

Graph Powers of Groups II: The RA Matrix

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Abstract

For a graph Γ and group G , G^Γ is the subgroup of $G^{|\Gamma|}$ generated by elements with g in the coordinates corresponding to v and its neighbors in Γ . There is a natural epimorphism $G^\Gamma \rightarrow (G/[G, G])^\Gamma$ with kernel $[G, G]^n \cap G^\Gamma$. When $[G, G]^n \leq G^\Gamma$, the structure of G^Γ is easily described from $(G/[G, G])^\Gamma$. Fixing Γ , if $[G, G]^{|\Gamma|} \leq G^\Gamma$ for all G , we say that Γ is RA (reducible to abelian). We showed in [2] that wide classes of graphs are RA, including graphs of girth 5 or more. The key tool is the *RA matrix* C_Γ , and we showed that Γ is RA if and only if the row space $\text{Row}(C_\Gamma) = \mathbb{Z}^{|\Gamma|}$.

Here, we study the possibilities for the elementary divisors of C_Γ ; the more nontrivial elementary divisors we get, the further Γ is from being RA (and the harder G^Γ is to describe). We show that while many graphs, including those of girth 4, cartesian products, and most tensor products have at most one nontrivial elementary divisor, one can construct a graph of girth 3 with any prescribed set of elementary divisors and $\mathbb{Z}$ -nullity.

1 Introduction

Given a graph Γ and a group G , the graph power G^Γ is the subgroup of $G^{|\Gamma|}$ generated by elements g^v which have g in coordinates corresponding to v and vertices adjacent to v in Γ , and the identity element 1 in all other coordinates. This is a natural generalization of the parallel product of group actions (see [19]) and a vast generalization of the original Lights Out Puzzle by Tiger Electronics, where the group G was $\mathbb{Z}/2\mathbb{Z}$ and the graph was fixed as the 5×5 grid graph. The short exact sequence

$$1 \rightarrow [G, G]^{|\Gamma|} \cap G^\Gamma \rightarrow G^\Gamma \rightarrow (G/[G, G])^\Gamma \rightarrow 1$$

shows us that discovering the structure of G^Γ really amounts to two things: (1) understanding the subgroup $[G, G]^{|\Gamma|} \cap G^\Gamma$ and (2) doing some linear algebra over $\mathbb{Z}$ and $\mathbb{Z}/k\mathbb{Z}$ to compute the structure of $(G^{\text{Ab}})^\Gamma$ for many graphs Γ , as done by many authors [2, 3, 4, 5, 6, 8, 11, 14, 16, 18] in exploring the Lights Out Puzzle over an abelian group (and primarily over $\mathbb{Z}/2\mathbb{Z}$, where it can be shown the all-ones vector is always in $(\mathbb{Z}/2\mathbb{Z})^\Gamma$ for any Γ [15]).

In [2] we defined Γ to be *G-RA* (“reducible to abelian”) if $[G, G]^{|\Gamma|} \subseteq G^\Gamma$, that is, if, by a series of clicks g^v we can place any commutator $[g, h] \in G$ on any single vertex, leaving the rest of the vertices’ states unchanged. If Γ is *G-RA* for all groups G , we simply say Γ is RA. If Γ is *G-RA*, understanding the structure of G^Γ reduces to the abelian question of understanding $(G/[G, G])^\Gamma$, which has already been widely investigated. In [2] we illustrated the importance of considering the chain of subgroups

$$\text{Comm}_b(G, \Gamma) \leq [G^\Gamma, G^\Gamma] \leq [G, G]^{|\Gamma|} \cap G^\Gamma \leq [G, G]^{|\Gamma|}$$

where $\text{Comm}_b(G, \Gamma)$ is generated by commutators $[g^v, h^w]$ for $g, h \in G$ and vertices v, w in Γ (possibly $v = w$).

For many graphs, we have proven even the smallest subgroup $\text{Comm}_b(G, \Gamma)$ gives all of $[G, G]^{|\Gamma|}$ for every G , making these RA, namely:

1. Any graph with girth 4 or more either (a) with a vertex of degree ≤ 2 or (b) that has edges $u - v - w$ that do not complete to a 4-cycle $u - v - w - x - u$. This includes:
 - a) $m \times n$ grid graphs with $m, n \geq 2$ and path graphs P_n with $n \geq 3$,
 - b) Cylindrical grids (the cartesian product of a cycle C_m with a path P_n with n vertices) with $m \geq 4$ and $n \geq 3$, and

- c) Torus grids (the cartesian product of C_m with C_n) with $m, n \geq 4$.
 - d) All graphs with girth 5 or more other than K_2 .
2. Complete bipartite graphs $K_{m,n}$.
 3. Cube graphs Q_d with d even.

On the other hand, in [2] we have shown that cube graphs Q_d with d odd are not RA, and neither are the folded cubes $\square_d$ with d odd. In Theorem 4.7, we expand on #1 above and show that for any product of path and cycle graphs (so any higher-dimensional version of grids, cylinders, and tori) the only examples that are not RA are the cubes Q_d with odd d .

The main tool used to tell us all this is what we call the “RA Matrix” C_Γ for a graph Γ . We start by building the so-called “activation matrix” A_Γ for a graph Γ by simply adding the identity matrix to its adjacency matrix; the rows (or columns) of A_Γ give the generators g^v when the 1’s in the rows are replaced by g and 0’s by the identity $1 \in G$. Then C_Γ is the matrix built by adding “intersections” of rows from A_Γ to the bottom of A_Γ , where the intersection of two rows is their bitwise AND; this comes from taking $[g^u, h^v]$ for vertices u and v and noticing that the result of this element of $[G, G]^{|\Gamma|}$ is that the state of all vertices in the intersection of the closed neighborhoods $N[u]$ and $N[v]$ of u and v respectively is multiplied by the commutator $[g, h]$ (by “closed neighborhood” $N[v]$ of v we mean all vertices adjacent to v and v itself). One of the main results in [2] was that if Γ is G -RA for every Heisenberg group, then Γ is RA. This fact, combined with the fact that a Heisenberg group G has central commutator subgroup $[G, G]$, implies that the $\mathbb{Z}$ -span of the rows of C_Γ tells us everything we need to know about whether Γ is RA or not. If the rows of C_Γ span all of $\mathbb{Z}^{|\Gamma|}$, we know that we can change the state of any one vertex by $[g, h]$ for any elements $g, h \in G$, for any group G . On the other hand, if C_Γ has any nontrivial elementary divisors in its Smith Normal Form¹, and p is a prime dividing one of the nontrivial elementary divisors (which may equal 0, in which case any prime p would do), then for the Heisenberg group $G = H(\mathbb{F}_p)$, Γ is not G -RA. In fact, if there are k elementary divisors of C_Γ that are divisible by p , then $[H(\mathbb{F}_p), H(\mathbb{F}_p)]^{|\Gamma|} \cap H(\mathbb{F}_p)^\Gamma$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{|\Gamma|-k}$. In general, the more nontrivial elementary divisors C_Γ has, the harder it is to analyze the puzzle. For example, it becomes more difficult to make a concrete theory to determine whether an element of $G^{|\Gamma|}$ is in G^Γ , or given an element shown by a computer to be in G^Γ , for a human to find a set of “clicks” $g_1^{v_1}, \dots, g_m^{v_m}$ that would result in this element.

In this paper, we give more criteria for certain graphs to be RA and explore graphs that are not, as well as how far graphs can be from being RA, and we essentially show they can be as far from RA as one could ask for in Theorem 6.8.

We say a graph Γ is $1/\mu$ -RA if C_Γ has a single nontrivial, nonzero elementary divisor and that divisor is μ (allowing the possibility of Γ being RA, in which case $\mu = 1$). In Section 3, we first develop the tools we will use throughout the paper to both show that a graph is $1/\mu$ -RA and to determine μ as the gcd of certain numbers related to the degrees of vertices and intersections of neighborhoods in the graph.

In Section 4 we explore strong products, cartesian (box) products, and tensor products of graphs. The strong product of two graphs produces the tensor product of their activation matrices, so they are easiest to understand. Cartesian products have particularly nice characterizations: they are $1/\mu$ -RA, where μ is described combinatorially in terms of the properties of the graphs involved in Theorem 4.2. This description, and in particular Theorem 4.7, allows us to extend to the non-abelian setting much work that has been done in the Lights Out Puzzle community related to grids, tori, and cylinders, as well as puzzles where clicking a square affects its entire row and column ($K_m \square K_n$), the higher-dimensional analogues of all of these, and others. Tensor products also behave well enough to be characterized as $1/\mu$ -RA for a well-described μ in many cases (see Theorem 4.10, Theorem 4.12, and Theorem 4.16), and particularly the tensor product of arbitrarily many complete graphs (Theorem 4.14).

Tensor products allow us to exhibit a family of graphs that are almost RA, namely the crown graphs $\text{Cr}(2n) = K_2 \times K_n$, in Section 5, where we continue the exploration of girth 4 graphs started in [2] and paint a fairly complete picture of them. A connected, neighborhood-distinguishable² girth 4 graph Γ must be $1/\mu$ -RA, where μ is described in Theorem 5.2. We demonstrate a family of graphs that are $1/\mu$ -RA for any μ , namely the crown graphs on $2\mu + 4$ vertices, and show that these are in fact the smallest girth-4 graphs that are $1/\mu$ -RA (Theorem 5.9). We also use these graphs and tensor products to construct, for any given μ , an infinite family of graphs that is $1/\mu$ -RA in Theorem 5.5.

In Section 6, a vastly different story about girth 3 graphs unfolds. While most girth 3 graphs (at least on small

¹We always ignore the rectangular nature of the matrix and think of just the $|\Gamma|$ elementary divisors along its main diagonal.

²This means the closed neighborhoods of any two distinct vertices must not be equal.

numbers of vertices) are RA and we can still build a family of $1/\mu$ -RA graphs that are girth 3 for each μ (by just taking the pyramid over crown graphs — see Theorem 6.1), we also find a family of graphs Γ for which C_Γ has arbitrarily many nontrivial nonzero elementary divisors (the Kneser graphs - see Theorem 6.2 and Theorem 6.3), as well as a family of graphs Γ that have arbitrarily large $\mathbb{Z}$ -nullity (see Theorem 6.6 and Theorem 6.7). The latter is not a family that is known in the House of Graphs [1] and may still be unexplored; we call these “binary graphs” because their activation matrix is written down using binary representations of the row numbers. We end the section with Theorem 6.8, which says that we can make a graph Γ for which $\mathbb{Z}^{C_\Gamma}$ has whatever nontrivial elementary divisors we prescribe, both zero and nonzero, essentially confirming that Γ can be as far from RA as desired. This paves the way for puzzle makers to explore puzzles over such graphs as they exhibit non-abelian behavior in the largest sense possible so that the order of operations makes the biggest difference.

Most graphs (at least on a small number of vertices) are RA, and we show the relevant data we have gathered in Section 7. That means most families of standard graphs do not exhibit interesting non-RA examples, but one family - the Kneser graphs - do give us something to explore, and somehow, in each section of the paper there is something interesting to say about these graphs.

2 Notation and Background

2.1 Graphs

We will assume all the graphs in this paper to be finite simple graphs, so that each edge is uniquely determined by its endpoints. We will refer to many standard graphs, such as the cycle graph on n vertices C_n , the complete graph on n vertices K_n , and the complete bipartite graph $K_{m,n}$. We will use P_n to refer to the path graph with n vertices.

We will also encounter the following graphs:

1. The cube graph Q_n on 2^n vertices, labeled with n -bit strings, and with vertices adjacent whenever their labels differ in a single bit.
2. The crown graph $\text{Cr}(2n)$ on $2n$ vertices; the vertices may be labeled $\{1, 2, \dots, n, 1', 2', \dots, n'\}$, where i is adjacent to j' and i' to j precisely when $i \neq j$. This can also be understood as the tensor product $K_2 \times K_n$ (see Section 4.3).
3. The Kneser graph $\text{Kn}(n, k)$ on k -sized subsets of $\{1, 2, \dots, n\}$, where two vertices are adjacent if their sets are disjoint.

We use $N(v)$ to denote the neighborhood of v (the vertices adjacent to v). We more commonly will want to refer to the closed neighborhood $N[v] = N(v) \cup \{v\}$. As shorthand, when we have a product of graphs, we write $N[u, v]$ for $N[(u, v)]$. We write $d(u, v)$ to mean the distance between u and v , i.e., the number of edges in a shortest path between u and v .

Let Γ be a graph on n vertices $\{v_1, \dots, v_n\}$. If $v = v_i$ is a vertex of Γ , then $\vec{e}_v \in \mathbb{Z}^n$ is defined to be the standard basis vector $\vec{e}_i$. If S is subset of the vertices, then we define

$$\vec{S} = \sum_{v \in S} \vec{e}_v,$$

so that $\vec{S}$ is essentially the bit vector that represents S . We will most commonly use this notation for $\vec{N}[v]$. If $\vec{x}$ and $\vec{y}$ are vectors whose components are only 0 and 1, then we will use $\vec{x} \cap \vec{y}$ to mean the bitwise AND of $\vec{x}$ and $\vec{y}$. Thus, $\vec{N}[u] \cap \vec{N}[v]$ means the same as $\vec{S}$ where $S = N[u] \cap N[v]$.

2.2 G^Γ and the RA property

The definition of G^Γ is inspired by the metaphor of playing Lights Out on Γ , but where the state of each vertex is given by an element of G . Let us make this precise. Suppose that Γ is a graph on n vertices. For any vector $\vec{x} = \langle x_1, \dots, x_n \rangle \in \mathbb{Z}^n$, we define $g^{\vec{x}}$ to be the element $(g^{x_1}, \dots, g^{x_n}) \in G^n$. Then for each vertex $v \in \Gamma$, we define g^v to be $g^{\vec{N}[v]}$. In other words, g^v is the element of G^n that is g in the coordinates corresponding to v and its neighbors, and is the identity of G everywhere else. Then we define

$$G^\Gamma = \langle g^v \mid g \in G, v \in \Gamma \rangle.$$

If we imagine that Γ starts out with the identity of G at every vertex, and that “clicking v with g ” multiplies v and its neighbors on the right by g , then G^Γ is precisely the state space of this puzzle.

If G is abelian, then the structure of G^Γ is in some sense straightforward to obtain from linear algebra. When G is non-abelian, however, the analysis of G^Γ is more difficult. To help understand G^Γ , we can consider the natural epimorphism to $(G/[G, G])^\Gamma$, giving rise to the short exact sequence

$$1 \rightarrow [G, G]^{|\Gamma|} \cap G^\Gamma \rightarrow G^\Gamma \rightarrow (G/[G, G])^\Gamma \rightarrow 1.$$

The easiest possibility to understand is when $[G, G]^{|\Gamma|} \leq G^\Gamma$, which says that given any element of G^Γ , we may multiply each coordinate independently by any product of commutators and the result will still lie in G^Γ . In this case, the description of G^Γ essentially only depends on the structure of the abelian group $(G/[G, G])^\Gamma$. We say that Γ is G -RA (short for “reducible to abelian”) if $[G, G]^{|\Gamma|} \leq G^\Gamma$, and Γ is RA if it is G -RA for each group G .

