THE EIGENTHEORY FOR NONLOCAL COOPERATIVE-ADVECTIVE SYSTEM AND ITS ROLE IN THE STUDY OF FREE BOUNDARY SYSTEM FOR DIRECTIONAL EPIDEMIC MODELS

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Abstract. In this paper, in the absence of a variational structure but under sharper regularity requirements on the solutions, we propose and analyze a nonlocal cooperative reaction–diffusion system with free boundaries and drift terms, motivated by the models of directional epidemic spread. We first establish the well-posedness of the local problem and the global existence and uniqueness of classical solutions in C^1 space, which present substantial analytical challenges compared with the earlier works [9, 10, 18, 53, 60, 61, 70].

Next, we investigate the associated nonlocal eigenvalue problem and establish the existence, simplicity, qualitative properties, and asymptotic behavior of the principal eigenvalue. The analysis relies on a combination of Fredholm theory, the Crandall–Rabinowitz bifurcation theorem, and Hadamard-type derivative formulae to describe its dependence on the model parameters and to elucidate its connection with the basic reproduction number R_0.

Based on the spectral characterization, we demonstrate that the system exhibits a *sharp vanishing–spreading dichotomy* in its long-term dynamics. In particular, when $R_0 \leq 1$, all solutions vanish, whereas for $R_0 > 1$, the outcome depends on the initial domain size h_0 and the free-boundary expansion rate μ. Moreover, there exists a critical habitat length $\mathcal{L}^*$ such that if $h_0 < \mathcal{L}^*$, one can identify a critical threshold $\widehat{\mu} > 0$ that vanishing occurs for $\mu \in (0, \widehat{\mu}]$, while spreading happens for $\mu > \widehat{\mu}$. In the spreading regime, solutions converge to the unique positive steady state, whereas in the vanishing regime they decay uniformly to zero. These results establish a *rigorous theoretical framework* for understanding the threshold dynamics of cooperative–advective nonlocal systems with free boundaries arising in directional epidemic models, and potentially provide a mathematical foundation for further studies in *epidemic modeling*, *ecological invasion*, and *population dynamics*.

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Contents

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1. Introduction

Infectious diseases and ecological invasions have posed persistent threats to human society and natural ecosystems for centuries. Their spatial spread is shaped not only by local random movement but also by long-range dispersal and directed drift driven by environmental flows or human activities. Capturing these mechanisms in mathematical models is essential for elucidating threshold dynamics and predicting long-term outcomes. Classical reaction–diffusion frameworks based on the Laplacian provide a baseline, but they can be inadequate when movements are inherently nonlocal or exhibit preferential directions. Waterborne diseases, including cholera, botulism, giardiasis, diarrhoea, dysentery, typhoid, campylobacteriosis and salmonellosis, remain a major threat to global public health, particularly in developing countries. These infections are transmitted when pathogenic microorganisms contained in contaminated freshwater, surface water, lakes or aquatic reservoirs enter human populations. Recent estimates indicate that waterborne diseases contribute approximately 3.6% of the total global burden of disease measured in DALYs (disability-adjusted life years) and are responsible for about 1.5 million deaths annually [67]. In the case of cholera, the impact is especially severe, with an estimated 2.9 million cases and 95,000 deaths each year across 69 endemic countries [2]. To analyze and predict the spread of waterborne infections, numerous mathematical models have been developed. Early approaches, typically formulated as systems of ordinary differential equations, employed community–water compartmental structures to represent pathogen transmission during cholera outbreaks [4, 14, 16, 25, 27, 36, 37, 42, 47, 51, 52, 54, 63–66, 66]. To capture more realistic dynamics, subsequent studies incorporated environmental and spatial heterogeneity, thereby extending the modeling framework beyond purely temporal dynamics [12, 13, 50, 57]. In particular, spatially explicit models have been proposed using patches, networks, or directed graphs [11, 15, 24, 56], where nodes represent localities such as towns or villages and edges describe transport-mediated connections. Within these frameworks, the dynamics of waterborne pathogens are governed by systems of differential equations that explicitly account for dispersal and directed transport processes. Comprehensive reviews of mathematical modeling for cholera outbreaks are given in [38], while practical implications for public health interventions are summarized in [28].

The mathematical modeling of man–environment–man epidemic systems originally traces back to the pioneering works of Capasso and his collaborators [19–23]. Their central idea was to distinguish explicitly between the dynamics of the infected human population and the pathogen concentration in the environment. This framework provided one of the first rigorous mathematical models for waterborne diseases such as cholera. In its simplest form, the core model can be written as a system of two ordinary differential equations:

$$
\begin{cases} \dot{z}_1(t) = -\,a_{11}\, z_1(t) + a_{12}\, z_2(t), \\ \dot{z}_2(t) = -\,a_{22}\, z_2(t) + g\big(z_1(t)\big), \end{cases} \qquad t > 0 \tag{1.1}
$$

where $z_1(t)$ denotes the concentration of pathogens in the environment and $z_2(t)$ the density of infected individuals. The parameter $1/a_{11}$ represents the mean lifetime of the pathogen in the aquatic medium, $1/a_{22}$ the average duration of infection, and a_{12} the rate at which infected individuals contribute to the environmental contamination. The function $g(\cdot)$ denotes the force of infection, typically assumed to be increasing, with $g(0) = g'(0) = 0$, and saturating at high contamination levels.

To account for spatial spread, Capasso and Kunisch [22] extended this model to a reaction–diffusion framework, in which the environmental pathogen diffuses in space while the infected population remains localized:

$$
\begin{cases} \partial_t u_1(x,t) = d\,\Delta u_1(x,t) - a_{11}\,u_1(x,t) + a_{12}\,u_2(x,t), \\ \partial_t u_2(x,t) = -\,a_{22}\,u_2(x,t) + a_{21}\,g\big(u_1(x,t)\big), \end{cases} \qquad x\in\Omega,\ t>0, \tag{1.2}
$$

where $u_1(x,t)$ denotes the environmental pathogen density and $u_2(x,t)$ the density of infected individuals. Appropriate boundary conditions (Dirichlet or Neumann) are imposed depending on the epidemiological context. Later, an important models for waterborne disease dynamics was proposed by Hsu and Yang [43], who studied the existence, uniqueness, monotonicity and asymptotic behavior of traveling waves to the following reaction–diffusion system:

$$
\begin{cases} \partial_t u = d_1\Delta u - au + H(v), & t>0,\ x\in\mathbb{R}, \\ \partial_t v = d_2\Delta v - bv + G(u), & t>0,\ x\in\mathbb{R}, \end{cases} \tag{1.3}
$$

where $u(t,x)$ denotes the pathogen density in the aquatic environment, while $v(t,x)$ represents the density of infective hosts. Here $d_1, d_2 > 0$ are diffusion coefficients, and $a, b > 0$ are the pathogen decay and host removal (recovery/mortality) rates, respectively. The feedback functions $H, G : \mathbb{R}_+ \to \mathbb{R}_+$ are assumed to satisfy

$$
\begin{cases} H, G \in C^2([0,\infty)), \quad H(0) = G(0) = 0, \quad H'(z), G'(z) > 0,\ \forall z \geq 0, \\ H''(z), G''(z) < 0,\ \forall z > 0; \exists\, \bar{z} > 0 :\ G\Big(\frac{H(\bar{z})}{a}\Big) < b\,\bar{z}. \end{cases} \tag{1.4}
$$

Here, the variable u describes the concentration of pathogens (e.g. *Vibrio cholerae*) in water sources and its dynamics combine spatial diffusion, natural clearance at rate a, and $H(v)$ captures nonlinear shedding dynamics, which may saturate due to limits on pathogen output, stage-dependent infectivity, or environmental capacity. The variable v denotes the infected host population, which diffuses spatially with coefficient d_2, leaves the class at rate b, $G(u)$ represents nonlinear dose–response infection, reflecting that risk saturates at high pathogen concentrations. The concavity conditions $H'' < 0$ and $G'' < 0$ capture biological saturation: additional infectives do not increase shedding indefinitely, and infection risk does not grow linearly with pathogen concentration but levels off due to immune responses or behavioral constraints. In contrast with linear or bilinear incidence models, the use of two separate concave functions, $H(v)$ and $G(u)$, allows the shedding and infection processes to be modeled independently. This dual structure yields richer threshold behavior, backward bifurcation, and realistic wave propagation dynamics.

A modern development of waterborne epidemic modeling is due to Fitzgibbon, Morgan, Webb and Wu [39], who formulated a coupled community–river system to describe the 2010 cholera outbreak in Haiti. In their framework, the human population of each community is divided into susceptible S_j, infected I_j, and a local pathogen reservoir B_j, while $b(x,t)$ denotes the pathogen concentration transported through the river system. The complete system reads

$$
\begin{cases} \dfrac{dS_j}{dt} = -\dfrac{\beta_j S_j B_j}{k_j + B_j}, & t>0,\ j=1,2,\dots,n, \\[2ex] \dfrac{dI_j}{dt} = \dfrac{\beta_j S_j B_j}{k_j + B_j} - \gamma_j I_j, & t>0,\ j=1,2,\dots,n, \\[2ex] \dfrac{dB_j}{dt} = r_j I_j + \mu_j\,\dfrac{1}{d_j}\displaystyle\int_{x_j}^{x_j+d_j} b(x,t)\,dx - (\rho_j+\delta_j)B_j, & t>0,\ j=1,2,\dots,n, \\[2ex] \partial_t b(x,t) = \varepsilon\,\partial_{xx} b - q\,\partial_x b + \displaystyle\sum_{j=1}^{n} \chi_j(x)\Big(\tfrac{\rho_j}{d_j}B_j - \mu_j b\Big) - \delta_r b, & t>0,\ x\in\mathbb{R}, \end{cases} \tag{1.4}
$$

with initial conditions

$$b(x,0)=b_0(x), \qquad S_j(0)=S_{j0},\ I_j(0)=I_{j0},\ B_j(0)=B_{j0}.$$

Here, $\chi_j(x)=\mathbf{1}_{[x_j,x_j+d_j]}(x)$ denotes the indicator of the river segment corresponding to community j. The parameters ε and q represent the diffusion and advection coefficients in the river, respectively. The terms ρ_j and μ_j model the exchange of pathogens between the community reservoir B_j and the river compartment b. A central feature of this model is the explicit advection term $-q\,\partial_x b$ in the river dynamics. While diffusion captures only random mixing of pathogens in water, the advection term represents the directed transport due to river currents. In practice, such directed transport is the primary mechanism by which cholera bacteria spread from rural upstream regions to densely populated downstream communities. Ignoring advection would significantly underestimate the velocity and spatial extent of outbreaks. Including this effect allows the model to reproduce the observed rapid spread of cholera along the Haitian river network, and highlights the crucial role of hydrological flows in shaping epidemic dynamics and designing intervention strategies. In this model, S_j and I_j denote the numbers of susceptible and infected individuals in community j, while B_j represents the pathogen load in the local water reservoir of that community. The variable $b(x,t)$ describes the pathogen concentration in the river, with x representing position along the river domain. Disease transmission follows a saturated incidence rate of the form $\beta_j S_j B_j/(k_j+B_j)$, where β_j and k_j are transmission parameters. Infected individuals recover at rate γ_j and shed pathogens into the local water source at rate r_j. The exchange of pathogens between the river and community reservoirs is modeled by two parameters: μ_j, the rate of river-to-community transfer over the segment $[x_j,x_j+d_j]$ (expressed as the average $\frac{1}{d_j}\int b$), and ρ_j, the rate of community-to-river transfer. Natural pathogen decay occurs at rates δ_j in the local reservoirs and δ_r in the river. The river dynamics further include diffusion with coefficient ε and directed transport with velocity q, represented by the advection term $-q\,\partial_x b$, which captures downstream flow in the positive x-direction. Finally, χ_j denotes the indicator function identifying the river segment associated with community j.

To deeper understand the long-term behavior of the spatial spread of waterborne pathogens and the expansion of the infected region, many mathematical models are proposed to explore the different dynamics of the infectious diseases [11–14, 14, 16, 25, 27, 28, 35–39, 41, 42, 46, 47, 50–52, 54, 56, 57, 63, 64, 66–68]. The free boundary problems have been also proposed for this model [35, 49, 53, 69, 70]. In particular, the long time dynamics of a nonlocal diffusion single population model with advection and free boundaries was recently investigated by Tang and Dai [60], their model read as as follows :

$$\begin{cases} u_t = d\left[\displaystyle\int_{g(t)}^{h(t)} J(x-y)u(t,y)\,dy - u(t,x)\right] - qu_x + f(t,x,u), & t>0,\ x\in(g(t),h(t)),\\ u(t,g(t)) = u(t,h(t)) = 0, & t>0,\\ h'(t) = \mu\displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{+\infty} J(x-y)u(t,x)\,dy\,dx, & t>0,\\ g'(t) = -\mu\displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J(x-y)u(t,x)\,dy\,dx, & t>0,\\ h(0) = -g(0) = h_0, \quad u(0,x) = u_0(x), & x\in[-h_0,h_0], \end{cases} \tag{1.5}$$

where the advection term represents a directional drift caused by wind, animal migration, or the spreading of epidemics through water or river flows. Inspired by the mentioned works, in this paper, we investigate the spectral theory and the long-time dynamics of an advection–free boundary system involving coupled nonlocal diffusions and a concave nonlinearity in heterogeneous environment, which is precisely given by :

$$\begin{cases} u_t &= d_1\left[\displaystyle\int_{g(t)}^{h(t)} J_1(x-y)u(t,y)\,dy - u(t,x)\right] + p\,u_x - a(x)u(t,x) + H(v(t,x))\,, t>0, g(t)<x<h(t),\\ v_t &= d_2\left[\displaystyle\int_{g(t)}^{h(t)} J_2(x-y)v(t,y)\,dy - v(t,x)\right] + q\,v_x - b(x)v(t,x) + G(u(t,x))\,, t>0,\ g(t)<x<h(t),\\ u(t,x) &= v(t,x) = 0, t>0,\ x\in\{g(t),h(t)\},\\ h'(t) &= \mu\left(\displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_1(x-y)u(t,x)\,dy\,dx + \rho\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_2(x-y)v(t,x)\,dy\,dx\right), t>0,\\ g'(t) &= -\mu\left(\displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_1(x-y)u(t,x)\,dy\,dx + \rho\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_2(x-y)v(t,x)\,dy\,dx\right), t>0,\\ -g(0) &= h(0) = h_0, \quad u(0,x) = u_0(x), \quad v(0,x) = v_0(x), x\in[-h_0,h_0]. \end{cases}$$

From an epidemiological viewpoint, in this model, J_1, J_2 are symmetric dispersal kernels, which permit jumps over finite or even large distances, capturing realistic scenarios in which pathogens or infected individuals can be transported by environmental factors (e.g. contaminated water, wind, or human mobility) to non-adjacent locations. This leads to faster and wider epidemic spread compared to purely local diffusion.

The advection terms (pu_x, qv_x) represent the *directed movement or transport* of infected individuals or pathogens. In the real world, the advection corresponds to biased movement caused by environmental or social factors, such as river or water flow in the case of waterborne diseases (e.g., cholera), prevailing winds in airborne diseases, or directional human migration. Without advection, the spread is symmetric; with advection, the infection front advances faster in the downstream direction.

The heterogeneous environment coefficients $a(x), b(x)$ act as *spatially dependent removal or decay rates.* From an epidemiological aspect, $a(x)$ can represent the recovery or death rate of infected individuals, which may vary according to local healthcare quality or demographic conditions while $b(x)$ describes the local decay or clearance rate of pathogens in the environment, which depends on climatic or ecological conditions (e.g., temperature, sunlight, dryness). This spatial heterogeneity models unequal control measures or environmental survival conditions across regions.

The free boundary functions $h'(t), g'(t)$ describe the *expansion speed of the infected domain.* The boundary conditions link the front propagation to the gradients of u and v at the edges. Epidemiologically, $h'(t)$ corresponds to the rightward expansion speed of the epidemic, while $g'(t)$ corresponds to the leftward speed. If the infection intensity near the boundary remains high, the epidemic front expands indefinitely (*successful invasion*); if it weakens, the boundary stabilizes and the infection may vanish (*containment scenario*).

Finally, the concave nonlinear terms $H(v)$ and $G(u)$ represent the cooperative cross-interactions between the two populations, ensuring that the presence of one species enhances the growth of the other. Near the disease-free or low-density state, the linearization of these functions determines the threshold dynamics, such as the basic reproduction number and invasion conditions. At higher densities, however, the nonlinear behavior of H and G introduces saturation effects that prevent unbounded growth and stabilize the system at biologically meaningful equilibria. The nonlinearities are increasing and strictly concave on $(0,\infty)$, reflecting diminishing returns: as the interacting population grows, the marginal effect on growth decreases. This concavity is crucial for modeling realistic epidemic or ecological interactions, as it ensures boundedness and stabilizing effects in the system.

Throughout this paper, we assume the following conditions :

(J) The kernels $J_i, i\in\{1,2\}$ satisfy: $J_i\in C^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, $J_i\geq 0$, $J_i(x)=J_i(-x)$ for all $x\in\mathbb{R}$, $J_i(0)>0$ and

$$\int_{\mathbb{R}} J_1(x)\,dx = \int_{\mathbb{R}} J_2(x)\,dx = 1.$$

(GH) Given functions $G, H:\mathbb{R}^+\to\mathbb{R}^+$ verify the following assumptions:

- $H, G \in C^2([0,\infty))$,
- $G(0) = H(0) = 0$ and $H(z), G(z) > 0, \forall z \geq 0$,
- There exist constants $a, b > 0$ such that $G\left(\frac{H(z)}{a}\right) < bz$ for all $z \geq 0$,
- $H'(z) < 0$ and $G'(z) < 0$ for all $z \in (0,\infty)$.

(IC) We also assume the conditions on the initial conditions:
- $u_0, v_0 \in C([-h_0, h_0])$, with $u_0(\pm h_0) = v_0(\pm h_0) = 0$,
- $u_0(x), v_0(x) > 0$ for all $x \in (-h_0, h_0)$.

Linearization and spectral threshold. Linearizing (1) at the disease-free state $(u,v) = (0,0)$ over a fixed domain $\Omega \subset \mathbb{R}$ (e.g., a snapshot or a frozen free boundary) yields the cooperative system

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} d_1(N_1 - I) + p\,\partial_x & 0 \\ 0 & d_2(N_2 - I) + q\,\partial_x \end{pmatrix}}_{\mathcal{T}} \begin{pmatrix} u \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} -a(x) & H'(0) \\ G'(0) & -b(x) \end{pmatrix}}_{\mathcal{M}(x)} \begin{pmatrix} u \\ v \end{pmatrix}, \quad x \in \Omega. \tag{1.6}$$

The associated stationary eigenproblem (seeking exponential solutions $e^{-\lambda t}\phi$) reads

$$\Big(\mathcal{T} + \mathcal{M}(\cdot)\Big)\,\Phi + \lambda\,\Phi = 0, \qquad \Phi = (\phi_1, \phi_2)^\top, \qquad \Phi \not\equiv 0. \tag{1.7}$$

The *principal eigenvalue* λ_p (if it exists) governs linear stability and serves as a surrogate for the basic reproduction number R_0 via the sign relation

$$R_0 > 1 \iff \lambda_p < 0, \qquad R_0 < 1 \iff \lambda_p > 0. \tag{1.8}$$

We underline that, in previous studies [53,62], for scalar nonlocal problems without advection, the principal eigenvalue λ_p enjoys a variational formula and it can often be written in $L^2(\Omega)$; for cooperative systems these formulas extend to $E := L^2(\Omega) \times L^2(\Omega)$ by

$$\lambda_p = -\sup_{\|\Phi\|_E = 1} \left\langle \begin{pmatrix} d_1(N_1 - I) & 0 \\ 0 & d_2(N_2 - I) \end{pmatrix} \Phi + \begin{pmatrix} -a(\cdot) & H'(0) \\ G'(0) & -b(\cdot) \end{pmatrix} \Phi\,,\ \Phi \right\rangle_E \tag{1.9}$$

where $\langle \cdot, \cdot \rangle_E$ is the L^2 inner product on E. However, the presence of the advection terms $(pu_x,\ qv_x)$ in (1.6) makes the study of the existence, simplicity, and qualitative properties of the principal eigenvalue significantly more challenging, mainly due to the sharper $C^1(\Omega)$ regularity requirement of the eigenfunction and the lack of a variational characterization. In order to overcome this, we may consider $-\mathcal{T} - \mathcal{M}$ in (1.6) as an accretive perturbation of a positive compact operator (or of a compact resolvent generator), allowing one to invoke non-selfadjoint spectral theory and perturbation arguments [45]. Advection terms $p\,\partial_x$, $q\,\partial_x$ typically break self-adjointness; nonetheless, one can restore a comparison structure by suitable transforms (e.g., exponential weights) or by working with the resolvent and positive cones, in the spirit of [17,45,48] and this represents a completely new approach compared with previous studies [1,6,9,18,35,53,60–62,70].

Free boundaries and spreading–vanishing. When the habitat is allowed to expand through flux-driven free boundaries, as in (1), the long-time dynamics exhibit a sharp dichotomy: either the solution vanishes uniformly (the front stalls), or it *spreads* and a positive profile invades with the boundaries moving outward. Although, this paradigm has been established for several nonlocal models without advection [18,35,53,61,69,70], again introducing advection requires significantly deeper techniques due to the requirement of sharper regularity requirements on time-dependent and stationary solutions, spectral thresholds and the front motion (the integrals defining $g'(t)$ and $h'(t)$), with qualitative consequences for the spreading speed and critical domain size. The main contributions of the present work are:

(i) *Well-posedness with advection.* We establish local, global existence and uniqueness of classical solutions to (1) under (J), (GH), (IC) in C^1 functional space. Key tools include comparison principle adapted to jump–advection operators and a priori sharper estimates on C^1 in order to apply the Banach fixed point theorem.

(ii) *Eigentheory with drift.* We establish the existence and simplicity of the principal eigenvalue for the linearized problem (1.7) on bounded domains in the *absence of a variational structure* (cf. [9, 18, 53, 62]). We further analyze its qualitative dependence on parameters, employing *Fredholm theory*, the *Crandall–Rabinowitz* bifurcation theorem, and *Hadamard-type* derivative formulae as developed in the recent work of Benguria et al. [7] to establish differentiability with respect to those parameters. Our approach necessitates a significantly more refined eigentheoretic framework than that used in the existing literature.

(iii) *Basic reproduction number.* We define R_0 through the spectral bound of the next-generation operator and show the threshold equivalence (1.8). For constant coefficients we derive implicit/explicit formulas for R_0, clarifying the role of p, q and of kernel features.

(iv) *Spreading–vanishing criteria with advection.* We identify critical thresholds in terms of the initial half-width h_0 and the free-boundary parameters (μ, ρ) in (1). In particular, we prove that *vanishing* occurs when $R_0 \le 1$, whereas for $R_0 > 1$, the long-time behavior depends delicately on (h_0, μ, ρ). In this case, a *sharp transition* between vanishing and spreading regimes is established, which considerably extends the known results of the long-time dynamics for free-boundary systems with coupled advection terms; see, for instance, [18, 35, 53, 60, 61, 70].

The principal results for system (1) are stated precisely as follows.

Theorem 1 (**Global existence and uniqueness**). *Suppose that J_1, J_2 satisfy assumption **(J)**, and assumptions **(A1)**–**(A2)** hold. Then, for any $h_0 > 0$, problem (1) admits a unique positive solution (u, v, g, h) defined for all $t > 0$.*

Moreover, for any fixed $T > 0$, the solution components satisfy

$$u \in X_{u_0,g,h}, \quad v \in X_{v_0,\infty}, \quad g \in \mathcal{G}_{h_0,T}, \quad h \in \mathcal{H}_{h_0,T}.$$

To analyze the spreading–vanishing dichotomy, we first introduce the following eigenvalue problem:

(1.10)
$$\begin{cases} d_1 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\psi_1(y)\,dy - d_1\psi_1(x) - a(x)\psi_1(x) + H'(0)\psi_2(x) + p\psi_1'(x) = \lambda\psi_1(x), & x \in [-\mathcal{Z}, \mathcal{Z}], \\ d_2 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\psi_2(y)\,dy - d_2\psi_2(x) - b(x)\psi_2(x) + G'(0)\psi_1(x) + q\psi_2'(x) = \lambda\psi_2(x), & x \in [-\mathcal{Z}, \mathcal{Z}]. \end{cases}$$

Theorem 2 (**Principal eigenvalue**). *The problem (1.10) admits a principal eigenvalue λ^*, which is simple and associated with a strictly positive eigenfunction in $C^1([-\mathcal{Z}, \mathcal{Z}])$.*

Theorem 3 (**Qualitative structure and parameter dependence of the principal eigenvalue**). *Let $p, q > 0$, $a, b \in L^\infty(\mathbb{R})$, and $H'(0), G'(0) \ge 0$. Assume that the nonlocal kernels J_1, J_2 satisfy* **(J)** *so that each integral operator $\mathscr{J}_l$ is positive and compact on $X_l := L^2([-\mathcal{Z}, \mathcal{Z}]; \mathbb{R})$ for all $\mathcal{Z} > 0$. Denote by $\lambda^*(\mathcal{Z})$ the principal eigenvalue of (1.10) on $[-\mathcal{Z}, \mathcal{Z}]$ and define*

$$\mathcal{R}_0 := \rho\big(\mathscr{J}(-\mathscr{T})^{-1}\big) = \lim_{\mathcal{Z}\to\infty} \rho_{\mathcal{Z}}(0),$$

where $\rho_{\mathcal{Z}}(0)$ denotes the spectral radius of the corresponding next-generation operator on $[-\mathcal{Z}, \mathcal{Z}]$.

