leq 2(1 - \alpha) + \lambda$	$\check{\Psi}_{(z_\lambda \star \hat{v}_i)} \geq 4(1 - \alpha) - 2\lambda$

Table 5: Calibrated MMS bounds under various reduction sequences. If an instance satisfies both preconditions, the stated guarantee holds for the calibrated MMS after applying the reductions.

$f_\lambda(1 - \frac{\alpha}{3} + \lambda) = 1 - \frac{\alpha}{3} - 2\lambda$. Furthermore, since $\mathcal{R}^2$ is not applicable, we have $\hat{v}_i(\hat{g}_x) \leq \frac{\alpha}{3}$, and $f_\lambda(\frac{\alpha}{3}) = \frac{\alpha}{3} - \lambda$. Hence, $(f \star \hat{v}_i)(\{g_1, g_x\}) \leq 1 - \frac{\alpha}{3} - 2\lambda + \frac{\alpha}{3} - \lambda = 1 - 3\lambda$, and by Observation 2, this reduction does not decrease MMS under $(f_\lambda \star \hat{v}_i)$. Therefore, in the final instance $\check{M}$, we have $\check{\Psi}_{(f_\lambda \star \hat{v}_i)} \geq 1 - 3\lambda$.

Case 2: Since $\hat{\Psi}_{\hat{v}_i} \geq 1$, by Lemma 19, we have $\hat{\Psi}_{(w_\lambda \star \hat{v}_i)} \geq 1 - 2\lambda$. Therefore it suffices to show that no reduction makes the MMS under $(w_\lambda \star \hat{v}_i)$ less than $1 - 2\lambda$. By Observation 1, the only reduction that can decrease the MMS value of an agent is $\tilde{\mathcal{R}}^2$. Suppose that the rule allocates goods $\hat{g}_1$ and $\hat{g}_x$, where $x \geq 2$. As Precondition 2 holds, we have $\hat{v}_i(\hat{g}_1) \leq \frac{1}{2} + \lambda$, and $w_\lambda(\frac{1}{2} + \lambda) = \frac{1}{2} - \lambda$. Hence, $(w \star \hat{v}_i)(\{g_1, g_x\}) \leq 2(\frac{1}{2} - \lambda) = 1 - 2\lambda$, and by Observation 2, this reduction does not decrease MMS under $(w_\lambda \star \hat{v}_i)$. Therefore, in the final instance $\check{M}$, we have $\check{\Psi}_{(w_\lambda \star \hat{v}_i)} \geq 1 - 2\lambda$.

Case 3: Since $\hat{\Psi}_{\hat{v}_i} \geq 4(1 - \alpha)$, by Lemma 20, we have $\hat{\Psi}_{(z_\lambda \star \hat{v}_i)} \geq 4(1 - \alpha) - 2\lambda$. Therefore it suffices to show that no reduction makes the MMS under $(z_\lambda \star \hat{v}_i)$ less than $4(1 - \alpha) - 2\lambda$. By Observation 1, the only reduction that can decrease the MMS value of an agent is $\tilde{\mathcal{R}}^2$. Suppose that the rule allocates goods $\hat{g}_1$ and $\hat{g}_x$, where $x \geq 2$. As Precondition 2 holds, we have $\hat{v}_i(\hat{g}_1) \leq 2(1 - \alpha) + \lambda$, and $z_\lambda(2(1 - \alpha) + \lambda) = 2(1 - \alpha) - \lambda$. Hence, $(z \star \hat{v}_i)(\{g_1, g_x\}) \leq 4(1 - \alpha) - 2\lambda$, and by Observation 2, this reduction does not decrease MMS under $(z_\lambda \star \hat{v}_i)$. Therefore, in the final instance $\check{M}$, we have $\check{\Psi}_{(z_\lambda \star \hat{v}_i)} \geq 4(1 - \alpha) - 2\lambda$. $\square$

E Proofs for Section 7 (Algorithm 3: Frequent Green Agents)

Lemma 8. *Every green agent $a_i \in \check{N}$ with $\check{\Psi}_{v_i} \geq 1$ after the primary reductions, and $\check{\Psi}_{v_i} \geq 1$ after the secondary reductions, receives a bundle in Algorithm 5.*

Proof. Note that by Lemma 17, we have $\check{\Psi}_{(f \star v_i)} \geq 4(1 - \alpha)$, therefore by Lemma 18, we conclude $\check{\Psi}_{(h \star f \star v_i)} \geq 4(2 - \frac{7\alpha}{3})$.

If $v_i(\{\check{g}_{3\check{n}}\}) \leq \frac{4\alpha}{3} - 1$, we show that function $(h \star f \star v_i)$ satisfies conditions of Lemma 6:

- Inequality (10) follows directly by Definition 4.
- For Inequality (11) we have:

$$\begin{aligned}
\alpha + (h \star f \star v_i)(\{\check{g}_{3\check{n}+1}\}) &< \alpha + \frac{4\alpha}{3} - 1 \\
&\leq 4 \left(2 - \frac{7\alpha}{3} \right) & \alpha \leq \frac{27}{35}, \\
&\leq \check{\Psi}_{(h \star f \star v_i)}.
\end{aligned}$$

- For Inequality (12) we have:

$$\begin{aligned}
& (h \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}\}) \\
& \leq \left(3 - \frac{7\alpha}{2}\right) + \left(3 - \frac{7\alpha}{2}\right) + \left(2 - \frac{7\alpha}{3}\right) \quad \text{Observation 5,} \\
& = 4 \left(2 - \frac{7\alpha}{3}\right).
\end{aligned}$$

- To prove Inequality (13), first we show $(h \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) < \frac{7\alpha}{3} - 1$:

$$\begin{aligned}
& (h \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) \\
& < \alpha + (h \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_{3\ddot{n}}\}) \quad \widetilde{\mathcal{R}}^2 \text{ is not applicable,} \\
& \leq \alpha + \left(\frac{4\alpha}{3} - 1\right) \\
& = \frac{7\alpha}{3} - 1.
\end{aligned}$$

Since $\alpha \leq \frac{17}{22}$, we have $\frac{7\alpha}{3} - 1 + \alpha \leq 2(4(2 - \frac{7\alpha}{3}))$, therefore we can verify Inequality (13).

Thus, we assume $v_i(\{\ddot{g}_{3\ddot{n}}\}) > \frac{4\alpha}{3} - 1$, in the rest of the proof.