Note that if $g, h \in G$ and $u, v \in \Gamma$, then $[g^u, h^v] = [g, h]^{\vec{N}[u] \cap \vec{N}[v]}$. For many graphs, it turns out that these commutators already generate all of $[G, G]^{|\Gamma|}$, making Γ RA. In general, to help understand the subgroup of $[G, G]^{|\Gamma|} \cap G^\Gamma$ that is generated by commutators $[g^u, h^v]$, we define the RA matrix C_Γ , whose rows are $\vec{N}[v]$ and $\vec{N}[u] \cap \vec{N}[v]$ over all vertices u and v . We showed in [2, Theorem 6.11] that Γ is RA if and only if C_Γ has n elementary divisors of 1 – in other words, if the integer row space (denoted $\mathbb{Z}^{C_\Gamma}$) is all of $\mathbb{Z}^n$. Furthermore, if C_Γ has an elementary divisor divisible by p , then it is not $H(\mathbb{F}_p)$ -RA, where $H(\mathbb{F}_p)$ is the Heisenberg group of upper-triangular 3×3 integer matrices modulo p .

If Γ has connected components $\Gamma_1, \dots, \Gamma_k$, then $G^\Gamma \cong G^{\Gamma_1} \times \dots \times G^{\Gamma_k}$, so that we fully understand G^Γ just from knowing each G^{Γ_i} . Hence we will always assume that the graphs we start with are connected, and the only time we will deal with a disconnected graph is when we take the tensor product of two connected bipartite graphs; see Theorem 4.12.

There is also a natural definition of G^M for an integer matrix M : the group generated by $g^{\vec{x}}$ where $\vec{x}$ ranges over the rows of M . If M and M' are row-equivalent, then their rows generate the same integer lattice, so $G^M = G^{M'}$. In other words, we may as well put M into Hermite form. It is also clear that if M'' is obtained from M by permuting the columns (but performing no other column operations), then $G^M \cong G^{M''}$. Thus, we may in principle permute the columns of M as we wish, and then reduce it to Hermite form. However, it turns out that the pivots of the resulting matrix are not an invariant of M ; they may depend on how we permute the columns. For example, the matrix $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is already in Hermite form, with pivots $[2, 2]$. If we switch the columns and then put the new matrix in Hermite form, we get $\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$, and so now the pivots are $[1, 4]$.

3 Neighborly and almost RA graphs

Throughout this paper, we will see that many classes of graphs Γ have an RA Matrix C_Γ with a single nontrivial elementary divisor. By the theory developed in [2], if the nontrivial elementary divisor is a prime p , then the index of $[G^\Gamma, G^\Gamma]$ in $[G, G]^{|\Gamma|}$ is precisely p when $G = H(\mathbb{F}_p)$, the Heisenberg group of order p^3 . (Indeed, in this case, $[G, G] = \mathbb{Z}/p\mathbb{Z}$ and $[G^\Gamma, G^\Gamma] \cong (\mathbb{Z}/p\mathbb{Z})^{|\Gamma|-1}$.) In fact, we will be able to say something more about all these classes of graphs: their Hermite form only has at most one element $k \neq 1$ along the diagonal, no matter the rearrangement of the columns of the RA matrix, and $k \neq 0$.

Definition 3.1. For an integer $k \geq 1$, we will say the graph Γ is $1/k$ -RA if the diagonal of the Hermite form of C_Γ has elements $[1^{|\Gamma|-1}, k]$ for every arrangement of the columns of C_Γ . We will call Γ almost RA if it is $1/k$ -RA for some $k \geq 1$.

Note that, by a slight abuse of English, every RA graph is almost RA (with $k = 1$); it will be more convenient to allow this than to rule it out.

Remark 3.2. If Γ is $1/k$ -RA, then k is the only nontrivial elementary divisor of C_Γ and the determinant of the largest minor of C_Γ .

The following property is clear.

Proposition 3.3. If Γ is $1/k$ -RA, then k is the smallest positive integer such that $k\vec{e}_v \in \mathbb{Z}^{C_\Gamma}$ for any vertex v . Furthermore, if $a\vec{e}_v \in \mathbb{Z}^{C_\Gamma}$, then $k \mid a$.

We will see later that some broad classes of graphs all share the following property which makes it simple to show that they are almost RA.

Definition 3.4. We will call a pair of vertices $\{u, v\}$ positive (resp. negative) for a graph Γ if $\vec{e}_u + \vec{e}_v \in \mathbb{Z}^{C_\Gamma}$ (resp. $\vec{e}_u - \vec{e}_v \in \mathbb{Z}^{C_\Gamma}$). A pair is signed if it is positive or negative. A graph Γ is neighborly if every **edge** is signed. (That is, every pair $\{u, v\}$ with u and v adjacent.) We say that Γ is positively (resp. negatively) neighborly if every edge is positive (resp. negative) for Γ .

Proposition 3.5. A connected graph Γ is neighborly (resp. negatively neighborly) if and only if every pair is signed (resp. negative). Furthermore, if Γ is positively neighborly, then whenever u and v are connected by a path of length m , $\vec{e}_u + (-1)^{m-1}\vec{e}_v \in \mathbb{Z}^{C_\Gamma}$.

Proof. The first part follows easily from the connectedness of the graph; for example, if there is a path $u - v - w$ in the graph and $\vec{e}_u + \vec{e}_v$ and $\vec{e}_v - \vec{e}_w$ are both in $\mathbb{Z}^{C_\Gamma}$, then so is $(\vec{e}_u + \vec{e}_v) - (\vec{e}_v - \vec{e}_w) = \vec{e}_u + \vec{e}_w$.

For the second part, consider a path of length m from u to v , say $u = u_0, u_1, \dots, u_m = v$. Then

$$\vec{e}_u + (-1)^{m-1}\vec{e}_v = (\vec{e}_{u_0} + \vec{e}_{u_1}) - (\vec{e}_{u_1} + \vec{e}_{u_2}) + \dots + (-1)^{m-1}(\vec{e}_{u_{m-1}} + \vec{e}_{u_m}) \in \mathbb{Z}^{C_\Gamma}. \quad \square$$

Proposition 3.6. If Γ is neighborly and $a\vec{e}_v \in \mathbb{Z}^{C_\Gamma}$ for some $a > 0$ and some vertex v , then Γ is $1/k$ -RA for some divisor k of a .

Proof. Let w be the vertex corresponding to the last column of C_Γ . Since Γ is neighborly, for each vertex $u \neq w$, one of the vectors $\vec{e}_u \pm \vec{e}_w \in \mathbb{Z}^{C_\Gamma}$. It follows that we can use row operations to reduce the matrix C_Γ to an echelon form where the diagonal has a 1 in every column but the last. Since $a\vec{e}_v \in \mathbb{Z}^{C_\Gamma}$, we also have $a\vec{e}_v - a(\vec{e}_v \pm \vec{e}_w) = \mp a\vec{e}_w \in \mathbb{Z}^{C_\Gamma}$ so the last number in the diagonal must be nonzero, and a divisor of a (if $a\vec{e}_w \in \mathbb{Z}^{C_\Gamma}$ and $b\vec{e}_w \in \mathbb{Z}^{C_\Gamma}$, then $\gcd(a, b)\vec{e}_w \in \mathbb{Z}^{C_\Gamma}$ by the Euclidean Algorithm). $\square$

In light of Theorem 3.6, we would like to find an easy way to establish that some $a\vec{e}_v$ is in $\mathbb{Z}^{C_\Gamma}$. The following technical lemma will usually suffice by taking $S = N[v]$ or $S = N[u] \cap N[v]$.

Lemma 3.7. Suppose Γ is neighborly. Let S be a set of vertices such that $\vec{S} \in \mathbb{Z}^{C_\Gamma}$. Fix $v \in S$, and suppose that $S = \{v\} \sqcup A \sqcup B$ such that $\vec{e}_v + \vec{e}_u \in \mathbb{Z}^{C_\Gamma}$ for $u \in A$ and $\vec{e}_v - \vec{e}_w \in \mathbb{Z}^{C_\Gamma}$ for $w \in B$. Then $a\vec{e}_v \in \mathbb{Z}^{C_\Gamma}$, where $a = 1 - |A| + |B|$.

Proof. This follows from

$$(1 - |A| + |B|)\vec{e}_v = \vec{S} - \sum_{u \in A} (\vec{e}_v + \vec{e}_u) + \sum_{w \in B} (\vec{e}_v - \vec{e}_w) \in \mathbb{Z}^{C_\Gamma}. \quad \square$$

Combining Theorem 3.6 with Theorem 3.7, we see that a neighborly graph Γ will be almost RA unless, in Theorem 3.7, every $1 - |A| + |B|$ is 0. This puts strong restrictions on the closed neighborhoods ($S = N[v]$) and their pairwise intersections ($S = N[u] \cap N[v]$), and we have not yet found such a graph.

Now we introduce the main tool that will show us many classes of graphs are particularly nice, being either $1/2$ -RA or RA (see Theorems 3.11, 4.2, 4.5, 4.6, 4.13, 4.14 and 5.2).

Theorem 3.8. If Γ is neighborly, then the following are equivalent.

- (1) For some u and v , both vectors $\vec{e}_u \pm \vec{e}_v$ are in $\mathbb{Z}^{C_\Gamma}$.
- (2) $2\vec{e}_u \in \mathbb{Z}^{C_\Gamma}$ for some vertex u .
- (3) Γ is $1/2$ -RA or RA.

Furthermore, in this case, Γ is $1/2$ -RA if and only if every vertex has odd degree and every pair of vertices has an even number of common neighbors (possibly zero).

Proof. (1) clearly implies (2) since $(\vec{e}_u + \vec{e}_v) + (\vec{e}_u - \vec{e}_v) = 2\vec{e}_u$. Conversely, if $2\vec{e}_u \in \mathbb{Z}^{C_\Gamma}$, then $\vec{e}_u + \vec{e}_v = 2\vec{e}_u - (\vec{e}_u - \vec{e}_v)$, and so (2) implies (1). Finally, in light of Theorem 3.3 and Theorem 3.6, (2) and (3) are equivalent.

Now, suppose that Γ is $1/2$ -RA or RA. If every vertex has odd degree and every pair of vertices has an even number of common neighbors, then every row of C_Γ has an even sum, and so 2 must divide the largest elementary divisor of Γ , which means the determinant of the largest minor is divisible by 2, hence the last entry in the diagonal of the Hermite form must be 2, i.e. Γ is $1/2$ -RA. Conversely, if any row had an odd sum k , then by fixing u and adding vectors of the form $\vec{e}_u - \vec{e}_v$ and subtracting vectors of the form $\vec{e}_u + \vec{e}_v$, where v ranges over all other vertices whose coordinates are 1 in that row, we would get $c\vec{e}_u \in \mathbb{Z}^{C_\Gamma}$ for some odd number c . Combined with the fact that $2\vec{e}_u \in \mathbb{Z}^{C_\Gamma}$, we see that $\vec{e}_u \in \mathbb{Z}^{C_\Gamma}$, which means Γ is RA. $\square$

We will use the following fact in the proof of Theorem 3.10, where we could assume the graph Γ is neighborly and the matrix $A = C_\Gamma$ has only a single nontrivial elementary divisor, making the proof easier. However, for the sake of generality and completeness, we prove a much more general result.

Proposition 3.9. *Let A be an integer matrix and $A\vec{x} \equiv \vec{0} \pmod{m}$ for some integer $m > 1$ and nonzero vector $\vec{x}$. Then A has an elementary divisor divisible by m .*

Proof. Let³ $PAQ = D$, with P and Q unimodular and D the diagonal matrix representing the Smith normal form of A , and the elementary divisors are $d_1 \mid d_2 \mid \dots \mid d_n$. Then $\pmod{m}$ we have $DQ^{-1}\vec{x} = PA\vec{x} = P\vec{0} = \vec{0}$. Letting p be a prime divisor of m and p^k the highest power of p dividing m , and further letting x_i be the i -th entry of $Q^{-1}\vec{x}$, we must have $d_i x_i \equiv 0 \pmod{p^k}$ for all i . If there is no elementary divisor d_i that is divisible by p^k , then x_i must be divisible by p for each i . But that means $\vec{x}$ is in the right kernel of $Q^{-1} \pmod{p}$, which has determinant ± 1 - a contradiction. Thus, there must be an elementary divisor d_i divisible by p^k and in particular d_n is divisible by p^k . Such is true for all prime powers dividing m , so d_n must be divisible by m (possibly $d_n = 0$). $\square$

Note that [2, Proposition 2.1] is the special case of this statement where the $\vec{x} = \vec{1}$ and m is prime.

Now we can establish that many neighborly graphs are almost RA, and we can even determine the value of k such that they are $1/k$ -RA.

Theorem 3.10. *Let Γ be neighborly. Suppose that $\Gamma = U \sqcup V$ is a partition of the graph such that $\vec{e}_u - \vec{e}_v \in \mathbb{Z}^{C_\Gamma}$ if u and v are in the same part, and otherwise $\vec{e}_u + \vec{e}_v \in \mathbb{Z}^{C_\Gamma}$. Let*

$$\delta = \gcd(\{|N[v] \cap U| - |N[v] \cap V| : v \in \Gamma\}), \text{ and}$$

$$\kappa = \gcd(\{|N[u] \cap N[v] \cap U| - |N[u] \cap N[v] \cap V| : u, v \in \Gamma, u \neq v\}).$$

If δ and κ are not both zero, then Γ is $1/\mu$ -RA, where $\mu = \gcd(\delta, \kappa)$.

Proof. Let us assume that $v \in U$ and let $S = N[v]$. Let $A = S \cap V$ and $B = (S \setminus \{v\}) \cap U$, so that $S = \{v\} \sqcup A \sqcup B$. Then by Theorem 3.7, $(1 - |A| + |B|)\vec{e}_v \in \mathbb{Z}^{C_\Gamma}$. On the other hand, $1 - |A| + |B| = 1 - |S \cap V| + (|S \cap U| - 1) = |S \cap U| - |S \cap V|$. So $d\vec{e}_v \in \mathbb{Z}^{C_\Gamma}$, with $d = |S \cap U| - |S \cap V|$. A similar calculation shows that the same is true if $v \in V$ instead. It follows that $\delta\vec{e}_v \in \mathbb{Z}^{C_\Gamma}$.

Similarly, if $S = N[u] \cap N[v]$ and $w \in S$, then we can take $A = S \cap V$ and $B = (S \setminus \{w\}) \cap U$, and we again find that $k\vec{e}_w \in \mathbb{Z}^{C_\Gamma}$, where $k = |N[u] \cap N[v] \cap U| - |N[u] \cap N[v] \cap V|$. So $\kappa\vec{e}_w \in \mathbb{Z}^{C_\Gamma}$.

By Theorem 3.6 and Theorem 3.3, if δ and κ are both nonzero, then Γ is $1/\mu$ -RA for some μ dividing $\gamma = \gcd(\delta, \kappa)$. On the other hand, our definition of δ and κ ensures that $|S \cap U| - |S \cap V|$ is a multiple of γ for every $S = N[v]$ and $S = N[u] \cap N[v]$. That means for every row of C_Γ , the sum of the coordinates corresponding to vertices in U , less the sum of the coordinates corresponding to vertices in V , is divisible by γ . In other words, $\vec{x} = \sum_{u \in U} \vec{e}_u - \sum_{v \in V} \vec{e}_v$ is in the right kernel of $C_\Gamma \pmod{\gamma}$, so by Theorem 3.9 there is an elementary divisor divisible by γ . Since the only nontrivial elementary divisor is μ , it follows that $\mu = \gamma$. $\square$

Corollary 3.11. *Suppose that Γ is positively neighborly.*

1. *If Γ is not bipartite, then Γ is $1/2$ -RA or RA, and it is $1/2$ -RA if and only if every vertex has odd degree and every pair of vertices has an even number of common neighbors (possibly zero).*
2. *If Γ is bipartite, then it is $1/\mu$ -RA, where $\mu = \gcd(\delta, \kappa)$ with*

$$\delta = \gcd(\{\deg(v) - 1 : v \in \Gamma\}) \text{ and}$$

$$\kappa = \gcd(\{|N[u] \cap N[v]| : u, v \in \Gamma, u \neq v, d(u, v) = 2\}).$$

Proof. First, suppose that Γ is not bipartite. Then given any two vertices u and v , there is a path of odd length and a path of even length between them. Then since Γ is positively neighborly, Theorem 3.5 implies that both vectors $\vec{e}_u \pm \vec{e}_v$ are in $\mathbb{Z}^{C_\Gamma}$, and the result follows from Theorem 3.8.