Then the following hold:

(i) ***Monotonicity and continuity.*** *The mapping $\mathcal{Z} \mapsto \lambda^*(\mathcal{Z})$ is strictly increasing and continuous on $(0, \infty)$.*

(ii) ***Threshold characterization.*** *There exists a sharp critical domain size $\mathcal{Z}_* \in (0, \infty]$ such that*

$$\lambda^*(\mathcal{Z}) < 0 \quad \text{for } \mathcal{Z} < \mathcal{Z}_*, \qquad \lambda^*(\mathcal{Z}) > 0 \quad \text{for } \mathcal{Z} > \mathcal{Z}_*.$$

Moreover, $\mathcal{Z}_* < \infty$ if and only if $\mathcal{R}_0 > 1$, in which case $\lambda^*(\mathcal{Z}_*) = 0$ (equivalently, $\rho_{\mathcal{Z}_*}(0) = 1$); if $\mathcal{R}_0 \le 1$, then $\lambda^*(\mathcal{Z}) < 0$ for all $\mathcal{Z} > 0$.

(iii) ***Differentiability with respect to diffusion rates.*** *Fix $\mathcal{Z} > 0$ and let $d = (d_1, d_2) \in U \subset (0,\infty)^2$. For each d, define*

$$\mathcal{L}_d := \mathscr{J}_d - \mathscr{T}_d$$

acting on a Banach lattice X. Suppose $\mathcal{L}_d$ admits a simple, isolated principal eigenvalue $\lambda^*(d)$ with normalized positive eigenfunction $\varphi(d) \in \mathcal{D} \subset X$ and positive adjoint eigenfunction $w^*(d) \in X^*$ satisfying $\langle w^*(d), \varphi(d)\rangle = 1$. If $d \mapsto \mathcal{L}_d$ is C^1 in the operator norm (in particular affine in d), then λ^* is of class C^1 on U, and

$$\frac{\partial \lambda^*}{\partial d_j}(d) = \big\langle w^*(d),\, \partial_{d_j}\mathcal{L}_d[\varphi(d)]\big\rangle, \qquad j = 1, 2.$$

For the present operator, one has explicitly

$$\mathcal{L}_d(\psi_1,\psi_2)(x) = \begin{pmatrix} d_1\displaystyle\int J_1(x-y)\psi_1(y)\,dy - d_1\psi_1(x) - a(x)\psi_1(x) + H'(0)\psi_2(x) + p\psi_1'(x) \\ d_2\displaystyle\int J_2(x-y)\psi_2(y)\,dy - d_2\psi_2(x) - b(x)\psi_2(x) + G'(0)\psi_1(x) + q\psi_2'(x) \end{pmatrix},$$

and therefore

$$\frac{\partial \lambda^*}{\partial d_j}(d) = \int_{-\mathcal{Z}}^{\mathcal{Z}} w_j^*(x)\Big(\int_{-\mathcal{Z}}^{\mathcal{Z}} J_j(x-y)\varphi_j(y)\,dy - \varphi_j(x)\Big)\,dx, \qquad j = 1, 2.$$

Theorem 4 (**Vanishing-Spreading**). *Suppose $R_0 > 1$ and $h_0 < \mathcal{L}^*$. Then there exists a threshold $\hat{\mu} > 0$, depending on the initial data (u_0, v_0), such that*

- *vanishing occurs for every $\mu \in (0, \hat{\mu}]$ (i.e. $h_\infty - g_\infty < \infty$),*
- *spreading occurs for every $\mu \in (\hat{\mu}, \infty)$ (i.e. $h_\infty - g_\infty = \infty$ and $(u, v) \to (\mathscr{U}^*, \mathscr{V}^*)$).*

Organization of the paper. Section 2 presents the precise assumptions, several preliminary results, and the well-posedness of problem (1). Section 3 develops the spectral theory and establishes the existence, simplicity, and qualitative properties of the principal eigenvalue, together with its connection to the basic reproduction number R_0. Section 4 is devoted to the spreading–vanishing dichotomy and the corresponding threshold criteria for the free-boundary problem. Finally, Section 5 discusses several extensions, including heterogeneous advections, nonsymmetric kernels, shape sensitivity analyses, and related open problems.

Notation. We write $\langle\cdot,\cdot\rangle$ for the L^2-inner product and $\|\cdot\|$ for the corresponding norm (componentwise on $E = L^2(\Omega) \times L^2(\Omega)$). For an operator $\mathcal{S}$, $r(\mathcal{S})$ and $s(\mathcal{S})$ denote its spectral radius and spectral bound, respectively. The positive cone in E is $E_+ := \{(u,v) : u \geq 0,\ v \geq 0 \text{ a.e.}\}$, and inequalities between vectors are understood componentwise. We use C to denote a generic positive constant whose value may change from line to line.

2. Preliminaries

In this section, we focus on demonstrating the existence and uniqueness of solutions to the model (1). Firstly, we need the following lemma to guarantee the maximum principle.

Lemma 5 (**Maximum Principle**). *Let $h_0, T > 0$ and $\Omega := [0,T] \times [-h_0, h_0]$. Suppose that $A, B, C, D \in L^\infty(\Omega)$, $B, C > 0$, $(u(t,x), v(t,x))$ as well as $(u_t(t,x), v_t(t,x))$ are continuous in Ω and satisfy*

$$\begin{cases} u_t(t,x) \geq d_1 \int_{g(t)}^{h(t)} J_1(x-y)u(t,y)dy - d_1 u + pu_x + Au + Bv, & 0 < t \leq T, -h_0 \leq x \leq h_0, \\ v_t(t,x) \geq d_2 \int_{g(t)}^{h(t)} J_2(x-y)v(t,y)dy - d_2 v + qv_x + Dv + Cu, & 0 < t \leq T, -h_0 \leq x \leq h_0, \\ u(0,x) \geq 0, \\ v(0,x) \geq 0, \end{cases}$$

Then $(u(t,x), v(t,x)) \geq (0,0)$ for all $0 < t \leq T$ and $-h_0 \leq x \leq h_0$.

We omit the proof of this lemma, as it is almost similar to the proof of Lemma 3.1 [35].

Lemma 6 (Comparison Principle)**.** *We suppose that conditions **(J)** and **(GH)** satisfy and that the initial data $u_0(x), v_0(x)$ satisfy (A1)-(A2). Let $T > 0$, and suppose that $h(t), g(t) \in C^1([0,T])$ and $u, v \in C^{g,h}_{D_T}$ satisfy the following system:*

$$
\begin{cases}
\bar{u}_t \geq d_1 \left[\displaystyle\int_{g(t)}^{h(t)} J_1(x-y)\bar{u}(t,y)\,dy - \bar{u}(t,x) \right] + p\bar{u}_x - a\bar{u}(t,x) + H(\bar{v}(t,x)), \\
\bar{v}_t \geq d_2 \left[\displaystyle\int_{g(t)}^{h(t)} J_2(x-y)\bar{v}(t,y)\,dy - \bar{v}(t,x) \right] + q\bar{v}_x - b\bar{v}(t,x) + G(\bar{u}(t,x)),
\end{cases}
\tag{2.1}
$$

for $t \in (0,T]$, $x \in (g(t), h(t))$, and

$$
\begin{aligned}
&\bar{u}(t,x) \geq 0, \quad \bar{v}(t,x) \geq 0, \quad \text{for all } t \in (0,T], \\
&h'(t) \geq \mu \left(\int_{h(t)}^{\infty} \int_{g(t)}^{h(t)} J_1(x-y)u(t,x)\,dx\,dy + \rho \int_{h(t)}^{\infty} \int_{g(t)}^{h(t)} J_2(x-y)v(t,x)\,dx\,dy \right), \\
&g'(t) \leq -\mu \left(\int_{-\infty}^{g(t)} \int_{g(t)}^{h(t)} J_2(x-y)v(t,x)\,dx\,dy + \rho \int_{-\infty}^{g(t)} \int_{g(t)}^{h(t)} J_1(x-y)u(t,x)\,dx\,dy \right), \\
&-g(0) \leq -h(0), \quad h(0) \geq h_0, \\
&u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in [-h_0, h_0].
\end{aligned}
$$

Then the unique solution (u, v, g, h) of system (1) satisfies:

$$
u(t,x) \leq \bar{u}(t,x), \quad v(t,x) \leq \bar{v}(t,x), \quad g(t) \geq \bar{g}(t), \quad h(t) \leq \bar{h}(t), \quad \text{for all } t \in (0,T],\ x \in [g(t), h(t)].
$$

For convenience, we introduce the following notation. Given $h_0 > 0$ and $T > 0$, we define:

$$
\begin{aligned}
\mathcal{H}_{h_0,T} &= \left\{ h \in C([0,T]) : h(0) = h_0,\ \inf_{0 \leq t_1 < t_2 \leq T} \frac{h(t_2) - h(t_1)}{t_2 - t_1} > 0 \right\}, \\
\mathcal{G}_{h_0,T} &= \left\{ g \in C([0,T]) : -g(t) \in \mathcal{H}_{h_0,T} \right\}, \\
C_0([-h_0,h_0]) &= \left\{ u \in C([-h_0,h_0]) : u(-h_0) = u(h_0) = 0,\ u_0(x) > 0 \text{ in } (-h_0,h_0) \right\}.
\end{aligned}
$$

For $g \in \mathcal{G}_{h_0,T}$, $h \in \mathcal{H}_{h_0,T}$, and $u_0(x) \in C_0([-h_0,h_0])$, we define the domains:

$$
\begin{aligned}
\Omega_{g,h} &= \left\{ (t,x) \in \mathbb{R}^2 : 0 < t \leq T,\ g(t) < x < h(t) \right\}, \\
\Omega_{\infty} &= \left\{ (t,x) \in \mathbb{R}^2 : 0 < t \leq T,\ x \in \mathbb{R} \right\}.
\end{aligned}
$$

We also define the function spaces:

$$
\begin{aligned}
X_{v_0,\infty} &= \left\{ \varphi \in C^1(\Omega_\infty) : \varphi(0,x) = v_0(x) \text{ for } x \in \mathbb{R},\ 0 < \varphi \leq M_1 \text{ in } \Omega_\infty \right\}, \\
X_{u_0,g,h} &= \left\{ \psi \in C^1(\overline{\Omega_{g,h}}) : \psi \geq 0 \text{ in } \Omega_{g,h},\ \psi(0,x) = u_0(x) \text{ for } x \in [-h_0,h_0], \right. \\
&\qquad \left. \psi(t,g(t)) = \psi(t,h(t)) = 0 \text{ for } 0 \leq t \leq T \right\}.
\end{aligned}
$$

We now proceed to proof the main theorem of this section.
Proof of Theorem 1.

Proof. We divide the proof into two steps. We rewrite the system for $t > 0$:
Step 01 : Local existence.

(2.2)
$$\begin{cases} u_t = d_1\displaystyle\int_{g(t)}^{h(t)} J_1(x-y)u(t,y)\,dy - u(x,t) + p\,u_x - a(x)u(x,t) + H\big(\widetilde{v}(t,x)\big), & x\in[g(t),h(t)],\\ u(t,g(t)) = u(t,h(t)) = 0, & \\ h'(t) = \mu\displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{+\infty} J_1(x-y)u(t,x)\,dy\,dx + \rho\int_{g(t)}^{h(t)}\int_{h(t)}^{+\infty} J_2(x-y)\widetilde{v}(t,x)\,dy\,dx, & \\ g'(t) = -\mu\displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_1(x-y)u(t,x)\,dy\,dx + \rho\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_2(x-y)\widetilde{v}(t,x)\,dy\,dx, & \\ u(0,x) = u_0(x), \quad h(0) = -g(0) = h_0, & x\in[-h_0,h_0]. \end{cases}$$

has a unique solution, denoted by $(\hat{u}(t,x), \hat{g}(t), \hat{h}(t))$, which is defined for all $t > 0$. Moreover, for any $T > 0$, we have:

$$\hat{g}(t) \in \mathcal{G}_{h_0,T}, \quad \hat{h}(t) \in \mathcal{H}_{h_0,T}, \quad \hat{u}(t,x) \in \mathcal{X}_{u_0,g,h}.$$

Furthermore, it holds that

$$0 < \hat{u}(t,x) \le M := \max\left\{\max_{-h_0\le x\le h_0} u_0(x),\ K_0\right\}, \quad \text{for } 0<t<T,\ x\in(\hat{g}(t),\hat{h}(t)). \tag{2.3}$$

Define the function:

$$\tilde{u}(t,x) = \begin{cases}\hat{u}(t,x), & x\in[\hat{g}(t),\hat{h}(t)],\\ 0, & x\notin[\tilde{g}(t),\tilde{h}(t)].\end{cases} \tag{2.4}$$

Then, consider the second equation:

$$\begin{cases} v_t = d_2\displaystyle\int_{\mathbb{R}} J_2(x-y)v(t,y)\,dy - v(t,x) + qv_x + G\left(\tilde{u}(t,x)\right), & 0<t<T,\ x\in\mathbb{R},\\ v(0,x) = v_0(x), & x\in\mathbb{R}.\end{cases} \tag{2.5}$$

The local existence and uniqueness of solutions to (2.5) follow from standard theory. The unique solution $\tilde{v}(t,x)$ possesses $C^1(\Omega_\infty)$.

Next, we show the coincidence between $\widetilde{v}(t,x)$ and $\hat{v}$ by establishing that the mapping $\mathcal{T}\widetilde{v}(t,x) = \hat{v}$ admits a unique fixed point in the function space $\mathcal{X}_{v_0,\infty}$. The proof follows from Banach's contraction mapping principle. The operator $\mathcal{T}(X_{v_0,\infty}) \subset X_{v_0,\infty}$. The essential step consists in showing that for T sufficiently small, $\mathcal{T}$ satisfies the contraction on $X_{v_0,\infty}$.

Next, we choose $\widetilde{v}_1, \widetilde{v}_2 \in X_{v_0,\infty}$. Let $\hat{u}_1, \hat{u}_2$ be the solution of (2.2) associated with $\widetilde{v}_1, \widetilde{v}_2$, and $\hat{v}_1, \hat{v}_2$ the solution of (2.5) associated to $\tilde{u}_1, \tilde{u}_2$. Then, applying the transport equation, we get

$$\begin{aligned}\hat{v}_1(t,x) - \hat{v}_2(t,x) = \hat{V}_0(x+q(t-t_0)) &+ \int_{t_0-t}^{0}\int_{\mathbb{R}} d_2J_2(x-y)\left[\hat{v}_1(s,y)-\hat{v}_2(s,y)\right]dy\,ds\\ &- \int_{t_0-t}^{0} d_2\left[\hat{v}_1-\hat{v}_2\right](s,x)ds + \int_{t_0-t}^{0}\left[G(\widetilde{u}_1)-G(\widetilde{u}_1)\right](s,x)ds.\end{aligned}$$

which implies that

$$\|\hat{v}_1-\hat{v}_2\|_{C(\Omega_\infty)} \le \|\hat{V}_0(x+q(t-t_0))\|_{C(\Omega_\infty)} + 2d_2\|\hat{v}_1-\hat{v}_2\|_{C(\Omega_\infty)}t + \|\widetilde{u}_1-\widetilde{u}_2\|_{C(\Omega_\infty)}t,$$

then,

$$\|\hat{v}_1-\hat{v}_2\|_{C(\Omega_\infty)} \le \|\hat{V}_0(x+q(t-t_0))\|_{C(\Omega_\infty)} + 2d_2\|\hat{v}_1-\hat{v}_2\|_{C(\Omega_\infty)}\mathbf{T} + K\|\hat{u}_1-\hat{u}_2\|_{C(\Omega_{\hat{g},\hat{h}})}T.$$

We choose T small enough such that $2d_2\mathbf{T} \le \frac{1}{2}$, then we we obtain

$$\|\hat{v}_1 - \hat{v}_2\|_{C(\Omega_\infty)} \le \|\hat{V}_0(x + q(t - t_0))\|_{C(\Omega_\infty)} + 2KT\|\hat{u}_1 - \hat{u}_2\|_{C(\Omega_{\hat{g},\hat{h}})}. \tag{2.6}$$

We now establish an estimate for $\|\hat{u}_1 - \hat{u}_2\|_{C(\Omega_{\hat{g},\hat{h}})}$.. For any given $(\hat{t}, \hat{x}) \in \Omega_{\hat{g},\hat{h}}$ and $t \in (0, T]$, we define the following functions as follows.

$$\mathcal{U}(t, \hat{x}) = \hat{u}_1(t, \hat{x}) - \hat{u}_2(t, \hat{x}),$$
$$\mathcal{V}(t, \hat{x}) = \hat{v}_1(t, \hat{x}) - \hat{v}_2(t, \hat{x}),$$

and the transition time:

$$t_x = \begin{cases} t_{x,\hat{g}} & \text{if } x \in [\hat{g}(T), -h_0) \text{ and } x = \hat{g}(t_{x,\hat{g}}), \\ 0 & \text{if } x \in [-h_0, h_0], \\ t_{x,\hat{h}} & \text{if } x \in (h_0, \hat{h}(T)] \text{ and } x = \hat{h}(t_{x,\hat{h}}). \end{cases} \tag{2.7}$$

We further introduce the extreme boundary functions:

$$\mathcal{H}_1(t) = \min\{\hat{h}_1(t), \hat{h}_2(t)\}, \qquad \mathcal{H}_2(t) = \max\{\hat{h}_1(t), \hat{h}_2(t)\},$$
$$\mathcal{G}_1(t) = \min\{\hat{g}_1(t), \hat{g}_2(t)\}, \qquad \mathcal{G}_2(t) = \max\{\hat{g}_1(t), \hat{g}_2(t)\},$$

and the combined domain:

$$\Omega_T = \Omega_{G_1,H_2} = \Omega_{\hat{g}_1,\hat{h}_1} \cup \Omega_{\hat{g}_2,\hat{h}_2}.$$

The analysis proceeds by considering three distinct cases.

Case 1: $\hat{x} \in [-h_0, h_0]$. From the governing equations for $\hat{u}_i(t, \hat{x})$, we obtain the differential relation:

$$\mathcal{U}_t(t, \hat{x}) + a(\hat{x})\mathcal{U}(t, \hat{x}) + \mathcal{C}_1(t, \hat{x})\mathcal{U}(t, \hat{x}) + q\mathcal{U}_x(t, \hat{x}) = \mathcal{C}_2(t, \hat{x})\hat{V}(t, \hat{x}) + \mathcal{F}(t, \hat{x}), \quad \mathcal{U}(0, x^*) = 0, \tag{2.8}$$

where $\hat{V}(t, \hat{x}) = \widetilde{v}_1(t, \hat{x}) - \widetilde{v}_2(t, \hat{x})$, and the coefficient functions are defined by:

$$\begin{aligned} \mathcal{C}_1(t, \hat{x}) &= d_1 - \frac{H(\widetilde{v}_1(t, \hat{x})) - H(\widetilde{v}_2(t, \hat{x}))}{\hat{u}_1(t, \hat{x}) - \hat{u}_2(t, \hat{x})}, \\ \mathcal{C}_2(t, \hat{x}) &= \frac{G(\hat{u}_1(t, \hat{x})) - G(\hat{u}_2(t, \hat{x}))}{\widetilde{v}_1(t, \hat{x}) - \widetilde{v}_2(t, \hat{x})}, \end{aligned} \tag{2.9}$$

with the interaction term:

$$\mathcal{F}(t, \hat{x}) = d_1 \left(\int_{\hat{g}_1(t)}^{\hat{h}_1(t)} J_1(\hat{x} - y)\hat{u}_1(t, y)\, dy - \int_{\hat{g}_2(t)}^{\hat{h}_2(t)} J_1(\hat{x} - y)\hat{u}_2(t, y)\, dy \right). \tag{2.10}$$

We decompose the term $\mathcal{F}(t, \hat{x})$ as follows:

$$\begin{aligned} \mathcal{F}(t, \hat{x}) &= d_1 \int_{\hat{h}_1(t)}^{\hat{g}_1(t)} J_1(\hat{x} - y)\hat{u}_1(t, y)dy - d_1 \int_{\hat{h}_2(t)}^{\hat{g}_2(t)} J(\hat{x} - y)\hat{u}_2(t, y)dy \\ &= d_1 \left(\int_{\hat{h}_1(t)}^{\hat{g}_1(t)} J_1(\hat{x} - y)[\hat{u}_1(t, y) - \hat{u}_2(t, y)]dy \right) \\ &\quad + d_1 \left(\int_{\tilde{h}_1(t)}^{\hat{g}_1(t)} J_1(\hat{x} - y)\hat{u}_2(t, y)dy - \int_{\hat{h}_2(t)}^{\hat{g}_2(t)} J(\hat{x} - y)\tilde{u}_2(t, y)dy \right), \end{aligned} \tag{2.11}$$

for the first term of (2.11) , we have

$$\left| \int_{\hat{h}_1(t)}^{\hat{g}_1(t)} J_1(\hat{x} - y)\mathcal{U}(t, y)dy \right| \le \|J_1\|_\infty \|\mathcal{U}\|_{C(\Omega_T)}(\hat{g}_1(t) - \hat{h}_1(t)), \tag{2.12}$$

For the second term of (2.11), we have

$$
\left|\int_{\hat{h}_1(t)}^{\hat{g}_1(t)} J(\hat{x}-y)\hat{u}_2(t,y)dy - \int_{\hat{h}_2(t)}^{\hat{g}_2(t)} J(\hat{x}-y)\hat{u}_2(t,y)dy\right| \tag{2.13}
$$
$$
\le \|J_1\|_\infty \mathbb{M}\big(|\hat{g}_1(t)-\hat{g}_2(t)|+|\hat{h}_1(t)-\hat{h}_2(t)|\big)
$$

where $\mathbb{M} := \max\{\|u_0\|_\infty, \|v_0\|_\infty, \mathbb{K}\}$ bounds $\|\hat{u}_2\|_\infty$ uniformly.

Combining (2.12) with (2.13), we obtain the following:

$$
|\mathcal{F}(t,\hat{x})| \le d_1\|J_1\|_\infty\big(\|\mathcal{U}\|_{C(\Omega_T)} + \mathbb{M}\mathbb{H}_{C([0,T])}\big), \tag{2.14}
$$

with

$$
\mathbb{H}_{C([0,T])} := \|\hat{h}_1-\hat{h}_2\|_{C([0,T])} + \|\hat{g}_1-\hat{g}_2\|_{C([0,T])}.
$$

Next, we analyze the terms $\mathcal{C}_1, \mathcal{C}_2$. Under the assumption that H, G are Lipschitz continuous with constant $K_1(\mathbb{M}^*)$ for $\|u\|_\infty, \|v\|_\infty \le \mathbb{M}^*$, we find

$$
\begin{aligned}
|\mathcal{C}_1(t,\hat{x})| &\le d_1 + \left|\frac{H(\tilde{v}_1(t,\hat{x})) - (H(\tilde{v}_1(t,\hat{x}))}{\hat{u}_1-\hat{u}_2}\right| \le d_1 + K(\mathbb{M}^*)\\
|\mathcal{C}_2(t,x^*)| &\le \left|\frac{G(\hat{u}_1(t,\hat{x}) - G(\tilde{u}_2)}{\tilde{v}_1-\tilde{v}_2}\right| \le K(\mathbb{M}^*)
\end{aligned} \tag{2.15}
$$

Hence, we obtain the combined estimate,

$$
\max\left(\|\mathcal{C}_1\|_\infty, \|\mathcal{C}_2\|_\infty\right) \le d_1 + K(M) =: \mathbb{K}, \tag{2.16}
$$

Next, we apply the characteristic method on the (2.8), we obtain the solution: Integrating from 0 to t and using the initial condition,

$$
\begin{aligned}
\mathcal{U}(t,\hat{x}) = \int_0^t &\exp\left(-\int_s^t [a(\hat{x}-q(t-\tau)) + \mathcal{C}_1(\tau,\hat{x}-q(t-\tau))]\,d\tau\right)\Big[\mathcal{C}_2(s,\hat{x}-q(t-s))\hat{V}(s,\hat{x}-q(t-s))\\
&+\mathcal{F}(s,\hat{x}-q(t-s))\Big]\,ds,
\end{aligned} \tag{2.17}
$$

employing (2.15)-(2.16)-(2.14), we conclude that

$$
|\mathcal{U}(t,\hat{x})| \le \int_0^t \left[\mathbb{K}\|\hat{V}\|_{C(\Omega_T)} + d_1\|J_1\|_\infty\left(\|\mathcal{U}\|_{C(\Omega_T)} + \mathbb{M}\mathbb{H}_{C([0,T])}\right)\right] ds.
$$

Therefore,

$$
\|\mathcal{U}\|_{C(\Omega_T)} \le T\left[\mathbb{K}\|\hat{V}\|_{C(\Omega_T)} + d_1\|J_1\|_\infty\left(\|\mathcal{U}\|_{C(\Omega_T)} + \mathbb{M}\mathbb{H}_{C([0,T])}\right)\right],
$$

Bringing the term $\|\mathcal{U}\|$ to the left-hand side,

$$
\|\mathcal{U}\|_{C(\Omega_T)}\left(1 - Td_1\|J_1\|_\infty\right) \le T\left[\mathbb{K}\|\hat{V}\|_{C(\Omega_T)} + d_1\|J_1\|_\infty\mathbb{M}\mathbb{H}_{C([0,T])}\right], \tag{2.18}
$$

Suppose that $T < \frac{1}{d_1\|J_1\|_\infty}$, we have the following:

$$
\|\mathcal{U}\|_{C(\Omega_T)} \le \frac{T}{1-Td_1\|J_1\|_\infty}\left[\mathbb{K}\|\hat{V}\|_{C(\Omega_T)} + d_1\|J_1\|_\infty\mathbb{M}\mathbb{H}_{C([0,T])}\right]. \tag{2.19}
$$

Case 2:

In this case, we consider

$$
\hat{x} \in (h_0, \mathcal{H}_1(t)),
$$

and we set $\hat{t}_1, \hat{t}_2 \in (0,t)$ such that $\hat{t}_1 \le \hat{t}_2$ and

$$
\hat{h}_1(\hat{t}_1) = \hat{h}_1(\hat{t}_2) = \hat{x}.
$$

Using a similar approach as in Case 1, we obtain

$$
\begin{aligned}
\mathcal{U}(t,\hat{x}) &= \mathcal{U}(\hat{t}_2,\hat{x}-q(t-\hat{t}_2))\exp\Big(-\int_{\hat{t}_2}^{t}\Big[a(\hat{x}-q(t-\tau))+\mathcal{C}_1(\tau,\hat{x}-q(t-\tau))\Big]d\tau\Big)\\
&\quad+\int_{\hat{t}_2}^{t}\exp\Big(-\int_{s}^{t}\Big[a(\hat{x}-q(t-\tau))+\mathcal{C}_1(\tau,\hat{x}-q(t-\tau))\Big]d\tau\Big)\\
&\qquad\times\Big[\mathcal{C}_2(s,\hat{x}-q(t-s))\hat{V}(s,\hat{x}-q(t-s))+\mathcal{F}(s,\hat{x}-q(t-s))\Big]ds.
\end{aligned}\tag{2.20}
$$

then,

$$
\begin{aligned}
|\mathcal{U}(t,\hat{x})| &\le |\mathcal{U}(\hat{t}_2,\hat{x}-q(t-\hat{t}_2))|\exp\left(-\int_{\hat{t}_2}^{t}\left[a(\hat{x}-q(t-\tau))+\mathcal{C}_1(\tau,\hat{x}-q(t-\tau))\right]d\tau\right)\\
&\quad+\int_{\hat{t}_2}^{t}\exp\left(-\int_{s}^{t}\left[a(\hat{x}-q(t-\tau))+\mathcal{C}_1(\tau,\hat{x}-q(t-\tau))\right]d\tau\right)\\
&\qquad\times\left[\mathbb{K}\|\hat{V}\|_{C(\Omega_T)}+d_1\|J_1\|_\infty\left(\|\mathcal{U}\|_{C(\Omega_T)}+\mathbb{M}\mathbb{H}_{C([0,T])}\right)\right]ds.
\end{aligned}\tag{2.21}
$$

Assuming $a(\cdot)+\mathcal{C}_1\ge 0$, and setting $\hat{y}=\hat{x}-q(t-\hat{t}_2)$ it then follows that

$$
\exp\left(-\int_{\hat{t}_2}^{t}\left[a(\hat{x}-q(t-\tau))+\mathcal{C}_1(\tau,\hat{x}-q(t-\tau))\right]d\tau\right)\le 1.
$$

We observe that $\mathcal{U}(\hat{t}_2,\cdot)$ can be decomposed as

$$
\begin{aligned}
\mathcal{U}(\hat{t}_2,\hat{y}) &= \big[\hat{u}_1(\hat{t}_2,\hat{y})-\hat{u}_1(\hat{t}_1,\hat{y})\big]+\big[\hat{u}_1(\hat{t}_1,\cdot)-\hat{u}_2(\hat{t}_2,\hat{y})\big]\\
&= \big[\hat{u}_1(\hat{t}_2,\hat{y})-\hat{u}_1(\hat{t}_1,\hat{y})\big]+\big[\hat{u}_1(\hat{t}_1,\hat{h}_1(\hat{t}_1))-\hat{u}_2(\hat{t}_2,\hat{h}_1(\hat{t}_1))\big]\\
&= \hat{u}_1(\hat{t}_2,\hat{y})-\hat{u}_1(\hat{t}_1,\hat{y})\\
&= \int_{\hat{t}_1}^{\hat{t}_2}\frac{\partial\hat{u}}{\partial t}\hat{u}_1(t,\hat{y})\,dt.
\end{aligned}\tag{2.22}
$$

which leads to

$$
\begin{aligned}
|\mathcal{U}(\hat{t}_2,\hat{y})| &\le \int_{\hat{t}_1}^{\hat{t}_2}\left|d_1\int_{\hat{h}_1(t)}^{\hat{g}_1(t)}J_1(\hat{t}_1-y)\hat{u}_1(t,y)dy-d_1\hat{u}_1(t,\hat{y})-a(\hat{y})\hat{u}_1(t,\hat{y})+H(\hat{v})-q\partial_x\hat{u}_1(t,\hat{y})\right|dt\\
&\le \mathsf{C}\big(\mathbb{M},\partial_x\hat{u}_1,H\big)|\hat{t}_2-\hat{t}_1|,
\end{aligned}\tag{2.23}
$$

Clearly, if we assume that $\hat{t}_2=\hat{t}_1$, we can easily find $\mathcal{U}(\hat{t}_2,\hat{y})=0$. Next, we suppose that $\hat{t}_1<\hat{t}_2$, using the definition of $\mathcal{H}_{h_0,T}$, we can find a positive constant $c>0$ such that

$$
\frac{\hat{h}_1(\hat{t}_2)-\hat{h}_1(\hat{t}_2)}{\hat{t}_2-\hat{t}_2}>c,\tag{2.24}
$$

we arrive at

$$
\hat{t}_2-\hat{t}_2<\frac{\hat{h}_1(\hat{t}_2)-\hat{h}_1(\hat{t}_2)}{c},
$$

we also have $\hat{h}_1(\hat{t}_2)-h_2(\hat{t}_1)=0$, we conclude that $\hat{h}_1(\hat{t}_2)-\hat{h}_2(\hat{t}_1)+\hat{h}_1(\hat{t}_2)-\hat{h}_1(\hat{t}_2)=0$, then $\hat{h}_1(\hat{t}_2)-\hat{h}_1(\hat{t}_2)=\hat{h}_1(\hat{t}_2)-h_2(\hat{t}_1)$, by substituting into (2.24), we derive the following result,

$$
\hat{t}_2-\hat{t}_1\le\frac{\hat{h}_1(\hat{t}_2)-\hat{h}_2(\hat{t}_1)}{c},
$$

it then follows

$$
\left|\hat{t}_2-\hat{t}_1\right|\le\left|\hat{h}_1(\hat{t}_2)-\hat{h}_2(\hat{t}_1)\right|c^{-1},
$$

we set $\mathrm{C} = c^{-1}\mathsf{C}\big(\mathbb{M}, \partial_x \hat{u}_1, H\big)$, by replacing in (2.21), we find

$$
\begin{aligned}
|\mathcal{U}(t,\hat{x})| \leq\ & \mathrm{C}\left|\hat{h}_1(\hat{t}_2) - \hat{h}_2(\hat{t}_1)\right| + T\left[\mathbb{K}\|\hat{V}\|_{C(\Omega_T)} + d_1\|J_1\|_\infty\left(\|\mathcal{U}\|_{C(\Omega_T)} + \mathbb{M}\mathbb{H}_{C([0,T])}\right)\right], \\
& \leq \mathrm{C}\left\|\hat{h}_1 - \hat{h}_2\right\| + T\left[\mathbb{K}\|\hat{V}\|_{C(\Omega_T)} + d_1\|J_1\|_\infty\left(\|\mathcal{U}\|_{C(\Omega_T)} + \mathbb{M}\mathbb{H}_{C([0,T])}\right)\right].
\end{aligned} \tag{2.25}
$$

Case 3, we assume that $\hat{x} \in [\mathcal{H}_1(t), \mathcal{H}_2(t)]$, given $\mathcal{H}_1 = \hat{h}_1$ and $\mathcal{H}_2 = \hat{h}_2$ and we also suppose that $\hat{u}(t_1, \hat{x}) = 0$ if $t \in [\hat{t}_2, \hat{t}]$ since we neglect $\hat{t}_1$, we suppose that $\hat{u}_1(\hat{t}_1, \hat{x}) = 0$, and since $\hat{h}_1 < \hat{h}_2$, then $\hat{h}_2(\hat{t}) - \hat{h}_2(t_2^*) = \hat{h}_2(\hat{t}) - \hat{h}_1(\hat{t}) + \hat{h}_1(\hat{t}) - \hat{h}_2(t_2^*) \leq \hat{h}_2(\hat{t}) - \hat{h}_1(\hat{t})$. we compute the following computation as follows

$$
\hat{u}_2(\hat{t},\hat{x}) = \int_{\hat{t}_2}^{\hat{t}}\left[d_1\int_{\hat{h}_1(t)}^{\hat{h}_2(t)} g_2(t)J(\hat{x}-y)\hat{u}_2(t,y)\,dy - d_1\hat{u}_2(t,\hat{x}) - a(\hat{x})\hat{u}_2(t,\hat{x}) + H(\tilde{v}) - q\partial_x\hat{u}_2(t,\hat{x})\right]dt. \tag{2.26}
$$

Using bounds, we obtain the following

$$
\hat{u}_2\left(\hat{t},\hat{x}\right) \leq \mathbb{M}[\mathfrak{D}_s + K(\mathbb{M})]\left(\hat{t} - \hat{t}_2\right) \leq \mathbb{M}\mathbb{K}\cdot\frac{\hat{h}_2(\hat{t}) - \hat{h}_2(\hat{t}_2)}{c^{-1}} \leq \mathbb{M}\mathbb{K}\cdot\frac{\hat{h}_2(\hat{t}) - \hat{h}_1(\hat{t})}{c^{-1}} \leq \mathsf{C}^*\|\hat{h}_1 - \hat{h}_2\|_{C([0,T])}.
$$

where $\mathsf{C}^* = \dfrac{\mathbb{M}\mathbb{K}}{c^{-1}}$. Given that $\hat{u}_1(\hat{t}_1, \hat{x}) = 0$,, we find

$$
|\mathcal{U}(\hat{t},\hat{x})| = |\hat{u}_2\left(\hat{t},\hat{x}\right)| \leq \mathsf{C}^*\|\hat{h}_1 - \hat{h}_2\|_{C([0,T])}. \tag{2.27}
$$

we set $\mathbb{C} = \max\{\mathsf{C}^*, \mathrm{C}, \mathbb{K}, d_1\|J_1\|_\infty \mathbb{M},\}$ by combining (2.19)-(2.25) and (2.27), and by using the definition of $\mathbb{H}$, we find that

$$
|\mathcal{U}(\hat{t},\hat{x})| = |\hat{u}_2\left(\hat{t},\hat{x}\right)| \leq \mathbb{C}\left[\|\mathcal{H}\|_{C([0,T])} + T\left\|\hat{V}\right\|_{C_{\Omega_T}} + \|\mathcal{U}\|_{C_{\Omega_T}}\right], \tag{2.28}
$$

Given that $T < \dfrac{1}{d_1\|J_1\|}$, we conclude that

$$
\|\mathcal{U}\| =\leq \mathbb{C}\left[\|\mathcal{H}\|_{C([0,T])} + T\left\|\hat{V}\right\|_{C_{\Omega_T}} + T\|\mathcal{U}\|_{C_{\Omega_T}}\right], \tag{2.29}
$$

For the case where $\hat{x} \in (\mathcal{G}_2(s), -h_0)$ or $\hat{x} \in (\mathcal{G}_1(s), \mathcal{G}_2(s)]$, one can proceed with an argument similar to that used in **Case 2** and **Case 3**.

We now focus on analyzing the term $\|\mathcal{H}\|_{C([0,T])}$. By the definition of $\mathcal{H}$, we have that

$$
\|\mathcal{H}\|_{C([0,T])} = \|\hat{h}_1 - \hat{h}_2\| + \|\hat{g}_1 - \hat{g}_2\|,
$$

on other hand, we know for $i = 1, 2$,

$$
\begin{aligned}
\hat{h}_i'(t) &= \mu\left(\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_1(x-y)u(t,x)dydx + \rho\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_2(x-y)v(t,x)dydx\right), \qquad t > 0, \\
\hat{g}_i'(t) &= -\mu\left(\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_1(x-y)u(t,x)dydx + \rho\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_2(x-y)v(t,x)dydx\right).
\end{aligned} \tag{2.30}
$$

Next, by integrating and computing the following difference as follows

(2.31)

$$|\hat{h}_1(t)-\hat{h}_2(t)| \le \mu\int_0^t \left| \int_{\hat{g}_1(\tau)}^{\hat{h}_1(\tau)}\int_{\hat{h}_1(\tau)}^{\infty} J_1(x-y)\,\hat{u}_1(\tau,x)\,dy\,dx + \rho\int_{\hat{g}_1(\tau)}^{\hat{h}_1(\tau)}\int_{\hat{h}_1(\tau)}^{\infty} \widetilde{h}_1(\tau)\,J_2(x-y)\,\tilde{v}_1(\tau,x)\,dy\,dx \right.$$

$$\left. -\mu\int_{\hat{g}_2(\tau)}^{\hat{h}_2(\tau)}\int_{\hat{h}_2(\tau)}^{\infty} J_1(x-y)\,\hat{u}_2(\tau,x)\,dy\,dx + \rho\int_{\hat{g}_2(\tau)}^{\hat{h}_2(\tau)}\int_{\hat{h}_2(\tau)}^{\infty} \widetilde{h}_1(\tau)\,J_2(x-y)\,\tilde{v}_2(\tau,x)\,dy\,dx \right| d\tau$$

$$\le \mu\int_0^t \left| \int_{\hat{g}_1(\tau)}^{\hat{h}_1(\tau)}\int_{\hat{h}_1(\tau)}^{\infty} J_1(x-y)\,(\hat{u}_1(\tau,x)-\hat{u}_2(\tau,x))\,dy\,dx \right.$$

$$\left. +\rho\int_{\hat{g}_1(\tau)}^{\hat{h}_1(\tau)}\int_{\hat{h}_1(\tau)}^{\infty} \widetilde{h}_1(\tau)\,J_2(x-y)\,(\tilde{v}_1(\tau,x)-\tilde{v}_2(\tau,x))\,dy\,dx \right| d\tau$$

$$+\mu\int_0^t \left[\int_{\hat{g}_1(\tau)}^{\hat{g}_2(\tau)}\int_{\tilde{h}_1(\tau)}^{\infty} + \int_{\hat{h}_1(\tau)}^{\hat{h}_2(\tau)}\int_{\hat{h}_1(\tau)}^{\infty} + \int_{\hat{g}_2(\tau)}^{\hat{h}_2(\tau)}\int_{\hat{h}_2(\tau)}^{\tilde{h}_1(\tau)} \right] J_1(x-y)\,\hat{u}_2(\tau,x)\,dy\,dx\,d\tau$$

$$+\rho\int_0^t \left[\int_{\hat{g}_1(\tau)}^{\hat{g}_2(\tau)}\int_{\tilde{h}_1(\tau)}^{\infty} + \int_{\hat{h}_1(\tau)}^{\hat{h}_2(\tau)}\int_{\hat{h}_1(\tau)}^{\infty} + \int_{\hat{g}_2(\tau)}^{\hat{h}_2(\tau)}\int_{\hat{h}_2(\tau)}^{\tilde{h}_1(\tau)} \right] J_2(x-y)\,\tilde{v}_2(\tau,x)\,dy\,dx\,d\tau$$

$$\le \mathbb{C}^* T\Big(\|\hat{u}_1-\hat{u}_2\|_{C(\Omega_T)} + \|\tilde{v}_1-\tilde{v}_2\|_{C(\Omega_T)} + \|\hat{h}_1-\hat{h}_2\|_{C([0,T])} + \|\hat{g}_1-\hat{g}_2\|_{C([0,T])}\Big).$$

where $\mathbb{C}^*$ is a positive constant that depends on $(\mu,\rho,\mathbb{M},J_1,J_2)$,
by applying a similar approach as (2.31) to $|\hat{g}_1-\hat{g}_2|$,, we get

$$|\hat{g}_1-\hat{g}_2| \le \mathbb{C}^* T\Big(\|\hat{u}_1-\hat{u}_2\|_{C(\Omega_T)} + \|\tilde{v}_1-\tilde{v}_2\|_{C(\Omega_T)} + \|\hat{h}_1-\hat{h}_2\|_{C([0,T])} + \|\hat{g}_1-\hat{g}_2\|_{C([0,T])}\Big).$$

by the definition of $\|\mathcal{H}\|_{C([0,T])}$, we get

$$\|\mathcal{H}\|_{C([0,T])} \le 2\mathbb{C}^* T\Big(\|\hat{u}_1-\hat{u}_2\|_{C(\Omega_T)} + \|\tilde{v}_1-\tilde{v}_2\|_{C(\Omega_T)} + \|\hat{h}_1-\hat{h}_2\|_{C([0,T])} + \|\hat{g}_1-\hat{g}_2\|_{C([0,T])}\Big), \tag{2.32}$$

Then, assuming the existence of a constant $\eta>0$ such that $1-2\mathbb{C}^*\eta_1 > \frac{1}{2}$ and $2\mathbb{C}^*\eta_1 \le \frac{1}{2}$, From the previous step one has the inequality

$$S := \|\hat{h}_1-\hat{h}_2\|_{C([0,T])} + \|\hat{g}_1-\hat{g}_2\|_{C([0,T])} \le 2\mathbb{C}^* T\,\|\hat{u}_1-\hat{u}_2\|_{C(\Omega_T)} + \,\|\tilde{v}_1-\tilde{v}_2\|_{C(\Omega_T)} + 2\mathbb{C}^* T\,S.$$

Bringing the $2\mathbb{C}^*TS$ term to the left-hand side gives

$$(1-2\mathbb{C}^*T)S \le 2\mathbb{C}^*T\,\|\hat{u}_1-\hat{u}_2\|_{C(\Omega_T)}+,\|\tilde{v}_1-\tilde{v}_2\|_{C(\Omega_T)}$$

Now choose T small enough so that $2\mathbb{C}^*T \le \frac{1}{2}$ (this is the choice δ_3 in the text). Then $1-2\mathbb{C}^* \ge \frac{1}{2}$, and therefore

(2.33)

$$S \le \frac{2\mathbb{C}^*T}{1-2\mathbb{C}^*T}\,\|\tilde{u}_1-\tilde{u}_2\|_{C(\Omega_T)} \le \frac{2\mathbb{C}^*T}{1/2}\Big(\|\hat{u}_1-\hat{u}_2\|_{C(\Omega_T)}+\|\tilde{v}_1-\tilde{v}_2\|_{C(\Omega_T)}\Big) = 4\mathbb{C}^*T\,\Big(\|\hat{u}_1-\hat{u}_2\|_{C(\Omega_T)}+\|\tilde{v}_1-\tilde{v}_2\|_{C(\Omega_T)}\Big).$$

by substituting (2.33) into (2.28), we find

$$\|\mathcal{U}\|_{C_{\Omega_T}} =\le \mathbb{C}T\left[(1+4\mathbb{C}^*)\left\|\hat{V}\right\|_{C_{\Omega_T}} + (1+4\mathbb{C}^*)\,\|\mathcal{U}\|_{C_{\Omega_T}}\right], \tag{2.34}$$

assuming that there exists η_2 such that $\mathbb{C}(1+4\mathbb{C}^*)\eta_2 < \frac{1}{2}$, we have

$$\|\mathcal{U}\|_{C_{\Omega_T}} \le \frac{1}{2}\left\|\hat{V}\right\|_{C_{\Omega_T}} \le \left\|\hat{V}\right\|_{C_{\Omega_T}},\ T\le\eta_2, \tag{2.35}$$

Following the approach used in (2.35), we can simplify (2.19) and (2.25) to obtain that

$$\|\mathcal{U}\|_{C_{\Omega_T}} \leq \left\|\hat{V}\right\|_{C_{\Omega_T}}, \ T \leq \eta_2. \tag{2.36}$$

Assuming that $\hat{v}_{1,0} = \hat{v}_{2,0}$, and replacing (2.36) with (2.38), we find

$$\|\hat{v}_1 - \hat{v}_2\|_{C(\Omega_\infty)} \leq 2KT\|\hat{u}_1 - \hat{u}_2\|_{C(\Omega_{\hat{g},\hat{h}})}. \tag{2.37}$$

by (2.36), we have

$$\|\hat{v}_1 - \hat{v}_2\|_{C(\Omega_\infty)} \leq 2KT\|\tilde{v}_1 - \tilde{v}_2\|_{C(\Omega_{\hat{g},\hat{h}})}. \tag{2.38}$$

Supposing that $T < \min\{\eta_1, \eta_2, \frac{1}{4K}, 1\}$, we deduce that $\mathcal{T}$ verifies the contraction on $X_{v_0,\infty}$.

Step 02: Global existence and uniqueness

In light of Step 1, we conclude that problem (1) admits a unique solution $(\tilde{u}, \hat{v}, \hat{h}, \hat{g})$ defined for $t \in (0, T)$. If we take $u(\sigma, \cdot)$ and $v(\sigma, \cdot)$ as initial functions and repeat Step 1, the solution of (1) can be extended from $t = \sigma$ to some $T^* \geq T$ Additionally, for any given $\sigma \in (0, T)$, we get that $u(\sigma, x) > 0$ for all $x \in (g(\sigma), h(\sigma))$ and $v(s, x) > 0$ for all $x \in \mathbb{R}$. Moreover, $u(\sigma, \cdot)$ and $v(\sigma, \cdot)$ are continuous in $[g(\sigma), h(\sigma)]$ and $\mathbb{R}$, respectively.

By iterating this extension procedure, we assume that $(0, T_{\max})$ is the maximal existence interval of the solution $(\tilde{u}, \hat{v}, \hat{g}, \hat{h})$. We shall establish $T^{\max} = \infty$.

Proceeding by contradiction, we assume that $T^{\max}$ is finite, and we have

$$\hat{h}'(t) - \hat{g}'(t) = \mu \int_{\hat{g}(t)}^{\hat{h}(t)} \left[\int_{\hat{h}(t)}^{\infty} + \int_{-\infty}^{\hat{g}(t)}\right] J_1(x-y)\hat{u}(t,x)\,dy\,dx + \rho \int_{\hat{g}(t)}^{\hat{h}(t)} \left[\int_{\hat{h}(t)}^{\infty} + \int_{-\infty}^{\hat{g}(t)}\right] J_2(x-y)\tilde{v}(t,x)\,dy\,dx \tag{2.39}$$

$$\leq \mathbb{M}\,(\mu+\rho)\left(\hat{h}(t) - \hat{g}(t)\right), \tag{2.40}$$

which implies

$$\hat{h}(t) - \hat{g}(t) \leq 2h_0 e^{\mathbb{M}(\mu+\rho)t} \leq 2h_0 e^{\mathbb{M}(\mu+\rho)T^{\max}}.$$

where $h_0 = \hat{h}(0) = -\hat{g}(0)$.

Given $\hat{h}(t)$ and $\hat{g}(t)$ are monotone on $[0, T^{\max})$, we obtain

$$\hat{h}(T^{\max}) := \lim_{t \to T^{\max}} \hat{h}(t), \quad \hat{g}(T^{\max}) := \lim_{t \to T^{\max}} \hat{g}(t),$$

so,

$$\hat{h}(T^{\max}) - \hat{g}(T^{\max}) \leq 2h_0 e^{\mathbb{M}(\mu+\rho)T^{\max}}.$$

then, we conclude that $0 < \hat{u}(t,x), \hat{v}(t,x) \leq \mathbb{M}$, which gives $\hat{h}'(t), \hat{g}'(t) \in L^\infty([0, T_{\max}))$. Therefore, $\hat{h}, \hat{g} \in C([0, T^{\max}])$.

Furthermore, we define the following sets as follows

$$\mathcal{E}_{T^{\max},B} := \{(t,x) \in \mathbb{R}^2 \mid t \in [0, T^{\max}],\ x \in (\hat{g}(t), \hat{h}(t))\},$$

and

$$\mathcal{E}_{T^{\max},\infty} := \{(t,x) \in \mathbb{R}^2 \mid t \in [0, T^{\max}],\ x \in \mathbb{R}\}.$$

since $u \in L^\infty(\mathcal{E}_{T^{\max},B}), v \in L^\infty(\mathcal{E}_{T^{\max},\infty})$, we get

$$d_1 \int_{\hat{g}(t)}^{\hat{h}(t)} J_1(x-y)u(t,y)\,dy - u(t,x) + \in L^\infty - a(x)u(t,x) + H(v)(\mathcal{E}^F_{T^{\max}}),$$

$$d_2 \int_{\mathbb{R}} J_2(x-y)v(t,y)\,dy - v(t,x) - b(x)v(t,x) + G(u) \in L^\infty(\mathcal{E}_{T^{\max},\infty}),$$

and by using the same approach as Theorem 2.11 in [60], we can obtain that u_x and v_x are uniformly bounded. Therefore, we deduce that the right term of the first and second equations are bounded in $\mathcal{E}_{T^{\max},B}$, and $\mathcal{E}_{T^{\max},\infty}$, respectively. This implies that $\hat{u}_t \in L^\infty(\mathcal{E}_{T^{\max},B})$ and $\hat{v}_t \in L^\infty(\mathcal{E}_{T^{\max},\infty})$. Hence, for each $x \in (\hat{g}(T^{\max}), \hat{h}(T^{\max}))$, we get

$$\hat{u}(T^{\max}, x) := \lim_{t \uparrow T_{\max}} \hat{u}(t,x) \quad \text{exists},$$

and for each $x \in \mathbb{R}$,

$$\tilde{v}(T^{\max}, x) := \lim_{t \uparrow T^{\max}} \tilde{v}(t,x) \quad \text{exists}.$$

Furthermore, $\hat{u}(\cdot, x)$ and $\tilde{v}(\cdot, x)$ are continuous for $t = T^{\max}$.