Now, consider for all $2 \leq k \leq \ddot{n}$, $v_i(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}\}) < 1$ holds. Under this assumption, we show that function v_i satisfies conditions of Lemma 6:

- Inequality (10) trivially holds.
- To prove Inequality (11), we have:

$$\begin{aligned}
\alpha + v_i(\{\ddot{g}_{3\ddot{n}+1}\}) & < \alpha + \frac{\alpha}{4} \quad \mathcal{R}^3 \text{ is not applicable,} \\
& \leq 1 \quad \alpha \leq \frac{4}{5}, \\
& \leq \ddot{\Psi}_{v_i}.
\end{aligned}$$

- Inequality (12) holds by the assumption that for each $k > 1$ we have $v_i(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}\}) < 1$.
- To prove Inequality (13), first we show $v_i(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) < \frac{4\alpha}{3}$:

$$\begin{aligned}
& v_i(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) \\
& = v_i(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}\}) + v_i(\{\ddot{g}_{3\ddot{n}}\}) \\
& < \alpha + v_i(\{\ddot{g}_{3\ddot{n}}\}) \quad \widetilde{\mathcal{R}}^2 \text{ is not applicable,} \\
& \leq \alpha + \frac{\alpha}{3} \quad \text{Observation 5,} \\
& = \frac{4\alpha}{3}.
\end{aligned}$$

For $\alpha \leq \frac{6}{7}$ we have $\frac{4\alpha}{3} + \alpha \leq 2$.

Next we show if for some $2 \leq k \leq \ddot{n}$, $v_i(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}\}) \geq 1$ holds, then for all $1 \leq k \leq \ddot{n}$ we have $v_i(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}\}) \geq \alpha$ which implies agent a_i receives a bundle. We first show that if $v_i(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}\}) \geq 1$, then

$$v_i(\{\ddot{g}_{\ddot{n}+k}\}) \geq 1 - \frac{5\alpha}{6} \quad \text{and} \quad v_i(\{\ddot{g}_{3\ddot{n}-k+1}\}) \geq 1 - \alpha.$$

Since $\widetilde{\mathcal{R}}^2$ is not applicable, we have $v_i(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}\}) \leq \alpha$. Therefore, $v_i(\{\ddot{g}_{3\ddot{n}-k+1}\}) \geq 1 - \alpha$. By Observation 5 we have

$$v_i(\{\ddot{g}_k\}) \leq \frac{\alpha}{2} \quad \text{and} \quad v_i(\{\ddot{g}_{3\ddot{n}-k+1}\}) \leq \frac{\alpha}{3}.$$

Hence, $v_i(\{\ddot{g}_{\ddot{n}+k}\}) \geq 1 - \frac{5\alpha}{6}$.

Next, we prove that for every index $j > k$ we have $v_i(\{\ddot{g}_j, \ddot{g}_{\ddot{n}+j}, \ddot{g}_{3\ddot{n}-j+1}\}) \geq \alpha$:

$$\begin{aligned} & v_i(\{\ddot{g}_j, \ddot{g}_{\ddot{n}+j}, \ddot{g}_{3\ddot{n}-j+1}\}) \\ & \geq v_i(\{\ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}, \ddot{g}_{3\ddot{n}-k+1}\}) \\ & \geq \left(1 - \frac{5\alpha}{6}\right) + (1 - \alpha) + (1 - \alpha) \\ & \geq \alpha \end{aligned} \quad \alpha \leq \frac{18}{23}.$$

Finally, we show that for every index $j < k$ it holds that

$$\begin{aligned} & v_i(\{\ddot{g}_j, \ddot{g}_{\ddot{n}+j}, \ddot{g}_{3\ddot{n}-j+1}\}) \\ & \geq v_i(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}}\}) \\ & = \left(v_i(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}\}) - v_i(\{\ddot{g}_{3\ddot{n}+1-k}\})\right) + v_i(\{\ddot{g}_{3\ddot{n}}\}) \\ & \geq \left(1 - v_i(\{\ddot{g}_{3\ddot{n}+1-k}\})\right) + v_i(\{\ddot{g}_{3\ddot{n}}\}) \\ & \geq \left(1 - v_i(\{\ddot{g}_{3\ddot{n}+1-k}\})\right) + \frac{4\alpha}{3} - 1 \\ & \geq \left(1 - \frac{\alpha}{3}\right) + \frac{4\alpha}{3} - 1 \quad \text{Observation 5,} \\ & = \alpha. \end{aligned}$$

Completing the proof. $\square$

Lemma 9. *Every green agent $a_i \in \ddot{N}$ with $\dot{\Psi}_{v_i} < 1$ after the primary reductions, and $\ddot{\Psi}_{(f \star v_i)} \geq 4(1 - \alpha)$ after the secondary reductions, receives a bundle in Algorithm 5.*

Proof. If $v_i(\{\ddot{g}_{3\ddot{n}}\}) \leq \frac{4\alpha}{3} - 1$, we show that function $(h \star \mathring{f} \star v_i)$ satisfies conditions of Lemma 6.

- Inequality (10) follows directly by Definition 4.
- For Inequality (11) we have:

$$\begin{aligned} \alpha + (h \star \mathring{f} \star v_i)(\{\ddot{g}_{3\ddot{n}+1}\}) & < \alpha + \frac{4\alpha}{3} - 1 \\ & \leq 4 \left(2 - \frac{7\alpha}{3}\right) \quad \alpha \leq \frac{27}{35}, \\ & \leq \ddot{\Psi}_{(h \star \mathring{f} \star v_i)} \quad \text{Lemma 18.} \end{aligned}$$

- For Inequality (12) we have:

$$\begin{aligned} & (h \star \mathring{f} \star v_i)(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}\}) \\ & \leq \left(3 - \frac{7\alpha}{2}\right) + \left(3 - \frac{7\alpha}{2}\right) + \left(2 - \frac{7\alpha}{3}\right) \quad \text{Observation 5,} \\ & = 4 \left(2 - \frac{7\alpha}{3}\right) \\ & \leq \ddot{\Psi}_{(h \star \mathring{f} \star v_i)} \quad \text{Lemma 18.} \end{aligned}$$

- To prove Inequality (13), first we show $(h \star \mathring{f} \star v_i)(\{\mathring{g}_1, \mathring{g}_{\bar{n}+1}, \mathring{g}_{3\bar{n}}\}) < \frac{7\alpha}{3} - 1$:

$$\begin{aligned}
& (h \star \mathring{f} \star v_i)(\{\mathring{g}_1, \mathring{g}_{\bar{n}+1}, \mathring{g}_{3\bar{n}}\}) \\
& < \alpha + (h \star \mathring{f} \star v_i)(\{\mathring{g}_{3\bar{n}}\}) \quad \widetilde{\mathcal{R}}^2 \text{ is not applicable,} \\
& \leq \alpha + \left(\frac{4\alpha}{3} - 1 \right) \\
& = \frac{7\alpha}{3} - 1.
\end{aligned}$$

Since $\alpha \leq \frac{17}{22}$ we have $\frac{7\alpha}{3} - 1 + \alpha \leq 2(4(2 - \frac{7\alpha}{3}))$, therefore by Lemma 18 we can verify Inequality (13).

Thus, we assume $v_i(\{\mathring{g}_{3\bar{n}}\}) > \frac{4\alpha}{3} - 1$ in the rest of the proof.