Now suppose that Γ is bipartite with bipartition $U \sqcup V$. Then Theorem 3.5 implies that U and V satisfy the conditions of Theorem 3.10. If $v \in V$, then

$$|N[v] \cap U| - |N[v] \cap V| = \deg(v) - 1,$$

³Argument adapted from a solution generated by OpenAI's ChatGPT, August 24, 2025.

and otherwise $|N[v] \cap U| - |N[v] \cap V| = 1 - \deg(v)$. If u and v are neighbors, then one of them is in U and one is in V , so that $|N[u] \cap N[v] \cap U| - |N[u] \cap N[v] \cap V| = 0$. If u and v are two apart, then $N[u] \cap N[v]$ is contained in either U or V . Then the result follows directly from Theorem 3.10. $\square$

Corollary 3.12. *Suppose that Γ is negatively neighborly. Then Γ is $1/\mu$ -RA, where $\mu = \gcd(\delta, \kappa)$ with*

$$\delta = \gcd(\{\deg(v) + 1 : v \in \Gamma\}) \text{ and}$$

$$\kappa = \gcd(\{|N[u] \cap N[v]| : u, v \in \Gamma, u \neq v\}).$$

In particular, if Γ has an edge $u - v$ that is not part of a 3-cycle $u - v - w - u$, e.g. if Γ is bipartite or has girth 4, then $\mu \leq 2$.

Proof. The first part follows directly from Theorem 3.10 by taking $V = \emptyset$. Then, if Γ is bipartite, we have $|N[u] \cap N[v]| = 2$ for any neighboring vertices u and v , and the second part follows. $\square$

4 Exploring products of graphs

In this section, we will apply the techniques developed in Section 3 to the strong product, cartesian product, and tensor product of graphs. In many cases, we can characterize the elementary divisors of the RA matrix of the product of two graphs in terms of the elementary divisors of the RA matrix of the factors.

4.1 Strong Products

Recall the strong product $\Gamma_1 \boxtimes \Gamma_2$ of two graphs Γ_1 and Γ_2 is the graph whose vertices are (u_1, u_2) with $u_1 \in \Gamma_1, u_2 \in \Gamma_2$ where (u_1, u_2) is adjacent to (v_1, v_2) if and only if:

- i. $u_1 = v_1$ and u_2 is adjacent to v_2 , or
- ii. u_1 is adjacent to v_1 and $u_2 = v_2$, or
- iii. u_1 is adjacent to v_1 and u_2 is adjacent to v_2 .

Note that in the product $\Gamma_1 \boxtimes \Gamma_2$, we have $N[u_1, u_2] = N[u_1] \times N[u_2]$, and given the importance of the closed neighborhoods to G^Γ , this makes the strong product perhaps the most “natural” product of two graphs for our consideration. We find the following properties.

Theorem 4.1. *Let Γ_1, Γ_2 be undirected simple graphs with $|\Gamma_1| = m$ and $|\Gamma_2| = n$.*

- (a) *The activation matrix $A_{\Gamma_1 \boxtimes \Gamma_2}$ for $\Gamma_1 \boxtimes \Gamma_2$ is the tensor product (or Kronecker product) $A_{\Gamma_1} \otimes A_{\Gamma_2}$ of the activation matrices A_{Γ_1} for Γ_1 and A_{Γ_2} for Γ_2 .*
- (b) *If A_{Γ_1} has elementary divisors $a_1 \mid a_2 \mid \cdots \mid a_m$ and A_{Γ_2} has $b_1 \mid b_2 \mid \cdots \mid b_n$ (where the last few a_i ’s and/or b_j ’s may be 0), then $A_{\Gamma_1 \boxtimes \Gamma_2}$ has elementary divisors $\{a_i b_j\}_{i=1, j=1}^{m, n}$, up to rearrangement of the prime factors.*
- (c) *$C_{\Gamma_1 \boxtimes \Gamma_2} = C_{\Gamma_1} \otimes C_{\Gamma_2}$, so the elementary divisors of the RA matrix $C_{\Gamma_1 \boxtimes \Gamma_2}$ are the products of the elementary divisors of the RA matrices C_{Γ_1} and C_{Γ_2} (up to rearrangement of the prime factors).*

Proof. (a) If $A_{\Gamma_1} = (a_{ij})$ and we label the vertices of $\Gamma_1 \boxtimes \Gamma_2$ in lexicographic order, then it is clear the activation matrix $A_{\Gamma_1 \boxtimes \Gamma_2}$ has the block form

$$\begin{pmatrix} A_{\Gamma_2} & a_{1,2}A_{\Gamma_2} & \cdots & a_{1,m}A_{\Gamma_2} \\ a_{2,1}A_{\Gamma_2} & A_{\Gamma_2} & \cdots & a_{2,m}A_{\Gamma_2} \\ \vdots & \vdots & & \vdots \\ a_{m,1}A_{\Gamma_2} & a_{m,2}A_{\Gamma_2} & \cdots & A_{\Gamma_2} \end{pmatrix}.$$

The well-known property $(A \otimes B)(C \otimes D) = AC \otimes BD$ guarantees that if $PA_{\Gamma_1}Q = D_1$ and $RA_{\Gamma_2}S = D_2$, with D_1 and D_2 the Smith normal forms of A_{Γ_1} and A_{Γ_2} , respectively, then $(P \otimes R)A_{\Gamma_1 \boxtimes \Gamma_2}(Q \otimes S) = D_1 \otimes D_2$. Now, $D_1 \otimes D_2$ has precisely the entries $a_i b_j$ for a_i in the diagonal of D_1 and b_j in that of D_2 , so up to rearrangement of the prime factors of $\{a_i b_j\}$, $D_1 \otimes D_2$ is the Smith normal form of $A_{\Gamma_1 \boxtimes \Gamma_2}$ and (b) follows.

Any row of $C_{\Gamma_1 \boxtimes \Gamma_2}$ not in $A_{\Gamma_1 \boxtimes \Gamma_2}$ is the intersection of two rows of the form

$$(a_{u_1,1}r_{v_1} \ a_{u_1,2}r_{v_1} \ \cdots \ a_{u_1,m}r_{v_1}) \text{ and } (a_{u_2,1}r_{v_2} \ a_{u_2,2}r_{v_2} \ \cdots \ a_{u_2,m}r_{v_2})$$

for some $u_1, u_2 \in \Gamma_1, v_1, v_2 \in \Gamma_2$, where r_v denotes the row of A_{Γ_2} corresponding to vertex $v \in G_2$ (by intersection of rows, we mean the element-wise product as usual). Letting $r_{v_1} \cap r_{v_2}$ denote the intersections of the rows r_{v_1} and r_{v_2} and similarly r_{u_1}, r_{u_2} denote the rows of A_{Γ_1} corresponding to the vertices u_1, u_2 , respectively, the result is

$$(a_{u_1,1}a_{u_2,1}(r_{v_1} \cap r_{v_2}) \ a_{u_1,2}a_{u_2,2}(r_{v_1} \cap r_{v_2}) \ \cdots \ a_{u_1,m}a_{u_2,m}(r_{v_1} \cap r_{v_2})),$$

which is precisely the tensor product $(r_{u_1} \cap r_{u_2}) \otimes (r_{v_1} \cap r_{v_2})$. Just as in (b), (c) follows. $\square$

4.2 Cartesian products

The cartesian product $\Gamma \square \Lambda$ of two graphs Γ and Λ has vertex set $V(\Gamma) \times V(\Lambda)$, and there is an edge between (u_1, u_2) and (v_1, v_2) whenever they agree in one coordinate and are adjacent in the other coordinate. That is, either $u_1 = v_1$ and u_2 and v_2 are neighbors, or $u_2 = v_2$ and u_1 and v_1 are neighbors. The *prism over* Γ is the graph $\Gamma \square K_2$.

We have seen that cube graphs are always either RA or 1/2-RA. In fact, there is a much more general phenomenon at play here: the cartesian product of any bipartite graphs is either RA or 1/2-RA. More generally, the cartesian product of any two graphs is almost RA:

Theorem 4.2. *The graph $\Gamma_1 \square \Gamma_2$ is neighborly. Furthermore,*

1. *If Γ_1 and Γ_2 are either both bipartite or both not bipartite, then $\Gamma_1 \square \Gamma_2$ is either 1/2-RA or RA, and it is 1/2-RA if and only if the degrees of vertices of Γ_1 and the degrees of vertices of Γ_2 have opposite parity, and every pair of vertices in each Γ_i has an even number of common neighbors.*
2. *Otherwise, if Γ_1 is not bipartite and Γ_2 is, then $\Gamma_1 \square \Gamma_2$ is $1/\mu$ -RA, where $\mu = \gcd(\delta, \kappa_1, \kappa_2)$, with*

$$\delta = \gcd(\{1 + \deg(u) - \deg(i) : u \in \Gamma_1, i \in \Gamma_2\}),$$

$$\kappa_1 = \gcd(\{|N[u] \cap N[v]| : u, v \in \Gamma_1, u \neq v\}), \text{ and}$$

$$\kappa_2 = \gcd(\{|N[i] \cap N[j]| : i, j \in \Gamma_2, d(i, j) = 2\}).$$

Proof. Let $\Gamma = \Gamma_1 \square \Gamma_2$. Consider neighbors (u, i) and (v, j) of $\Gamma_1 \square \Gamma_2$. Then either $u = v$ or $i = j$. Let us assume that $i = j = 1$. If Γ_1 is bipartite, then u and v have no common neighbors, so $\vec{e}_{(u,1)} + \vec{e}_{(v,1)} \in \mathbb{Z}^{C_r}$. If Γ_1 is not bipartite, let w be a neighbor of v other than u , and let 2 be a neighbor of 1. (If no such neighbor w exists, then again $N[u] \cap N[v] = \{u, v\}$ and thus $\vec{e}_{(u,1)} + \vec{e}_{(v,1)} \in \mathbb{Z}^{C_r}$.) Then

$$(\vec{N}[u, 2] \cap \vec{N}[v, 1]) - (\vec{N}[v, 1] \cap \vec{N}[w, 2]) = (\vec{e}_{(u,1)} + \vec{e}_{(v,2)}) - (\vec{e}_{(v,2)} + \vec{e}_{(w,1)}) = \vec{e}_{(u,1)} - \vec{e}_{(w,1)}.$$

Thus, whenever u and w are vertices connected by an even path in Γ_1 , we have $\vec{e}_{(u,1)} - \vec{e}_{(w,1)} \in \mathbb{Z}^{C_r}$. Since Γ_1 is not bipartite, every pair of vertices is connected by an even path, and so $\vec{e}_{(u,1)} - \vec{e}_{(w,1)} \in \mathbb{Z}^{C_r}$ for *all* vertices u and w of Γ_1 . Applying the same argument to Γ_2 shows that every $\vec{e}_{(u,i)} + \vec{e}_{(u,j)} \in \mathbb{Z}^{C_r}$ if Γ_2 is bipartite, and otherwise every $\vec{e}_{(u,i)} - \vec{e}_{(u,j)} \in \mathbb{Z}^{C_r}$. It follows that Γ is neighborly.

By the arguments above, if Γ_1 and Γ_2 are both not bipartite, then Γ is negatively neighborly. Similarly, if Γ_1 and Γ_2 are both bipartite, then Γ is positively neighborly. In either case, since $|N[u, 1] \cap N[v, 2]| = 2$, Theorem 3.11 or Theorem 3.12 says that Γ is either 1/2-RA or RA. Furthermore, by Theorem 3.8, Γ is half-RA if and only if every vertex has odd degree and every pair of vertices has an even number of common neighbors. Since $\deg(u, i) = \deg(u) + \deg(i)$, every vertex has odd degree if and only if every vertex of Γ_1 (resp. Γ_2) has odd degree and every vertex of Γ_2 (resp. Γ_1) has even degree. Now, two vertices (u, i) and (v, j) of Γ have common neighbors if and only if they agree in one coordinate or u is a neighbor of v and i is a neighbor of j . We already accounted for the latter case above (such vertices have two common neighbors), so every pair of vertices will have an even number of common neighbors if and only if that is true of both Γ_1 and Γ_2 .

Now suppose that Γ_1 is not bipartite and Γ_2 is. By the arguments above, if i and j are connected by a path of length k in Γ_2 , then $\vec{e}_{(u,i)} + (-1)^{k-1} \vec{e}_{(v,j)} \in \mathbb{Z}^{C_r}$ for every pair of vertices u and v in Γ_1 . The bipartition of Γ_2 induces a partition $U \sqcup V$ of $\Gamma_1 \square \Gamma_2$ by simply ignoring the first coordinate of each vertex, and this partition satisfies the conditions of Theorem 3.10. Each vertex (u, i) has $\deg(u) + \deg(i)$ neighbors, with $\deg(u)$ of them in the same part (with the same second coordinate) and $\deg(i)$ of them in the opposite part (with the same first coordinate). That shows that δ as defined here is the same as the δ in Theorem 3.10.

It remains to compute κ from Theorem 3.10. Consider vertices $(u, i) \neq (v, j)$ with common neighbors. If they do not agree in either coordinate, then u and v are neighbors, i and j are neighbors, and $N[u, i] \cap N[v, j] =$

$\{(u, j), (v, i)\}$. The two vertices in this intersection lie in different parts, and so they contribute a 0 to the set whose gcd is κ , leaving it unaffected. Thus, it suffices to consider vertices that agree in one coordinate. The vertices of $N[u, i] \cap N[v, i]$ all have second coordinate i and thus all lie in the same part, so the contribution of this set to κ is just $|N[u, i] \cap N[v, i]| = |N[u] \cap N[v]|$. Now, suppose that (u, i) and (u, j) have common neighbors. If they are adjacent, then since Γ_2 is bipartite (with bipartition induced by $U \sqcup V$), they are the only vertices of $N[u, i] \cap N[u, j]$, which thus has one vertex in U and one in V , contributing nothing to κ . Thus we may assume that (u, i) and (u, j) are non-adjacent, with all their common neighbors lying in the same part as each other. So again, the contribution of this set to κ is just $|N[u, i] \cap N[u, j]| = |N[i] \cap N[j]|$. Thus, $\kappa = \gcd(\kappa_1, \kappa_2)$, and the result follows. $\square$

Corollary 4.3. *If Γ is not bipartite, then the prism $\Gamma \square K_2$ is $1/\mu$ -RA, where $\mu = \gcd(\delta, \kappa)$ with*

$$\delta = \gcd(\{\deg(v) : v \in \Gamma\}), \quad \kappa = \gcd(\{|N[u] \cap N[v]| : u, v \in \Gamma\}).$$

Corollary 4.4. *If Γ_1 has non-adjacent vertices u and v that have a unique common neighbor, then $\Gamma_1 \square \Gamma_2$ is RA. In particular, if Γ_1 has girth 5 or more and $|\Gamma_1| > 2$, then $\Gamma_1 \square \Gamma_2$ is RA.*

Proof. If Γ_1 and Γ_2 are both bipartite or both not bipartite, then this follows directly from the first case of Theorem 4.2. Otherwise, in the second case, we see that $\kappa_1 = 1$ or, if Γ_1 is bipartite and Γ_2 is not, then we relabel them and $\kappa_2 = 1$. In each case, $\Gamma_1 \square \Gamma_2$ is RA. $\square$

Corollary 4.5. *If Γ_1 and Γ_2 are both girth 3 or both girth 4, then $\Gamma_1 \square \Gamma_2$ is either $1/2$ -RA or RA.*

Proof. This follows immediately from the statement of Theorem 4.2 except when Γ_1 is not bipartite and Γ_2 is (and both are girth 4). Then $\kappa_1 \leq 2$ since for adjacent vertices u and v in a girth 4 graph, $|N[u] \cap N[v]| = 2$. $\square$

Together Theorems 4.4 and 4.5 imply that the only way to get $\mu > 2$ in Theorem 4.2 is if either $\Gamma_2 = K_2$ or if Γ_1 has girth 3 and Γ_2 has girth 4.