For the known $(\hat{v}, \hat{g}, \hat{h})$ and t_x defined in (2.7) with T replaced by $T^{\max}$, if we suppose that $\hat{u}$ as the unique solution of the following ODE system:

$$\begin{cases} \hat{u}_t = d_1 \displaystyle\int_{\hat{g}(t)}^{\hat{h}(t)} J_1(x-y)\varphi(t,y)\,dy - d_1\hat{u}(t,x) + p\hat{u}_x - a(x)\hat{u} + H(\tilde{v}), & t_x < t \le T^{\max}, \\ \hat{u}(t_x, x) = \tilde{u}_0(x), & x \in (\hat{g}(T^{\max}), \hat{h}(T^{\max})), \end{cases}$$

with $\varphi = \hat{u}$ and

$$\hat{u}_0(x) = \begin{cases} u_0(x), & x \in [-h_0, h_0], \\ 0, & \text{otherwise}, \end{cases}$$

then, since t_x, $J(\cdot)$, and $H(V)$ are continuous in x, the continuous dependence of ODE solutions on initial data and parameters and by applying the Lemma 2.2 in [60], we obtain $\hat{u} \in C(\mathcal{E}_{T^{\max},B})$ for any $\sigma \in (0, T_{\max})$. A similar argument gives $\tilde{v} \in C(\Omega_s^\infty)$.

Our objective is to prove that $(\hat{u}, \tilde{v})$ is continuous at $t = T^{\max}$ to show this contradiction. For $\hat{u}$, it suffices to prove that

$$\hat{u}(t,x) \to 0 \quad \text{as} \quad (t,x) \to (T^{\max}, \hat{g}(T^{\max})) \quad \text{or} \quad (t,x) \to (T^{\max}, \hat{h}(T_{\max})).$$

We show only the first case $\hat{u}(t,x) \to 0$; the second is similar to $\tilde{v}(t,x) \to 0$. Indeed, as $(t,x) \to (T^{\max}, \hat{g}(T^{\max}))$, by applying the same approach as Lemma 2.2 in [60], we get

$$|\hat{u}(t,x)| \le e^{(\mathfrak{D} - d_1(t-\tilde{t}_x)}\,\mathbb{M}(t-\tilde{t}_x) \quad \longrightarrow \quad 0, \quad \text{as } (t,x) \to \big(\tilde{T}^-, \hat{h}(T^{\max})\big) \text{ or } \big(\tilde{T}^-, g(T^{\max})\big).$$

with $\mathfrak{D} = L\,\|\tilde{v}\|_\infty + \underline{a}$, since $t_x \to T^{\max}$ as $x \to \hat{g}(T_{\max})$. Similarly, for each $x \in \mathbb{R}$,

$$|\tilde{v}(t,x) - \tilde{v}(t_x, x)| \le (t - t_x)[2d_2 + K_1]M_0 \to 0.$$

Hence, $(\hat{u}, \tilde{v}) \in C^1(\mathcal{E}_{T^{\max},B}) \times C^1(\mathcal{E}_{T^{\max},\infty})$, and $(\tilde{u}, \tilde{v}, \tilde{g}, \tilde{h})$ solves problem (1.1) for $t \in (0, T_{\max})$.

By Lemma 2.1, we have $\hat{u}(T^{\max}, x) > 0$ and $\tilde{v}(T^{\max}, x) > 0$ in $(\hat{g}(T_{\max}), \hat{h}(T_{\max}))$. Using $(\tilde{u}(T_{\max}, \cdot), \tilde{v}(T_{\max}, \cdot))$ as initial data and applying Step 1 again, the solution of (1) can be extended to $(0, \hat{T})$ with $\hat{T} > T^{\max}$, contradicting the definition of $T^{\max}$. Hence, $T^{\max} = \infty$, which completes the proof. □

3. Spectral Properties of Integro-Differential Operators

In this section, we focus on the analysis of eigenvalue problems, which play a fundamental role in understanding the long-term dynamics of elliptic and parabolic equations. In particular, the sign of the principal eigenvalue associated with the linearized operator is crucial, as it determines the existence and stability of nontrivial steady states, distinguishes between spreading and vanishing phenomena of the system.

We give some previously known results concerning the principal eigenvalue of the linear nonlocal operator with drift

$$\mathcal{L}_{\Omega, J_1, p} + a,$$

which acts on functions in $C^1(\Omega)$ and is defined by

$$(\mathcal{L}_{\Omega,J,q}+a)[\psi(x)] := d\int_{\Omega} J(x-y)\psi(y)\,dy - q\psi'(x) + a(x)\psi(x),$$

where $a(x)\in C(\overline{\Omega})\cap L^{\infty}(\Omega)$.

We then consider the eigenvalue problem

$$d\int_{\Omega} J(x-y)\psi(y)\,dy - p\psi'(x) + a(x)\psi(x) + \lambda\psi(x) = 0, \quad x\in\Omega. \tag{3.1}$$

It is proved in [31] that admits a principal eigenvalue λ, which is simple and associated with a positive continuous eigenfunction ψ satisfying (3.1). Moreover, if $a(x)$ is bounded, one may further characterize it as the *generalized principal eigenvalue* of $\mathcal{L}_{\Omega,J,q}+a$ by

$$\lambda_p(\mathcal{L}_{\Omega,J,q}+a) := \sup\Big\{\lambda\in\mathbb{R} \;\Big|\; \exists\psi\in C^1(\Omega)\cap C(\Omega),\ \psi>0 \text{ in } \Omega,\ (\mathcal{L}_{\Omega,J,q}[\psi(x)] + a(x)\psi(x) + \lambda\psi(x) \le 0)\Big\}.$$

We are going to investigate the eigentheory for the nonlocal cooperative system with advections in this section.

3.1. **The associated eigenvalue system.** In this subsection, we focus on the analysis of the eigenvalue problem corresponding to the associated system, which can be formulated as follows:

$$\begin{cases} u_t = d_1\left[\displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)u(t,y)dy - u(t,x)\right] + pu_x - a(x)u(x,t) + H\left(v(t,x)\right), & t>0, \quad g(t)<x<h\left(t\right), \\ v_t = d_2\left[\displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)v(t,y)dy - v(t,x)\right] + qv_x - b(x)v(t,x) + G\left(u(t,x)\right), & t>0, \quad g(t)<x<h\left(t\right), \\ -g(0) = h(0) = ,\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x\in\left[-h_0,h_0\right], \end{cases} \tag{3.2}$$

where $-g(t) = h(t) \equiv \mathcal{Z} > 0$. We set

$$\mathcal{D} = H^2(\Omega;\mathbb{C}^2)\cap(\mathrm{BCs}), \qquad X = L^2(\Omega;\mathbb{R}^2).$$

Next, we introduce the associated eigenvalue problem, which is derived from the linearisation of the system at the trivial solution $(0,0)$. It takes the following form:

$$\begin{cases} d_1\displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\psi_1(y)\,dy - d_1\psi_1(x) - a(x)\psi_1(x) + H'(0)\psi_2(x) + p\psi_1'(x) = \lambda\psi_1(x), & x\in[-\mathcal{Z},\mathcal{Z}], \\ d_2\displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\psi_2(y)\,dy - d_2\psi_2(x) - b(x)\psi_2(x) + G'(0)\psi_1(x) + q\psi_2'(x) = \lambda\psi_2(x), & x\in[-\mathcal{Z},\mathcal{Z}]. \end{cases} \tag{3.3}$$

Let

$$\mathbf{D} := \left[C^1([-\mathcal{Z},\mathcal{Z}])\right]^2$$

be equipped with the norm

$$\|\psi\|_D = \sup_{x\in[-\mathcal{Z},\mathcal{Z}]} |\psi(x)|,$$

where $\psi = (\psi_1,\psi_2)\in\mathcal{D}$ and

$$|\psi(x)| = \sqrt{\psi_1^2(x)+\psi_2^2(x)}, \quad x\in[-\mathcal{Z},\mathcal{Z}].$$

We then define two operators $\mathscr{J} : \mathcal{D} \to \mathcal{D}$ by

$$\mathscr{J}(\psi_1, \psi_2)(x) = \left(d_1 \int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\psi_1(y)\, dy,\ d_2 \int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\psi_2(y)\, dy \right),$$

and

$$\mathscr{T}(\psi_1, \psi_2)(x) = \begin{pmatrix} -\left(d_1 + a(x)\right)\psi_1(x) + H'(0)\psi_2 + p\psi_1'(x) \\ -\left(d_2 + b(x)\right)\psi_2(x) + G'(0)\psi_1 + q\psi_2'(x) \end{pmatrix}, \quad \text{for } \psi = (\psi_1, \psi_2) \in \mathcal{D}.$$

Thus, we rewrite (3.3) as follows

$$(\mathscr{J} - \mathscr{T})\psi = \lambda\psi.$$

In the following analysis, we always assume that J_1, J_2, fulfill assumption (**J**), and that G fulfills assumption (**GH**).

In the forthcoming analysis, it is essential to investigate the existence of the principal eigenvalue of problem (3.3). To do this, we first investigate the following lemmas.

Lemma 7. *The following assertions are valid,*

(1) $(\mathscr{T} + \xi I)^{-1}$ *exists and is a bounded strongly positive operator for any* $\alpha > -\lambda_1$*;*

(2) $\|(\mathscr{T} + \xi I)^{-1}\| \to 0 \quad$ *as* $\xi \to \infty$*;*

(3) Let $K > 0$ *be the coercivity constant of* $\mathscr{T}$*. Then for any finite interval* $[\xi_1, \xi_2] \subset (K, \infty)$ *one has*

$$\sup_{\xi \in [\xi_1, \xi_2]} \|(\mathscr{T} + \xi I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\xi_1 - K} < \infty.$$

Proof. Firstly, we establish the existence of $(\mathscr{T} + \xi I)^{-1}$ by proving that the operator

$$\mathscr{T} + \xi I : \mathcal{D} \to \mathcal{D}$$

is bijective. More precisely, we show that it is *injective* (one-to-one) and *surjective* (onto).

Assuming that (ψ_1, ψ_2), such that

$$(\mathscr{T} + \xi I)(\psi_1, \psi_2) = 0$$

$$\mathscr{T}(\psi_1, \psi_2)(x) = \begin{pmatrix} -\big(d_1 + a(x)\big)\psi_1(x) + H'(0)\psi_2(x) + p\psi_1'(x) \\ -\big(d_2 + b(x)\big)\psi_2(x) + G'(0)\psi_1(x) + q\psi_2'(x) \end{pmatrix}, \quad \text{for } \psi = (\psi_1, \psi_2) \in \mathcal{D}.$$

We want to show that the operator $\mathscr{T} + \xi I$ is injective, i.e.,

$$(\mathscr{T} + \xi I)(\psi_1, \psi_2) = 0 \quad \Longrightarrow \quad \psi_1 = \psi_2 = 0.$$

The equation $(\mathscr{T} + \xi I)\psi = 0$ is equivalent to

$$\begin{cases} p\,\psi_1'(x) - \big(d_1 + a(x) - \xi\big)\psi_1(x) + H'(0)\,\psi_2(x) = 0, \\ q\,\psi_2'(x) - \big(d_2 + b(x) - \xi\big)\psi_2(x) + G'(0)\,\psi_1(x) = 0. \end{cases}$$

Step 2. Energy identity. Take the $L^2(-\mathcal{Z}, \mathcal{Z})$ inner product of $(\mathscr{T} + \xi I)\psi$ with ψ and integrate. Using $\Re(p\psi_1'\overline{\psi_1}) = \frac{p}{2}\partial_x|\psi_1|^2$ (which integrates to zero under the boundary/decay conditions), we obtain

$$\Re\langle(\mathscr{T} + \xi I)\psi, \psi\rangle = \int \Big((-d_1 - a(x) + \xi)|\psi_1|^2 + (-d_2 - b(x) + \xi)|\psi_2|^2\Big)\, dx$$
$$+\Re \int \left(H'(0)\psi_2\overline{\psi_1} + G'(0)\psi_1\overline{\psi_2}\right) dx.$$

with $\Re$ being a function of real value.

Step 3. Estimates for the cross terms. Applying Young's inequality with $\varepsilon > 0$ gives

$$\left|\Re(H'(0)\psi_2\overline{\psi_1})\right| \le |H'(0)|\left(\tfrac{\varepsilon}{2}|\psi_1|^2 + \tfrac{1}{2\varepsilon}|\psi_2|^2\right),$$
$$\left|\Re(G'(0)\psi_1\overline{\psi_2})\right| \le |G'(0)|\left(\tfrac{\varepsilon}{2}|\psi_2|^2 + \tfrac{1}{2\varepsilon}|\psi_1|^2\right).$$

Therefore,

$$\Re\langle(\mathscr{T}+\xi I)\psi,\psi\rangle \ge \int \Big((\,\xi - d_1 - a(x) - \tfrac{|H'(0)|}{2}\varepsilon - \tfrac{|G'(0)|}{2\varepsilon}\,)|\psi_1|^2$$
$$+(\,\xi - d_2 - b(x) - \tfrac{|G'(0)|}{2}\varepsilon - \tfrac{|H'(0)|}{2\varepsilon}\,)|\psi_2|^2\Big)\,dx.$$

Step 4. Coercivity and injectivity. Let

$$M_1 := \sup_x (d_1 + a(x)), \quad M_2 := \sup_x (d_2 + b(x)), \quad C_1 := |H'(0)|, \quad C_2 := |G'(0)|.$$

Then, choosing for example $\varepsilon = 1$, there exists a constant

$$K := \max\left\{M_1 + \tfrac{C_1}{2} + \tfrac{C_2}{2},\ M_2 + \tfrac{C_1}{2} + \tfrac{C_2}{2}\right\}$$

such that

$$\Re\langle(\mathscr{T}+\xi I)\psi,\psi\rangle \;\;\ge\;\; (\xi - K)\|\psi\|_{L^2}^2.$$

If $\xi > K$, this inequality implies coercivity. In particular, if $(\mathscr{T}+\xi I)\psi = 0$ then

$$0 = \Re\langle(\mathscr{T}+\xi I)\psi,\psi\rangle \ge (\xi - K)\|\psi\|_{L^2}^2,$$

which forces $\|\psi\|_{L^2} = 0$, hence $\psi \equiv 0$.

Thus, $\mathscr{T}+\xi I$ is injective.

Surjectivity. Define the sesquilinear form $a(\cdot,\cdot)$ on $\mathcal{V}\times\mathcal{V}$ by the same identity $a(\psi,\phi) = \langle(\mathscr{T}+\xi I)\psi,\phi\rangle_X$. Each term of $a(\cdot,\cdot)$ is continuous on $\mathcal{V}\times\mathcal{V}$ (use boundedness of $a, b, H'(0), G'(0)$ and the embedding $H^2 \hookrightarrow L^2$). From the coercivity estimate above and the inequality $\|\psi\|_{L^2} \le C\|\psi\|_{H^2}$ we obtain, for $\xi > K$, a coercivity constant $\alpha > 0$ so that

$$\Re a(\psi,\psi) \ge \alpha\|\psi\|_{\mathcal{V}}^2 \qquad \forall\psi\in\mathcal{V}.$$

For any $f \in X$ the anti-linear functional $\ell(\phi) := \langle f,\phi\rangle_X$ is continuous on $\mathcal{V}$, hence by Lax–Milgram there exists a unique $\psi \in \mathcal{V}$ such that

$$a(\psi,\phi) = \ell(\phi) \qquad \forall\phi\in\mathcal{V}.$$

This variational identity is the weak form of $(\mathscr{T}+\xi I)\psi = f$. Standard regularity (here immediate in 1D first-order system or by choosing $\mathcal{D} = \mathcal{V}$) yields $\psi \in \mathcal{D}$. Uniqueness follows from injectivity. Therefore $(\mathscr{T}+\xi I)$ is onto X.

Combining injectivity and surjectivity gives bijectivity for every $\xi > K$.

We need to show that the operator $\mathscr{T}+\xi I^{-1}$ is a bounded linear. Let $f = (f_1, f_2) \in X$ and $\psi = (\psi_1,\psi_2) \in \mathcal{V}$ solve

$$(\mathscr{T}+\xi I)\psi = f,$$

i.e.

$$\begin{cases} p\,\psi_1' - (d_1 + a - \xi)\psi_1 + H'(0)\psi_2 = f_1, \\ q\,\psi_2' - (d_2 + b - \xi)\psi_2 + G'(0)\psi_1 = f_2. \end{cases}$$

(i) L^2-bound. From the coercivity already established, there exists $c_0 > 0$ such that

$$c_0\|\psi\|_{L^2}^2 \le \Re\langle(\mathscr{T}+\xi I)\psi,\psi\rangle = \Re\langle f,\psi\rangle \le \|f\|_{L^2}\,\|\psi\|_{L^2}.$$

Hence $\|\psi\|_{L^2} \le c_0^{-1}\|f\|_{L^2}$.

(ii) Derivative bound. Solve each equation for the derivative and take L^2-norms:

$$\|\psi_1'\|_{L^2} \le \tfrac{1}{p}\Big(\|d_1 + a - \xi\|_{L^\infty}\|\psi_1\|_{L^2} + |H'(0)|\,\|\psi_2\|_{L^2} + \|f_1\|_{L^2}\Big),$$
$$\|\psi_2'\|_{L^2} \leqq \tfrac{1}{q}\Big(\|d_2 + b - \xi\|_{L^\infty}\|\psi_2\|_{L^2} + |G'(0)|\,\|\psi_1\|_{L^2} + \|f_2\|_{L^2}\Big).$$

Using the L^2-bound from (i) and boundedness of the coefficients, we find

$$\|\psi_1'\|_{L^2} + \|\psi_2'\|_{L^2} \le C_1 \|f\|_{L^2}.$$

Combining (i) and this estimate yields

$$\|\psi\|_{H^2} \le C\, \|f\|_{L^2}.$$

Thus $(\mathscr{T} + \xi I)^{-1} : X \to \mathcal{V}$ is bounded, hence also bounded as $L^2 \to L^2$.

Assuming that the coupling constants satisfy $H'(0) \ge 0$ and $G'(0) \ge 0$ (cooperative system), let $f \in X$ with $f \ge 0$ and set $\psi = R(\xi) f \in \mathcal{V}$. The variational (weak) formulation is

$$a(\psi, \phi) := \langle (\mathscr{T} + \xi I)\psi, \phi\rangle_X = \langle f, \phi\rangle_X \qquad \forall \phi \in \mathcal{V}.$$

Recall the bilinear form (writing components)

$$a(\psi,\phi) = \int_\Omega \Big(p\psi_1'\overline{\phi_1} + q\psi_2'\overline{\phi_2} - (d_1 + a)\psi_1\overline{\phi_1} - (d_2 + b)\psi_2\overline{\phi_2} + H'(0)\psi_2\overline{\phi_1} + G'(0)\psi_1\overline{\phi_2} + \xi\psi \cdot \overline{\phi} \Big)\, dx.$$

Define the negative part of the vector ψ componentwise:

$$\psi_i^- := \min\{\psi_i, 0\} \qquad (i = 1, 2),$$

and take the test function $\phi = \psi^- = (\psi_1^-, \psi_2^-) \in \mathcal{V}$ in the weak formulation. Because $f \ge 0$ and $\psi^- \le 0$ a.e., the right-hand side satisfies

$$\langle f, \psi^-\rangle_X = \int_\Omega f_1\psi_1^- + f_2\psi_2^- \le 0.$$

Thus

(1) $$a(\psi, \psi^-) \le 0.$$

We now estimate the left-hand side from below in terms of $\|\psi^-\|_{L^2}^2$.

First, integration by parts kills the derivative-terms because $\psi_i'\psi_i^-$ is a total derivative and boundary terms vanish by the BCs/decay, hence

$$\Re \int_\Omega p\psi_1'\overline{\psi_1^-}\, dx = 0, \qquad \Re \int_\Omega q\psi_2'\overline{\psi_2^-}\, dx = 0.$$

Therefore (taking real parts)

$$\begin{aligned}\Re\, a(\psi, \psi^-) = &\int_\Omega \Big(\xi(|\psi_1^-|^2 + |\psi_2^-|^2) - (d_1 + a)|\psi_1^-|^2 - (d_2 + b)|\psi_2^-|^2 \Big)\, dx \\ &+ \Re \int_\Omega \Big(H'(0)\psi_2\overline{\psi_1^-} + G'(0)\psi_1\overline{\psi_2^-} \Big)\, dx.\end{aligned}$$

Handle the coupling integrals using the decomposition $\psi_i = \psi_i^+ + \psi_i^-$ (with $\psi_i^+ := \max\{\psi_i, 0\}$). Observe the algebraic sign properties:

$$\psi_2\psi_1^- = \psi_2^+\psi_1^- + \psi_2^-\psi_1^-$$

where $\psi_2^+\psi_1^- \le 0$ and $\psi_2^-\psi_1^- \ge 0$. Since $H'(0) \ge 0$,

$$\Re \int H'(0)\psi_2\overline{\psi_1^-} \le H'(0) \int \psi_2^-\psi_1^-.$$

Similarly,

$$\Re \int G'(0)\psi_1\overline{\psi_2^-} \le G'(0) \int \psi_1^-\psi_2^-.$$

Summing these,

$$\Re \int \Big(H'(0)\psi_2\overline{\psi_1^-} + G'(0)\psi_1\overline{\psi_2^-} \Big) \le (H'(0) + G'(0)) \int \psi_1^-\psi_2^-.$$

Apply Young's inequality to the last term: for any $\varepsilon > 0$,

$$\psi_1^-\psi_2^- \le \tfrac{\varepsilon}{2}|\psi_1^-|^2 + \tfrac{1}{2\varepsilon}|\psi_2^-|^2.$$

Hence

$$\Re\, a(\psi,\psi^-) \geq \int_\Omega \Big((\xi-(d_1+a))|\psi_1^-|^2+(\xi-(d_2+b))|\psi_2^-|^2\Big)\,dx$$
$$-\,(H'(0)+G'(0))\int_\Omega \Big(\tfrac{\varepsilon}{2}|\psi_1^-|^2+\tfrac{1}{2\varepsilon}|\psi_2^-|^2\Big)\,dx.$$

Choose $\varepsilon>0$ (fixed) and set

$$M_1 := \sup_{x\in\Omega}(d_1+a(x)), \qquad M_2 := \sup_{x\in\Omega}(d_2+b(x)),$$

and $C := H'(0)+G'(0)$. Then there exists a constant K (depending on M_1, M_2, C, ε) such that

$$\Re\, a(\psi,\psi^-) \geq (\xi-K)\|\psi^-\|_{L^2}^2. \tag{2}$$

Combine (1) and (2). Since $a(\psi,\psi^-)$ is real (we took real parts), we have

$$0 \geq a(\psi,\psi^-) = \Re a(\psi,\psi^-) \geq (\xi-K)\|\psi^-\|_{L^2}^2.$$

For $\xi>K$ the factor $\xi-K>0$, hence $\|\psi^-\|_{L^2}^2=0$. Therefore $\psi^-\equiv 0$, i.e. $\psi_i\geq 0$ a.e. in Ω for $i=1,2$.

This proves the positivity of the resolvent: $f\geq 0$ implies $R(\xi)f=\psi\geq 0$.

(B) Strong positivity (use compactness + Kreĭn–Rutman). On a bounded interval the embedding $H^2(\Omega)\hookrightarrow L^2(\Omega)$ is compact, hence the resolvent operator $R(\xi): X\to X$ is compact for ξ in the resolvent set (indeed $R(\xi): L^2\to H^2\hookrightarrow L^2$). We have just shown $R(\xi)$ is positive (order-preserving) and compact. Now apply the Kreĭn–Rutman theorem (or the infinite-dimensional analogue of Perron–Frobenius): a compact, positivity-preserving operator on a Banach lattice has a principal eigenvalue with associated eigenvector in the interior of the positive cone, provided the operator is *strongly positive* on some iterate or the operator is irreducible. Concretely, irreducibility of the cooperative system (for instance $H'(0)G'(0)>0$) implies that some power $R(\xi)^n$ maps nonzero nonnegative functions to functions strictly positive everywhere. Kreĭn–Rutman then yields that $R(\xi)$ itself maps every nonzero $f\geq 0$ to a strictly positive vector, i.e. $R(\xi)f\gg 0$.

Therefore $R(\xi)$ is strongly positive under the irreducibility hypothesis.

Next, we establish that for $\xi>k,\ (\mathscr{T}+\xi I)^{-1}\to 0$, as $\xi\to\infty$.

Taking the real part of the L^2 inner product with ψ and using the coercivity,

$$(\xi-K)\|\psi\|_X^2 \ \leq\ \Re\langle(\mathscr{T}+\xi I)\psi,\psi\rangle \ =\ \Re\langle f,\psi\rangle \ \leq\ \|f\|_X\,\|\psi\|_X.$$

If $\psi\neq 0$, divide both sides by $\|\psi\|_X$ to get

$$\|\psi\|_X \ \leq\ \frac{1}{\xi-K}\,\|f\|_X.$$

If $\psi=0$, the inequality is trivial. Taking the supremum over $\|f\|_X=1$ yields

$$\big\|(\mathscr{T}+\xi I)^{-1}\big\|_{\mathcal{L}(X)} \leq \frac{1}{\xi-K}.$$

Letting $\xi\to\infty$ gives the claimed limit.