Now, consider for all $2 \leq k \leq \bar{n}$, $(\mathring{f} \star v_i)(\{\mathring{g}_k, \mathring{g}_{\bar{n}+k}, \mathring{g}_{3\bar{n}}\}) < 4(1 - \alpha)$ holds. Under this assumption, we show that function $(\mathring{f} \star v)$ satisfies conditions of Lemma 6:

- Inequality (10) follows directly by Definition 4.
- For Inequality (11) we have:

$$\begin{aligned}
& \alpha + (f \star v_i)(\{\mathring{g}_{3\bar{n}+1}\}) \\
& < \alpha + \frac{4\alpha}{3} - 1 \quad \text{Observation 6,} \\
& \leq 4(1 - \alpha) \quad \alpha \leq \frac{15}{19}, \\
& \leq \ddot{\Psi}_{(\mathring{f} \star v_i)}.
\end{aligned}$$

- Inequality (12) holds by the assumption $(\mathring{f} \star v_i)(\{\mathring{g}_k, \mathring{g}_{\bar{n}+k}, \mathring{g}_{3\bar{n}-k+1}\}) < 4(1 - \alpha)$.
- To prove Inequality (13), we show first that

$$(\mathring{f} \star v_i)(\{\mathring{g}_1, \mathring{g}_{\bar{n}+1}, \mathring{g}_{3\bar{n}}\}) = (\mathring{f} \star v_i)(\{\mathring{g}_1, \mathring{g}_{\bar{n}+1}\}) + (\mathring{f} \star v_i)(\{\mathring{g}_{3\bar{n}}\}) < \alpha + (1 - \alpha) = 1,$$

where the inequality uses the fact that $\widetilde{\mathcal{R}}^2$ is not applicable and Observation 5. Since $\alpha \leq \frac{7}{9}$, it follows that

$$\alpha + (\mathring{f} \star v_i)(\{\mathring{g}_1, \mathring{g}_{\bar{n}+1}, \mathring{g}_{3\bar{n}}\}) \leq 8(1 - \alpha).$$

Now, we show that if for some $2 \leq k \leq \bar{n}$, $(\mathring{f} \star v_i)(\{\mathring{g}_k, \mathring{g}_{\bar{n}+k}, \mathring{g}_{3\bar{n}-k+1}\}) \geq 4(1 - \alpha)$ holds, then for all $1 \leq j \leq \bar{n}$ we have $(\mathring{f} \star v_i)(\{\mathring{g}_j, \mathring{g}_{\bar{n}+j}, \mathring{g}_{3\bar{n}-j+1}\}) \geq \alpha$ which implies agent a_i receives a bundle. We first show that $(\mathring{f} \star v_i)(\{\mathring{g}_{\bar{n}+k}\}) \geq 2 - \frac{13\alpha}{6}$. By Observation 5 we have

$$(\mathring{f} \star v_i)(\{\mathring{g}_k\}) \leq 1 - \frac{5\alpha}{6} \quad \text{and} \quad (\mathring{f} \star v_i)(\{\mathring{g}_{3\bar{n}-k+1}\}) \leq 1 - \alpha.$$

Therefore

$$\begin{aligned}
(\mathring{f} \star v_i)(\{\mathring{g}_{\bar{n}+k}\}) & \geq 4(1 - \alpha) - \left(1 - \frac{5\alpha}{6}\right) - (1 - \alpha) \\
& = 2 - \frac{13\alpha}{6}.
\end{aligned}$$

By the assumption $v_i(\{\ddot{g}_{3\ddot{n}}\}) > \frac{4\alpha}{3} - 1$, and by Lemma 3, we have $v_i(\{\ddot{g}_{3\ddot{n}}\}) > 1 - \alpha$, which by definition of $\mathring{f}$ we can conclude that $(\mathring{f} \star v_i)(\{\ddot{g}_{3\ddot{n}}\}) \geq 1 - \alpha$. Thus, for any $1 \leq j \leq \ddot{n}$, we have:

$$\begin{aligned}
& (\mathring{f} \star v_i)(\{\ddot{g}_j, \ddot{g}_{\ddot{n}+j}, \ddot{g}_{3\ddot{n}-j+1}\}) \\
& \geq (\mathring{f} \star v_i)(\{\ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}}, \ddot{g}_{3\ddot{n}}\}) \\
& \geq \left(2 - \frac{13\alpha}{6}\right) + (1 - \alpha) + (1 - \alpha) \\
& \geq \alpha
\end{aligned}
\qquad \alpha \leq \frac{24}{31}.$$

Completing the proof. $\square$

Lemma 10. *Every green agent a_i in $\ddot{N}$ such that $\dot{\Psi}_{v_i} \geq 1$ after the primary reductions and $\ddot{\Psi}_{v_i} < 1$ after the secondary reductions, receives a bundle in Algorithm 5.*

Proof. Note that by Lemma 4, there exists a number t_i satisfying conditions Inequality (4), Equation (5), and Inequality (6). We verify that valuation function v_i satisfies all conditions required by Lemma 7:

- Inequality (14) holds trivially.
- For Inequality (15), we have:

$$\begin{aligned}
\alpha + v_i(\{\ddot{g}_{4\ddot{n}+1}\}) & < \alpha + \frac{\alpha}{5} & \mathcal{R}^4 \text{ is not applicable,} \\
& \leq 1 - 2 \left(2\alpha - \frac{3}{2}\right) & \alpha \leq \frac{10}{13}, \\
& \leq 1 - 2t_i & \text{Inequality (4),} \\
& \leq \ddot{\Psi}_{v_i} & \text{Inequality (6).}
\end{aligned}$$

- For Inequality (16), first, we show that $v_i(\{\ddot{g}_2\}) \leq \alpha - \frac{1}{2} - t_i$.

$$\begin{aligned}
v_i(\{\ddot{g}_2\}) & \leq \frac{\alpha}{2} & \text{Observation 5,} \\
& \leq \frac{1}{2} - \left(2\alpha - \frac{3}{2}\right) & \alpha \leq \frac{4}{5}, \\
& \leq \frac{1}{2} - t_i & \text{Inequality (4).}
\end{aligned}$$

Equation (5) ensures no good's value lies in interval $[\alpha - \frac{1}{2} - t_i, \frac{1}{2} - t_i]$. Thus, since $\frac{\alpha}{2}$ is within this interval, we conclude $v_i(\{\ddot{g}_2\}) \leq \alpha - \frac{1}{2} - t_i$.