Recall that for every group G , the group G^{K_n} is the diagonal subgroup of G^n , isomorphic to just G itself. In particular, K_n is as far from being RA as possible – C_{K_n} has only a single nontrivial elementary divisor. It is perhaps surprising then that the cartesian product of complete graphs is always $1/2$ -RA or RA:

Corollary 4.6. *Let $K = K_{n_1} \square K_{n_2} \square \cdots \square K_{n_m}$, with $m \geq 2$ and each $n_i \geq 2$. Then K is $1/2$ -RA or RA, and it is RA if and only if m is even or at least one n_i is odd.*

Proof. Up to rearrangement, we may assume that $n_1 \geq n_2 \geq \cdots \geq n_m$.

First, suppose that each $n_i = 2$. Then K is the cube graph Q_m , which is RA if and only if m is even.

Next, suppose that $n_1 > 2$ and $n_2 = n_3 = \cdots = n_m = 2$. Then K_{n_1} is not bipartite and $(K_{n_2} \square \cdots \square K_{n_m})$ is. Applying Theorem 4.2 gives us $\delta = n_1 - m + 1$, $\kappa_1 = n_1$, and $\kappa_2 = 2$. Thus $\mu = \gcd(\delta, \kappa_1, \kappa_2)$ will be 2 if and only if n_1 is even and m is odd, and otherwise $\mu = 1$.

Finally, suppose that $n_1 > 2$ and $n_2 > 2$. Then K_{n_1} and $(K_{n_2} \square \cdots \square K_{n_m})$ are both not bipartite, so K is $1/2$ -RA or RA. Let $K' = K_{n_2} \square \cdots \square K_{n_m}$. The degree of each vertex of K_{n_1} is $n_1 - 1$, and the degree of each vertex of K' is $\sum_{i=2}^m n_i - (m - 1)$. Every pair of vertices of K_{n_1} has n_1 common neighbors (including each other), and some pairs of vertices of K' have 2 common neighbors, with others having n_i common neighbors for some $2 \leq i \leq m$. Thus, K will be $1/2$ -RA if and only if every n_i is even and m is odd. $\square$

Example 4.7. Let us consider the higher-dimensional analogue of grids / cylinders / tori

$$\Gamma = P_{m_1} \square \cdots \square P_{m_k} \square C_{n_1} \square \cdots \square C_{n_t},$$

where k or t could be 0 and all n_j are equal to 3 or at least 5 ($C_2 = P_2$ and $C_4 = P_2 \square P_2$ are lumped in with the path graphs). Paths with at least 3 vertices and cycles C_n with $n \geq 5$ have girth at least 5, so if some $m_i \geq 3$ or some $n_j \geq 5$ then Γ is RA by Theorem 4.4. Otherwise, each $m_i = 2$ and each $n_j = 3$ so we have a cartesian product of complete graphs and Theorem 4.6 tells us Γ is RA if and only if $t \geq 1$ or $t = 0$ (so $\Gamma = Q_k$) and k is even. Thus, the only case when such a graph Γ is not RA is when $\Gamma = Q_k$ with k odd, and then Γ is $1/2$ -RA.

Let us now show one fun example of an infinite family of non-RA graphs built from Kneser graphs.

Example 4.8. Consider the prism $\text{Kn}(n, k) \square K_2$. Let us show that, for infinitely many choices of n and k , the result is 1/3-RA. To do so, we will use Theorem 4.3 and prove that $\mu = 3$.

The degree of any vertex of the Kneser graph $\text{Kn}(n, k)$ is $\binom{n-k}{k}$. Any two non-adjacent vertices correspond to subsets of $\{1, 2, \dots, n\}$ that share j elements, with $1 \leq j \leq k-1$, and the number of vertices adjacent to such two vertices is $\binom{n-2k+j}{k}$. Any two adjacent vertices correspond to disjoint subsets of $\{1, 2, \dots, n\}$ and have $\binom{n-2k}{k}$ vertices in common (other than themselves). Thus, we are looking for pairs (n, k) such that

- (a) For any $j = 1, \dots, k$ we have $\binom{n-2k+j}{k} \equiv 0 \pmod{3}$ and
- (b) $\binom{n-2k}{k} \equiv 1 \pmod{3}$.

The key point will be the self-similarity of Pascal's Triangle $\pmod{p}$ for a prime p (see Fig. 1 and, e.g. [9] and [12]): take $\Delta =$ the first p rows of Pascal's triangle and replace each entry by that entry multiplied by Δ , putting in upside-down triangles of 0's in between the resulting triangles to get the first p^2 rows, then do the same, either multiplying the entries of the first p rows by the entire p^2 first rows or multiplying the entries of the first p^2 rows by Δ , to find the first p^3 rows, and so on. See Fig. 1.

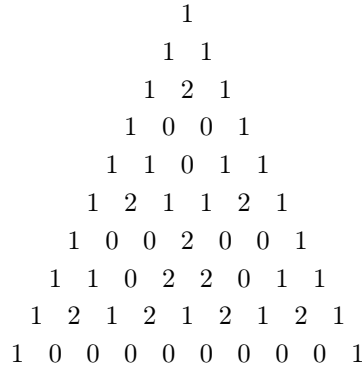


Figure 1: Rows 0 - 9 of Pascal's Triangle $\pmod{3}$

Note that for any $a \geq 0$, the 3^{a+1} -th row $(1, 0, \dots, 0, 1)$ marks the start of an upside down triangle of zeroes of size $3^{a+1} - 1$ (in width and height), so the zeroes extend from the $\binom{3^{a+1}}{r}$ entry down-left to the $\binom{3^{a+1}+r-1}{r}$ entry (inclusive) for $r = 1, 2, \dots, 3^{a+1} - 1$. Then if $n = 3^{a+1} + 2k - 1$ and $k < 3^{a+1}$, we get

$$\binom{n-2k+j}{k} = \binom{3^{a+1}+j-1}{k} \equiv 0 \pmod{3}$$

for each j with $1 \leq j \leq k$. This proves (a) above. Then note that

$$\binom{n-2k}{k} = \binom{3^{a+1}-1}{k},$$

which we want to be 1. The $(3^a - 1)$ st row follows the alternating pattern $1, 2, 1, 2, \dots, 2, 1$, meaning the entry before the zeroes start for even k is 1. So, picking k to be an even number between 3^a and 3^{a+1} satisfies (b). In other words, picking any nonnegative integers a and b , we can set $k = 3^a + 1 + 2b$ and $n = 3^{a+1} + 2k - 1$, and then $\text{Kn}(n, k)$ has properties (a) and (b). In fact, the numbers in column k repeat with period 3^{a+1} so the statements above still hold if we add $3^{a+1}i$ to n for any nonnegative integer i .

Example 4.9. Prisms over complements of Kneser graphs serve as an example of non-RA graphs as well. Let $n \geq 3$, K be the complement of $\text{Kn}(n, 2)$ and $\Gamma = K \square K_2$. Then Γ is 1/4-RA if $n \equiv 0 \pmod{4}$, 1/2-RA if $n \equiv 2 \pmod{4}$, and RA if n is odd. Indeed,

- (a) The degree of a vertex, say $\{1, 2\}$, is $2(n-2)$ since it has adjacent vertices $\{1, j\}$ and $\{2, j\}$ for $3 \leq j \leq n$.
- (b) The intersection $N(u) \cap N(v)$ for two adjacent vertices, say $\{1, 2\}$ and $\{1, 3\}$, has n vertices: u and v themselves, $\{2, 3\}$, and $\{1, j\}$ for $4 \leq j \leq n$.
- (c) The intersection $N(u) \cap N(v)$ for two non-adjacent vertices, say $\{1, 2\}$ and $\{3, 4\}$, has exactly 4 vertices: $\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}$.

Taking the gcd of these numbers and using Theorem 4.3 completes the proof.

4.3 Tensor Products

The tensor product $\Gamma_1 \times \Gamma_2$ of two graphs Γ_1 and Γ_2 is the graph whose vertices again are (u_1, u_2) with $u_1 \in \Gamma_1, u_2 \in \Gamma_2$, and where (u_1, u_2) is adjacent to (v_1, v_2) if and only if u_1 is adjacent to v_1 and u_2 is adjacent to v_2 . It is well-known that the tensor product of two connected graphs is disconnected if and only if they are both bipartite; first, we will deal with the case that one of the graphs is bipartite, then the case where both are, and last with the case that neither are.

Theorem 4.10. *Let Γ be a non-bipartite graph and $\Lambda = \Lambda_1 \sqcup \Lambda_2$ a bipartite graph. Then $\Gamma \times \Lambda$ is $1/\mu$ -RA with $\mu = \gcd(\delta, \kappa)$, where*

$$\delta = \gcd\{\deg(v)\deg(\lambda) - 1 : v \in \Gamma, \lambda \in \Lambda\} \text{ and}$$

$$\kappa = \gcd\{|N(u) \cap N(v)| \cdot |N(\lambda_1) \cap N(\lambda_2)| : u, v \in \Gamma, \lambda_1, \lambda_2 \in \Lambda_i, i = 1, 2, (u, \lambda_1) \neq (v, \lambda_2)\}.$$

In particular, $\Gamma \times K_2$ is $1/\mu$ -RA, where $\mu = \gcd(\delta, \kappa)$ with

$$\delta = \gcd\{\deg(v) - 1 : v \in \Gamma\}, \quad \kappa = \gcd\{|N(u) \cap N(v)| : u, v \in \Gamma\}.$$

Proof. Let 1 and 2 denote adjacent vertices in Λ . If u and v are neighbors in Γ , then $N[u, 1] \cap N[v, 2] = \{(u, 1), (v, 2)\}$ since 1 and 2 have no common neighbors. Then if we take a path $u - v - w$ in Γ , and take $(\vec{N}[u, 1] \cap \vec{N}[v, 2]) - (\vec{N}[v, 2] \cap \vec{N}[w, 1])$, we see that $\vec{e}_{(u,1)} - \vec{e}_{(w,1)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$. By similar arguments, if u and w are any two vertices of Γ that are an even distance apart, we have $\vec{e}_{(u,1)} - \vec{e}_{(w,1)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$. Now Γ is not bipartite so every pair of vertices of Γ is connected by an even-length path. Thus, for any $u, v \in \Gamma$ and $i = 1, 2$ we have $\vec{e}_{(u,i)} - \vec{e}_{(v,i)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$. Coming back to neighbors u and v , this, together with the fact that $\vec{e}_{(u,1)} + \vec{e}_{(v,2)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$, tells us $\vec{e}_{(u,1)} + \vec{e}_{(u,2)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$. Continuing along the edges of Λ , we see that $\vec{e}_{(u,\lambda_1)} - \vec{e}_{(u,\lambda_2)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$ if $\lambda_1, \lambda_2 \in \Lambda$ are an even distance apart (i.e. in the same part of Λ), whereas $\vec{e}_{(u,\lambda_1)} + \vec{e}_{(u,\lambda_2)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$ if λ_1 and λ_2 are an odd distance apart (i.e. in different parts of Λ). Thus, taking $U = \{(v, \lambda) : v \in \Gamma, \lambda \in \Lambda_1\}$ and $V = \{(v, \lambda) : v \in \Gamma, \lambda \in \Lambda_2\}$, we can apply Theorem 3.10 to see that $\Gamma \times \Lambda$ is almost RA, as follows.

For every $v \in \Gamma, \lambda \in \Lambda$, $N[v, \lambda]$ has 1 vertex in one part and $\deg(v)\deg(\lambda)$ vertices in the other, so $\delta = \gcd\{\deg(v)\deg(\lambda) - 1 : v \in \Gamma, \lambda \in \Lambda\} = 0$ only if $\deg(\lambda) = \deg(v) = 1$ for all $v \in \Gamma$ and $\lambda \in \Lambda$. But this means $\Gamma = K_2$ is bipartite — a contradiction, so $\delta \neq 0$ and $\Gamma \times \Lambda$ is $1/\mu$ -RA, where $\mu = \gcd(\delta, \kappa)$. To write down κ , note that if λ_1 and λ_2 are in different parts of Λ , then $N[u, \lambda_1] \cap N[v, \lambda_2]$ is $\emptyset$ if u and v are not adjacent, or else contains (u, λ_1) in one part and (v, λ_2) in the other, in which case the difference in the sizes of intersections of $N[u, \lambda_1] \cap N[v, \lambda_2]$ with U and V is $1 - 1 = 0$. Thus, the only parts that contribute to κ are where λ_1 and λ_2 are in the same part of Λ , and in that case all of $N[u, \lambda_1] \cap N[v, \lambda_2]$ is contained in one part, so the difference we seek is $|N(u) \cap N(v)| \cdot |N(\lambda_1) \cap N(\lambda_2)|$. Noting that $|N(\lambda_1) \cap N(\lambda_2)| = 1$ and $\deg(\lambda) = 1$ when $\Lambda = K_2$, this proves the theorem. $\square$

Taking the tensor product of a Kneser graph with K_2 often gives an example where Theorem 4.10 gives a nontrivial $\mathbb{Z}^{C_\Gamma}$. The following general result follows from our work and [10].