Lastly, we need to show (3). For $\xi>K$ and $f\in X$, let $\psi=(\mathscr{T}+\xi I)^{-1}f$. By coercivity,

$$(\xi-K)\|\psi\|_X^2 \leq \Re\langle f,\psi\rangle \leq \|f\|_X\|\psi\|_X.$$

Hence $\|\psi\|_X\leq \frac{1}{\xi-K}\|f\|_X$, which leads

$$\|(\mathscr{T}+\xi I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\xi-K}.$$

Taking the supremum over $\xi\in[\xi_1,\xi_2]$ yields

$$\sup_{\xi\in[\xi_1,\xi_2]}\|(\mathscr{T}+\xi I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\xi_1-K} < \infty.$$

$\square$

Lemma 8. *For $\xi > K$, setting by*
$$\rho(\xi) := \rho\Big(\mathscr{J}\,(\mathscr{T}+\xi I)^{-1}\Big)$$
the spectral radius of $\mathscr{J}\,(\mathscr{T}+\xi I)^{-1}$. Hence, we have the following statements

(1) $\rho\big((\mathscr{T}+\xi I)^{-1}J\big) = \rho(\xi)$,
(2) $\lim_{\xi\downarrow K}\rho(\xi) = \infty$,
(3) $\lim_{\xi\to\infty}\rho(\xi) = 0$,
(4) $\rho(\xi)$ *is continuous and strictly decreasing for $\xi > K$.*

Proof. (1) For $\xi > K$ the operators $R(\xi) = (\mathscr{T}+\xi I)^{-1}$ and $\mathscr{J}$ are compact and positive on X. Hence $\mathscr{J}\,R(\xi)$ is a compact, positive operator, so by the Kreĭn–Rutman theorem and lemma (7) its spectral radius $\rho(\xi)$ is a positive, algebraically simple eigenvalue with a corresponding nonnegative eigenfunction. If $\mathscr{J}\,R(\xi)$ is irreducible, the eigenfunction is strictly positive. The equality of spectral radius for $\mathscr{J}\,R(\xi)$ and $R(\xi)\,\mathscr{J}$ follows from the spectral identity for products of bounded operators.

(2),

We aim to prove that
$$\lim_{\xi\downarrow K}\rho(\xi) = \infty,$$
where
$$\rho(\xi) := \rho\big(\mathscr{J}\,(\mathscr{T}+\xi I)^{-1}\big), \qquad R(\xi) := (\mathscr{T}+\xi I)^{-1},$$
with is the resolvent.

First, recall that $R(\xi)$ is the resolvent operator of $\mathscr{T}$. Near the pole at $\xi = K$, standard spectral theory gives the expansion
$$(R(\xi)\psi)(x) \;\approx\; \frac{1}{\xi-K}\,\langle\psi, w^*\rangle\, w(x) \;+\; \text{bounded part},$$
where w is the positive eigenfunction of $\mathscr{T}$ associated to the eigenvalue $-K$, and w^* is the positive eigenfunction of the adjoint operator $\mathscr{T}^*$.

Now, let us act with $\mathscr{J}R(\xi)$ on a nonnegative vector $\psi = (\psi_1,\psi_2)$. By definition, for each component $i = 1, 2$,
$$\big(\mathscr{J}R(\xi)\psi\big)_i(x) = d_i\int_{-\mathcal{Z}}^{\mathcal{Z}} J_i(x-y)\,(R(\xi)\psi)_i(y)\,dy.$$
Using the above expansion of the resolvent, we obtain the lower bound
$$(R(\xi)\psi)_i(y) \;\geq\; \frac{c}{\xi-K}\,\psi_i(y), \qquad y\in[-\mathcal{Z},\mathcal{Z}],$$
for some constant $c>0$ depending only on w, w^*. Hence,
$$\big(\mathscr{J}R(\xi)\psi\big)_i(x) \;\geq\; \frac{d_i}{\xi-K}\int_{-\mathcal{Z}}^{\mathcal{Z}} J_i(x-y)\,\psi_i(y)\,dy,\; i = 1,2$$
This shows that, for every nonnegative $\psi\not\equiv 0$,
$$\mathscr{J}R(\xi)\psi \;\geq\; \frac{c}{\xi-K}\,\psi,$$
in the cone of nonnegative functions. Therefore, by the definition of the spectral radius,
$$\rho(\xi) \;\geq\; \frac{c}{\xi-K}.$$

Iterating the last inequality gives, for every integer $n\geq 1$,
$$\big(\mathscr{J}R(\xi)\big)^n\psi \;\geq\; \Big(\frac{c}{-K+\xi}\Big)^n\psi.$$

Taking norms (for instance the L^2-norm) and dividing by $\|\psi\| > 0$ gives

$$\left\|\left(\mathscr{J}R(\xi)\right)^n\right\| \;\geq\; \frac{\left\|\left(\mathscr{J}R(\xi)\right)^n\psi\right\|}{\|\psi\|} \;\leq\; \left(\frac{c}{-K+\xi}\right)^n.$$

By the spectral radius formula,

$$\rho(\xi) \;=\; \lim_{n\to\infty}\left\|\left(\mathscr{J}R(\xi)\right)^n\right\|^{1/n} \;\geq\; \lim_{n\to\infty}\left(\frac{c}{-K+\xi}\right) \;=\; \frac{c}{-K+\xi}.$$

Since $-K+\xi \to 0^+$ as $\xi \downarrow K$, the right-hand side diverges to $+\infty$. Hence,

$$\lim_{\xi\downarrow K}\rho(\xi) = \infty.$$

(3)

By the coercivity estimate proved earlier (in Lemma (7)) there exists $K \in \mathbb{R}$ such that for every $\xi > K$ we have the resolvent bound

$$\|R(\xi)\|_{\mathcal{L}(X)} \leq \frac{1}{\xi - K}.$$

Since $\mathscr{J}$ is a fixed bounded operator on X, the operator norm of the product satisfies

$$\|\mathscr{J}R(\xi)\|_{\mathcal{L}(X)} \leq \|\mathscr{J}\|_{\mathcal{L}(X)}\,\|R(\xi)\|_{\mathcal{L}(X)} \leq \frac{\|\mathscr{J}\|_{\mathcal{L}(X)}}{\xi - K}.$$

Because the spectral radius is bounded by the operator norm for bounded operators, we obtain

$$0 \leq \rho(\xi) = \rho\big(\mathscr{J}R(\xi)\big) \leq \|\mathscr{J}R(\xi)\|_{\mathcal{L}(X)} \leq \frac{\|\mathscr{J}\|_{\mathcal{L}(X)}}{\xi - K}.$$

Letting $\xi \to \infty$ in the last inequality yields

$$\lim_{\xi\to\infty}\rho(\xi) = 0.$$

To show continuity, for $\xi > K$ the resolvent $R(\xi) = (\mathscr{T} + \xi I)^{-1}$ is analytic (hence continuous) as a map $\xi \mapsto R(\xi) \in \mathcal{L}(X)$. Since $\mathscr{J} \in \mathcal{L}(X)$ is fixed, the map

$$\xi \longmapsto \mathscr{J}R(\xi) \in \mathcal{L}(X)$$

is continuous in the operator norm. The spectral radius of a bounded operator may be defined by $\rho(T) = \lim_{n\to\infty}\|T^n\|^{1/n}$. Because the operator norm $\|\cdot\|$ is continuous and the limit defining ρ is uniform under small perturbations, it follows that $\rho(T)$ depends continuously on T in the operator norm topology. Hence the composition $\xi \mapsto \rho(\mathscr{J}R(\xi)) = \rho(\xi)$ is continuous on (K, ∞).

Monotonicity. Fix ξ_1, ξ_2 with $K < \xi_1 < \xi_2$. The resolvent identity gives

$$R(\xi_1) - R(\xi_2) = (\xi_2 - \xi_1)\,R(\xi_1)R(\xi_2).$$

Since, for $\xi > K$, each $R(\xi)$ is a positive operator (order-preserving), the product $R(\xi_1)R(\xi_2)$ is also positive. Therefore

$$R(\xi_1) - R(\xi_2) \geq 0 \qquad \text{(operator order)}.$$

Multiplying on the left by the positive operator $\mathscr{J}$ preserves the order, so

$$\mathscr{J}R(\xi_1) - \mathscr{J}R(\xi_2) \geq 0,$$

i.e. $0 \leq \mathscr{J}R(\xi_2) \leq \mathscr{J}R(\xi_1)$.

For positive compact operators on a Banach lattice the spectral radius is monotone with respect to the order: if $0 \leq A \leq B$ then $\rho(A) \leq \rho(B)$. Applying this where $A = \mathscr{J}R(\xi_2)$, $B = \mathscr{J}R(\xi_1)$ yields

$$\rho(\xi_2) = \rho(\mathscr{J}R(\xi_2)) \leq \rho(\mathscr{J}R(\xi_1)) = \rho(\xi_1).$$

Thus, $\rho(\xi)$ is not increasing in (K, ∞).

Strict monotonicity (under irreducibility). If, in addition, the operators $\mathscr{J}R(\xi)$ are *irreducible* (or some power is strongly positive), then the order monotonicity is strict whenever the two operators are not equal. Indeed, for $\xi_1 < \xi_2$ we have $R(\xi_1) - R(\xi_2) = (\xi_2 - \xi_1)R(\xi_1)R(\xi_2) \not\equiv 0$, hence $\mathscr{J}R(\xi_1) - \mathscr{J}R(\xi_2)$ is a nonzero positive operator. Kreĭn–Rutman / Perron–Frobenius type results for irreducible compact positive operators imply that if $0 \le A \le B$ and $A \ne B$ (with A, B compact, positive and irreducible in the appropriate sense) then $\rho(A) < \rho(B)$. Applying this to $A = \mathscr{J}R(\xi_2)$ and $B = \mathscr{J}R(\xi_1)$ gives the strict inequality $\rho(\xi_2) < \rho(\xi_1)$. Hence $\rho(\xi)$ is strictly decreasing on (K, ∞) under the irreducibility assumption. □

We are in a position to establish the principal result of this section, stated in the Theorem below.

Proof of Theorem (2).

Proof. Consider the operator family

$$R(\xi) := (\mathscr{T} + \xi I)^{-1}, \qquad \xi > -K,$$

and the compact positive operator

$$A(\xi) := \mathscr{J}R(\xi) = \mathscr{J}(\mathscr{T} + \xi I)^{-1}$$

on the Banach lattice $X = L^2([-\mathcal{Z}, \mathcal{Z}]; \mathbb{R}^2)$. Recall from the lemma (8) that the map $\xi \mapsto \rho(\xi) := \rho(A(\xi))$ is continuous and strictly decreasing on (K, ∞), and moreover

$$\lim_{\xi \downarrow -K} \rho(\xi) = \infty, \qquad \lim_{\xi \to \infty} \rho(\xi) = 0.$$

By the intermediate value theorem there exists a unique $\xi_0 > -K$ such that

$$\rho(\xi_0) = 1.$$

By Kreĭn–Rutman applied to the compact positive operator $A(\xi_0)$ there exists a nontrivial vector $\varphi \in X$, $\varphi \ge 0$, $\varphi \not\equiv 0$, such that

$$A(\xi_0)\varphi = \varphi.$$

Moreover, under the irreducibility/strong-positivity hypothesis this eigenvector may be chosen strictly positive and the eigenvalue 1 is algebraically simple and isolated in the spectrum of $A(\xi_0)$.

Set

$$\psi := R(\xi_0)\varphi = (\mathscr{T} + \xi_0 I)^{-1}\varphi.$$

Then $\psi \in \mathfrak{D}$ and by the definition of $R(\xi_0)$ we have

$$(\mathscr{T} + \xi_0 I)\psi = \varphi.$$

Using $A(\xi_0)\varphi = \varphi$ we get $\mathscr{J}\psi = \varphi$. Substituting yields

$$(\mathscr{T} + \xi_0 I)\psi = \mathscr{J}\psi.$$

Rearranging,

$$\mathscr{T}\psi - \mathscr{J}\psi = -\xi_0\psi.$$

Thus ψ is an eigenfunction of the operator $\mathscr{T} - \mathscr{J}$ with eigenvalue $-\xi_0$. Equivalently, writing $\lambda^* := -\xi_0$, we obtain the eigenpair

$$(\lambda^*, \psi), \qquad \psi = (\psi_1, \psi_2) \gg 0,$$

which satisfies the system (3.3)

$$\begin{cases} d_1 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\psi_1(y)\,dy - d_1\psi_1(x) - a(x)\psi_1(x) + H'(0)\psi_2(x) + p\psi_1'(x) = \lambda^*\psi_1(x), \\ d_2 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\psi_2(y)\,dy - d_2\psi_2(x) - b(x)\psi_2(x) + G'(0)\psi_1(x) + q\psi_2'(x) = \lambda^*\psi_2(x). \end{cases}$$

It remains to show uniqueness (simplicity) of λ^* among eigenvalues with a positive eigenfunction. Suppose $(\tilde{\lambda}, \tilde{\psi})$ is another eigenpair with $\tilde{\psi} \ge 0$, $\tilde{\psi} \not\equiv 0$. Then the same construction (set $\tilde{\xi} := -\tilde{\lambda}$) shows that

1 is in the spectrum of $A(\tilde{\xi})$. But by monotonicity of $\rho(\xi)$ the equation $\rho(\xi) = 1$ has the unique solution ξ_0. Hence $\tilde{\xi} = \xi_0$ and therefore $\tilde{\lambda} = \lambda^*$.

Finally, Kreĭn–Rutman also gives algebraic simplicity of the eigenvalue 1 of $A(\xi_0)$ (under irreducibility), which translates into algebraic simplicity of λ^* for the original problem: no other linearly independent eigenfunction corresponds to λ^* in the positive cone. This proves that system (3.3) admits a unique principal eigenvalue λ^* with an associated positive eigenfunction ψ, and that λ^* is simple and isolated. $\square$

Next, we have the following property on the principal eigenvalue λ^*.

Proposition 9 (**Monotonicity with respect to domain size**). *Assume $p, q > 0$, $a, b \in L^\infty$, $H'(0), G'(0) \geq 0$, and that the kernels J_1, J_2 satisfy assumption **(J)** so that each $\mathscr{J}_l$ is a positive compact integral operator on $X_l := L^2([-\mathcal{Z}, \mathcal{Z}]; \mathbb{R}^2)$. Let $\lambda^*(\mathcal{Z})$ denote the principal eigenvalue of (1.10) on $[-\mathcal{Z}, \mathcal{Z}]$. Then*

$$\mathcal{Z} \longmapsto \lambda^*(\mathcal{Z})$$

is strictly increasing and continuous on $(0, \infty)$.

Proof. Fix $0 < \mathcal{Z}_1 < \mathcal{Z}_2$. We compare $A_{\mathcal{Z}_1}(\xi)$ and $A_{\mathcal{Z}_2}(\xi)$ for a fixed $\xi > K$.

View $X_{\mathcal{Z}_1} := L^2([-\mathcal{Z}_1, \mathcal{Z}_1]; \mathbb{R}^2)$ as a subspace of $X_{\mathcal{Z}_2} := L^2([-\mathcal{Z}_2, \mathcal{Z}_2]; \mathbb{R}^2)$ by extending functions by 0 on $[-\mathcal{Z}_2, -\mathcal{Z}_1) \cup (\mathcal{Z}_1, \mathcal{Z}_2]$. Under our hypotheses, for $\xi > K$ the resolvent $R_{\mathcal{Z}}(\xi)$ is positive on $X_{\mathcal{Z}}$ and $\mathscr{J}_{\mathcal{Z}}$ is a positive integral operator with kernels $J_i \geq 0$. For $f \in X_{\mathcal{Z}_1}$, $f \geq 0$, we then have (pointwise a.e.)

$$\big(A_{\mathcal{Z}_2}(\xi)f\big)_i(x) = d_i \int_{-\mathcal{Z}_2}^{\mathcal{Z}_2} J_i(x-y)\, \big(R_{\mathcal{Z}_2}(\xi)f\big)_i(y)\, dy \;\geq\; d_i \int_{-\mathcal{Z}_1}^{\mathcal{Z}_1} J_i(x-y)\, \big(R_{\mathcal{Z}_1}(\xi)f\big)_i(y)\, dy = \big(A_{\mathcal{Z}_1}(\xi)f\big)_i(x),$$

because: (i) $R_{\mathcal{Z}_2}(\xi)f \geq 0$ and integrates J_i over a larger set; (ii) restricting the integral to $[-\mathcal{Z}_1, \mathcal{Z}_1]$ recovers $A_{\mathcal{Z}_1}(\xi)f$. Thus, in the operator order on the cone $X^+_{\mathcal{Z}_1}$,

$$0 \leq A_{\mathcal{Z}_1}(\xi) \;\leq\; A_{\mathcal{Z}_2}(\xi) \quad \text{on } X_{\mathcal{Z}_1}.$$

Step 2: monotonicity of spectral radius in the cone. For positive compact operators on a Banach lattice, $0 \leq A \leq B$ implies $\rho(A) \leq \rho(B)$. Applying this to the restriction on $X_{\mathcal{Z}_1}$ yields

$$\rho_{\mathcal{Z}_1}(\xi) \;\leq\; \rho_{\mathcal{Z}_2}(\xi) \qquad \text{for every } \xi > K.$$

Moreover, under our standard irreducibility/strong positivity hypothesis (e.g. kernels J_i strictly positive on a set of positive measure and the resolvent kernel positive), the inequality is *strict*:

$$\rho_{\mathcal{Z}_1}(\xi) \;<\; \rho_{\mathcal{Z}_2}(\xi), \qquad \xi > K.$$

Step 3: solve $\rho_{\mathcal{Z}}(\xi) = 1$ and compare $\lambda^*(\mathcal{Z})$. For each fixed $\mathcal{Z}$, we have shown previously that $\xi \mapsto \rho_{\mathcal{Z}}(\xi)$ is continuous and strictly decreasing on (K, ∞), with $\rho_{\mathcal{Z}}(\xi) \to +\infty$ as $\xi \downarrow K$ and $\rho_{\mathcal{Z}}(\xi) \to 0$ as $\xi \to \infty$. Hence there is a unique $\lambda^*(\mathcal{Z}) > K$ such that $\rho_{\mathcal{Z}}(\lambda^*(\mathcal{Z})) = 1$.

Now fix $\mathcal{Z}_1 < \mathcal{Z}_2$ and set $\lambda_1^* := \lambda^*(\mathcal{Z}_1)$. Using Step 2 at $\xi = \lambda_1^*$ gives

$$1 = \rho_{\mathcal{Z}_1}(\lambda_1^*) \;<\; \rho_{\mathcal{Z}_2}(\lambda_1^*).$$

Since $\rho_{\mathcal{Z}_2}(\xi)$ is strictly decreasing in ξ, there exists a unique $\lambda_2^* := \lambda^*(\mathcal{Z}_2)$ with $\rho_{\mathcal{Z}_2}(\lambda_2^*) = 1$ and necessarily

$$\lambda_2^* \;>\; \lambda_1^*.$$

Next, we prove the continuity of $\lambda^*(\mathcal{Z})$, let $\mathcal{Z}_0 > 0$ be arbitrary and fix $\varepsilon > 0$. We show there exists $\delta > 0$ such that

$$|\lambda^*(\mathcal{Z}) - \lambda^*(\mathcal{Z}_0)| < \varepsilon \quad \text{whenever } |\mathcal{Z} - \mathcal{Z}_0| < \delta.$$

Recall $\lambda^*(\mathcal{Z}) = -\xi(\mathcal{Z})$ where $\xi(\mathcal{Z}) > K$ is the unique solution of

$$\rho_{\mathcal{Z}}\big(\xi(\mathcal{Z})\big) = 1, \qquad \rho_{\mathcal{Z}}(\xi) := \rho\big(\mathscr{J}_{\mathcal{Z}}(\mathscr{T}_{\mathcal{Z}} + \xi I)^{-1}\big).$$

Since $\xi \mapsto \rho_{\mathcal{Z}_0}(\xi)$ is continuous and strictly decreasing and $\rho_{\mathcal{Z}_0}(\xi(\mathcal{Z}_0)) = 1$, there exists $\eta > 0$ such that

$$\rho_{\mathcal{Z}_0}\big(\xi(\mathcal{Z}_0) - \eta\big) > 1 \quad \text{and} \quad \rho_{\mathcal{Z}_0}\big(\xi(\mathcal{Z}_0) + \eta\big) < 1.$$

Because $\rho_{\mathcal{Z}_0}$ is strictly decreasing these two inequalities are strict and the interval $[\,\xi(\mathcal{Z}_0) - \eta,\ \xi(\mathcal{Z}_0) + \eta\,]$ contains no other root of $\rho_{\mathcal{Z}_0}(\xi) = 1$.

By the joint continuity of $(\mathcal{Z}, \xi) \mapsto \rho_{\mathcal{Z}}(\xi)$ on compact sets, there exists $\delta_1 > 0$ such that for all $\mathcal{Z}$ with $|\mathcal{Z} - \mathcal{Z}_0| < \delta_1$ and for all $\xi \in [\xi(\mathcal{Z}_0) - \eta, \xi(\mathcal{Z}_0) + \eta]$ we have

$$\big|\rho_{\mathcal{Z}}(\xi) - \rho_{\mathcal{Z}_0}(\xi)\big| < \gamma,$$

where $0 < \gamma < \min\{\rho_{\mathcal{Z}_0}(\xi(\mathcal{Z}_0) - \eta) - 1,\ 1 - \rho_{\mathcal{Z}_0}(\xi(\mathcal{Z}_0) + \eta)\}$. Choosing such a γ is possible because the two differences in the min are positive.

Hence for all $\mathcal{Z}$ with $|\mathcal{Z} - \mathcal{Z}_0| < \delta_1$ we obtain

$$\rho_{\mathcal{Z}}(\xi(\mathcal{Z}_0) - \eta) \geq \rho_{\mathcal{Z}_0}(\xi(\mathcal{Z}_0) - \eta) - \gamma > 1,$$

and

$$\rho_{\mathcal{Z}}(\xi(\mathcal{Z}_0) + \eta) \leq \rho_{\mathcal{Z}_0}(\xi(\mathcal{Z}_0) + \eta) + \gamma < 1.$$

Fix any $\mathcal{Z}$ with $|\mathcal{Z} - \mathcal{Z}_0| < \delta_1$. The function $\xi \mapsto \rho_{\mathcal{Z}}(\xi)$ is continuous and strictly decreasing and we just showed it is > 1 at $\xi = \xi(\mathcal{Z}_0) - \eta$ and < 1 at $\xi = \xi(\mathcal{Z}_0) + \eta$. By the intermediate value theorem there exists a root $\xi(\mathcal{Z}) \in (\xi(\mathcal{Z}_0) - \eta, \xi(\mathcal{Z}_0) + \eta)$ of $\rho_{\mathcal{Z}}(\xi) = 1$. By uniqueness of the root (for fixed $\mathcal{Z}$) this must be the unique $\xi(\mathcal{Z})$. Therefore

$$|\xi(\mathcal{Z}) - \xi(\mathcal{Z}_0)| < \eta.$$

Recalling $\lambda^*(\mathcal{Z}) = -\xi(\mathcal{Z})$, we obtain

$$|\lambda^*(\mathcal{Z}) - \lambda^*(\mathcal{Z}_0)| = |\xi(\mathcal{Z}) - \xi(\mathcal{Z}_0)| < \eta.$$

Finally choose η small enough so that $\eta < \varepsilon$ and set $\delta := \delta_1$. Then for $|\mathcal{Z} - \mathcal{Z}_0| < \delta$ we have $|\lambda^*(\mathcal{Z}) - \lambda^*(\mathcal{Z}_0)| < \varepsilon$, proving continuity at $\mathcal{Z}_0$.

Since $\mathcal{Z}_0 > 0$ was arbitrary, $\lambda^*(\mathcal{Z})$ is continuous on $(0, \infty)$.

□

Lemma 10. *Let the global basic reproduction number be*

$$\mathcal{R}_0 \ := \ \rho\big(\mathscr{J}(-\mathscr{T})^{-1}\big) = \lim_{\mathcal{Z}\to\infty} \rho_{\mathcal{Z}}(0),$$

(where the last equality is maintained because $A_{\mathcal{Z}}(0)$ increases in $\mathcal{Z}$ and converges to the next-generation operator in the full space). Then the following hold:

1. *If $\mathcal{R}_0 \leq 1$, then for every $\mathcal{Z} > 0$ one has $\lambda^*(\mathcal{Z}) < 0$.*
2. *If $\mathcal{R}_0 > 1$, then there exists a unique $\mathcal{Z}_* \in (0, \infty)$ such that $\lambda^*(\mathcal{Z}_*) = 0$ (equivalently $\rho_{\mathcal{Z}_*}(0) = 1$). Moreover $\lambda^*(\mathcal{Z}) < 0$ for $\mathcal{Z} < \mathcal{Z}_*$ and $\lambda^*(\mathcal{Z}) > 0$ for $\mathcal{Z} > \mathcal{Z}_*$.*

Lemma 11 (**Threshold characterization**)**.** *Define the global basic reproduction number by*

$$\mathcal{R}_0 \ := \ \rho\big(\mathscr{J}(-\mathscr{T})^{-1}\big) = \lim_{\mathcal{Z}\to\infty} \rho_{\mathcal{Z}}(0),$$

where the limit holds since $A_{\mathcal{Z}}(0)$ increases with $\mathcal{Z}$ and converges to the next-generation operator in the full space. Then:

1. *If $\mathcal{R}_0 \leq 1$, then for every $\mathcal{Z} > 0$, one has $\lambda^*(\mathcal{Z}) < 0$.*
2. *If $\mathcal{R}_0 > 1$, then there exists a unique $\mathcal{Z}_* \in (0, \infty)$ such that $\lambda^*(\mathcal{Z}_*) = 0$ (equivalently $\rho_{\mathcal{Z}_*}(0) = 1$). Moreover, $\lambda^*(\mathcal{Z}) < 0$ for $\mathcal{Z} < \mathcal{Z}_*$ and $\lambda^*(\mathcal{Z}) > 0$ for $\mathcal{Z} > \mathcal{Z}_*$.*

Proof. The proof is based on monotonicity and continuity arguments for $\rho_{\mathcal{Z}}(\xi)$.

For fixed $\mathcal{Z} > 0$ the function $\xi \mapsto \rho_{\mathcal{Z}}(\xi)$ is continuous and strictly decreasing on (K, ∞) and satisfies

$$\lim_{\xi \downarrow K} \rho_{\mathcal{Z}}(\xi) = \infty, \qquad \lim_{\xi\to\infty} \rho_{\mathcal{Z}}(\xi) = 0.$$

Hence, for each $\mathcal{Z}$ there is a unique $\xi(\mathcal{Z}) > K$ such that $\rho_{\mathcal{Z}}(\xi(\mathcal{Z})) = 1$, and we set $\lambda^*(\mathcal{Z}) = -\xi(\mathcal{Z})$.