Now we bound $v_i(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\})$:

$$\begin{aligned}
& v_i(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) \\
&= v_i(\{\ddot{g}_1\}) + v_i(\{\ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) \\
&\leq v_i(\{\ddot{g}_1\}) + 2 v_i(\{\ddot{g}_2\}) \\
&\leq \left(t_i + \frac{1}{2}\right) + 2 \left(\alpha - \frac{1}{2} - t_i\right) && \text{Inequality (4),} \\
&= 2\alpha - \frac{1}{2} - t_i.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \alpha + v_i(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) \\
&\leq 3\alpha - \frac{1}{2} - t_i \\
&\leq 2 - 3 \left(2\alpha - \frac{3}{2}\right) - t_i && \alpha \leq \frac{7}{9}, \\
&\leq 2(1 - 2t_i) && \text{Inequality (4),} \\
&\leq 2\ddot{\Psi}_{v_i} && \text{Inequality (6).}
\end{aligned}$$

- To prove Inequality (17), for all $2 \leq k \leq \ddot{n}$, we have:

$$\begin{aligned}
& v_i(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}, \ddot{g}_{3\ddot{n}+k}\}) \\
&\leq 3 \left(\alpha - \frac{1}{2} - t_i\right) + v_i(\{\ddot{g}_{3\ddot{n}+k}\}) \\
&\leq 3 \left(\alpha - \frac{1}{2} - t_i\right) + \frac{\alpha}{4} && \mathcal{R}^3 \text{ is not applicable,} \\
&= 1 - 2t_i - \left(\frac{5}{2} - \frac{13\alpha}{4} + t_i\right) \\
&< 1 - 2t_i - \left(\frac{5}{2} - \frac{13\alpha}{4}\right) && t_i > 0, \\
&\leq 1 - 2t_i && \alpha \leq \frac{10}{13}, \\
&\leq \ddot{\Psi}_{v_i} && \text{Inequality (6).}
\end{aligned}$$

- For Inequality (18), we have:

$$\begin{aligned}
& 2\alpha + v_i(\{\ddot{g}_{3\ddot{n}+1}\}) < 2\alpha + \frac{\alpha}{4} && \mathcal{R}^3 \text{ is not applicable,} \\
&= 2(1 - 2t_i) - \left(2(1 - 2t_i) - \frac{9\alpha}{4}\right) \\
&\leq 2(1 - 2t_i) - \left(2\left(1 - 2\left(2\alpha - \frac{3}{2}\right)\right) - \frac{9\alpha}{4}\right) && \text{Inequality (4),} \\
&\leq 2(1 - 2t_i) && \alpha \leq \frac{32}{41}, \\
&\leq 2\ddot{\Psi}_{v_i} && \text{Inequality (6).}
\end{aligned}$$

- To prove Inequality (19), note that we have already shown that $v_i(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) \leq 2\alpha - \frac{1}{2} - t_i$. Therefore we have:

$$\begin{aligned}
& 2\alpha + v_i(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}, \ddot{g}_{3\ddot{n}+1}\}) \\
& \leq 2\alpha + (2\alpha - \frac{1}{2} - t_i) + v_i(\{\ddot{g}_{3\ddot{n}+1}\}) \\
& \leq 2\alpha + (2\alpha - \frac{1}{2} - t_i) + \frac{\alpha}{4} & \mathcal{R}^3 \text{ is not applicable,} \\
& = 3(1 - 2t_i) - \left(\frac{7}{2} - 5t_i - \frac{17\alpha}{4}\right) \\
& \leq 3(1 - 2t_i) - \left(\frac{7}{2} - 5\left(2\alpha - \frac{3}{2}\right) - \frac{17\alpha}{4}\right) & \text{Inequality (4),} \\
& \leq 3(1 - 2t_i) & \alpha \leq \frac{44}{57}, \\
& \leq 3\ddot{\Psi}_{v_i} & \text{Inequality (6).}
\end{aligned}$$

Since all conditions of Lemma 7 are satisfied, the proof is complete. $\square$

Lemma 11. *Every green agent a_i in $\ddot{N}$ such that $\ddot{\Psi}_{v_i} < 1$ after the primary reductions and $\ddot{\Psi}_{(\mathring{f} \star v_i)} < 4(1 - \alpha)$ after the secondary reductions, receives a bundle in Algorithm 5.*

Proof. Note that by Lemma 4, there exists a number t_i satisfying conditions Inequality (7), Equation (8), and Inequality (9). We verify that valuation function $(z_{t_i} \star \mathring{f} \star v)$ satisfies all conditions required by Lemma 7:

- Inequality (14) follows directly by Definition 4.
- To prove Inequality (15), we show:

$$\begin{aligned}
& \alpha + (z_{t_i} \star \mathring{f} \star v_i)(\{\ddot{g}_{4\ddot{n}+1}\}) \\
& < \alpha + \frac{4\alpha}{3} - 1 & \text{Observation 6,} \\
& \leq 4(1 - \alpha) - 2\left(2\alpha - \frac{3}{2}\right) & \alpha \leq \frac{24}{31}, \\
& \leq 4(1 - \alpha) - 2t_i & \text{Inequality (7),} \\
& \leq \ddot{\Psi}_{(z_{t_i} \star \mathring{f} \star v_i)} & \text{Inequality (9).}
\end{aligned}$$

- To prove Inequality (16), first, we show that $(\mathring{f} \star v_i)(\{\ddot{g}_2\}) \leq \frac{5\alpha}{3} - 1 - t_i$.

$$\begin{aligned}
& (\mathring{f} \star v_i)(\{\ddot{g}_2\}) \leq 1 - \frac{5\alpha}{6} & \text{Observation 5,} \\
& \leq 2(1 - \alpha) - \left(2\alpha - \frac{3}{2}\right) & \alpha \leq \frac{15}{19}, \\
& \leq 2(1 - \alpha) - t_i & \text{Inequality (7).}
\end{aligned}$$

Equation (8) ensures no good's value lies in interval $\left[\frac{5\alpha}{3} - 1 - t_i, 2(1 - \alpha) - t_i\right]$. Thus, since $1 - \frac{5\alpha}{6}$ is within this interval, we conclude $(\mathring{f} \star v_i)(\{\ddot{g}_2\}) \leq \frac{5\alpha}{3} - 1 - t_i$.