Corollary 4.11. *Let $\Gamma = Kn(n, k) \times K_2$, with $n > 2k$. Let $L = \text{lcm}\{1, 2, \dots, k\}$ and $n' = n - 2k$. Then Γ is $1/\mu$ -RA, where $\mu = n' / \gcd(L, n')$.*

Proof. Recall from Theorem 4.8 that the degree of every vertex v is $\binom{n-k}{k}$ and the number of common neighbors of two vertices that share i numbers in their label, say $\{1, \dots, k\}$ and $\{1, \dots, i, k+1, \dots, 2k-i\}$, is given by $\binom{n-2k+i}{k}$, where $i \in \{0, 1, \dots, k-1\}$. Thus, by Theorem 4.10,

$$\mu = \gcd\left\{\binom{n-2k}{k}, \dots, \binom{n-k-1}{k}, \binom{n-k}{k} - 1\right\}.$$

Repeatedly applying the fact that $\gcd(b, a+b) = \gcd(a, b)$ to the binomial coefficients in the manner of

$$\gcd\left(\binom{m}{k}, \binom{m+1}{k}\right) = \gcd\left(\binom{m}{k}, \binom{m}{k-1} + \binom{m}{k}\right) = \gcd\left(\binom{m}{k}, \binom{m}{k-1}\right)$$

and

$$\gcd\left(\binom{m}{k}, \binom{m+1}{k} - 1\right) = \gcd\left(\binom{m}{k}, \binom{m}{k-1} + \binom{m}{k} - 1\right) = \gcd\left(\binom{m}{k}, \binom{m}{k-1} - 1\right)$$

shows that the desired is actually the gcd of

$$\binom{n-2k}{k}, \dots, \binom{n-2k}{2}, \binom{n-2k}{1}, \binom{n-2k}{0} - 1 = 0.$$

The result follows from [10, Theorem 1], which states the gcd of the nonzero numbers above equals $d(n-2k; 1, k) = (n-2k)/\gcd(L, n-2k)$. $\square$

Now let us consider the case where both $\Gamma = \Gamma_1 \sqcup \Gamma_2$ and $\Lambda = \Lambda_1 \sqcup \Lambda_2$ are bipartite. In this case, their tensor product has two connected components: the first is the bipartite graph with parts $\Gamma_1 \times \Lambda_1$ and $\Gamma_2 \times \Lambda_2$, and the second is bipartite with parts $\Gamma_1 \times \Lambda_2$ and $\Gamma_2 \times \Lambda_1$. We will work with the first connected component and keep in mind that the same arguments can be applied to the second. Just as in the proof of Theorem 4.10, if $u, v \in \Gamma$ are neighbors and $\lambda_1, \lambda_2 \in \Lambda$ are neighbors, then $N[u, 1] \cap N[v, 2] = \{(u, 1), (v, 2)\}$ and if we have paths $u - v - w$ in Γ and $\lambda_1 - \lambda_2 - \lambda_3$ in Λ , then $(\vec{N}[u, \lambda_1] \cap \vec{N}[v, \lambda_2]) - (\vec{N}[v, \lambda_2] \cap \vec{N}[w, \lambda_3]) = \vec{e}_{(u, \lambda_1)} - \vec{e}_{(w, \lambda_3)}$. But for either $i \in \{1, 2\}$ and $j \in \{1, 2\}$, if $u, w \in \Gamma_i$ and $\lambda_1, \lambda_3 \in \Lambda_j$ then there is an even-length path between (u, λ_1) and (w, λ_3) (take any even length path between u and w in Γ and follow it with second coordinates swapping between λ_1 and its neighbor in Λ , and then do the same for going from λ_1 to λ_3 , swapping between w and its neighbor in Γ for the first coordinate). Similarly, there is an odd-length path between (u, λ_1) and (v, λ_2) if u and v are in opposite parts of Γ and λ_1, λ_2 are in opposite parts of Λ . So again, $\vec{e}_{(u, \lambda_1)} + \vec{e}_{(v, \lambda_2)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$ if (u, λ_1) and (v, λ_2) are in opposite parts of a connected component of $\Gamma \times \Lambda$, and $\vec{e}_{(u, \lambda_1)} - \vec{e}_{(w, \lambda_3)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$ if (u, λ_1) and (w, λ_3) are in the same part of a connected component. That means we can apply Theorem 3.10 to each of the connected components of $\Gamma \times \Lambda$. By the same considerations as in the proof of Theorem 4.10, this gives us the following.

Theorem 4.12. *Let $\Gamma = \Gamma_1 \sqcup \Gamma_2$ and $\Lambda = \Lambda_1 \sqcup \Lambda_2$ be bipartite graphs. Then $\Gamma \times \Lambda$ is the disjoint union of two connected components, one $1/\mu_1$ -RA and the other $1/\mu_2$ -RA, with $\mu_i = \gcd(\delta_i, \kappa_i)$, where*

$$\delta_i = \gcd\{\deg(v)\deg(\lambda) - 1 : v \in \Gamma_a, \lambda \in \Lambda_b, a \in \{1, 2\}\},$$

$$\kappa_i = \gcd\{|N(u) \cap N(v)| \cdot |N(\lambda_1) \cap N(\lambda_2)| : u, v \in \Gamma_a, \lambda_1, \lambda_2 \in \Lambda_b, a \in \{1, 2\}, (u, \lambda_1) \neq (v, \lambda_2)\},$$

and $b = a$ (resp. $b = 3 - a$) if $i = 1$ (resp. $i = 2$).

For example, C_Γ for $\Gamma = K_2 \times \text{Cr}(2k+4)$ has two elementary divisors equal to k for any $k \geq 1$. This makes sense because, as we will see in Theorem 5.4, the crown graph $\text{Cr}(2k+4) \equiv K_2 \times K_{k+2}$ is $1/k$ -RA and Γ has two disconnected components, each isomorphic to $\text{Cr}(2k+4)$. In general, for $\Gamma = K_2 \times \Lambda$ where Λ is a bipartite graph, Γ is isomorphic to two disjoint copies of Λ and C_Γ either has all 1's as elementary divisors (if Λ is RA) or has two elementary divisors equal to μ if Λ is $1/\mu$ -RA.

Finally, we explore the tensor product of two non-bipartite graphs.

Theorem 4.13. *If Γ is not bipartite and $m \geq 3$, then $\Gamma \times K_m$ is $1/2$ -RA or RA, and it is $1/2$ -RA if and only if m is even, $\deg v$ is odd for every $v \in \Gamma$, and $|N[u] \cap N[v]|$ is even for every $u, v \in \Gamma$.*

Proof. First, note that for any $u \in \Gamma$,

$$\vec{N}[u, 1] - \vec{N}[u, 2] + (\vec{N}[u, 2] \cap \vec{N}[u, 3]) - (\vec{N}[u, 1] \cap \vec{N}[u, 3]) = \vec{e}_{(u, 1)} - \vec{e}_{(u, 2)} \in \mathbb{Z}^{C_{\Gamma \times K_m}}, \quad (1)$$

which means we can move the coefficient of any vertex in a row of $C_{\Gamma \times K_m}$ to a vertex with any other second coordinate.

Next, note that

$$(m-2)\vec{N}[u, 1] - (\vec{N}[u, 1] \cap \vec{N}[u, 2]) - \dots - (\vec{N}[u, 1] \cap \vec{N}[u, m]) = (m-2)\vec{e}_{(u, 1)} \in \mathbb{Z}^{C_{\Gamma \times K_m}}. \quad (2)$$

Furthermore, for any neighbors $u, v \in \Gamma$, we have

$$(m-2)(\vec{N}[u, 1] \cap \vec{N}[v, 1]) - (\vec{N}[u, 1] \cap \vec{N}[v, 2]) - \dots - (\vec{N}[u, 1] \cap \vec{N}[v, m]) = (1-m)\vec{e}_{(u, 1)} - \sum_{i=2}^m \vec{e}_{(v, i)} \in \mathbb{Z}^{C_{\Gamma \times K_m}}. \quad (3)$$

From (1), moving all the (v, i) coefficients to that of $(v, 1)$ by adding $\sum_{i=2}^m (\vec{e}_{(v, i)} - \vec{e}_{(v, 1)})$, we get

$$(1-m)(\vec{e}_{(u, 1)} + \vec{e}_{(v, 1)}) \in \mathbb{Z}^{C_{\Gamma \times K_m}}.$$

Adding this to $(m-2)\vec{e}_{(u,1)} + (m-2)\vec{e}_{(v,1)}$ from (2), we find that $\vec{e}_{(u,1)} + \vec{e}_{(v,1)} \in \mathbb{Z}^{C_{\Gamma \times K_m}}$. This means the edges along the first coordinate are “positive”, so overall $\Gamma \times K_m$ is neighborly. But Γ is not bipartite, so going around an odd cycle also gives us $\vec{e}_{(u,1)} - \vec{e}_{(v,1)} \in \mathbb{Z}^{C_{\Gamma \times K_m}}$. That means, not only is $\Gamma \times K_m$ negatively neighborly but also, combined with what we already had gives us $2\vec{e}_{(u,1)} \in \mathbb{Z}^{C_{\Gamma \times K_m}}$, so $\Gamma \times K_m$ is 1/2-RA or RA.

Applying Theorem 3.12, we first look at $\delta = \gcd\{\deg(v,i) + 1 : v \in \Gamma, i \in K_m\}$. We have $\deg(v,i) = (\deg v)(m-1) + 1$, which is even iff m is even and $\deg v$ is odd. Next, considering $\kappa = \gcd\{|N[u,i] \cap N[v,j]| : u, v \in \Gamma, i, j \in K_m, (u,i) \neq (v,j)\}$, if $i \neq j$ then $|N[u,i] \cap N[v,j]| = |N[u] \cap N[v]|(m-2) + 2b$, where $b = 1$ if u and v are neighbors or else $b = 0$. If instead $i = j$, then we have $|N[u,i] \cap N[v,i]| = |N[u] \cap N[v]|(m-1)$, so we see that $|N[u] \cap N[v]|$ is multiplied by both $m-1$ and $m-2$ (individually) in this set, which means κ is even iff $|N[u] \cap N[v]|$ is always even. Thus, $\mu = \gcd(\delta, \kappa)$ is even iff all these conditions hold. $\square$

Note in particular that Theorem 4.13 states that if Γ is not bipartite, then $\Gamma \times K_m$ is always RA for odd m .

Now we turn our attention to the general tensor product of an arbitrary number of complete graphs. Note that $K_2 \times K_2$ is disconnected, and in general the tensor product $K_2 \times K_2 \times \cdots \times K_2$ of n copies of K_2 has 2^{n-1} disconnected components; these components are just the pairs of opposite vertices of an n -dimensional cube. The tensor product of n copies of K_2 and a non-bipartite graph Γ results in 2^{n-1} isomorphic disjoint copies of Γ , so it suffices to consider tensor products $K_{m_1} \times K_{m_2} \times \cdots \times K_{m_n}$, with $m_1 \leq m_2 \leq \cdots \leq m_n$, where either $m_1 > 2$ or $m_1 = 2$ and $m_2 > 2$.

Theorem 4.14. *The graph $\Gamma = K_{m_1} \times K_{m_2} \times \cdots \times K_{m_n}$, for $2 \leq m_1 \leq m_2 \leq \cdots \leq m_n$ and $m_2 \geq 3$ is always 1/ μ -RA, where*

- (a) if $m_1 = 2$, then $\mu = \gcd\{m_i - 2 \mid i = 2, \dots, n\}$;
- (b) otherwise, $\mu \leq 2$ with $\mu = 2$ if and only if all m_i are even.

Proof. For the case (a) $m_1 = 2$, we could write down a bit of a messy proof using Theorem 4.10, but the cleanest way is to simply use Theorem 5.5 below, so we defer until then (see Theorem 5.6). Now, for (b), let $m_1 \geq 3$. By Theorem 4.13, Γ is 1/2-RA or RA, and if at least one m_i is odd, the graph is RA. If all the m_i are even, the degree of a vertex is $\prod_{i=1}^n (m_i - 1)$, which is odd, and the intersection of the closed neighborhoods of two distinct vertices has size $\prod_{i=1}^n (m_i - a_i) + 2b$, where $a_i = 1$ if the vertices agree in coordinate i or else $a_i = 2$, and $b = 0$ unless the vertices differ in all coordinates, in which case $b = 1$ (to account for the two vertices themselves being in the intersection). The vertices must differ in at least one coordinate, so at least one a_i must be even and hence $\prod_{i=1}^n (m_i - a_i)$ is even so Γ is 1/2-RA. $\square$

We have been unable to show that an arbitrary tensor product of two non-bipartite graphs is almost RA, but the following leads to a partial result in this direction (Theorem 4.16).

Lemma 4.15. *If Λ is a graph with an edge $\lambda_1 - \lambda_2$ that is not part of a 3-cycle, then for any non-bipartite graph Γ , any two vertices $u, v \in \Gamma$ and any $\lambda \in \Lambda$ we have $\vec{e}_{(u,\lambda)} - \vec{e}_{(v,\lambda)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$.*

Proof. If the edge $\lambda_1 - \lambda_2$ is not part of a 3-cycle, then for any edge $u - v \in \Gamma$, (u, λ_1) and (v, λ_2) have no common neighbors, so $\vec{N}[u, \lambda_1] \cap \vec{N}[v, \lambda_2] = \vec{e}_{(u,\lambda_1)} + \vec{e}_{(v,\lambda_2)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$. For any other edge $v - w$ in Γ , then, we have

$$(\vec{e}_{(u,\lambda_1)} + \vec{e}_{(v,\lambda_2)}) - (\vec{e}_{(v,\lambda_2)} + \vec{e}_{(w,\lambda_1)}) = \vec{e}_{(u,\lambda_1)} - \vec{e}_{(w,\lambda_1)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}.$$

Now Γ is non-bipartite, meaning there is an odd cycle we can go around so any two vertices are connected by an even-length path and for any two vertices $u, v \in \Gamma$ we have $\vec{e}_{(u,\lambda_1)} - \vec{e}_{(v,\lambda_1)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$.

Furthermore, if $u - v$ is an edge in Γ and $\lambda_1 - \lambda_2$ an edge in Λ , we have the vector

$$\times(u, v, \lambda_1, \lambda_2) := (\vec{e}_{(u,\lambda_1)} - \vec{e}_{(v,\lambda_1)}) - (\vec{e}_{(u,\lambda_2)} - \vec{e}_{(v,\lambda_2)}) = \vec{N}[u, \lambda_1] \cap \vec{N}[v, \lambda_2] - \vec{N}[u, \lambda_2] \cap \vec{N}[v, \lambda_1] \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}.$$

Using the identity

$$\times(u, v, \lambda_1, \lambda_2) + \times(u, v, \lambda_2, \lambda_3) = \times(u, v, \lambda_1, \lambda_3)$$

along any path from λ_1 to $\lambda \in \Lambda$, we find $\times(u, v, \lambda_1, \lambda) \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$, and subtracting this from $\vec{e}_{(u,\lambda_1)} - \vec{e}_{(v,\lambda_1)}$ gives $\vec{e}_{(u,\lambda)} - \vec{e}_{(v,\lambda)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$. $\square$

Proposition 4.16. *If Γ and Λ are non-bipartite graphs that each have an edge that is not part of a 3-cycle, then $\Gamma \times \Lambda$ is 1/2-RA or RA.*

Proof. Apply Theorem 4.15 to both graphs shows that $\Gamma \times \Lambda$ is negatively neighborly and if $u - v$ is an edge in Γ that is not part of a triangle, then for any edge $\lambda_1 - \lambda_2$ in Λ we have $\vec{N}[u, \lambda_1] \cap \vec{N}[v, \lambda_2] = \vec{e}_{(u, \lambda_1)} + \vec{e}_{(v, \lambda_2)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$ and again applying Theorem 4.15 twice to this expression, we have $2\vec{e}_{(u, \lambda_1)} \in \mathbb{Z}^{C_{\Gamma \times \Lambda}}$ so $\Gamma \times \Lambda$ is 1/2-RA or RA (by Theorem 3.8). $\square$

In conclusion, we have shown that various tensor products are 1/ μ -RA and described μ : the tensor product of a bipartite graph with a non-bipartite graph in Theorem 4.10, that of two bipartite graphs in Theorem 4.12, an arbitrary tensor product of complete graphs in Theorem 4.14, and more. We have not been able to deduce such a general result for arbitrary tensor products of two non-bipartite graphs, though we will return to this question again briefly in Theorem 5.7, where we show that the result is 1/2-RA or RA if at least one of the graphs has girth 4 or more. We leave the general question an open problem (Problem 1).

5 Girth 4⁺ graphs are almost RA

In [2, Cor. 5.13], we showed that every graph of girth 5 or higher is RA. On the other hand, [2, Cor. 6.14] showed that the cube graph Q_n is not RA when n is odd. In this section, we will further explore graphs of girth 4. Note that a connected girth 4 graph must automatically be neighborhood-distinguishable: if u and v have the same closed neighborhood, then they must be adjacent and have a common neighbor w (since otherwise there would be only two vertices), which would give us a triangle.

With the theory already developed, girth 4 graphs will be easy to study due to the following simple result.