Moreover, the family $\mathcal{Z} \mapsto A_{\mathcal{Z}}(0) = \mathscr{J}_{\mathcal{Z}}(\mathscr{T}_{\mathcal{Z}})^{-1}$ is monotone in the sense that if $0 < \mathcal{Z}_1 < \mathcal{Z}_2$ then, viewing functions on $[-\mathcal{Z}_1, \mathcal{Z}_1]$ as extended by zero to $[-\mathcal{Z}_2, \mathcal{Z}_2]$,

$$0 \le A_{\mathcal{Z}_1}(0) \le A_{\mathcal{Z}_2}(0),$$

so by monotonicity of the spectral radius for positive operators,

$$\mathcal{Z} \mapsto \rho_{\mathcal{Z}}(0) \quad \text{is nondecreasing,}$$

and (by irreducibility / strong positivity) strictly increasing unless degenerate kernels occur. We also assume (as is standard) that

$$\lim_{\mathcal{Z}\to\infty} \rho_{\mathcal{Z}}(0) = \mathcal{R}_0,$$

so $\rho_{\mathcal{Z}}(0)$ increases to the global reproduction number $\mathcal{R}_0$.

(i) The case $\mathcal{R}_0 \le 1$. Since $\mathcal{Z} \mapsto \rho_{\mathcal{Z}}(0)$ is nondecreasing and its limit as $\mathcal{Z} \to \infty$ equals $\mathcal{R}_0 \le 1$, we have for every finite $\mathcal{Z} > 0$

$$\rho_{\mathcal{Z}}(0) \le \mathcal{R}_0 \le 1.$$

But $\xi \mapsto \rho_{\mathcal{Z}}(\xi)$ is strictly decreasing, so for $\xi \ge 0$ we have $\rho_{\mathcal{Z}}(\xi) \le \rho_{\mathcal{Z}}(0) \le 1$. In particular the unique root $\xi(\mathcal{Z})$ of $\rho_{\mathcal{Z}}(\xi) = 1$ must satisfy $\xi(\mathcal{Z}) > 0$. Equivalently

$$\lambda^*(\mathcal{Z}) = -\xi(\mathcal{Z}) < 0,$$

for every $\mathcal{Z} > 0$. This proves (i).

(ii) The case $\mathcal{R}_0 > 1$. Since $\rho_{\mathcal{Z}}(0)$ is nondecreasing in $\mathcal{Z}$ and tends to $\mathcal{R}_0 > 1$ as $\mathcal{Z} \to \infty$, there are two possibilities for small $\mathcal{Z}$: either $\rho_{\mathcal{Z}}(0) < 1$ for all small $\mathcal{Z}$ (this is typical: for very small habitat the next-generation operator is small), or in pathological degenerate cases equality might occur at a single point — under the irreducibility assumptions one gets strict inequality for sufficiently small $\mathcal{Z}$. Hence, there exists $\mathcal{Z}$ small enough with $\rho_{\mathcal{Z}}(0) < 1$, while for large $\mathcal{Z}$ we have $\rho_{\mathcal{Z}}(0)$ arbitrarily close to $\mathcal{R}_0 > 1$ and in particular > 1 for $\mathcal{Z}$ large.

Now consider the continuous function $h(\mathcal{Z}) := \rho_{\mathcal{Z}}(0)$ of $\mathcal{Z}$ (it is monotone and hence continuous almost everywhere; under our joint continuity hypotheses it is continuous). We have

$$\lim_{\mathcal{Z}\downarrow 0} h(\mathcal{Z}) = 0 \quad \text{and} \quad \lim_{\mathcal{Z}\to\infty} h(\mathcal{Z}) = \mathcal{R}_0 > 1,$$

so by the intermediate value theorem there exists at least one $\mathcal{Z}_* > 0$ with $h(\mathcal{Z}_*) = 1$; that is,

$$\rho_{\mathcal{Z}_*}(0) = 1.$$

Equivalently $\xi(\mathcal{Z}_*) = 0$ and $\lambda^*(\mathcal{Z}_*) = 0$.

To prove the uniqueness of $\mathcal{Z}_*$, note that $\mathcal{Z} \mapsto \rho_{\mathcal{Z}}(0)$ is strictly increasing (under the irreducibility / strong-positivity hypothesis). Therefore, the equation $\rho_{\mathcal{Z}}(0) = 1$ has at most one root, so $\mathcal{Z}_*$ is unique.

Finally, monotonicity of $\lambda^*(\mathcal{Z})$ (equivalently monotonicity of $\xi(\mathcal{Z})$) gives the sign pattern: for $\mathcal{Z} < \mathcal{Z}_*$ we have $\rho_{\mathcal{Z}}(0) < 1$ so $\xi(\mathcal{Z}) > 0$ and $\lambda^*(\mathcal{Z}) < 0$; for $\mathcal{Z} > \mathcal{Z}_*$ we have $\rho_{\mathcal{Z}}(0) > 1$ so $\xi(\mathcal{Z}) < 0$ and $\lambda^*(\mathcal{Z}) > 0$. This completes the proof. □

Lemma 12 (Upper / lower eigenvalue bounds in the C^1 setting)**.** *Let $\mathcal{Z} > 0$ be fixed and consider the linear operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset C^1([-\mathcal{Z}, \mathcal{Z}]; \mathbb{R}^2) \to C([-\mathcal{Z}, \mathcal{Z}]; \mathbb{R}^2)$ given componentwise by*

$$\mathcal{L}(\psi_1, \psi_2)(x) = \begin{pmatrix} d_1 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\psi_1(y)\,dy - d_1\psi_1(x) - a(x)\psi_1(x) + H'(0)\psi_2(x) + p\psi_1'(x) \\ d_2 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\psi_2(y)\,dy - d_2\psi_2(x) - b(x)\psi_2(x) + G'(0)\psi_1(x) + q\psi_2'(x) \end{pmatrix}.$$

Assume $\mathcal{L}$ generates a positive C_0-semigroup on $C([-\mathcal{Z}, \mathcal{Z}]; \mathbb{R}^2)$ and has a principal eigenvalue λ^* (simple, real) with a strictly positive eigenfunction in $\mathcal{D}(\mathcal{L})$.

Suppose there exist nontrivial test functions $\underline{\psi} = (\underline{\psi}_1, \underline{\psi}_2)$, $\overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2) \in \mathcal{D}(\mathcal{L})$ *satisfying* $\underline{\psi}, \overline{\psi} \geq 0$ *(not identically zero) and real numbers* $\underline{\lambda}, \overline{\lambda}$ *such that the componentwise inequalities*

$$\begin{cases} d_1 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\underline{\psi}_1(y)\,dy - d_1\underline{\psi}_1(x) - a(x)\underline{\psi}_1(x) + H'(0)\underline{\psi}_2(x) + p\underline{\psi}_1'(x) \;\geq\; \underline{\lambda}\,\underline{\psi}_1(x), \\ d_2 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\underline{\psi}_2(y)\,dy - d_2\underline{\psi}_2(x) - b(x)\underline{\psi}_2(x) + G'(0)\underline{\psi}_1(x) + q\underline{\psi}_2'(x) \;\geq\; \underline{\lambda}\,\underline{\psi}_2(x), \end{cases} \tag{3.4}$$

and

$$\begin{cases} d_1 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\overline{\psi}_1(y)\,dy - d_1\overline{\psi}_1(x) - a(x)\overline{\psi}_1(x) + H'(0)\overline{\psi}_2(x) + p\overline{\psi}_1'(x) \;\leq\; \overline{\lambda}\,\overline{\psi}_1(x), \\ d_2 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\overline{\psi}_2(y)\,dy - d_2\overline{\psi}_2(x) - b(x)\overline{\psi}_2(x) + G'(0)\overline{\psi}_1(x) + q\overline{\psi}_2'(x) \;\leq\; \overline{\lambda}\,\overline{\psi}_2(x), \end{cases} \tag{3.5}$$

hold pointwise for all $x \in [-\mathcal{Z}, \mathcal{Z}]$.

Then

$$\underline{\lambda} \;\leq\; \lambda^* \;\leq\; \overline{\lambda}.$$

Proof. We prove the upper statement; the lower one is analogous (or may be obtained by the same argument applied to $\mathscr{J} - \mathscr{T}$ with reversed inequalities).

Assume by contradiction that $\lambda^* > \overline{\lambda}$. Choose a number ξ_2 such that

$$\lambda^* > \xi_2 > \max\{-K,\ \overline{\lambda}\},$$

where K is the pole threshold introduced earlier (so that the resolvent $R(\xi) = (\mathscr{T} + \xi I)^{-1}$ exists and is positive for $\xi > K$).

Rewriting (3.5) as $\mathscr{J}\varphi \leq (\mathscr{T} + \overline{\lambda} I)\varphi$ and applying the positive operator $(\mathscr{T} + \xi_2 I)^{-1}$ (which exists and is positive since $\xi_2 > -K$) we obtain

$$(\mathscr{T} + \xi_2 I)^{-1}\mathscr{J}\varphi \;\leq\; (\mathscr{T} + \xi_2 I)^{-1}(\mathscr{T} + \overline{\lambda} I)\varphi.$$

But

$$(\mathscr{T} + \xi_2 I)^{-1}(\mathscr{T} + \overline{\lambda} I) = I - (\xi_2 - \overline{\lambda})(\mathscr{T} + \xi_2 I)^{-1},$$

so the right-hand side is

$$\varphi - (\xi_2 - \overline{\lambda})(\mathscr{T} + \xi_2 I)^{-1}\varphi \leq \varphi.$$

Hence, we deduce the cone-comparison

$$(\mathscr{T} + \xi_2 I)^{-1}\mathscr{J}\varphi \;\leq\; \varphi. \tag{*}$$

Set

$$S(\xi_2) := (\mathscr{T} + \xi_2 I)^{-1}\mathscr{J},$$

a positive compact operator on the Banach lattice $\mathcal{D}$. Let $S(\xi_2)^*$ denote its dual (conjugate) operator acting on the dual space $\mathcal{D}^*$. By Kreĭn–Rutman the spectral radius

$$\rho(\xi_2) := \rho\big(S(\xi_2)\big) = \rho\big(S(\xi_2)^*\big)$$

is the principal (simple) eigenvalue of $S(\xi_2)^*$ and there exists a strongly positive dual eigenvector $E_{\xi_2} \in \mathcal{D}^*_+$ such that

$$S(\xi_2)^* E_{\xi_2} = r(\xi_2) E_{\xi_2}.$$

Pairing the dual eigenvector with the inequality $(*)$ and using positivity yields

$$\langle E_{\xi_2},\, S(\xi_2)\varphi\rangle \leq \langle E_{\xi_2},\, \varphi\rangle.$$

But the left-hand side equals

$$\langle S(\xi_2)^* E_{\xi_2},\, \varphi\rangle = \rho(\xi_2)\,\langle E_{\xi_2},\, \varphi\rangle.$$

Since E_{ξ_2} is strictly positive and $\varphi \geq 0$, $\langle E_{\xi_2}, \varphi \rangle > 0$. Cancelling this positive scalar gives

$$r(\xi_2) \leq 1.$$

On the other hand, by construction $\lambda^* > \xi_2$. From the characterization of λ^* we know that at $\xi = \lambda^*$ the operator $\mathscr{J}(\mathscr{T} + \xi I)^{-1}$ has spectral radius 1; equivalently the function $\xi \mapsto \rho\big(\mathscr{J}(\mathscr{T} + \xi I)^{-1}\big)$ is strictly decreasing and satisfies $\rho(\mathscr{J}(\mathscr{T} + \lambda^* I)^{-1}) = 1$. Therefore, since $\xi_2 < \lambda^*$, monotonicity gives

$$\rho(\xi_2) = \rho\big(S(\xi_2)\big) > \rho\big(S(\lambda^*)\big) = 1,$$

a contradiction to $\rho(\xi_2) \leq 1$. This contradiction shows $\lambda^* \leq \overline{\lambda}$.

If equality holds in (3.5) (so $\mathscr{J}\varphi - \mathscr{T}\varphi = \overline{\lambda}\varphi$), then the above chain gives

$$S(\xi_2)\varphi = \varphi$$

for $\xi_2 = \overline{\lambda}$, and hence 1 is an eigenvalue of $S(\overline{\lambda})$. By uniqueness (Krein–Rutman) the principal eigenvalue equals 1, so $\lambda^* = \overline{\lambda}$ and φ (up to scaling) is the positive principal eigenfunction.

For completeness, we reproduce the short iterative argument that yields the same contradiction in the "lower" style lemma used later. If instead one assumes

$$\mathscr{J}\eta - \mathscr{T}\eta \;\geq\; \lambda\eta$$

with $\lambda > \lambda^*$, then applying $S(\lambda) := (\mathscr{T} + \lambda I)^{-1}\mathscr{J}$ gives

$$\eta \leq S(\lambda)\eta,$$

hence by induction $\eta \leq S(\lambda)^n\eta$ for every $n \geq 1$. Taking norms and using the spectral radius formula gives

$$1 \leq \lim_{n\to\infty} \|S(\lambda)^n\|^{1/n} = \rho\big(S(\lambda)\big).$$

But monotonicity of $\rho(S(\xi))$ in ξ and the fact $\rho(S(\lambda^*)) = 1$ imply $\rho(S(\lambda)) < 1$ for $\lambda > \lambda^*$, contradiction. Thus $\lambda \leq \lambda^*$. $\square$

Lemma 13. *Assume the standing hypotheses hold (positivity and compactness of $\mathscr{J}_{\mathcal{Z}}$, monotonicity in $\mathcal{Z}$, and the resolvent estimates for $\mathscr{T}$). For each $\mathcal{Z} > 0$ define*

$$\rho_{\mathcal{Z}}(\xi) := \rho\big(\mathscr{J}_{\mathcal{Z}}(\mathscr{T} + \xi I)^{-1}\big), \qquad \xi > K,$$

and let $\lambda^*(\mathcal{Z}) > K$ denote the unique number satisfying

$$\rho_{\mathcal{Z}}\big(\lambda^*(\mathcal{Z})\big) = 1.$$

Let

$$\rho_\infty(\xi) := \rho\big(\mathscr{J}_\infty(\mathscr{T} + \xi I)^{-1}\big), \qquad \xi > K,$$

and define $R_0 := \rho_\infty(0)$. Then

(1) $\lim_{\mathcal{Z}\downarrow 0} \lambda^*(\mathcal{Z}) = K$.

(2) $\lim_{\mathcal{Z}\to\infty} \lambda^*(\mathcal{Z}) =: \lambda_\infty \in [K, \infty)$ *finite, where*

$$\lambda_\infty = \inf\{\xi > K : \rho_\infty(\xi) \leq 1\}.$$

Moreover, one has the sign relation

$$\operatorname{sign}(\lambda_\infty) = \operatorname{sign}(R_0 - 1).$$

Remark 1. *The constant K has already been introduced in Lemma (7). More explicitly, we set*

$$M_1 := \sup_x \big(d_1 + a(x)\big), \quad M_2 := \sup_x \big(d_2 + b(x)\big), \quad C_1 := |H'(0)|, \quad C_2 := |G'(0)|,$$

and, by choosing $\varepsilon = 1$, we define

$$K := \max\Big\{M_1 + \tfrac{C_1}{2} + \tfrac{C_2}{2},\; M_2 + \tfrac{C_1}{2} + \tfrac{C_2}{2}\Big\}\,.$$

Proof. We split the proof into two parts.

(i) Small-domain limit $\mathcal{Z} \downarrow 0$. Fix any $\xi > K$. As $\mathcal{Z} \downarrow 0$ the integration window in the operator $\mathscr{J}_{\mathcal{Z}}$ shrinks to a point, hence $\mathscr{J}_{\mathcal{Z}} \to 0$ in operator norm. Since $(\mathscr{T} + \xi I)^{-1}$ is bounded for this fixed ξ, we obtain

$$\lim_{\mathcal{Z}\downarrow 0} \rho_{\mathcal{Z}}(\xi) \le \lim_{\mathcal{Z}\downarrow 0} \left\| \mathscr{J}_{\mathcal{Z}}(\mathscr{T} + \xi I)^{-1} \right\| = 0.$$

Thus for any fixed $\xi > K$ and $\mathcal{Z}$ small enough we have $\rho_{\mathcal{Z}}(\xi) < 1$. Because $\xi \mapsto \rho_{\mathcal{Z}}(\xi)$ is strictly decreasing and $\rho_{\mathcal{Z}}(\xi) \to \infty$ as $\xi \downarrow K$, the unique root $\lambda^*(\mathcal{Z})$ of $\rho_{\mathcal{Z}}(\cdot) = 1$ must satisfy $\lambda^*(\mathcal{Z}) > \xi$ for all small $\mathcal{Z}$. Letting $\xi \downarrow K$ yields

$$\liminf_{\mathcal{Z}\downarrow 0} \lambda^*(\mathcal{Z}) \ge K.$$

On the other hand $\lambda^*(\mathcal{Z}) > K$ for every $\mathcal{Z}$, so the limit exists and equals K: $\lim\limits_{\mathcal{Z}\downarrow 0} \lambda^*(\mathcal{Z}) = K$.

(ii) Large-domain limit $\mathcal{Z} \to \infty$ and finiteness of λ_∞. Fix $\xi > K$. By monotone convergence of the truncation operators we have $\mathscr{J}_{\mathcal{Z}} \uparrow \mathscr{J}_\infty$ (strongly) as $\mathcal{Z} \to \infty$. Since $(\mathscr{T} + \xi I)^{-1}$ is a fixed bounded positive operator, it follows that

$$\mathscr{J}_{\mathcal{Z}}(\mathscr{T} + \xi I)^{-1} \longrightarrow \mathscr{J}_\infty(\mathscr{T} + \xi I)^{-1} \quad \text{in the strong operator topology,}$$

and therefore

$$\lim_{\mathcal{Z}\to\infty} \rho_{\mathcal{Z}}(\xi) = \rho_\infty(\xi) \qquad (\text{pointwise in } \xi > K).$$

For each $\mathcal{Z} > 0$ the function $\xi \mapsto \rho_{\mathcal{Z}}(\xi)$ is continuous, strictly decreasing on (K, ∞), and crosses the level 1 exactly once at $\xi = \lambda^*(\mathcal{Z})$. Because $\mathscr{J}_{\mathcal{Z}}$ is increasing in $\mathcal{Z}$, the family $\rho_{\mathcal{Z}}(\xi)$ is increasing in $\mathcal{Z}$ for each fixed ξ; consequently $\lambda^*(\mathcal{Z})$ is nondecreasing in $\mathcal{Z}$. Hence the pointwise limit

$$\lambda_\infty := \lim_{\mathcal{Z}\to\infty} \lambda^*(\mathcal{Z}) \in [K, \infty]$$

exists (possibly infinite), and by passing to the limit in the equation $\rho_{\mathcal{Z}}(\lambda^*(\mathcal{Z})) = 1$ we obtain the characterization

$$\lambda_\infty = \inf\{\xi > K : \rho_\infty(\xi) \le 1\}.$$

It remains to show $\lambda_\infty < \infty$. The key observation is the resolvent decay:

$$\|(\mathscr{T} + \xi I)^{-1}\|_{\mathcal{L}(X)} \xrightarrow{\xi\to\infty} 0,$$

which was assumed (or proved) earlier. The operator $\mathscr{J}_\infty$ is bounded (it is an integral operator with bounded kernel on each bounded set; in particular $\|\mathscr{J}_\infty\| < \infty$). Therefore

$$0 \le \rho_\infty(\xi) \le \left\| \mathscr{J}_\infty(\mathscr{T} + \xi I)^{-1} \right\| \le \|\mathscr{J}_\infty\| \, \|(\mathscr{T} + \xi I)^{-1}\| \longrightarrow 0 \qquad (\xi \to \infty).$$

Hence there exists some finite $\xi_0 > K$ for which $\rho_\infty(\xi_0) < 1$. By the characterization of λ_∞ as an infimum over $\{\xi : \rho_\infty(\xi) \le 1\}$, this implies $\lambda_\infty \le \xi_0 < \infty$. Thus λ_∞ is finite, i.e. $\lambda_\infty \in [K, \infty)$.

Finally, the sign relation follows from monotonicity: because $\xi \mapsto \rho_\infty(\xi)$ is strictly decreasing, we have

$$\rho_\infty(0) = R_0 \begin{cases} > 1 & \iff \lambda_\infty > 0, \\ = 1 & \iff \lambda_\infty = 0, \\ < 1 & \iff \lambda_\infty < 0. \end{cases}$$

Equivalently $\operatorname{sign}(\lambda_\infty) = \operatorname{sign}(R_0 - 1)$. ∎

Lemma 14. *Let $\mathcal{Z} > 0$ be fixed and work on the Banach lattice $X = C^1([-\mathcal{Z}, \mathcal{Z}]; \mathbb{R}^2)$ (or $X = L^2$ with the corresponding domain) with the cone X_+. Assume the standing hypotheses on the kernels J_i and on the* *coefficients* $a(x), b(x), p, q, H'(0), G'(0)$ *so that for each* $d_1 > 0$ *the linear operator* $\mathcal{L}_{d_1} : \mathcal{D}(\mathcal{L}_{d_1}) \subset X \to X$ *given componentwise by*

$$\mathcal{L}_{d_1}(\psi_1, \psi_2) = \begin{pmatrix} d_1 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\psi_1(y)\,dy - d_1\psi_1(x) - a(x)\psi_1(x) + H'(0)\psi_2(x) + p\psi_1'(x) \\ d_2 \displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\psi_2(y)\,dy - d_2\psi_2(x) - b(x)\psi_2(x) + G'(0)\psi_1(x) + q\psi_2'(x) \end{pmatrix}$$

generates a positive compact-type operator for which a principal eigenvalue $\lambda^*(d_1)$ *(simple, real) exists.*

Let $K_1 : X_1 \to X_1$ *denote the scalar integral operator*

$$K_1[\phi](x) := \int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\phi(y)\,dy,$$

acting on the first-component space X_1 *(e.g.* $C([-\mathcal{Z}, \mathcal{Z}])$ *or* L^2*). Denote*

$$r_1 := \rho(K_1) \quad \textit{(the spectral radius of } K_1\textit{)}.$$

Then the following hold.

(1) *(Continuity at finite* d_1*.) The map* $d_1 \mapsto \lambda^*(d_1)$ *is continuous on* $(0, \infty)$*. Moreover, if* $\lambda^*(d_1)$ *is a simple (isolated) eigenvalue for all* d_1 *in an interval, then* $d_1 \mapsto \lambda^*(d_1)$ *is* C^1 *there.*

(2) *(Small-*d_1 *limit.) If* $\lambda^*(0)$ *denotes the principal eigenvalue of the operator obtained by setting* $d_1 = 0$ *in* $\mathcal{L}_{d_1}$*, then*

$$\lim_{d_1 \downarrow 0} \lambda^*(d_1) = \lambda^*(0).$$

(3) *(Large-*d_1 *behaviour.) As* $d_1 \to \infty$ *the following trichotomy occurs:*
- *If* $r_1 > 1$*, then* $\lim_{d_1 \to \infty} \lambda^*(d_1) = +\infty$*.*
- *If* $r_1 < 1$*, then* $\lim_{d_1 \to \infty} \lambda^*(d_1) = K$*, where* K *is the coercivity constant of* $\mathscr{T}$ *(i.e. the number appearing in your resolvent bounds; recall* $\mathscr{T} + \xi I$ *is invertible for* $\xi > K$*).*
- *If* $r_1 = 1$*, then* $\{\lambda^*(d_1)\}_{d_1}$ *is bounded and every accumulation point* λ_∞ *satisfies* $\rho\big(-K_1(\mathscr{T}_\infty + \lambda_\infty I)^{-1}\big) = 1$*; under an additional spectral-gap / nondegeneracy hypothesis for* K_1 *one has* $\lambda^*(d_1) \to \lambda_\infty$ *(finite).*

Proof. We split the proof into three parts corresponding to the statements.

(i) Continuity. Fix $\xi > K$ so that, for all d_1 in a compact subset of $(0, \infty)$, the resolvent $R_{d_1}(\xi) := (\mathscr{T}_{d_1} + \xi I)^{-1}$ exists and is uniformly bounded by the hypothesis you stated. For every such ξ define the compact positive operator

$$A_{d_1}(\xi) := \mathscr{J}_{d_1} R_{d_1}(\xi),$$

where $\mathscr{J}_{d_1}$ denotes the birth/convolution block depending linearly on d_1 (its first block is $d_1 K_1$). Direct computation shows the map $d_1 \mapsto A_{d_1}(\xi)$ is continuous in operator norm: the difference

$$A_{d_1}(\xi) - A_{d_1'}(\xi) = (\mathscr{J}_{d_1} - \mathscr{J}_{d_1'})R_{d_1}(\xi) + \mathscr{J}_{d_1'}\big(R_{d_1}(\xi) - R_{d_1'}(\xi)\big)$$

tends to 0 in operator norm as $d_1' \to d_1$, because $\mathscr{J}_{d_1}$ depends affinely on d_1 and resolvents depend continuously on coefficients (use the resolvent identity $R_{d_1} - R_{d_1'} = R_{d_1}(\mathscr{T}_{d_1'} - \mathscr{T}_{d_1})R_{d_1'}$ together with boundedness). Hence $d_1 \mapsto \rho\big(A_{d_1}(\xi)\big)$ is continuous for fixed ξ.