Now we bound $(z_{t_i} \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\})$:

$$\begin{aligned}
& (z_{t_i} \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) \\
& \leq (z_{t_i} \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_1\}) + (\overset{\circ}{f} \star v_i)(\{\ddot{g}_{\ddot{n}+1}\}) + (\overset{\circ}{f} \star v_i)(\{\ddot{g}_{3\ddot{n}}\}) \\
& \leq (z_{t_i} \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_1\}) + (\overset{\circ}{f} \star v_i)(\{\ddot{g}_2\}) + (1 - \alpha) && \text{Observation 5,} \\
& \leq (z_{t_i} \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_1\}) + \left(\frac{5\alpha}{3} - 1 - t_i\right) + (1 - \alpha) \\
& \leq z_{t_i}(2(1 - \alpha) + t_i) + \left(\frac{5\alpha}{3} - 1 - t_i\right) + (1 - \alpha) && \text{Inequality (7),} \\
& = (2(1 - \alpha) - t_i) + \left(\frac{5\alpha}{3} - 1 - t_i\right) + (1 - \alpha) \\
& = 2 - \frac{4\alpha}{3} - 2t_i.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \alpha + (z_{t_i} \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) \\
& \leq 2 - \frac{\alpha}{3} - 2t_i \\
& = 2(4(1 - \alpha) - 2t_i) + \left(-6 + \frac{23\alpha}{3} + 2t_i\right) \\
& \leq 2(4(1 - \alpha) - 2t_i) + \left(-6 + \frac{23\alpha}{3} + 2\left(2\alpha - \frac{3}{2}\right)\right) && \text{Inequality (7),} \\
& \leq 2(4(1 - \alpha) - 2t_i) && \alpha \leq \frac{27}{35}, \\
& \leq 2\ddot{\Psi}_{(z_{t_i} \star \overset{\circ}{f} \star v_i)} && \text{Inequality (9).}
\end{aligned}$$

- To prove Inequality (17), for all $2 \leq k \leq \ddot{n}$, we show:

$$\begin{aligned}
& (z_{t_i} \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_k, \ddot{g}_{\ddot{n}+k}, \ddot{g}_{3\ddot{n}-k+1}, \ddot{g}_{3\ddot{n}+k}\}) \\
& \leq 2(\overset{\circ}{f} \star v_i)(\{\ddot{g}_2\}) + (\overset{\circ}{f} \star v_i)(\{\ddot{g}_{3\ddot{n}-k+1}, \ddot{g}_{3\ddot{n}+k}\}) && k \geq 2, \\
& \leq 2\left(\frac{5\alpha}{3} - 1 - t_i\right) + (\overset{\circ}{f} \star v_i)(\{\ddot{g}_{3\ddot{n}-k+1}, \ddot{g}_{3\ddot{n}+k}\}) \\
& \leq 2\left(\frac{5\alpha}{3} - 1 - t_i\right) + 1 - \alpha + (\overset{\circ}{f} \star v_i)(\{\ddot{g}_{3\ddot{n}+k}\}) && \text{Observation 5,} \\
& \leq 2\left(\frac{5\alpha}{3} - 1 - t_i\right) + 1 - \alpha + \frac{4\alpha}{3} - 1 && \text{Observation 6,} \\
& \leq 4(1 - \alpha) - 2t_i && \alpha \leq \frac{18}{23}, \\
& \leq \ddot{\Psi}_{(z_{t_i} \star \overset{\circ}{f} \star v_i)} && \text{Inequality (9).}
\end{aligned}$$

- To prove Inequality (18), we show:

$$\begin{aligned}
& 2\alpha + (z_{t_i} \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_{3\ddot{n}+1}\}) \\
& < 2\alpha + \frac{4\alpha}{3} - 1 && \text{Observation 6,} \\
& = 2(4(1 - \alpha) - 2t_i) + \frac{34\alpha}{3} - 9 + 4t_i \\
& \leq 2(4(1 - \alpha) - 2t_i) + \frac{34\alpha}{3} - 9 + 4\left(2\alpha - \frac{3}{2}\right) && \text{Inequality (7),} \\
& \leq 2(4(1 - \alpha) - 2t_i) && \alpha \leq \frac{45}{58}, \\
& \leq 2\ddot{\Psi}_{(z_{t_i} \star \overset{\circ}{f} \star v_i)} && \text{Inequality (9).}
\end{aligned}$$

- To prove Inequality (19), note that we have already shown that $(z_{t_i} \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}\}) \leq 2 - \frac{4\alpha}{3} - 2t_i$. Therefore we have:

$$\begin{aligned}
& 2\alpha + (z_{t_i} \star \overset{\circ}{f} \star v_i)(\{\ddot{g}_1, \ddot{g}_{\ddot{n}+1}, \ddot{g}_{3\ddot{n}}, \ddot{g}_{3\ddot{n}+1}\}) \\
& \leq 2\alpha + (2 - \frac{4\alpha}{3} - 2t_i) + v_i(\{\ddot{g}_{3\ddot{n}+1}\}) \\
& \leq 2\alpha + (2 - \frac{4\alpha}{3} - 2t_i) + \frac{4\alpha}{3} - 1 && \text{Observation 6,} \\
& = (1 - \alpha) - 6t_i + 3\alpha + 4t_i \\
& \leq (1 - \alpha) - 6t_i + 3\alpha + 4(2\alpha - \frac{3}{2}) && \text{Inequality (7),} \\
& = 3(4(1 - \alpha) - 2t_i) + (22\alpha - 17) \\
& \leq 3(4(1 - \alpha) - 2t_i) && \alpha \leq \frac{17}{22}, \\
& \leq 3\ddot{\Psi}_{(z_{t_i} \star \overset{\circ}{f} \star v_i)} && \text{Inequality (9).}
\end{aligned}$$

Since all conditions of Lemma 7 are satisfied, the proof is complete. $\square$

F Proofs for Section 8 (Algorithm 6: Less Frequent Green Agents)

Lemma 14. *Every red agent a_i in $\dot{N}$ with $\dot{\Psi}_{v_i} \geq 1$ after the primary reductions, receives a bundle in Algorithm 6.*

Proof. Considering the following setup. Let $0 \leq y \leq \dot{n}$ be index such that

- $v_i(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \geq \alpha$ for $1 \leq k \leq y$.
- $v_i(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) < \alpha$ for $y < k \leq \dot{n}$.

Next, let $x \leq y$ be the smallest index satisfying $\dot{\Psi}_{v_i}^{\dot{n}-(y-x)}(\dot{M} \setminus \bigcup_{k=x+1}^y \{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \geq 1$. Since $\dot{\Psi}_{v_i} \geq 1$ such x exist. Let

$$n' = \dot{n} - (y - x) \quad \text{and} \quad M' = \dot{M} \setminus \bigcup_{k=x+1}^y \{\dot{g}_k, \dot{g}_{\dot{n}+k}\},$$

define $v_i^{\text{norm}} = \text{normalized}_1^{n'}(v_i, M')$, We verify that $0 \leq x \leq y \leq \dot{n}$ and valuation function v_i^{norm} satisfies all conditions required by Lemma 13. Inequality (21) and Inequality (22) hold by definition of y . Before verifying Inequality (23), we establish the following claim:

$$v_i^{\text{norm}}(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) > 1 \quad \text{for all } 1 \leq k \leq x.$$