Proposition 5.1. *A girth 4 graph is positively neighborly.*

Proof. This follows from the fact that if u and v are adjacent vertices in a graph of girth 4, the intersection of their closed neighborhoods consists solely of u and v . $\square$

Theorem 5.2. *Suppose Γ is a graph of girth 4.*

1. *If Γ is not bipartite, then it is either RA or 1/2-RA, and it is 1/2-RA if and only if every vertex has odd degree and every pair of vertices 2 apart have an even number of common neighbors.*
2. *If Γ is bipartite, then Γ is 1/ μ -RA, where $\mu = \gcd(\delta, \kappa)$, with*

$$\delta = \gcd(\{\deg(v) - 1 : v \in \Gamma\}) \text{ and}$$

$$\kappa = \gcd(\{|N[u] \cap N[v]| : u, v \in \Gamma, d(u, v) = 2\}).$$

Proof. This follows directly from Theorem 3.11 once we note that $N[u] \cap N[v] = N(u) \cap N(v)$ for non-adjacent vertices u and v . $\square$

In [2], we showed that if Γ does not satisfy the “square completion property”, that every path $u - v - w$ must be able to be completed to a 4-cycle $u - v - w - x - u$, then Γ is RA. Not having this property implies that $\kappa = 1$, so Theorem 5.2 may be thought of as a generalization of and addition to this fact. In particular, Theorem 5.2 implies that if, for example, a girth 4 graph has two vertices with coprime valency, then the graph is RA. It also puts strong restrictions on the structure of a girth 4 graph whose RA matrix C_Γ has an elementary divisor of k . In particular, such a graph must have the degree of every vertex congruent to 1 (mod k), and the number of common neighbors of any vertices a distance 2 apart must be divisible by k .

Example 5.3. The cube graph Q_d is 1/2-RA for odd $d \geq 3$, and the folded cube graph $\square_d$ is 1/2-RA for odd $d \geq 5$. Indeed, every vertex of each graph has odd degree, and vertices at a distance of 2 have exactly 2 common neighbors. In fact, as we will see in Theorem 5.10, the cube graph Q_3 is the only non-RA girth 4 graph with vertices of “small” degree

A natural question to ask at this point is whether there are graphs of girth 4 with an arbitrarily large elementary divisor. Indeed, we can now give a concrete example of a 1/ μ -RA graph of girth 4 for each $\mu \geq 2$.

Corollary 5.4. *The crown graph $\text{Cr}(2n)$ on $2n$ vertices (with $n \geq 4$) is a girth 4 graph that is 1/($n - 2$)-RA.*

Proof. This follows from Theorem 4.14 or Theorem 5.2, since the crown graph is a bipartite graph of girth 4, every vertex has degree $n - 1$, and vertices that are 2 apart have $n - 2$ common neighbors. $\square$

Let us now show that for any $\mu > 1$, any bipartite graph of girth 4 that is $1/\mu$ -RA gives rise to many infinite families of such graphs.

Corollary 5.5. *Suppose Λ is a bipartite girth 4 graph that is $1/\mu$ -RA. Then $\Gamma = \Lambda \times K_{\mu+2}$ is a connected, bipartite girth 4 graph that is $1/\gcd(\mu, \nu)$ -RA.*

Proof. The bipartition of $\Lambda = \Lambda_1 \sqcup \Lambda_2$ induces a bipartition on $\Lambda \times K_{\mu+2}$ as $(\Lambda_1 \times K_{\mu+2}) \sqcup (\Lambda_2 \times K_{\mu+2})$. In general, if Λ is any girth 4 graph, its tensor product with another graph has girth 4: the resulting graph Γ has no 3-cycles, and any 4-cycle in Λ induces a 4-cycle in Γ (by just alternating between two neighbors in the other coordinate), so Γ has girth 4. Now, Λ being $1/\mu$ -RA and girth 4 implies, by Theorem 5.2, that $\gcd(\delta_\Lambda, \kappa_\Lambda) = \mu$. Letting $\delta_\Gamma, \kappa_\Gamma$ be the numbers from Theorem 4.10 and $\delta_\Lambda, \kappa_\Lambda$ from Theorem 5.2, we have

$$\delta_\Gamma = \gcd\{(\nu + 1) \deg(\lambda) - 1 : \lambda \in \Lambda\}$$

since $\deg v = \nu + 1$ for all $v \in K_{\mu+2}$, and κ_Γ is the gcd of the two sets

$$S_1 = \{\nu \cdot |N(\lambda_1) \cap N(\lambda_2)| : \lambda_1, \lambda_2 \in \Lambda_i, i = 1, 2\},$$

$$S_2 = \{(\nu + 1)|N(\lambda_1) \cap N(\lambda_2)| : \lambda_1 \neq \lambda_2 \in \Lambda_i, i = 1, 2, d(\lambda_1, \lambda_2) = 2\}$$

since if we take $u \neq v$ in the expression for κ_Γ , we find $|N(u) \cap N(v)| = \nu$ but if $u = v$ then $|N(u) \cap N(v)| = \nu + 1$ and we cannot have $\lambda_1 = \lambda_2$, which implies that if they have common intersection of their neighborhoods, they must be a distance 2 apart and in the same part of Λ . Taking $\lambda = \lambda_1 = \lambda_2$ in S_1 gives $\nu \deg(\lambda)$ and combining that with δ_Γ , we see that $\gcd(\delta_\Gamma, \kappa_\Gamma)$ must divide $\gcd\{\deg(\lambda) - 1 : \lambda \in \Lambda\} = \delta_\Lambda$. Subtracting the expressions in S_1 and S_2 for $\lambda_1 \neq \lambda_2$, we find $\kappa_\Gamma \mid \gcd\{|N(\lambda_1) \cap N(\lambda_2)| : \lambda_1 \neq \lambda_2 \in \Lambda_i, i = 1, 2, d(\lambda_1, \lambda_2) = 2\}$, thus κ_Γ must divide κ_Λ . Also, taking any vertex $\lambda \in \Lambda$, we see that $\gcd(\delta_\Gamma, \kappa_\Gamma)$ divides $\nu \deg(\lambda)$ and also $\deg(\lambda) - 1$, hence also $\nu \deg(\lambda) - \nu(\deg(\lambda) - 1) = \nu$. Thus, $\gcd(\delta_\Gamma, \kappa_\Gamma) \mid \gcd(\mu, \nu)$, and it is easy to check the converse holds too so $\gcd(\delta_\Gamma, \kappa_\Gamma) = \gcd(\mu, \nu)$. $\square$

Remark 5.6. Theorem 5.5 shows that the tensor product $K_2 \times K_{m_2} \times \cdots \times K_{m_n}$ in part (a) of Theorem 4.14 is $1/\mu$ -RA, where $\mu = \gcd\{m_i - 2 : i = 2, \dots, n\}$ since $K_2 \times K_{m_2}$ is the crown graph $\text{Cr}(2m_2)$, which is $1/(m_2 - 2)$ -RA.

Remark 5.7. If Γ and Λ are non-bipartite, then $\Gamma \times \Lambda$ is non-bipartite, and if Λ is in addition girth 4 or more, then $\Gamma \times \Lambda$ is a non-bipartite graph of girth 4 (pick a 4-cycle $\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4$ in Λ and two neighbors $u, v \in \Gamma$; we then have the 4-cycle $(u, \lambda_1) - (v, \lambda_2) - (u, \lambda_3) - (v, \lambda_4) - (u, \lambda_1)$), which means we can apply Theorem 5.2 to it. That means $\Gamma \times \Lambda$ is RA or $1/2$ -RA. Since $\deg(v, \lambda) = \deg(v) \deg(\lambda)$ for $v \in \Gamma, \lambda \in \Lambda$ and for (u, λ_1) and (v, λ_2) a distance 2 away we have $|N(u, \lambda_1) \cap N(v, \lambda_2)| = |N(u) \cap N(v)| \cdot |N(\lambda_1) \cap N(\lambda_2)|$ so $\Gamma \times \Lambda$ is RA unless $\deg v$ and $\deg \lambda$ are always odd and at least one of Γ, Λ has every pair of vertices a distance 2 apart sharing an even number of common neighbors. Note that if Λ is a graph of girth 5 or higher, its cartesian product with any other connected graph is RA (see Theorem 4.4), while the same is not always true of tensor products.

Corollary 5.8. *For each $\mu \geq 2$, there are infinitely many bipartite graphs of girth 4 that are $1/\mu$ -RA.*

Note that the crown graph on 8 vertices is precisely the cube graph Q_3 , which is the smallest (connected, neighborhood-distinguishable) non-RA graph. Actually, the crown graphs are precisely the unique smallest $1/k$ -RA girth 4 graphs:

Theorem 5.9. *For each $k \geq 2$, the crown graph $\text{Cr}(2k + 4)$ is the unique smallest graph Γ of girth 4 that is $1/k$ -RA.*

Proof. Suppose Γ is a graph of girth 4 that is $1/k$ -RA. By Theorem 5.2, the degree of each vertex is congruent to 1 (mod k). If Γ has at most $2k + 4$ vertices, then the degree of each vertex is at most $2k + 3$, and thus the degree must be either $k + 1$ or $2k + 1$, or $3k + 1$ if $k = 2$. In this last case, we have $k = 2$ and the vertices have degree 7, so there must be exactly $2k + 4 = 8$ vertices; this implies that the graph is K_8 , which does not have girth 4. So each vertex has degree at most $2k + 1$.

Suppose there is a vertex v of degree $2k + 1$. Since each neighbor of v has degree at least $k + 1$, there must be at least k vertices that are distance 2 from v , and this gives us at least $1 + (2k + 1) + k = 3k + 2$ vertices, and if $k \geq 3$, then this is greater than $2k + 4$. If $k = 2$, then we have a graph with exactly $2k + 4 = 8$ vertices, where the vertices adjacent to v have degree 3 and are each adjacent to two vertices other than v . Since there are only 8 vertices, all neighbors of v must be adjacent to the same two other vertices. However, then both vertices that are distance 2 from v have 5 common neighbors with v , whereas Theorem 5.2 implies that the number of

common neighbors should be even. So if Γ has a vertex of degree $2k + 1$ and C_Γ has an elementary divisor of k , then Γ must have more than $2k + 4$ vertices.

We may now assume that every vertex has degree $k + 1$. Fix a vertex v . Its $k + 1$ neighbors each have k edges that lead to vertices that are 2 apart from v , so there are $k(k + 1)$ such edges. Now, by Theorem 5.2, the number of common neighbors between v and vertices that are 2 away is divisible by k , and in any case it must be at least 1 and smaller than the degree of v . Thus the vertices that are 2 apart from v must have exactly k common neighbors with v , and so there must be $k + 1$ of them. This gives us $2k + 3$ vertices so far, and the vertices that are distance 2 from v are still missing an edge. Clearly then, the only way to add only a single vertex to the graph is to make it adjacent to all vertices that are 2 away from v , and what we get is precisely the crown graph. $\square$

We finish the section with another application of Theorem 5.2 to show the cube Q_3 is the non-RA graph of girth 4 with the smallest degree of vertices.

Theorem 5.10. *If Γ is a connected graph of girth 4 whose vertices have degree ≤ 3 and Γ is not RA, then Γ is the 3-cube Q_3 .*

Proof. Since Γ has girth 4 and is not RA, Theorem 5.2 says that it must be $1/\mu$ -RA, where $\mu = \gcd(\delta, \kappa)$ as defined in Theorem 5.2. Then, since the degree of each vertex is at most 3, $\mu = 2$ and the degree of every vertex must be 3. Furthermore, $\kappa = 2$; that is, vertices that are 2 apart have exactly two common neighbors.

Now, pick an initial vertex of Γ and label it 000. It must have degree 3, so we'll call its neighbors 100, 010, and 001 in arbitrary order. Since vertices that are two apart have exactly two common neighbors, there must be another vertex adjacent to 100 and 010; we'll call it 110. Similarly, there must be a vertex 101 adjacent to 100 and 001, and a vertex 011 adjacent to 010 and 001. Note that these three new vertices are all distinct; if any two of them were the same, then that vertex (say 100) would have 3 common neighbors with 000 rather than 2.

Now 000, 100, 010, 001 all have degree 3 with the listed vertices. That means some vertex, which we'll call 111, must complete the 4-cycle $110 - 100 - 101 - 111 - 110$. Note that 111 can't be 011 because then $011 - 101 - 001$ would be a 3-cycle. To prove that 111 is connected to 011, note we must have a 4-cycle $011 - 010 - 110 - v - 011$, and since 110 already has degree 3, v must be either 111 or 100, but the latter already has degree 3. Now each of the listed vertices has degree 3 so there can't be any more connections, and there can't be any more vertices since Γ is connected. $\square$

6 Girth 3 graphs can be arbitrarily far from RA

We have seen that graphs of girth 5 or higher are RA, and graphs of girth 4 are never far off, with $\mathbb{Z}^{C_\Gamma}$ having full rank and at most one nontrivial elementary divisor. In this section, we will show that for girth 3 graphs Γ , $\mathbb{Z}^{C_\Gamma}$ can have arbitrarily many nontrivial, nonzero elementary divisors and arbitrarily large nullity over $\mathbb{Z}$. In fact, we will show in Theorem 6.8 that for any numbers $1 < d_1 \mid d_2 \mid \cdots \mid d_m$ and any nullity choice n , it is possible to create a graph Γ for which $\mathbb{Z}^{C_\Gamma}$ has the d_i as the elementary divisors and nullity n over $\mathbb{Z}$. The distinguishing fact for girth 4 graphs is the fact that adjacent vertices u and v have no common neighbors, so that $\vec{e}_u + \vec{e}_v \in \mathbb{Z}^{C_\Gamma}$ for any edge $u - v$. In girth 3 graphs, adjacent vertices may have any number of common neighbors, and so there are no rows of $\mathbb{Z}^{C_\Gamma}$ that are guaranteed to be simple.

We will start by showing that we can find small examples of graphs, namely having only $2k + 5$ vertices, that are $1/k$ -RA for any $k \geq 2$. In addition to the small examples it provides, the following construction and proof will be useful in proving Theorem 6.8. Recall that the join graph $\Gamma_1 + \Gamma_2$ has vertices that are the disjoint union of the vertices of the two graphs Γ_1 and Γ_2 and whose edges are the ones from the graphs Γ_i together with those joining every vertex of every Γ_1 to every vertex of Γ_2 . The pyramid over a graph Γ is just its join with the one-vertex graph K_1 .

Proposition 6.1. *The pyramid $\Delta = \text{Cr}(2k + 4) + K_1$ over the crown graph on $2k + 4$ vertices is $1/k$ -RA.*

Proof. Adding the apex vertex v to the crown graph $\text{Cr}(2k + 4)$ as the first vertex changes the RA matrix of $C_{\text{Cr}(2k+4)}$ precisely by adding a row of 1's to the top and a column of 1's to the left. To see this, note that the vertex v itself is connected to all vertices, so the top row, corresponding to $N[v]$, is all 1; now for every vertex u the row corresponding to $N[u]$ has a 1 in the leftmost column since u is adjacent to v , and furthermore $v \in N[u_1] \cap N[u_2]$ for any two vertices u_1, u_2 . No more rows are added to C_Γ because the intersections $N[u] \cap N[v]$ are just $N[u]$.