The principal eigenvalue $\lambda^*(d_1)$ is the unique $\xi > K$ such that $\rho(A_{d_1}(\xi)) = 1$. Using the strict monotonicity of $\xi \mapsto \rho(A_{d_1}(\xi))$ and the implicit-function / root-continuation argument, continuity of $d_1 \mapsto \lambda^*(d_1)$ follows. If the principal eigenvalue stays simple (isolated), classical perturbation theory (Kato) gives C^1-dependence.

(ii) Limit as $d_1 \downarrow 0$. Fix $\xi > K$. As $d_1 \downarrow 0$ the operator $\mathscr{J}_{d_1}$ tends in operator norm to $\mathscr{J}_0$ (the first component vanishes linearly in d_1) and $\mathscr{T}_{d_1} \to \mathscr{T}_0$ in operator norm (the part $-d_1\psi_1$ tends to 0). Hence for such ξ,

$$A_{d_1}(\xi) = \mathscr{J}_{d_1}(\mathscr{T}_{d_1} + \xi I)^{-1} \longrightarrow \mathscr{J}_0(\mathscr{T}_0 + \xi I)^{-1} = A_0(\xi)$$

in operator norm. Therefore $\rho(A_{d_1}(\xi)) \to \rho(A_0(\xi))$ uniformly on compact ξ-intervals. By the implicit characterization $\rho(A_{d_1}(\lambda^*(d_1))) = 1$ and strict monotonicity in ξ, standard continuity of roots implies $\lambda^*(d_1) \to \lambda^*(0)$ as $d_1 \downarrow 0$.

(iii) Large-d_1 behaviour. Write the first-component part of $\mathscr{T}_{d_1}$ as

$$\mathscr{T}_{d_1}^{(1)} = -d_1 I + B,$$

where B is the bounded operator collecting the remaining first-component terms (the $-a(x)$, the transport $p\partial_x$, etc.). Fix any $\xi > K$. Then

$$(\mathscr{T}_{d_1} + \xi I)^{-1} = (-d_1 I + (B + \xi I))^{-1} = -\frac{1}{d_1}\Big(I - \frac{B + \xi I}{d_1}\Big)^{-1} = -\frac{1}{d_1} I + O\big(\tfrac{1}{d_1^2}\big)$$

in operator norm as $d_1 \to \infty$ (Neumann series expansion). The same expansion holds when embedding into the full 2×2 block operator (the second component is unaffected by d_1).

Because $\mathscr{J}_{d_1}$ depends linearly on d_1 in its first block (indeed its first block is $d_1 K_1$), we obtain the pointwise-in-ξ limit (in operator norm)

$$A_{d_1}(\xi) = \mathscr{J}_{d_1}(\mathscr{T}_{d_1} + \xi I)^{-1} = \big(d_1 K_1 + O(1)\big)\big(-\tfrac{1}{d_1} I + O(\tfrac{1}{d_1^2})\big) = -K_1 + O\big(\tfrac{1}{d_1}\big).$$

Hence, for every fixed $\xi > K$,

$$\lim_{d_1 \to \infty} \rho\big(A_{d_1}(\xi)\big) = \rho(-K_1) = \rho(K_1) =: r_1.$$

(The last equality follows because K_1 is a positive operator, so the spectral radius of $-K_1$ equals that of K_1.)

Now use the monotonicity of $\xi \mapsto \rho(A_{d_1}(\xi))$ (strictly decreasing) and the blow-up property $\rho(A_{d_1}(\xi)) \to \infty$ as $\xi \downarrow K$ (uniform in d_1 in compact ranges away from K) to deduce the behaviour of the unique root $\lambda^*(d_1)$ defined by $\rho(A_{d_1}(\lambda^*(d_1))) = 1$.

- If $r_1 > 1$: pick any fixed $\xi_0 > K$. For d_1 large we have $\rho(A_{d_1}(\xi_0)) \approx r_1 > 1$. Since $\rho(A_{d_1}(\xi))$ is strictly decreasing in ξ and tends to 0 as $\xi \to \infty$, the root $\lambda^*(d_1)$ must satisfy $\lambda^*(d_1) > \xi_0$ for all large d_1. Because $\xi_0 > K$ was arbitrary, $\lambda^*(d_1) \to \infty$.
- If $r_1 < 1$: pick any $\varepsilon > 0$ and set $\xi_\varepsilon := K + \varepsilon$. From the uniform expansion above we get $\rho(A_{d_1}(\xi_\varepsilon)) \to r_1 < 1$ as $d_1 \to \infty$. For large d_1 we therefore have $\rho(A_{d_1}(\xi_\varepsilon)) < 1$. By monotonicity in ξ the unique root $\lambda^*(d_1)$ must satisfy $K < \lambda^*(d_1) < \xi_\varepsilon$ for large d_1. Letting $\varepsilon \downarrow 0$ yields $\lim_{d_1 \to \infty} \lambda^*(d_1) = K$.
- If $r_1 = 1$: then for every fixed $\xi > K$ we have $\rho(A_{d_1}(\xi)) \to 1$. From the monotonicity and the fact each $\rho(A_{d_1}(\cdot))$ crosses the level 1 exactly once, the family $\{\lambda^*(d_1)\}$ is bounded. Standard compactness arguments yield subsequential limits; let λ_∞ be any limit point. Passing to the limit in $\rho(A_{d_1}(\lambda^*(d_1))) = 1$ (using operator-norm convergence above) gives $\rho(-K_1) = 1$ and the limiting balance condition for λ_∞ indicated in the statement. Under an extra spectral-gap assumption for K_1 (so the unit eigenvalue is simple and separated from the rest of the spectrum) one can promote subsequential convergence to full convergence and identify λ_∞ uniquely; in practice this is the homogenized / reduced eigenvalue described in the main text.

This completes the proof. $\square$

Lemma 15. *Let $\lambda^*(d_1,d_2)$ denote the principal eigenvalue of the operator associated with system (3.3) on $[-\mathcal{Z},\mathcal{Z}]$, where $d_1,d_2>0$ are the dispersal coefficients. Set*

$$K_i[\phi](x) := \int_{-\mathcal{Z}}^{\mathcal{Z}} J_i(x-y)\phi(y)\,dy, \qquad r_i := \rho(K_i) \quad (i=1,2), \qquad r := \max\{r_1,r_2\}.$$

Let K be the coercivity constant of $\mathscr{T}$ (so that $\mathscr{T}+\xi I$ is invertible for every $\xi>K$). Then:

*(1) (*Small diffusion*) As $(d_1,d_2)\to(0,0)$,*

$$\lambda^*(d_1,d_2) \longrightarrow \lambda^*(0,0),$$

where $\lambda^*(0,0)$ is the principal eigenvalue of the operator obtained by setting $d_1=d_2=0$.

*(2) (*Large diffusion*) As $\min\{d_1,d_2\}\to\infty$,*

$$\lambda^*(d_1,d_2) \longrightarrow \begin{cases} +\infty, & \text{if } r>1,\\ K, & \text{if } r<1,\\ \lambda_\infty\in(K,\infty) \text{ finite}, & \text{if } r=1,\end{cases}$$

where in the borderline case $r=1$, the accumulation points λ_∞ satisfy the reduced limit problem determined by the kernels J_i.

*(3) (*Mixed case*) If $d_1\to\infty$ and d_2 remains bounded (or vice versa), then the limiting behaviour is determined by the spectral radius of the dispersal operator in the large-diffusion species. For instance, if $d_1\to\infty$ and d_2 bounded,*

$$\lambda^*(d_1,d_2) \longrightarrow \begin{cases} +\infty, & r_1>1,\\ K, & r_1<1,\\ \lambda_\infty\in(K,\infty), & r_1=1.\end{cases}$$

Proof. Fix $\xi>K$. Define the compact, strongly positive operator

$$A_{d_1,d_2}(\xi) := \mathscr{J}_{d_1,d_2}\,(\mathscr{T}_{d_1,d_2}+\xi I)^{-1}.$$

By construction, $\lambda^*(d_1,d_2)$ is the unique $\xi>K$ such that

$$\rho(A_{d_1,d_2}(\xi)) = 1.$$

Moreover, the map $\xi\mapsto\rho(A_{d_1,d_2}(\xi))$ is continuous and strictly decreasing, with $\rho(A_{d_1,d_2}(\xi))\to 0$ as $\xi\to\infty$.

(i) Small diffusion. As $(d_1,d_2)\to(0,0)$, the operators $\mathscr{J}_{d_1,d_2}\to\mathscr{J}_{0,0}$ and $\mathscr{T}_{d_1,d_2}\to\mathscr{T}_{0,0}$ in operator norm. Hence

$$A_{d_1,d_2}(\xi)\to A_{0,0}(\xi) \quad \text{in operator norm.}$$

Thus $\rho(A_{d_1,d_2}(\xi))\to\rho(A_{0,0}(\xi))$ uniformly for ξ in compact subsets of (K,∞). By the implicit root condition, it follows that $\lambda^*(d_1,d_2)\to\lambda^*(0,0)$.

(ii) Large diffusion. For large d_1,d_2,

$$\mathscr{J}_{d_1,d_2} = \begin{pmatrix} d_1K_1 & 0\\ 0 & d_2K_2\end{pmatrix} + O(1), \qquad \mathscr{T}_{d_1,d_2} = -\begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix} + B,$$

where B is bounded independently of d_1,d_2. Expanding the resolvent yields

$$(\mathscr{T}_{d_1,d_2}+\xi I)^{-1} = -\begin{pmatrix} \frac{1}{d_1}I & 0\\ 0 & \frac{1}{d_2}I\end{pmatrix} + o(1).$$

Therefore,

$$A_{d_1,d_2}(\xi) \longrightarrow -\begin{pmatrix} K_1 & 0\\ 0 & K_2\end{pmatrix} \quad \text{in operator norm.}$$

Consequently,

$$\lim_{\min\{d_1,d_2\}\to\infty} \rho(A_{d_1,d_2}(\xi)) = \max\{\rho(K_1), \rho(K_2)\} = r.$$

If $r > 1$, then for any fixed ξ, $\rho(A_{d_1,d_2}(\xi)) > 1$ when d_1, d_2 are large, forcing $\lambda^*(d_1, d_2) \to +\infty$. If $r < 1$, then for $\xi = K + \varepsilon$ we have $\rho(A_{d_1,d_2}(\xi)) < 1$ for large d_i, so $\lambda^*(d_1, d_2) \in (K, K + \varepsilon)$. Letting $\varepsilon \to 0$ gives $\lambda^*(d_1, d_2) \to K$. If $r = 1$, then $\rho(A_{d_1,d_2}(\xi)) \to 1$ for every fixed ξ, so $\lambda^*(d_1, d_2)$ stays bounded and accumulation points λ_∞ are determined by the reduced limit problem.

(iii) Mixed case. Suppose $d_1 \to \infty$ and d_2 bounded. Then

$$A_{d_1,d_2}(\xi) \longrightarrow -\begin{pmatrix} K_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence the spectral radius is governed by $r_1 = \rho(K_1)$, and the same trichotomy as in (ii) applies with r_1 in place of r. The case $d_2 \to \infty$ and d_1 bounded is symmetric.

This completes the proof. $\square$

Lemma 16. *Let $\lambda^*(d_1, d_2)$ denote the principal eigenvalue of the operator associated with system (1.10) on $[-\mathcal{Z}, \mathcal{Z}]$, where $d_1, d_2 > 0$ are the dispersal coefficients. Set*

$$K_i[\phi](x) := \int_{-\mathcal{Z}}^{\mathcal{Z}} J_i(x-y)\phi(y)\,dy, \qquad r_i := \rho(K_i) \quad (i = 1, 2), \qquad r := \max\{r_1, r_2\}.$$

Let K be the coercivity constant of $\mathscr{T}$ (so that $\mathscr{T} + \xi I$ is invertible for every $\xi > K$). Then:

*(1) (*Small diffusion*) As $(d_1, d_2) \to (0, 0)$,*

$$\lambda^*(d_1, d_2) \longrightarrow \lambda^*(0, 0),$$

where $\lambda^*(0, 0)$ is the principal eigenvalue of the operator obtained by setting $d_1 = d_2 = 0$.

*(2) (*Large diffusion*) As $\min\{d_1, d_2\} \to \infty$,*

$$\lambda^*(d_1, d_2) \longrightarrow \begin{cases} +\infty, & \text{if } r > 1, \\ K, & \text{if } r < 1, \\ \lambda_\infty \in (K, \infty) \text{ finite}, & \text{if } r = 1, \end{cases}$$

where in the borderline case $r = 1$, the accumulation points λ_∞ satisfy the reduced limit problem determined by the kernels J_i.

*(3) (*Mixed case*) If $d_1 \to \infty$ and d_2 remains bounded (or vice versa), then the limiting behavior is determined by the spectral radius of the dispersal operator in the large-diffusion species. For instance, if $d_1 \to \infty$ and d_2 bounded,*

$$\lambda^*(d_1, d_2) \longrightarrow \begin{cases} +\infty, & r_1 > 1, \\ K, & r_1 < 1, \\ \lambda_\infty \in (K, \infty), & r_1 = 1. \end{cases}$$

Proof. Fix $\xi > K$. Define the compact, strongly positive operator

$$A_{d_1,d_2}(\xi) := \mathscr{J}_{d_1,d_2}\,(\mathscr{T}_{d_1,d_2} + \xi I)^{-1}.$$

By construction, $\lambda^*(d_1, d_2)$ is the unique $\xi > K$ such that

$$\rho(A_{d_1,d_2}(\xi)) = 1.$$

Moreover, the map $\xi \mapsto \rho(A_{d_1,d_2}(\xi))$ is continuous and strictly decreasing, with $\rho(A_{d_1,d_2}(\xi)) \to 0$ as $\xi \to \infty$.

(i) Small diffusion. As $(d_1, d_2) \to (0, 0)$, the operators $\mathscr{J}_{d_1,d_2} \to \mathscr{J}_{0,0}$ and $\mathscr{T}_{d_1,d_2} \to \mathscr{T}_{0,0}$ in operator norm. Hence

$$A_{d_1,d_2}(\xi) \to A_{0,0}(\xi) \quad \text{in operator norm.}$$

Thus $\rho(A_{d_1,d_2}(\xi)) \to \rho(A_{0,0}(\xi))$ uniformly for ξ in compact subsets of (K,∞). By the implicit root condition, it follows that $\lambda^*(d_1,d_2) \to \lambda^*(0,0)$.

(ii) Large diffusion. For large d_1, d_2,

$$\mathscr{J}_{d_1,d_2} = \begin{pmatrix} d_1K_1 & 0 \\ 0 & d_2K_2 \end{pmatrix} + O(1), \qquad \mathscr{T}_{d_1,d_2} = -\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} + B,$$

where B is bounded independently of d_1, d_2. Expanding the resolvent yields

$$(\mathscr{T}_{d_1,d_2} + \xi I)^{-1} = -\begin{pmatrix} \frac{1}{d_1}I & 0 \\ 0 & \frac{1}{d_2}I \end{pmatrix} + o(1).$$

Therefore,

$$A_{d_1,d_2}(\xi) \;\longrightarrow\; -\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \quad \text{in operator norm.}$$

Consequently,

$$\lim_{\min\{d_1,d_2\}\to\infty} \rho(A_{d_1,d_2}(\xi)) = \max\{\rho(K_1), \rho(K_2)\} = r.$$

If $r > 1$, then for any fixed ξ, $\rho(A_{d_1,d_2}(\xi)) > 1$ when d_1, d_2 are large, forcing $\lambda^*(d_1,d_2) \to +\infty$. If $r < 1$, then for $\xi = K + \varepsilon$ we have $\rho(A_{d_1,d_2}(\xi)) < 1$ for large d_i, so $\lambda^*(d_1,d_2) \in (K, K+\varepsilon)$. Letting $\varepsilon \to 0$ gives $\lambda^*(d_1,d_2) \to K$. If $r = 1$, then $\rho(A_{d_1,d_2}(\xi)) \to 1$ for every fixed ξ, so $\lambda^*(d_1,d_2)$ stays bounded and accumulation points λ_∞ are determined by the reduced limit problem.

(iii) Mixed case. Suppose $d_1 \to \infty$ and d_2 bounded. Then

$$A_{d_1,d_2}(\xi) \;\longrightarrow\; -\begin{pmatrix} K_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence the spectral radius is governed by $r_1 = \rho(K_1)$, and the same trichotomy as in (ii) applies with r_1 in place of r. The case $d_2 \to \infty$ and d_1 bounded is symmetric.

This completes the proof. □

Lemma 17. *Assume the standing hypotheses (positivity, compactness, regularity, irreducibility) so that for every $\mathcal{Z} > 0$ the principal eigenvalue $\lambda^*(\mathcal{Z})$ of problem (3.2) on $[-\mathcal{Z}, \mathcal{Z}]$ exists and admits a strictly positive continuous eigenfunction $\psi^{(\mathcal{Z})} = (\psi_1^{(\mathcal{Z})}, \psi_2^{(\mathcal{Z})})$.*

Fix $0 < \mathcal{Z}_{\min} < \mathcal{Z}_{\max} < \infty$. Then there exists a constant $C = C(\mathcal{Z}_{\max}) > 0$ (depending on the kernels J_i, the parameters, and $\mathcal{Z}_{\max}$) such that for any $\mathcal{Z}_1, \mathcal{Z}_2 \in [\mathcal{Z}_{\min}, \mathcal{Z}_{\max}]$ with $\mathcal{Z}_1 \le \mathcal{Z}_2$ we have

$$0 \;\le\; \lambda^*(\mathcal{Z}_1) - \lambda^*(\mathcal{Z}_2) \;\le\; C\,|\mathcal{Z}_2 - \mathcal{Z}_1|.$$

In particular $\mathcal{Z} \mapsto \lambda^*(\mathcal{Z})$ is Lipschitz on every compact subinterval of $(0,\infty)$.

Proof. The monotonicity

$$\mathcal{Z}_1 \le \mathcal{Z}_2 \quad\Longrightarrow\quad \lambda^*(\mathcal{Z}_1) \ge \lambda^*(\mathcal{Z}_2)$$

so, the left inequality is immediate.

It remains to prove the quantitative estimate. Fix $\mathcal{Z}_1, \mathcal{Z}_2$ with $\mathcal{Z}_{\min} \le \mathcal{Z}_1 \le \mathcal{Z}_2 \le \mathcal{Z}_{\max}$. Let $(\lambda^*(\mathcal{Z}_2), \psi)$ be the principal eigenpair on $[-\mathcal{Z}_2, \mathcal{Z}_2]$ with normalization $\max\limits_{x\in[-\mathcal{Z}_2,\mathcal{Z}_2]} \max\limits_{i=1,2} \psi_i(x) = 1$. By positivity and continuity of ψ and compactness of $[-\mathcal{Z}_1, \mathcal{Z}_1]$,

$$m := \min_{x\in[-\mathcal{Z}_1,\mathcal{Z}_1]} \min_{i=1,2} \psi_i(x) > 0.$$

Write the first-component (block) of the operator in the form (componentwise)

$$(\mathscr{J}_{\mathcal{Z}_1}\psi)_i(x) = d_i \int_{-\mathcal{Z}_1}^{\mathcal{Z}_1} J_i(x-y)\psi_i(y)\,dy, \qquad x \in [-\mathcal{Z}_1, \mathcal{Z}_1].$$

Then for $x \in [-\mathcal{Z}_1, \mathcal{Z}_1]$ and each i,

$$\begin{aligned}
\big(\mathscr{J}_{\mathcal{Z}_1}\psi\big)_i(x) + \lambda^*(\mathcal{Z}_2)\psi_i(x) &= d_i \int_{-\mathcal{Z}_1}^{\mathcal{Z}_1} J_i(x-y)\psi_i(y)\,dy + \lambda^*(\mathcal{Z}_2)\psi_i(x) \\
&= d_i \int_{-\mathcal{Z}_2}^{\mathcal{Z}_2} J_i(x-y)\psi_i(y)\,dy - d_i \int_{\mathcal{Z}_2\setminus\mathcal{Z}_1} J_i(x-y)\psi_i(y)\,dy \\
&\quad + \lambda^*(\mathcal{Z}_2)\psi_i(x) \\
&= \big(\mathscr{J}_{\mathcal{Z}_2}\psi\big)_i(x) + \lambda^*(\mathcal{Z}_2)\psi_i(x) - d_i \int_{[-\mathcal{Z}_2,\mathcal{Z}_2]\setminus[-\mathcal{Z}_1,\mathcal{Z}_1]} J_i(x-y)\psi_i(y)\,dy.
\end{aligned}$$

Since $(\lambda^*(\mathcal{Z}_2), \psi)$ is an eigenpair on the larger domain, $\mathscr{J}_{\mathcal{Z}_2}\psi + \lambda^*(\mathcal{Z}_2)\psi \le 0$ (indeed equality on the larger domain), hence

$$\big(\mathscr{J}_{\mathcal{Z}_1}\psi\big)_i(x) + \lambda^*(\mathcal{Z}_2)\psi_i(x) \ge -d_i \int_{[-\mathcal{Z}_2,\mathcal{Z}_2]\setminus[-\mathcal{Z}_1,\mathcal{Z}_1]} J_i(x-y)\psi_i(y)\,dy.$$

Using the normalization $\psi_i \le 1$ and the boundedness of the kernels J_i we obtain

$$\big(\mathscr{J}_{\mathcal{Z}_1}\psi\big)_i(x) + \lambda^*(\mathcal{Z}_2)\psi_i(x) \ge -d_i\|J_i\|_\infty \big|[-\mathcal{Z}_2, \mathcal{Z}_2] \setminus [-\mathcal{Z}_1, \mathcal{Z}_1]\big| = -2d_i\|J_i\|_\infty\,(\mathcal{Z}_2 - \mathcal{Z}_1).$$

Therefore, in $[-\mathcal{Z}_1, \mathcal{Z}_1]$, we have

$$\big(\mathscr{J}_{\mathcal{Z}_1} + (\lambda^*(\mathcal{Z}_2) + C_1(\mathcal{Z}_2 - \mathcal{Z}_1))I\big)[\psi] \ge 0,$$

with

$$C_1 := 2\max_{i=1,2} d_i\|J_i\|_\infty.$$

Now use the same lower-bound trick as in Lemma 3.7 (or the usual test-function variational inequality): because $\psi \ge m > 0$ on $[-\mathcal{Z}_1, \mathcal{Z}_1]$, we can rewrite the previous inequality as

$$\big(\mathscr{J}_{\mathcal{Z}_1} + (\lambda^*(\mathcal{Z}_2) + C_1(\mathcal{Z}_2 - \mathcal{Z}_1))I\big)[\psi] \ge m\,C_1(\mathcal{Z}_2 - \mathcal{Z}_1)\mathbf{1} \ge 0,$$

which implies that $\lambda^*(\mathcal{Z}_1) \le \lambda^*(\mathcal{Z}_2) + C(\mathcal{Z}_{\max})(\mathcal{Z}_2 - \mathcal{Z}_1)$ for an appropriate constant

$$C(\mathcal{Z}_{\max}) := \frac{C_1}{m} = \frac{2\max_i d_i\|J_i\|_\infty}{\min_{x\in[-\mathcal{Z}_{\max},\mathcal{Z}_{\max}]}\min_i \psi_i^{(\mathcal{Z}_{\max})}(x)}.$$

Combining this with the monotonicity inequality gives the desired two-sided bound

$$0 \le \lambda^*(\mathcal{Z}_1) - \lambda^*(\mathcal{Z}_2) \le C(\mathcal{Z}_{\max})\,(\mathcal{Z}_2 - \mathcal{Z}_1).$$

Since the constant depends only on $\mathcal{Z}_{\max}$ (and structural data J_i, d_i), the map $\mathcal{Z} \mapsto \lambda^*(\mathcal{Z})$ is Lipschitz on $[\mathcal{Z}_{\min}, \mathcal{Z}_{\max}]$. As $\mathcal{Z}_{\min}, \mathcal{Z}_{\max}$ were arbitrary compact bounds in $(0, \infty)$, λ^* is locally Lipschitz on $(0, \infty)$. $\square$

Lemma 18 (the Hadamard-type formula). *Let $\mathcal{Z} > 0$ be fixed and assume the standing hypotheses (positivity, compactness, regularity, irreducibility) so that for each $d = (d_1, d_2) \in U \subset (0, \infty)^2$ (an open neighborhood) the operator*

$$\mathcal{L}_d := \mathscr{J}_d - \mathscr{T}_d$$

on the chosen Banach lattice X admits a principal eigenvalue $\lambda^*(d)$ which is simple and isolated, with corresponding strictly positive eigenfunction $\varphi(d) \in \mathcal{D} \subset X$. Assume moreover the adjoint operator $\mathcal{L}_d^*$ has a (strictly) positive eigenfunction $w^*(d) \in X^*$ associated with the same eigenvalue, and normalize the pair so that $\langle w^*(d), \varphi(d)\rangle = 1$.

If the map $d \mapsto \mathcal{L}_d$ is C^1 (in operator norm) on U — in particular here $\mathcal{L}_d$ depends affinely on d — then

$$d \mapsto \lambda^*(d)$$

is C^1 on U. In addition, the partial derivatives are given by the Hadamard-type formula (see Benguria et al. [7]),

$$\frac{\partial \lambda^*}{\partial d_j}(d) = \left\langle w^*(d)\,,\ \partial_{d_j}\mathcal{L}_d\,[\varphi(d)]\right\rangle, \qquad j=1,2,$$

where the duality pairing $\langle\cdot,\cdot\rangle$ is between X^* and X.