Indeed, if for some $k \leq x$ we had $v_i^{\text{norm}}(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \leq 1$, then $v_i^{\text{norm}}(\{\dot{g}_x, \dot{g}_{\dot{n}+x}\}) \leq 1$, since $\Psi_{v_i^{\text{norm}}}^{n'}(M') \geq 1$, we would get $\Psi_{v_i^{\text{norm}}}^{n'-1}(M' \setminus \{\dot{g}_x, \dot{g}_{\dot{n}+x}\}) \geq 1$, therefore $\Psi_{v_i^{\text{norm}}}^{\dot{n}-(y-(x-1))}(M \setminus \bigcup_{k=x}^y \{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \geq 1$, contradicting the minimality of x . Now we are ready to verify Inequality (23). We distinguish four cases according to the ratio $\frac{x}{n'}$:

- **Case 1.** $x = 0$:

$$\begin{aligned} & x v_i^{\text{norm}}(\{\dot{g}_1\}) + \sum_{k=1}^x v_i^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}) + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \\ &= n'(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \\ &< n'(\alpha + (1 - \alpha)) \quad a_i \in N^r, \\ &= n' \Psi_{v_i^{\text{norm}}}^{n'}(M'). \end{aligned}$$

- **Case 2.** $0 < \frac{x}{n'} \leq \frac{3}{5}$:

$$\begin{aligned} & x v_i^{\text{norm}}(\{\dot{g}_1\}) + \sum_{k=1}^x v_i^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}) + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \\ &< x(v_i^{\text{norm}}(\{\dot{g}_1\}) + \frac{\alpha}{2}) + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \quad \mathcal{R}^1 \text{ is not applicable,} \\ &\leq x(\alpha - v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\}) + \frac{\alpha}{2}) + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \quad \tilde{\mathcal{R}}^1 \text{ is not applicable,} \\ &= (n' - 2x)v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\}) + \frac{\alpha(2n' + x)}{2}. \end{aligned}$$

First, consider the case where $x < \frac{n'}{2}$. Since the coefficient of $v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})$ is $(n' - 2x) > 0$, we have:

$$\begin{aligned} & x v_i^{\text{norm}}(\{\dot{g}_1\}) + \sum_{k=1}^x v_i^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}) + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \\ &< (n' - 2x)v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\}) + \frac{\alpha(2n' + x)}{2} \\ &\leq (n' - 2x)(1 - \alpha) + \frac{\alpha(2n' + x)}{2} \quad a_i \in N^r, \\ &= x\left(\frac{5\alpha}{2} - 2\right) + n' \\ &\leq n' \quad \alpha \leq \frac{4}{5}, \\ &= n' \Psi_{v_i^{\text{norm}}}^{n'}(M'). \end{aligned}$$

Next, consider the case where $x \geq \frac{n'}{2}$:

$$\begin{aligned}
& x v_i^{\text{norm}}(\{\dot{g}_1\}) + \sum_{k=1}^x v_i^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}) + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \\
& < (n' - 2x)v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\}) + \frac{\alpha(2n' + x)}{2} \\
& \leq \frac{\alpha(2n' + x)}{2} \\
& \leq \frac{\alpha\left(2n' + \frac{3n'}{5}\right)}{2} & x \leq \frac{3n'}{5}, \\
& = \frac{13\alpha n'}{10} \\
& \leq n' & \alpha \leq \frac{10}{13}, \\
& = n' \Psi_{v_i^{\text{norm}}}^{n'}(M').
\end{aligned}$$

- **Case 3.** $\frac{3}{5} < \frac{x}{n'} \leq \frac{2}{3}$: Let $(P_1, \dots, P_{n'})$ be a partition of M' with

$$v_i^{\text{norm}}(P_k) = 1 \quad \text{for all } k.$$

Note that since $v_i^{\text{norm}}(\{\dot{g}_x, \dot{g}_{\dot{n}+x}\}) > 1$, the goods $\dot{g}_1$ to $\dot{g}_x$ lie in x distinct bundles and the goods $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$ lie in $n' - x$ remaining bundles. We claim that

$$\sum_{k=1}^x v_i^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}) \leq \frac{x}{3}. \quad (27)$$

To prove this claim, we aim to show that among the goods $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$, at least x goods belong to at most $x/3$ bundles; therefore, that sum is at most $\frac{x}{3}$ —since the goods $\dot{g}_{\dot{n}+1}$ to $\dot{g}_{\dot{n}+x}$ are the smallest x goods among these n' goods, the claim follows. Denote the $n' - x$ bundles by $P_1, P_2, \dots, P_{n'-x}$ and suppose that in P_k there are c_k goods among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$ (with $c_1 \geq c_2 \geq \dots \geq c_{n'-x}$). Let l be the largest index such that $\sum_{k=1}^l c_k \geq 3l$.

Note that by definition, l is the largest index such that the first l bundles satisfy $\sum_{k=1}^l c_k \geq 3l$. This means that for the first l bundles, the total number of goods among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$, is at least $3l$, but when we include the $(l+1)$ -th bundle, we no longer have this property; in fact, the total number of goods among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$, in the first $l+1$ bundles is less than $3(l+1)$. Moreover, for every bundle from index $l+1$ onward, each bundle can have at most 2 goods—if any such bundle contained 3 or more goods, then we could increase l , contradicting its maximality.

Thus, an upper bound for the total number of goods is given by assuming that the first $l+1$ bundles contain at most $3(l+1) - 1$ goods among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$, and that each of the remaining $(n' - x - l - 1)$ bundles contains at most 2 goods among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$. This leads to the inequality $3(l+1) - 1 + 2(n' - x - l - 1) \geq n'$, which represents an upper bound on the total number of goods among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$. We simplify it to obtain

$$\begin{aligned}
l & \geq 2x - n' \\
& > 2x - \frac{5x}{3} & x > \frac{3n'}{5}, \\
& = \frac{x}{3}.
\end{aligned}$$

Thus, we conclude that the sum of the values of the x smallest goods among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$, is at most $\frac{x}{3}$, which implies Inequality (27). We have:

$$\begin{aligned}
& x v_i^{\text{norm}}(\{\dot{g}_1\}) + \sum_{k=1}^x v_i^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}) + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \\
& \leq x v_i^{\text{norm}}(\{\dot{g}_1\}) + \frac{x}{3} + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \quad \text{Inequality (27),} \\
& < x(\alpha - v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) + \frac{x}{3} + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \quad \widetilde{\mathcal{R}}^1 \text{ is not applicable,} \\
& = v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})(n' - 2x) + \left(n'\alpha + \frac{x}{3}\right) \\
& \leq n'\alpha + \frac{x}{3} \quad n' - 2x < 0, \\
& \leq n'\alpha + \frac{2n'}{9} \quad x < \frac{2n'}{3}, \\
& \leq n' \quad \alpha \leq \frac{7}{9}, \\
& = n'\Psi_{v_i^{\text{norm}}}^{n'}(M').
\end{aligned}$$