Label the vertices of $\text{Cr}(2k+4)$ as $\{1, 2, \dots, (k+2), 1', 2', \dots, (k+2)'\}$ so the vertex labelled i is adjacent to all j' , and i' is adjacent to all j , such that $j \neq i$. To find the Smith normal form of C_Δ , we first note that the rows corresponding to $N[1]$ and $N[1']$ only intersect in the first column. Using this row of C_Δ , we can thus zero out the entire rest of the first column. Now these rows add up to a row of all-1's (with a 0 in the first column), so subtracting these two rows from the top row makes that row zero. Now, we can work with the submatrix of C_Δ with the first row and column removed, find its Smith normal form, and just note that we get an extra elementary divisor of 1 from the left out row and column, leaving us with just one nontrivial elementary divisor equal to k by Theorem 5.4. $\square$

Using SageMath and data from [13], we found that the pyramid over $\text{Cr}(8) = Q_3$ is one of three 1/2-RA graphs of girth 3 with 9 vertices, and no 1/2-RA girth 3 graphs have fewer vertices. The other two graphs have graph6 strings `H?zTb_{}` and `HCOFFz~`. It is unclear whether the pyramid over the crown graph is always one of the minimal girth 3 $(1/k)$ -RA graphs; we leave this question as an open problem - see Problem 2.

Unlike with girth 4, we can certainly find graphs Γ of girth 3 where C_Γ has multiple nontrivial elementary divisors. For example, if Γ is the Kneser graph $\text{Kn}(6, 2)$ on 15 vertices, then C_Γ has elementary divisors $[1^{11}, 2^4]$. In fact, we now show that Kneser graphs $\text{Kn}(kp, p)$ have an unbounded number of nontrivial elementary divisors as the prime $p \rightarrow \infty$. We start by exhibiting a family of vectors in the kernel of C_Γ for $\Gamma = \text{Kn}(kp, p)$.

Theorem 6.2. *Suppose p is a prime number and $k \geq 3$. Let $\Gamma = \text{Kn}(kp, p)$. For each vertex $x \in \Gamma$, define the $\binom{kp}{p}$ -dimensional vector $\vec{x} = \sum_{v \in \Gamma} |x \cap v| \vec{e}_v$ (recalling that x and v are p -subsets of $\{1, \dots, kp\}$). Then $\vec{x}$ lies in the kernel of C_Γ over $\mathbb{Z}/p\mathbb{Z}$.*

Proof. Without loss of generality, assume $x = \{1, 2, \dots, p\}$. What we need to show is that for each row $\vec{y}$ of C_Γ , $\vec{y} \cdot \vec{x} = 0$ (in $\mathbb{Z}/p\mathbb{Z}$). Now, each $\vec{y}$ is a $(0, 1)$ -vector, so it corresponds to a subset S of the vertices. Thus, $\vec{y} \cdot \vec{x}$ is simply $\sum_{v \in S} |x \cap v|$, which is to say that it counts the total number of occurrences of numbers 1 through p among the vertices of S . Thus, our goal is to show that for each row of C_Γ , the total number of occurrences of numbers 1 through p is a multiple of p .

The first type of row corresponds to the closed neighborhood of some vertex u . Suppose that $|x \cap u| = r$; without loss of generality we will assume that $u \cap \{1, 2, \dots, p\} = \{1, 2, \dots, r\}$. Now, u has $\binom{kp-p}{p}$ neighbors, and the numbers in $\{1, \dots, kp\} \setminus u$ each occur the same number of times among the neighbors. Since each vertex is a set of size p , and there are $kp - p$ numbers that can occur, we see that the number of times each number appears is

$$\binom{kp-p}{p} \cdot \frac{p}{kp-p} = \frac{(kp-p-1)(kp-p-2) \cdots (kp-2p+1)}{(p-1)(p-2) \cdots 1} \equiv 1 \pmod{p}.$$

Since there are $p-r$ numbers $r+1, r+2, \dots, p$, it follows that the total number of occurrences of these numbers is congruent to $p-r \pmod{p}$. Finally, u itself contains $1, 2, \dots, r$, contributing another r to the total, so that the total number of occurrences is congruent to 0 $\pmod{p}$.

The second type of row of C_Γ describes the intersection of the neighborhoods of adjacent vertices u and v . Without loss of generality, let us assume that $u \cap \{1, \dots, p\} = \{1, \dots, j\}$ and $v \cap \{1, \dots, p\} = \{j+1, \dots, j+r\}$. Similar to before, u and v have $\binom{kp-2p}{p}$ common neighbors, with each of the $kp-2p$ possible elements occurring the same number of times. Thus, the number of times each number appears is

$$\binom{kp-2p}{p} \cdot \frac{p}{kp-2p} = \frac{(kp-2p-1)(kp-2p-2) \cdots (kp-3p+1)}{(p-1)(p-2) \cdots 1} \equiv 1 \pmod{p}.$$

Thus, the total number of occurrences of the numbers $j+r+1, \dots, p$ is congruent to $p-(j+r) \pmod{p}$, and then including the fact that u and v themselves contribute $j+r$ to the total, we again see that the total is divisible by p .

Finally, the third type of row describes the intersection of neighborhoods of vertices u and v that are distance 2 apart. Let us assume that $|u \cap v| = m$. Then the number of common neighbors of u and v is $\binom{kp-2p+m}{p}$, and the number of times each number appears among these vertices is

$$\binom{kp-2p+m}{p} \cdot \frac{p}{kp-2p+m} = \frac{(kp-2p+m-1)(kp-2p+m-2) \cdots (kp-3p+m+1)}{(p-1)!} \equiv 0 \pmod{p},$$

since $1 \leq m \leq p-1$ guarantees exactly one factor in the numerator is divisible by p (whereas nothing in the denominator is since p is prime). Thus the total number of occurrences of the numbers between 1 and p is divisible by p . $\square$

Now we know some of the contents of the kernel of C_Γ for $\Gamma = \text{Kn}(kp, p)$. In the following lemma, we generalize slightly to $\text{Kn}(n, p)$ for any $n > p$ and show that it is not hard to find the number of dimensions that the vectors $\vec{x}$ described in Theorem 6.2 span in general.

Theorem 6.3. *The vectors described in Theorem 6.2 for the Kneser graph $\text{Kn}(n, p)$, with $n > p$, span a vector space $\text{KnV}(n, p)$ of dimension m , where $m = n - 2$ if $p \mid n$, and $m = n - 1$ otherwise.*

Proof. We define the vector space $\text{Tup}(n, p)$ of all tuples $(a_1, \dots, a_n)$ where each $a_i \in \mathbb{Z}/p\mathbb{Z}$ and $\sum_{i=1}^n a_i \equiv 0 \pmod{p}$. This is naturally an $(n - 1)$ -dimensional $\mathbb{Z}/p\mathbb{Z}$ -vector space (since the last coordinate must simply ensure $\sum_{i=1}^n a_i \equiv 0 \pmod{p}$).

One obvious spanning set for $\text{Tup}(n, p)$ is $\{\vec{b}_i\}_{i=1}^{n-1}$, where $\vec{b}_i$ has $a_i = 1, a_n = -1$, and $a_j = 0$ for $j \notin \{i, n\}$. Another spanning set is the set of vertices of $\text{Kn}(n, p)$, where the vertex $\{i_1, \dots, i_p\}$ is identified with its characteristic tuple, with $a_{i_1} = \dots = a_{i_p} = 1$ and $a_j = 0$ for $j \notin \{i_1, \dots, i_p\}$. This is because for $i < n$, we have $\vec{b}_i = u - v$, where u is a vertex containing i and $p - 1$ other numbers and v contains those same $p - 1$ numbers and n instead of i .

There is a surjective linear map $q : \text{Tup}(n, p) \rightarrow \text{KnV}(n, p)$ which takes a tuple $t = (a_1, \dots, a_n)$ to the vector $\vec{v}$ defined by declaring its coordinate corresponding to a vertex u to be given by $\sum_{i \in u} a_i \pmod{p}$ (the image of the map lands in $\text{KnV}(n, p)$ because the image of each vertex $v \in \text{Kn}(n, p)$ does, and the vertices span $\text{Tup}(n, p)$). This shows that we always have $\dim(\text{KnV}(n, p)) \leq n - 1$, and furthermore when $p \mid n$, the tuple $\vec{1} = (1, \dots, 1)$ is in the kernel, so in this case the dimension is reduced by one and we have $\dim(\text{KnV}(n, p)) \leq n - 2$.

Finally, let us show that there is no kernel if $p \nmid n$ and that $\langle \vec{1} \rangle$ is the entire kernel when $p \mid n$. Suppose that $t = (a_1, \dots, a_n) \in \ker q$. Considering the entries of $q(t)$ corresponding to $\{1, \dots, p\}$ and $\{1, \dots, p - 1, p + 1\}$ shows that $a_p \equiv a_{p+1} \pmod{p}$. Similarly we see that $a_i \equiv a_j \pmod{p}$ for all $i \neq j$, so only a multiple of $\vec{1}$ can be in the kernel, so $\vec{1} \in \text{Tup}(n, p)$ precisely when $p \mid n$. $\square$

Remark 6.4. One can show that a basis for $\text{KnV}(n, p)$ is given by $\{\vec{v}_p, \vec{v}_{p+1}, \dots, \vec{v}_{m+1}, \widehat{\vec{v}_{p-1}}, \widehat{\vec{v}_{p-2}}, \dots, \vec{v}_2\}$, the vectors corresponding to vertices $v_i = \{1, \dots, p - 1, i\}$, $v_{\hat{j}} = \{1, \dots, \hat{j}, \dots, p + 1\}$ (the numbers $1, \dots, p + 1$ with j missing).

The preceding two lemmas show that we can find girth-3 graphs with an arbitrarily large number of elementary divisors divisible by any fixed prime p :

Corollary 6.5. *For each prime p and each positive integer m , there is a graph Γ such that C_Γ has at least m elementary divisors that are multiples of p .*

Note that we are not describing the entire kernel of C_Γ in Theorems 6.2 and 6.3. For example, if $\Gamma = \text{Kn}(8, 2)$, then we can compute that the elementary divisors of C_Γ are $[1^{21}, 2^7]$, so we have only accounted for $1/2$ of the vectors in the kernel. The following table shows the nontrivial elementary divisors of C_Γ for some small Kneser graphs $\text{Kn}(n, p)$ for multiples n of p .

(n, p)	$(6, 2)$	$(8, 2)$	$(10, 2)$	$(12, 2)$	$(14, 2)$	$(16, 2)$	$(18, 2)$	$(20, 2)$	$(9, 3)$	$(12, 3)$	$(15, 3)$
$C_{\text{Kn}(n, p)}$:	2^4	2^7	2^8	$2^{10}, 4^1$	2^{12}	2^{15}	2^{16}	$2^{18}, 4^1$	3^7	3^{10}	$3^1, 9^{13}$

Table 1: The elementary divisors > 1 of C_Γ for the Kneser graphs $\text{Kn}(kp, p)$

Girth 3 graphs can in fact get still much farther from being RA and the C_Γ matrix can have arbitrarily many elementary divisors that are 0; that is, its right kernel can have arbitrarily large dimension, as we proceed to show now. For any integer $n \geq 2$, let $r = \lceil \log_2 n \rceil$ and $M = M(n)$ be the $(n + r) \times (n + r)$ block matrix⁴

$$\begin{pmatrix} J_n & B \\ B^T & J_r \end{pmatrix}$$

where J_i is the all-ones $i \times i$ matrix and B is the $n \times r$ matrix whose k -th row (consisting of 0's and 1's) is the number $k - 1$ written out in binary. Since M is symmetric and has 1's along the main diagonal, $M - I$ is the adjacency matrix for a graph $\text{BF}(n)$ (BF stands for “binary graph”).

We can also think of $\text{BF}(n)$ in terms of an incidence graph. Consider a bipartite graph where one part consists of vertices $1, 2, \dots, r$ and the other part consists of the first n subsets of $\{1, 2, \dots, r\}$ (in an ordering where $A < B$

⁴Idea adapted from ChatGPT (OpenAI), conversation with Igor Minevich, August 16, 2025.

if $\max(A) \leq \max(B)$ and A is lexicographically before B). Then put an edge between each number and the sets that contain it. Finally, we convert each part of the bipartition into a clique.

Theorem 6.6. *Let $z(n)$ be the dimension of the right kernel of $C_{\text{BF}(n)}$. Then, $z(n+1) = z(n) + 1$ if n has at least three 1's in its binary representation, and otherwise $z(n+1) = z(n)$.*

Proof. First we show that any vector $\vec{v}$ in the kernel must be zero in the last r coordinates. Row 1 has 1's in only the first n places, so the sum of the first n entries of $\vec{v}$ must be 0. But rows $2, 3, 5, 9, \dots, 2^{i-1} + 1, \dots, 2^{r-1} + 1$ have 1's in the first n entries, a single 1 in the i -th place, and 0's elsewhere in the last r entries, for $i = 1, 2, \dots, r$. This forces the i -th coordinate of any vector in the kernel to be 0 (for $n+1 \leq i \leq n+r$).

Now, a vector $\vec{v}$ in the kernel of $C_{\text{BF}(n)}$ must simply have the sum of its elements equal to 0 and dot product 0 with every intersection of two (not necessarily distinct) rows of B^T (where intersection is the bitwise AND operation).

If a vector $\vec{v}$ is in the kernel of $C_{\text{BF}(n)}$, then extending it by a 0 on the right gives a vector in the kernel of $C_{\text{BF}(n+1)}$. This is because the entries in the first n columns of B^T do not change as n increases, as more columns are added for the binary representations of further rows and eventually more rows are added to B^T that have 0's in the first n entries, so if the dot product of the first n entries of $\vec{v}$ with the intersections of rows of B^T was 0 in $M(n)$, it will remain zero for B^T in $M(n+1)$.

If n has only one 1 in its binary representation, so will the $(n+1)$ st column of B^T in $M(n+1)$. The rightmost column of B^T obviously always has a 1 at the top since a nonzero binary number starts with 1, so this 1 is in the top row. In this case, n is a power of 2 and no previous number has been as large as that power of 2, so there are only 0's to the left of this 1. Any vector $\vec{v}$ in the kernel must have a zero dot product with this row, which means its entry in that $(n+1)$ st column must be 0 and it must come from adding 0's to the right of vectors that were in the kernel for $C_{\text{BF}(n)}$.

If n has only two 1's in its binary representation, then one of the 1's is in the top row and by the time the 1's begin to appear in the top row, there are only 0's to the left of the bottom 1, so the two rows corresponding to the two 1's only intersect in the $(n+1)$ st column. (To be explicit, this is because n must equal $2^{r-1} + 2^i$ for some $i, 0 \leq i \leq r-1$, the only ones in the top $(r$ -th) row of B^T are the ones in columns $2^{r-1} + 1$ through $2^{r-1} + 2^i$, and none of the numbers between 2^r and $2^r + 2^i - 1$ (inclusive) have a 1 in the i -th binary digit.) Again, any vector $\vec{v}$ in the kernel must have dot product 0 with this intersection, so its $(n+1)$ st entry must be 0 and it comes from the kernel for $C_{\text{BF}(n)}$. This proves that if n has less than three 1's in its binary representation, then $z(n+1) = z(n)$.

There is another way to see this, relying on the same reasoning. We will restrict our attention only to the first n coordinates of rows in $C_{\text{BF}(n)}$ for all n , as these are really responsible for the right kernel of $C_{\text{BF}(n)}$; we will denote by $C'_{\text{BF}(n)}$ the first n columns of $C_{\text{BF}(n)}$. As it turns out, the number of distinct nonzero rows in $C'_{\text{BF}(n)}$ increases by 1 if n has one or two 1's in its binary representation and it does not increase if it has at least three; what's more, these are all linearly independent. If n has only one 1 in its binary representation, then $C'_{\text{BF}(n+1)}$ gains a new row in B^T which is just 1 in the $(n+1)$ st column and zeroes elsewhere; this has trivial intersection with all other rows in $C'_{\text{BF}(n+1)}$ so only one vector is added, and is indeed linearly independent to those that came before. If n has two 1's in its binary representation, then rows in $C'_{\text{BF}(n)}$ are extended to longer rows in $C'_{\text{BF}(n+1)}$, and if they were already linearly independent, they remain so. Now by the reasoning above, $C_{\text{BF}(n+1)}$ has a new row with a 1 in the $(n+1)$ st column and zeroes elsewhere; again, this is clearly independent of any vector whose nonzero entries were confined to the first n columns. These are linearly independent, so by the rank-nullity theorem we have $z(n+1) = z(n)$ if n has one or two 1's in its binary representation.