- **Case 4.** $\frac{2}{3} < \frac{x}{n'} \leq 1$: Let $(P_1, \dots, P_{n'})$ be a partition of M' with

$$v_i^{\text{norm}}(P_k) = 1 \quad \text{for all } k.$$

Note that since $v_i^{\text{norm}}(\{\dot{g}_x, \dot{g}_{\dot{n}+x}\}) > 1$, the goods $\dot{g}_1$ to $\dot{g}_x$ lie in x distinct bundles and the goods $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$ lie in $n' - x$ remaining bundles. Since the goods $\dot{g}_{\dot{n}+1}$ to $\dot{g}_{\dot{n}+x}$ form the smallest x -element subset among these, their total sum under v_i^{norm} is at most $\frac{x(n'-x)}{n'}$. We have:

$$\begin{aligned}
& x v_i^{\text{norm}}(\{\dot{g}_1\}) + \sum_{k=1}^x v_i^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}) + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \\
& \leq x v_i^{\text{norm}}(\{\dot{g}_1\}) + \frac{x(n' - x)}{n'} + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \\
& < x(\alpha - v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) + \frac{x(n' - x)}{n'} + (n' - x)(\alpha + v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \quad \widetilde{\mathcal{R}}^1 \text{ is not applicable,} \\
& = v_i^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})(n' - 2x) + \left(n'\alpha + \frac{x(n' - x)}{n'}\right) \\
& \leq n'\alpha + \frac{x(n' - x)}{n'} \quad n' - 2x < 0, \\
& \leq n'\alpha + \frac{\frac{2n'}{3}\left(n' - \frac{2n'}{3}\right)}{n'} \quad x > \frac{2n'}{3}, \\
& = n'\left(\alpha + \frac{2}{9}\right) \\
& < n' \quad \alpha < \frac{7}{9}, \\
& = n'\Psi_{v_i^{\text{norm}}}^{n'}(M').
\end{aligned}$$

Completing the proof in all 4 cases. $\square$

Lemma 15. *Every red agent a_i in $\dot{N}$ with $\dot{\Psi}_{v_i} < 1$ after the primary reductions, receives a bundle in Algorithm 6.*

Proof. Considering the following setup. Let $0 \leq y \leq \dot{n}$ be index such that

- $(f_{s_i} \star v_i)(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \geq \alpha$ for $1 \leq k \leq y$.
- $(f_{s_i} \star v_i)(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) < \alpha$ for $y < k \leq \dot{n}$.

Next, let $x \leq y$ be the smallest index satisfying $\Psi_{(f_{s_i} \star v_i)}^{\dot{n}-(y-x)}(\dot{M} \setminus \bigcup_{k=x+1}^y \{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \geq 1 - 3s_i$. Since $\dot{\Psi}_{(f_{s_i} \star v_i)} \geq 1 - 3s_i$ such x exist. Let

$$n' = \dot{n} - (y - x) \quad \text{and} \quad M' = \dot{M} \setminus \bigcup_{k=x+1}^y \{\dot{g}_k, \dot{g}_{\dot{n}+k}\}$$

define $(f_{s_i} \star v_i)^{\text{norm}} = \text{normalized}_{1-3s_i}^{n'}((f_{s_i} \star v_i), M')$, We verify that $0 \leq x \leq y \leq \dot{n}$ and valuation function $(f_{s_i} \star v_i)^{\text{norm}}$ satisfies all conditions required by Lemma 13. Inequality (21) and Inequality (22) hold by definition of y . Before verifying Inequality (23), we establish the following claim:

$$(f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) > 1 - 3s_i \quad \text{for all } 1 \leq k \leq x.$$

Indeed, if for some $k \leq x$ we had $(f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \leq 1 - 3s_i$, then $(f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_x, \dot{g}_{\dot{n}+x}\}) \leq 1 - 3s_i$, since $\Psi_{(f_{s_i} \star v_i)^{\text{norm}}}^{n'}(M') \geq 1 - 3s_i$, we would get $\Psi_{(f_{s_i} \star v_i)^{\text{norm}}}^{n'-1}(M' \setminus \{\dot{g}_x, \dot{g}_{\dot{n}+x}\}) \geq 1 - 3s_i$, therefore $\Psi_{(f_{s_i} \star v_i)}^{\dot{n}-(y-(x-1))}(\dot{M} \setminus \bigcup_{k=x}^y \{\dot{g}_k, \dot{g}_{\dot{n}+k}\}) \geq 1 - 3s_i$, contradicting the minimality of x . Now we are ready to verify Inequality (23). We distinguish four cases according to the ratio $\frac{x}{n'}$:

- **Case 1.** $\frac{x}{n'} \leq \frac{1}{2}$:

$$\begin{aligned} & x(f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_1\}) + \sum_{k=1}^x (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}) \\ & + (n' - x)(\alpha + (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \\ & \leq x(f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_1, \dot{g}_{\dot{n}+1}\}) + (n' - x)(\alpha + (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \\ & \leq x\left(1 - \frac{\alpha}{3} - 2s_i + \frac{\alpha}{2} - s_i\right) + (n' - x)\left(\alpha + \frac{4\alpha}{3} - 1 - s_i\right) \quad \text{Observation 7,} \\ & = n'(1 - 3s_i) + (2n' - 2x)s_i - \frac{2n'(6 - 7\alpha) + x(13\alpha - 12)}{6} \\ & \leq n'(1 - 3s_i) + (2n' - 2x)\left(\frac{4\alpha}{3} - 1\right) - \frac{2n'(6 - 7\alpha) + x(13\alpha - 12)}{6} \quad \text{Inequality (1),} \\ & = n'(1 - 3s_i) + x\left(4 - \frac{29\alpha}{6}\right) + n'(5\alpha - 4) \\ & \leq n'(1 - 3s_i) + n'\left(2 - \frac{29\alpha}{12}\right) + n'(5\alpha - 4) \quad x \leq \frac{n'}{2}, \\ & = n'(1 - 3s_i) + \frac{n'(31\alpha - 24)}{12} \\ & < n'(1 - 3s_i) \quad \alpha < \frac{24}{31}, \\ & = n'\Psi_{(f_{s_i} \star v_i)^{\text{norm}}}^{n'}(M'). \end{aligned}$$

- **Case 2.** $\frac{1}{2} < \frac{x}{n'} \leq \frac{3}{5}$: Let $(P_1, \dots, P_{n'})$ be a partition of M' with

$$(f_{s_i} \star v_i)^{\text{norm}}(P_k) = 1 - 3s_i \quad \text{for all } k.$$

Note that since $(f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_x, \dot{g}_{\dot{n}+x}\}) > 1 - 3s_i$, the goods $\dot{g}_1$ to $\dot{g}_x$ lie in x distinct bundles and the goods $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$ lie in $n' - x$ remaining bundles. We now show that it is impossible for any of these $n' - x$ bundles to contain 4 goods among the n' goods. Suppose that one of these bundles does contain at least 4 goods among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$. Then $(f_{s_i} \star v_i)^{\text{norm}}\{\dot{g}_{\dot{n}+x}\} \leq \frac{1-3s_i}{4}$. It follows that:

$$\begin{aligned} 1 - 3s_i &< (f_{s_i} \star v_i)^{\text{norm}}\{\dot{g}_x, \dot{g}_{\dot{n}+x}\} \\ &\leq \left(1 - \frac{\alpha}{3} - 2s_i\right) + \frac{1 - 3s_i}{4} \end{aligned} \quad \text{Observation 7.}$$

Which implies $s_i > \frac{4\alpha}{3} - 1$, that's a contradiction. Hence, it is impossible for any of the $n' - x$ bundles to contain 4 goods among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$. Assume that among $n' - x$ bundles, exactly r of them contains exactly 3 goods among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$. Then we must have $2(n' - x - r) + 3r \geq n'$, which implies $r \geq 2x - n'$. Since the total value of each bundle is exactly $1 - 3s_i$, it follows that the total value of the $3(2x - n')$ minimum goods (among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$) is at most $(1 - 3s_i)(2x - n')$. On the other hand, since $x \leq \frac{3n'}{5}$ we have $3(2x - n') \leq x$, hence these $3(2x - n')$ goods fall in the range from $\dot{g}_{\dot{n}+1}$ to $\dot{g}_{\dot{n}+x}$.

Therefore we can obtain

$$\begin{aligned} \sum_{k=1}^x (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}) &\leq (2x - n')(1 - 3s_i) + \left(x - 3(2x - n')\right) \left(\frac{\alpha}{2} - s_i\right) \quad \text{Observation 7,} \\ &= (2x - n')(1 - 3s_i) + (3n' - 5x) \left(\frac{\alpha}{2} - s_i\right) \\ &= \frac{\alpha(3n' - 5x)}{2} + (2x - n') - x s_i. \end{aligned}$$

Therefore

$$\begin{aligned}
& x (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_1\}) + \sum_{k=1}^x (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_{\dot{n}+k}\}) \\
& + (n' - x)(\alpha + (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\})) \\
& \leq x((f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_1\}) + \left(\frac{\alpha(3n' - 5x)}{2} + (2x - n') - x s_i\right) \\
& \quad + (n' - x)(\alpha + (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_{2\dot{n}+1}\}))) \\
& \leq x\left(1 - \frac{\alpha}{3} - 2s_i\right) + \frac{\alpha(3n' - 5x)}{2} \\
& \quad + (2x - n') - x s_i + (n' - x)\left(\alpha + \frac{4\alpha}{3} - 1 - s_i\right) \quad \text{Observation 7,} \\
& = n'(1 - 3s_i) + (2n' - 2x)s_i - \frac{n'(18 - 23\alpha) + x(31\alpha - 24)}{6} \\
& \leq n'(1 - 3s_i) + (2n' - 2x)\left(\frac{4\alpha}{3} - 1\right) - \frac{n'(18 - 23\alpha) + x(31\alpha - 24)}{6} \quad \text{Inequality (1),} \\
& = n'(1 - 3s_i) + \frac{39n' - 47x}{6} \alpha + (6x - 5n') \\
& \leq n'(1 - 3s_i) + \frac{39n' - 47x}{6} \frac{10}{13} + (6x - 5n') \quad \alpha \leq \frac{10}{13}, \\
& = n'(1 - 3s_i) - \frac{x}{39} \\
& < n'(1 - 3s_i) \\
& = n' \Psi_{(f_{s_i} \star v_i)^{\text{norm}}}^{n'}(M').
\end{aligned}$$

- **Case 3.** $\frac{3}{5} < \frac{x}{n'} \leq \frac{2}{3}$: Let $(P_1, \dots, P_{n'})$ be a partition of M' with

$$(f_{s_i} \star v_i)^{\text{norm}}(P_k) = 1 - 3s_i \quad \text{for all } k.$$

In Case 2, we showed that among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$ there are at least $3(2x - n')$ goods, which are grouped into at most $(2x - n')$ bundles (each bundle containing exactly 3 goods among $\dot{g}_{y+1}$ to $\dot{g}_{\dot{n}+x}$). Since $x > \frac{3n'}{5}$, we have $3(2x - n') > x$, it follows that the total value of the goods $\dot{g}_{\dot{n}+1}$ to $\dot{g}_{\dot{n}+x}$ is at most $\frac{x(1-3s_i)}{3}$. We have:

$$\begin{aligned}
& x(f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_1\}) + \sum_{k=1}^x (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_{\hat{n}+k}\}) \\
& + (n' - x)(\alpha + (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_{2\hat{n}+1}\})) \\
& \leq x((f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_1\}) + \frac{x(1-3s_i)}{3} + (n' - x)(\alpha + (f_{s_i} \star v_i)^{\text{norm}}(\{\dot{g}_{2\hat{n}+1}\}))) \\
& < x\left(1 - \frac{\alpha}{3} - 2s_i\right) + \frac{x(1-3s_i)}{3} + (n' - x)\left(\alpha + \frac{4\alpha}{3} - 1 - s_i\right) \quad \text{Observation 7,} \\
& = n'(1-3s_i) + (2n' - 2x)s_i - \left[n' - \frac{n'(7\alpha - 3) + x(7 - 8\alpha)}{3}\right] \\
& \leq n'(1-3s_i) + (2n' - 2x)\left(\frac{4\alpha}{3} - 1\right) - \left[n' - \frac{n'(7\alpha - 3) + x(7 - 8\alpha)}{3}\right] \quad \text{Inequality (1),} \\
& = n'(1-3s_i) + \frac{13-16\alpha}{3}x + n'(5\alpha - 4) \\
& \leq n'(1-3s_i) + \frac{13-16\alpha}{3} \frac{2n'}{3} + n'(5\alpha - 4) \quad x \leq \frac{2n'}{3}, \\
& = n'(1-3s_i) + \frac{n'(13\alpha - 10)}{9} \\
& \leq n'(1-3s_i) \quad \alpha \leq \frac{10}{13}, \\
& = n'\Psi_{(f_{s_i} \star v_i)^{\text{norm}}}^{n'}(M').
\end{aligned}$$

- **Case 4.** $\frac{2}{3} < \frac{x}{n'} \leq 1$: Let $(P_1, \dots, P_{n'})$ be a partition of M' with

$$(f_{s_i} \star v_i)^{\text{norm}}(P_k) = 1 - 3s_i \quad \text{for all } k.$$

As established in case 2, among $\dot{g}_{y+1}$ to $\dot{g}_{\hat{n}+x}$, no four of them can be placed together in any of the $n' - x$ bundles. This implies that the total number of goods (among $\dot{g}_{y+1}$ to $\dot{g}_{\hat{n}+x}$) in these bundles must satisfy $3(n' - x) \geq n'$. Rearranging gives $x \leq \frac{2n'}{3}$. However, this contradicts the assumption of Case 4 that $x > 2n'/3$. Hence, this case is impossible.

Completing the proof. □