If n has at least three 1's in its binary representation, then there are 1's to the left of all of these 1's in the $(n+1)$ st column, and furthermore we shall show that there is no intersection of these rows that yields a 1 only in the $(n+1)$ st column. Suppose the 1's in the $(n+1)$ st column occur in rows $i_1, i_2, \dots, i_m, r$ of B^T (in order from the bottom up). Any two rows i_a and i_b with $i_a < i_b < r$ intersect at least once again in the $(n+1 - 2^{r-1})$ -th column. Indeed, the pattern of the first 2^{r-1} columns and first $r-1$ rows repeats starting at the $(2^{r-1} + 1)$ st column. For the same reason, for each $a < m$, the top row intersects the i_a -th row at least once again in the $(n+1 - 2^{i_m})$ -th column. Finally, the top row intersects with row i_m in the n -th column since the existence of a third 1 in the $(n+1)$ st column guarantees n is not the first number to have a 1 in the i_m -th digit after the numbers have started having a 1 in the r -th digit.

Thus, no row of $C_{\text{BF}(n+1)}$ has a single 1 in the $(n+1)$ st column and zeroes elsewhere. Every intersection of two rows of B^T in $M(n+1)$ thus already has nonzero entries in the first n columns so comes from a row in $C_{\text{BF}(n)}$

and adds no new rows to $C_{\text{BF}(n+1)}$. Since n has increased by exactly 1 and the kernel depends only on n , not r , the rank-nullity theorem says that the dimension of the right kernel of $C_{\text{BF}(n+1)}$ is one more than that of $C_{\text{BF}(n)}$. $\square$

Since infinitely many numbers have at least three 1's in their binary representation, we have the following theorem.

Corollary 6.7. *As $n \rightarrow \infty$, the number of zero elementary divisors $z(n)$ of $C_{\text{BF}(n)}$ increases without bound, and the function $n \mapsto z(n)$ is surjective onto the set of nonnegative integers. In particular, the dimension of the kernel is*

$$z(n) = n - 1 - r - \binom{r-1}{2} - \lceil \log_2(n - 2^{r-1}) \rceil.$$

Proof. Of course, $\lim_{n \rightarrow \infty} z(n) = \infty$ since there are infinitely many numbers with at least 3 1's in binary. The function $n \mapsto z(n)$ is surjective because $z(2) = 0$ and as n goes up by 1, $z(n)$ goes up by at most 1. The value of $z(n)$ is simply how many positive integers less than n have at least three 1's in their binary representation. Starting with the $n-1$ numbers 1 through $n-1$, we have exactly r of them that have a single 1 in their binary expansion: $1, 2, 4, \dots, 2^{r-1}$. There are $\binom{r}{2}$ numbers less than 2^r with exactly two 1's in their binary expansion, but some of these may not be less than n . Certainly, since $n > 2^{r-1}$, every way of having two 1's that does not use the leftmost bit will be less than n , so that gives us $\binom{r-1}{2}$ numbers. For the remainder, we simply subtract off the leftmost bit from n by computing $n - 2^{r-1}$, and then $\lceil \log_2(n - 2^{r-1}) \rceil$ gives us how many numbers less than $n - 2^{r-1}$ have a single 1 in their binary expansion. $\square$

We note that the smallest graphs with a zero elementary divisor of C_Γ have 10 vertices: they were found by exhaustive search using SageMath and the data in [13]. They have graph6 strings `I?otQji\0` and `ICQrThix_`, and both have C_Γ with elementary divisors $1^9, 0$. The smallest n such that $C_{\text{BF}(n)}$ has nontrivial kernel is $n = 8$, giving a graph with 11 vertices.

We have shown in Theorem 5.4 that C_Γ can have any single elementary divisor we wish, and in Theorem 6.7 that we can make C_Γ have any nullity. Let us now use the crown graphs and binary graphs as building blocks to show that we can make C_Γ as “bad” as we want.

Theorem 6.8. *Given any list of numbers $d_1 \mid d_2 \mid \dots \mid d_n$ with $d_1 \geq 2$, and any number $r \geq 0$, there is a graph Γ whose RA matrix C_Γ has nullity r and nontrivial, nonzero elementary divisors precisely $d_1, \dots, d_n$.*

Proof. Let $\Gamma_i = \text{Cr}(2d_i + 4)$ be the crown graph on $2d_i + 4$ vertices, and (if $r > 0$) let N be an integer such that $z(N) = r$ (from Theorem 6.7) and $\text{BF} = \text{BF}(N)$. Our graph Γ is $K_1 + \bigsqcup \Gamma_i \sqcup \text{BF}$, the pyramid over the disjoint union of all of these (if $r = 0$, we do not include BF). Let A_i be the activation matrix of $\text{Cr}(2d_i + 4)$ and A the activation matrix for BF . Then, ordering the vertices with the pyramid's apex first, the activation matrix A_Γ of Γ takes the form

$$\begin{pmatrix} 1 & 1 \dots 1 & 1 \dots 1 & \dots & 1 \dots 1 \\ \vec{1} & A_1 & 0 & \dots & 0 \\ \vec{1} & 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{1} & 0 & 0 & \dots & A \end{pmatrix}$$

The intersections of rows are either (a) the intersection of the first row with a row that comes from some A_i or A , which is just the same row as in A_Γ , (b) the intersection of two rows in the same block, which form a row of C_{Γ_i} or C_{BF} and a 1 in the leftmost column, or (c) the intersection of two rows from two different blocks, which just form a 1 in the leftmost column.

Note that, as in the proof of Theorem 6.1, for every i there are two rows of A_i that have no intersection and add up to the all-ones row of size $2d_i + 4$. The intersection of these two rows in A_1 gives just a 1 in the leftmost column and 0's elsewhere, so we can use row operations to zero out the entire left column (excluding one row equal to $\vec{e}_1^T$) and then the top row above each A_i . In the activation matrix A for $\text{BF}(N)$, we have two particular rows, say row x which looks like $1 \dots 110 \dots 0$ with $N+1$ repeated 1's and row y which looks like $1 \dots 101 \dots 1$, with N 1's followed by a 0 and the rest 1's. Their intersection is just a row with N 1's followed by 0's, so subtracting rows x and y from the first row and then adding their intersection cancels out the first row completely. Now we are just left with a single 1 in the first column and blocks of the form C_{Γ_i} and C_{BF} , so using the column and row operations for each of these gives us diagonals with a single nonzero elementary divisor d_i for each

$i = 1, \dots, n$ and r zeros in the diagonal corresponding to BF . Since each $d_i \mid d_{i+1}$ already by assumption, these are precisely the nonzero elementary divisors of C_Γ . $\square$

Using similar techniques, one can also show that if we build a pyramid over each $\text{Cr}(2d_i + 4)$ and $\text{BF}(N)$, and then take the disjoint union of all these, adding a single vertex connected to just the apices of these pyramids, this too gives a graph Γ with the same elementary divisors of C_Γ . In fact, so does simply the join $\text{Cr}(2d_1 + 4) + \text{Cr}(2d_2 + 4) + \dots + \text{Cr}(2d_n + 4) + \text{BF}(N)$.

Note that Theorem 6.8 certainly does not give the smallest graph satisfying the desired properties. For example, taking the join of four crown graphs $\text{Cr}(8)$ gives a 32-vertex graph Γ whose $\mathbb{Z}^{C_\Gamma}$ has four elementary divisors equal to 2, but Theorem 6.3 says the Kneser graph $\text{Kn}(6, 2)$ on just 15 vertices already has four elementary divisors of $\mathbb{Z}^{C_\Gamma}$ divisible by 2, and the computations in Table 1 showed that indeed they are all exactly 2 and there are no more.

7 Further insights from data

We collect here some information about small connected graphs; see Table 2. Graphs of girth 5 or more are always RA by Theorem 4.4, as are graphs on 7 or fewer vertices, as discussed in [2]. Graphs of girth 4 are always neighborhood-distinguishable and may be RA or $1/k$ -RA for some k . Graphs of girth 3 may have vertices with identical neighborhoods, and if not, may be RA or not. Looking at connected graphs with up to 10 vertices, the proportion of graphs that are neighborhood-distinguishable rapidly increases, from about 67% of 8-vertex graphs to about 84% of 10-vertex graphs. The vast majority of small neighborhood-distinguishable graphs are RA, with about 99.9997% of neighborhood-distinguishable graphs on 10 vertices being RA.

The data suggest that being RA is extremely common. However, it is possible that this is just an artifact of dealing with small graphs. In fact, the proportion of graphs on n vertices that are girth 3 tends to 1, and girth 3 is precisely where there seems to be plenty of opportunity to find obstructions to being RA. We leave the question of the proportion of graphs that are RA as an open problem - see Problem 3.

Girth	Category	8 vertices	9 vertices	10 vertices
3	Nbhd-Indist.	3675	63308	1908362
	Nbhd-Dist. RA	7175	196389	9798347
	Nbhd-Dist. not RA	0	3	30
4	RA	219	1243	9367
	not RA	1	0	1
5+	(all RA)	47	137	464
Total		11117	261080	11716571

Table 2: Counts of connected graphs on a fixed number of vertices by girth, category, and RA status.

8 Conclusion and Open Problems

We have fairly comprehensively explored the possibilities for the elementary divisors of the RA matrix C_Γ of a graph Γ . Here we collect a variety of open problems for future research.

In our explorations of products of graphs, we were able to characterize the elementary divisors of the RA matrix of strong products and cartesian products. For tensor products of graphs, our characterization is complete except for the tensor product of graphs of girth 3.

Problem 1. *Describe the elementary divisors of C_Γ for the tensor product Γ of two graphs of girth 3.*

In Theorem 5.9, we showed that the crown graphs yield the smallest $1/k$ -RA graphs of girth 4, and Theorem 6.1 showed that the pyramids over crown graphs are $1/k$ -RA graphs of girth 3. It is not clear whether the pyramids over crown graphs are always among the minimal girth 3 $1/k$ -RA graphs.

Problem 2. *For each $k \geq 2$, determine the smallest graph(s) of girth 3 that are $1/k$ -RA.*

The data in Table 2, though limited in scope, suggests that almost every graph is RA. Furthermore, the examples of non-RA graphs we have found so far have been highly structured, providing further evidence that a random graph is almost surely RA.

Problem 3. *Does the proportion of n -vertex graphs that are RA tend toward 1 as $n \rightarrow \infty$?*

When Γ is RA, the structure of G^Γ essentially boils down to the structure of $(G^{\text{Ab}})^\Gamma$ (which itself can be obtained from $\mathbb{Z}^\Gamma$). Explicitly, in this case, the full preimage of each element of $(G^{\text{Ab}})^\Gamma$ is contained in G^Γ . If Γ is not RA, then this is no longer the case, though perhaps a similar description of G^Γ is possible if C_Γ has few elementary divisors. Since we have shown that many graphs are $1/\mu$ -RA for some μ , it would be useful to more concretely describe G^Γ for such graphs.

Problem 4. *Describe G^Γ in terms of $(G^{\text{Ab}})^\Gamma$ in the case where Γ is $1/\mu$ -RA.*

Theorem 6.8 shows a simple construction for building graphs where C_Γ has any desired nullity and any sequence of nontrivial elementary divisors. Knowing the elementary divisors of C_Γ tells us everything we need to know about $[G, G]^{|\Gamma|} \cap G^\Gamma$ if G is a Heisenberg group $H(\mathbb{F}_p)$. For other groups G , however, the picture is much more complicated. Recall the chain of subgroups

$$\text{Comm}_b(G, \Gamma) \leq [G^\Gamma, G^\Gamma] \leq [G, G]^{|\Gamma|} \cap G^\Gamma \leq [G, G]^{|\Gamma|}$$

where $\text{Comm}_b(G, \Gamma)$ is generated by commutators $[g^v, h^w]$ for $g, h \in G$ and vertices v, w in Γ (possibly $v = w$). The group we are most interested in is $[G, G]^{|\Gamma|} \cap G^\Gamma$, so that we can use the short exact sequence

$$1 \rightarrow [G, G]^{|\Gamma|} \cap G^\Gamma \rightarrow G^\Gamma \rightarrow (G/[G, G])^\Gamma \rightarrow 1$$

to understand G^Γ . What the RA matrix C_Γ tells us is the structure of $\text{Comm}_b(G, \Gamma)$; in fact, $\text{Comm}_b(G, \Gamma) = [G, G]^{C_\Gamma}$. Thus, understanding $\text{Comm}_b(G, \Gamma)$ from the elementary divisors of C_Γ is a special case of understanding G^M from the elementary divisors of an arbitrary integer matrix M . Certainly the elementary divisors do not tell the whole story; for example, if $G = D_8$, $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, and $M_2 = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$, then M_1 and M_2 have the same elementary divisors, but $G^{M_1} \cong D_8$ while $G^{M_2} \cong C_2 \times D_8$.

Problem 5. *Determine to what extent we can describe G^M knowing only the elementary divisors of M .*

For Heisenberg groups, it turns out that $\text{Comm}_b(G, \Gamma) = [G^\Gamma, G^\Gamma] = [G, G]^{|\Gamma|} \cap G^\Gamma$ (see [2, Theorem 6.10]). More generally, any group G with a central commutator subgroup and with *faithful abelian generators* (see [2, Definition 6.7]) will also satisfy this property. What other groups have this property? Is there a family of graphs that can be used to determine whether a group has this property?

Problem 6. *Characterize groups G such that $\text{Comm}_b(G, \Gamma) = [G, G]^{|\Gamma|} \cap G^\Gamma$ for all graphs Γ .*

Problem 7. *Is there a family of graphs $\mathcal{F}$ such that, for a fixed group G , $\text{Comm}_b(G, \Gamma) = [G, G]^{|\Gamma|} \cap G^\Gamma$ for all graphs Γ if and only if $\text{Comm}_b(G, \Lambda) = [G, G]^{|\Lambda|} \cap G^\Lambda$ for some $\Lambda \in \mathcal{F}$ (or for all $\Lambda \in \mathcal{F}$)?*

Attacking the same problem from a different direction, if we fix a graph Γ such that C_Γ has many nontrivial elementary divisors, can we find a group G that makes the gap between $\text{Comm}_b(G, \Gamma)$ and $[G, G]^{|\Gamma|} \cap G^\Gamma$ as large as possible?

Problem 8. *For each graph Γ such that C_Γ has k nontrivial elementary divisors, is it possible to find a group G such that $\text{Comm}_b(G, \Gamma) \cong [G, G]^{|\Gamma|-k}$ but Γ is G -RA (that is, $[G, G]^{|\Gamma|} \leq G^\Gamma$)?*

We have seen that the Heisenberg group $H(\mathbb{F}_p)$ acts as a witness of the number of elementary divisors of C_Γ that are divisible by p . Is there a group that would serve this same function for prime powers?

Problem 9. *For each prime power p^k , is there a group G whose commutator subgroup is cyclic of order p^k and such that $\text{Comm}_b(G, \Gamma) = [G, G]^{|\Gamma|} \cap G^\Gamma$ for every Γ ?*

We plan to explore these and related problems in subsequent papers.

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