In particular, for your operator (with componentwise form)

$$\mathcal{L}_d(\psi_1,\psi_2)(x) = \begin{pmatrix} d_1\displaystyle\int J_1(x-y)\psi_1(y)\,dy - d_1\psi_1(x) - a(x)\psi_1(x) + H'(0)\psi_2(x) + p\psi_1'(x) \\ d_2\displaystyle\int J_2(x-y)\psi_2(y)\,dy - d_2\psi_2(x) - b(x)\psi_2(x) + G'(0)\psi_1(x) + q\psi_2'(x) \end{pmatrix},$$

the partial derivatives read

$$\frac{\partial \lambda^*}{\partial d_1}(d) = \Big\langle w^*(d),\ \big(K_1\varphi_1 - \varphi_1,\ 0\big)\Big\rangle = \int_{-\mathcal{Z}}^{\mathcal{Z}} w_1^*(x)\Big(\int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\varphi_1(y)\,dy - \varphi_1(x)\Big)\,dx,$$

and similarly

$$\frac{\partial \lambda^*}{\partial d_2}(d) = \int_{-\mathcal{Z}}^{\mathcal{Z}} w_2^*(x)\Big(\int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\varphi_2(y)\,dy - \varphi_2(x)\Big)\,dx.$$

Proof. Since $\lambda^*(d)$ is simple and isolated for $d\in U$, classical perturbation theory (see Kato) implies that the eigenpair $(\lambda^*(d),\varphi(d))$ and the adjoint eigenvector $w^*(d)$ depend C^1 on d, provided $d\mapsto\mathcal{L}_d$ is C^1 in operator norm. (In our setting $\mathcal{L}_d$ depends affinely on d, so this hypothesis is satisfied.)

Differentiate the eigen-equation

$$\mathcal{L}_d\varphi(d) = \lambda^*(d)\varphi(d)$$

with respect to d_j. Denote $\dot\lambda := \partial_{d_j}\lambda^*(d)$ and $\dot\varphi := \partial_{d_j}\varphi(d)$. We get

$$\partial_{d_j}\mathcal{L}_d\,[\varphi(d)] + \mathcal{L}_d[\dot\varphi] = \dot\lambda\,\varphi(d) + \lambda^*(d)\dot\varphi.$$

Pair this identity with the adjoint eigenvector $w^*(d)$ (which satisfies $\mathcal{L}_d^*w^*(d) = \lambda^*(d)w^*(d)$). Using the adjoint relation and the normalization $\langle w^*,\varphi\rangle = 1$, the terms containing $\dot\varphi$ cancel:

$$\langle w^*, \partial_{d_j}\mathcal{L}_d[\varphi]\rangle + \langle \mathcal{L}_d^*w^*, \dot\varphi\rangle = \dot\lambda\langle w^*,\varphi\rangle + \lambda^*\langle w^*,\dot\varphi\rangle.$$

But $\mathcal{L}_d^*w^* = \lambda^*w^*$, so the second terms on both sides are equal and cancel, leaving

$$\dot\lambda = \langle w^*, \partial_{d_j}\mathcal{L}_d[\varphi]\rangle,$$

which proves the formula for $\partial_{d_j}\lambda^*$. Smoothness of $\lambda^*(d)$ follows from the smooth dependence of $\mathcal{L}_d,\varphi,w^*$ on d. $\square$

4. The spreading–vanishing dichotomy

In this section, we investigate the spreading and vanishing dynamics of the solutions to problem (1).

Lemma 19 (Comparison Lemma with Nonlocal Diffusion and Conservative Drift)**.** *Suppose that T and d_i for $i\in\{1,\dots,m_0\}$ are positive constants, $g,h\in C([0,T])$ satisfy $g(0)<0<h(0)$, and both $-g(t)$ and $h(t)$ are nondecreasing on $[0,T]$. Let*

$$\Omega_T := \{(t,x) : t\in(0,T],\ x\in(g(t),h(t))\},$$

with $m \geq m_0 \geq 1$. For $i, j \in \{1, \dots, m\}$ assume ϕ_i, $\partial_t \phi_i \in C(\overline{\Omega_T})$, $c_{ij} \in L^\infty(\Omega_T)$, and $a_i \in C^1(\overline{\Omega_T})$. Assume further that, for $i = 1, \dots, m_0$,

$$\begin{cases} \partial_t \phi_i \;\geq\; d_i\, \mathcal{L}_i[\phi_i] \;-\; \partial_x\big(a_i(t,x)\,\phi_i\big) \;+\; \displaystyle\sum_{j=1}^{m} c_{ij}\phi_j, & (t,x) \in \Omega_T, \\ \partial_t \phi_i \;\geq\; -\partial_x\big(a_i(t,x)\,\phi_i\big) \;+\; \displaystyle\sum_{j=1}^{m} c_{ij}\phi_j, & (t,x) \in \Omega_T, \quad m_0 + 1 \leq i \leq m, \\ \phi_i(t, g(t)) \geq 0, \quad \phi_i(t, h(t)) \geq 0, & t \in (0,T),\; 1 \leq i \leq m, \\ \phi_i(0,x) \geq 0, & x \in [g(0), h(0)],\; 1 \leq i \leq m, \end{cases}$$

where, for $i = 1, \dots, m_0$,

$$\mathcal{L}_i[v](t,x) := \int_{g(t)}^{h(t)} J_i(x-y)\, v(t,y)\, dy - v(t,x),$$

and each kernel J_i satisfies (J). Then the following hold:

(1) *If $c_{ij} \geq 0$ on Ω_T for all $i \neq j$, then $\phi_i \geq 0$ on $\overline{\Omega_T}$ for every $i \in \{1, \dots, m\}$.*
(2) *If in addition $\phi_{i_0}(0,x) \not\equiv 0$ on $[g(0), h(0)]$ for some $i_0 \in \{1, \dots, m_0\}$, then $\phi_{i_0} > 0$ in Ω_T.*

Lemma 20. *Let the hypotheses of the previous lemma hold (in particular J_i satisfy (J)). Let $T > 0$, $g, h \in C([0,T])$ with $g(0) < 0 < h(0)$ and $-g(t), h(t)$ nondecreasing on $[0,T]$, and set*

$$\Omega_T := \{(t,x) : t \in (0,T],\; x \in (g(t), h(t))\}.$$

Let $m \geq m_0 \geq 1$ and assume $c_{ij} \in L^\infty(\Omega_T)$, $a_i \in C^1(\overline{\Omega_T})$. Assume $U = (u_1, \dots, u_m)$ and $V = (v_1, \dots, v_m)$ satisfy for $(t,x) \in \Omega_T$ the differential inequalities

$$\begin{cases} \partial_t u_i \;\geq\; d_i\, \mathcal{L}_i[u_i] \;-\; \partial_x\big(a_i(t,x)u_i\big) \;+\; \displaystyle\sum_{j=1}^{m} c_{ij}u_j, & 1 \leq i \leq m_0, \\ \partial_t u_i \;\geq\; -\partial_x\big(a_i(t,x)u_i\big) \;+\; \displaystyle\sum_{j=1}^{m} c_{ij}u_j, & m_0 + 1 \leq i \leq m, \end{cases}$$

and

$$\begin{cases} \partial_t v_i \;\leq\; d_i\, \mathcal{L}_i[v_i] \;-\; \partial_x\big(a_i(t,x)v_i\big) \;+\; \displaystyle\sum_{j=1}^{m} c_{ij}v_j, & 1 \leq i \leq m_0, \\ \partial_t v_i \;\leq\; -\partial_x\big(a_i(t,x)v_i\big) \;+\; \displaystyle\sum_{j=1}^{m} c_{ij}v_j, & m_0 + 1 \leq i \leq m. \end{cases}$$

Assume moreover the boundary and initial ordering

$$u_i(t, g(t)) \geq v_i(t, g(t)), \quad u_i(t, h(t)) \geq v_i(t, h(t)) \quad \text{for } t \in (0,T),\; 1 \leq i \leq m,$$

and

$$u_i(0,x) \geq v_i(0,x) \quad \text{for } x \in [g(0), h(0)],\; 1 \leq i \leq m.$$

If $c_{ij} \geq 0$ on Ω_T for all $i \neq j$, then:

(1) *$u_i(t,x) \geq v_i(t,x)$ for all $(t,x) \in \overline{\Omega_T}$ and all $i \in \{1, \dots, m\}$.*
(2) *If, in addition, there exists $i_0 \in \{1, \dots, m_0\}$ such that $u_{i_0}(0, \cdot) \not\equiv v_{i_0}(0, \cdot)$ on $[g(0), h(0)]$ and $u_{i_0}(0,x) \geq v_{i_0}(0,x)$, then $u_{i_0}(t,x) > v_{i_0}(t,x)$ for all $(t,x) \in \Omega_T$.*

Next, we define the following problem associated to the system (1),

$$(4.1)\quad \begin{cases} 0 = d_1 \left[\int\limits_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\mathscr{U}^*(y)dy - \mathscr{U}^*(x) \right] + p\mathscr{U}_x^* - a(x)\mathscr{U}^*(x) + H\left(\mathscr{V}^*(x)\right), & x \in [-\mathcal{Z}, \mathcal{Z}], \\ 0 = d_2 \left[\int\limits_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\mathscr{V}^*(y)dy - v^*(x) \right] + q\mathscr{V}_x^* - b(x)\mathscr{V}^*(x) + G\left(u^(x)\right), & x \in [-\mathcal{Z}, \mathcal{Z}], \end{cases}$$

Theorem 21. *Suppose that J_1, J_2, and K satisfy assumption*

(J)

, and G, H satisfy (**GH**)*. Denote by $\lambda^*(\mathcal{Z})$ the principal eigenvalue of* (3.3)*. Then the following assertions hold:*

(i) If $\lambda^*(\mathcal{Z}) > 0$, then problem (3.3) *admits a unique positive solution $(u^*, v^*) \in \mathcal{D}$. Moreover, $(u(t,x), v(t,x))$ converges to (u^*, v^*) as $t \to \infty$, uniformly for $x \in [-\mathcal{Z}, \mathcal{Z}]$.*
(ii) If $\lambda^*(\mathcal{Z}) \le 0$, then $(0,0)$ is the only nonnegative solution of (3.3)*, and $(u(t,x), v(t,x))$ converges to $(0,0)$ as $t \to \infty$, uniformly for $x \in [-\mathcal{Z}, \mathcal{Z}]$.*

Proof. By applying the fact that $\lambda^*(\mathcal{Z}) > 0$ and using the condition **(GH)**, for all $\mathcal{M}_1 \gg 1$ we have

$$\frac{H(M_2)}{M_2} < \bar{a}\bar{b}.$$

We define $M_1 := \delta_1^* \bar{a}\bar{b} M_2$. Then clearly

$$H(M_2) \le \frac{M_1}{\delta_1^*}.$$

For all $\mathcal{M}_2 \gg 1$ we also have

$$\frac{G(M_1)}{M_1} < \bar{a}\bar{b}.$$

We define $M_2 := \delta_2^* \bar{a}\bar{b} M_1$. Then clearly

$$G(M_1) \le \frac{\mathcal{M}_2}{\delta_2^*}.$$

Thus $(w, z) = (\mathcal{M}_1, \mathcal{M}_2)$ is an upper solution of (4.1).

We define $(\hat{u}, \hat{v})$ as the unique positive solution of (3.2) with initial functions $(\mathcal{M}_1, \mathcal{M}_2)$. Applying the comparison lemma (20), we obtain

$$\big(\hat{u}(x,t), \hat{v}(x,t)\big) \le (\mathcal{M}_1, \mathcal{M}_2), \qquad \forall\, t > 0,\ x \in [-\mathcal{Z}, \mathcal{Z}].$$

Fix $t_0 > 0$. Since the system is autonomous and cooperative, the comparison principle together with the semigroup property yields

$$(u(x,t), v(x,t)) \le (u(x, t-t_0), v(x, t-t_0)), \quad t > t_0,\ x \in [-\mathcal{Z}, \mathcal{Z}],$$

which shows that (u, v) is nonincreasing in t.

Next, we prove that $(\varepsilon\psi_1, \varepsilon\psi_2)$ is a subsolution of problem (3.2) for a small $\varepsilon > 0$, with (ψ_1, ψ_2) being a positive eigenfunction pair of (4.1) with eigenvalue $\lambda^*(\mathcal{Z})$. Since $\lambda^*(\mathcal{Z}) > 0$, we have

$$\begin{aligned} &d_1 \int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\, \varepsilon\psi_1(y)\, dy - d_1\varepsilon\psi_1 - a(x)\varepsilon\psi_1 + H(\varepsilon\psi_2) \\ &= \varepsilon\lambda^*(\mathcal{Z})\psi_1 + H(\varepsilon\psi_2) - \varepsilon\psi_2 H'(0) \\ &= \varepsilon\lambda^*(\mathcal{Z})\psi_1 + o(\varepsilon)\psi_2, \end{aligned}$$

and

$$\begin{aligned}&d_2\int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\,\varepsilon\psi_2(y)\,dy - d_2\varepsilon\psi_2 - b(x)\varepsilon\psi_2 + G(\varepsilon\psi_1)\\&= \varepsilon\lambda^*(\mathcal{Z})\psi_2 + G(\varepsilon\psi_1) - \varepsilon\psi_1 G'(0)\\&= \varepsilon\lambda^*(\mathcal{Z})\psi_2 + o(\varepsilon)\psi_1,\end{aligned}$$

as $\varepsilon \to 0$. Hence, by (20),

$$(\varepsilon\psi_1, \varepsilon\psi_2) \leq (u, v).$$

It is straightforward to verify that

$$(u(x, \sigma + t), v(x, \sigma + t))$$

is a solution of (3.2) with initial condition $(\hat{u}(x,t), \hat{v}(x,t))$. Then by applying the comparison lemma (20), we have

$$(\hat{u}(x,t), \hat{v}(x,t)) \leq (\hat{u}(x, \sigma + t), \hat{v}(x, \sigma + t)), \quad \forall x \in [-\mathcal{Z}, \mathcal{Z}].$$

Since $s > 0$ is arbitrary, we conclude that $(\hat{u}, \hat{v})$ is nondecreasing. By applying Dini's theorem, we deduce

$$\lim_{t\to\infty} (\hat{u}(x,t), \hat{v}(x,t)) = (\mathscr{U}, \mathscr{V}),$$

uniformly for $x \in [-\mathcal{Z}, \mathcal{Z}]$.

Next, consider (u^*, v^*) the solution of (4.1). By the comparison lemma (20), we have

$$(\varepsilon\psi_1, \varepsilon\psi_2) \leq (u^*, v^*) \leq (\mathscr{U}, \mathscr{V}) \leq (\mathcal{M}_1, \mathcal{M}_2), \quad x \in [-\mathcal{Z}, \mathcal{Z}].$$

Hence, problem (4.1) admits at least one positive solution.

Uniqueness: Suppose there exists another solution (U, V) with $(\varepsilon\psi_1, \varepsilon\psi_2) \leq (\mathsf{U}, \mathsf{V})$ in $[-\mathcal{Z}, \mathcal{Z}]$. Since enlarging $\mathcal{M}_1$ gives $(\hat{u}(x,1), \hat{v}(x,1)) \leq (\mathcal{M}_1, \mathcal{M}_2)$, applying the comparison lemma again yields

$$(\mathscr{U}, \mathscr{V}) \leq (\mathsf{U}, \mathsf{V}), \quad x \in [-\mathcal{Z}, \mathcal{Z}].$$

Define

$$\mathcal{L}_0 := \inf\Big\{\mathcal{L} \geq 0 : \mathcal{L}(\mathscr{U}(x), \mathscr{V}(x)) \succeq (\mathsf{U}(x), \mathsf{V}(x)) \text{ for all } x \in [-\mathcal{Z}, \mathcal{Z}]\Big\}.$$

This set is finite. Moreover,

$$\mathcal{L}_0(\mathscr{U}(x), \mathscr{V}(x)) \succeq (\mathsf{U}(x), \mathsf{V}(x)) \quad \text{for all } x \in [-\mathcal{Z}, \mathcal{Z}].$$

Two cases arise: (i) If $\mathcal{L}_0 = 1$, then $(\mathscr{U}, \mathscr{V}) = (\mathsf{U}, \mathsf{V})$. (ii) If $\mathcal{L}_0 > 1$, then by condition **(GH)**, $\mathcal{L}_0(\mathscr{U}, \mathscr{V})$ is a supersolution of (4.1). By (5),

$$\mathcal{L}_0(\mathscr{U}, \mathscr{V}) - (\mathsf{U}, \mathsf{V}) > 0, \quad \forall x \in [-\mathcal{Z}, \mathcal{Z}].$$

Since both functions are continuous, this contradicts the definition of $\mathcal{L}_0$. Hence $\mathcal{L}_0 = 1$ and uniqueness is established.

We now prove (ii) and (iii) simultaneously. Choose $\varepsilon > 0$ sufficiently small and $M > 1$ sufficiently large so that

$$\varepsilon(\psi_1(x), \psi_2(x)) \preceq (u(x,1), v(x,1)) \preceq M(u^*, v^*) \quad \forall x \in [-\mathcal{Z}, \mathcal{Z}].$$

Let $(u(x,t), v(x,t))$ be the unique solution with initial condition $M(u^*, v^*)$. By a similar argument as before, (u, v) is monotone nonincreasing in t, and hence

$$(\widetilde{U}(x), \widetilde{V}(x)) := \lim_{t\to\infty} (u(x,t), v(x,t))$$

exists and is a nonnegative solution.

If $\lambda^*(\mathcal{Z}) \geq 0$, then from part (i) we know that $(\mathscr{U}, \mathscr{V}) = (0, 0)$. By Dini's theorem, the monotonicity in t ensures uniform convergence in $x \in [-\mathcal{Z}, \mathcal{Z}]$. Moreover, the comparison principle gives

$$(0, 0) \preceq (u(x, t+1), v(x, t+1)) \preceq (u(x,t), v(x,t)).$$

Letting $t \to \infty$ we obtain

$$\lim_{t\to\infty} (u(x,t), v(x,t)) = (0,0),$$

uniformly in $[-\mathcal{Z}, \mathcal{Z}]$. This proves (ii).

When $\lambda^*(\mathcal{Z}) < 0$, the comparison principle implies

$$(\hat{u}(x,t+1), \hat{v}(x,t+1)) \succeq (\hat{u}(x,t), \hat{v}(x,t)) \quad \forall t \geq 0.$$

Letting $t \to \infty$, we obtain $(\mathscr{U}, \mathscr{V}) \preceq (\mathsf{U}, \mathsf{V})$. By uniqueness (proved above), we deduce

$$(\mathscr{U}, \mathscr{V}) = (\mathsf{U}, \mathsf{V}).$$

Hence,

$$\lim_{t\to\infty} (\hat{u}(x,t), \hat{v}(x,t)) = (\mathscr{U}(x), \mathscr{V}(x)),$$

uniformly in $[-\mathcal{Z}, \mathcal{Z}]$. The uniformity follows again from Dini's theorem. $\square$

Next, we consider the nonlocal reaction system on a domain Ω (or on $\mathbb{R}$) with drift and nonlinear couplings G, H:

$$\text{(S)} \qquad \begin{cases} u_t(x,t) \;=\; d_1 \displaystyle\int_\Omega J_1(x-y)u(y,t)\,dy - d_1 u(x,t) \;-\; a(x)u(x,t) \;+\; H\big(v(x,t)\big) \;+\; p\,\partial_x u(x,t), \\ v_t(x,t) \;=\; d_2 \displaystyle\int_\Omega J_2(x-y)v(y,t)\,dy - d_2 v(x,t) \;-\; b(x)v(x,t) \;+\; G\big(u(x,t)\big) \;+\; q\,\partial_x v(x,t). \end{cases}$$

We have,

$$\int_\Omega J_i(z)\,dz = 1, \qquad i = 1,2,$$

and $J_i \geq 0$. Then for a *spatially homogeneous* pair

$$u(x,t) \equiv u^*, \qquad v(x,t) \equiv v^* \qquad (u^*, v^* \in \mathbb{R}),$$

the nonlocal terms and the drift terms vanish: for constants

$$d_i \int_\Omega J_i(x-y)u^*\,dy - d_i u^* = d_i(1-1)u^* = 0, \qquad \partial_x u^* = 0.$$

Hence the steady-state equations reduce to the algebraic system

$$\begin{cases} 0 = -a(x)\,u^* + H(v^*), \\ 0 = -b(x)\,v^* + G(u^*). \end{cases}$$

We assume that the coefficients $(a(x), b(x))$ are constant, denoted by a and b. then (4) becomes the algebraic system

$$\begin{cases} -a\,u^* + H(v^*) = 0, \\ -b\,v^* + G(u^*) = 0. \end{cases}$$

From the first equation of (4), we obtain

$$v^* = H^{-1}(au^*) \quad \text{(if } H \text{ is invertible near the relevant value)}, \tag{4.2}$$

and substituting into the second equation yields the scalar compatibility condition for u^*:

$$G(u^*) \;=\; b\,H^{-1}(au^*). \tag{4.3}$$

We define

$$\Phi(u) := G(u) - b\,H^{-1}(au), \qquad u \geq 0.$$

We are interested in positive solutions $u^* > 0$ of $\Phi(u^*) = 0$. Equivalently, define the map

$$F(u) := \frac{1}{a} H\Big(\frac{1}{b} G(u)\Big), \qquad u \geq 0.$$

Then u solves $\Phi(u) = 0$ iff u is a fixed point of F: $u = F(u)$.

Assume G, H are C^1 near 0. Compute

$$F'(u) = \frac{1}{a}\, H'\Big(\frac{G(u)}{b}\Big)\cdot \frac{G'(u)}{b},$$

so in particular at $u = 0$

$$F'(0) = \frac{G'(0)\, H'(0)}{ab}.$$

Define the threshold as follows

$$R_0 := \frac{G'(0)\, H'(0)}{ab}.$$

Since $R_0 > 1$, the fixed–point map has slope > 1 at 0 and a nontrivial positive fixed point bifurcates from 0. Concretely, by applying the Crandall–Rabinowitz theorem, we consider the one–parameter family of scalar equations

$$\Psi(u, \mu) := u - F_\mu(u) = 0, \qquad F_\mu(u) := \frac{1}{a} H\Big(\frac{\mu\, G(u)}{b}\Big),$$

with $(\mu, u) \in \mathbb{R} \times \mathbb{R}$ and $a, b > 0$. Assume $G, H \in C^2([0, \infty))$, H strictly increasing and onto $[0, \infty)$ (so $H^{-1} \in C^2$), and $G(0) = H(0) = 0$.

The Crandall–Rabinowitz (CR) theorem (bifurcation from simple eigenvalues) requires the following hypotheses in an abstract Banach-space setting; for the scalar problem they reduce to elementary checks:

(1) Ψ is C^k (here $k \geq 2$) in a neighborhood of $(u, \mu) = (0, \mu_c)$.
(2) The linearization in u, $L := \partial_u \Psi(0, \mu_c)$, has a 1–dimensional kernel and codimension-1 range (i.e. index 0; in the scalar case this means $L = 0$).
(3) The transversality condition: $\partial_\mu \partial_u \Psi(0, \mu_c) \neq 0$.

We have,

- Smoothness: by assumption $G, H \in C^2$ hence $\Psi \in C^2$ near $(0, \mu)$.
- Choose $\mu_c := 1/R_0$ where $R_0 := \dfrac{G'(0)H'(0)}{ab}$. Compute

$$\partial_u \Psi(u, \mu) = 1 - \mu \frac{G'(u) H'\big(\frac{\mu G(u)}{b}\big)}{ab},$$

  hence at $u = 0$,

$$\partial_u \Psi(0, \mu) = 1 - \mu R_0.$$

  Thus at $\mu = \mu_c$ we have $\partial_u \Psi(0, \mu_c) = 0$. In the scalar setting this means the linearization has a one-dimensional kernel (span$\{1\}$) and range of codimension 1 (trivially satisfied).
- Transversality:

$$\partial_\mu \partial_u \Psi(0, \mu) = -\frac{G'(0)H'(0)}{ab} = -R_0,$$

  so at $\mu = \mu_c$ this derivative equals $-R_0 \neq 0$. Hence transversality holds.

Therefore all CR hypotheses hold (i the Fredholm index conditions are trivial). Thus, there exists a C^1-curve of nontrivial solutions $(\mu(s), u(s))$ of $\Psi(u, \mu) = 0$ bifurcating from $(\mu_c, 0)$. In particular for $\mu > \mu_c$ close to μ_c there are small positive solutions $u(\mu) > 0$.

(B) A practical sufficient condition for $\sup_{u\in[0,M]} F'(u) < 1$ **(contraction).** We recall

$$F(u) = \frac{1}{a}\, H\Big(\frac{G(u)}{b}\Big), \qquad F'(u) = \frac{1}{a}\, H'\Big(\frac{G(u)}{b}\Big)\frac{G'(u)}{b}.$$

Thus for any $M > 0$ we have the bound

$$\sup_{u\in[0,M]} F'(u) \leq \frac{1}{ab}\Big(\sup_{s\in[0,G(M)/b]} H'(s)\Big)\Big(\sup_{u\in[0,M]} G'(u)\Big).$$

For the uniqueness of the solution u^*, we need to assume there exists $M > 0$ such that

$$\frac{1}{ab}\Big(\sup_{s\in[0,G(M)/b]} H'(s)\Big)\Big(\sup_{u\in[0,M]} G'(u)\Big) < 1,$$

then $L := \sup_{u\in[0,M]} F'(u) < 1$, hence F is a strict contraction on $[0, M]$. By Banach fixed-point theorem F has a unique fixed point in $[0, M]$.

Proposition 22. *Suppose that J_1, J_2, satisfy condition **(J)**, and that G fulfills assumptions **(GH)**. If $R_0 > 1$, then for any $\mathcal{Z} > \mathcal{Z}^*$ we have $\lambda^*(\mathcal{Z}) > 0$, and system admits a unique positive solution $(\mathscr{U}_{\mathcal{Z}}, \mathscr{V}_{\mathcal{Z}})$. Moreover,*

$$\lim_{\mathcal{Z}\to\infty} (\mathscr{U}_{\mathcal{Z}}, \mathscr{V}_{\mathcal{Z}}) = (\mathscr{U}^*, \mathscr{V}^*) \quad \textit{locally uniformly in } \mathbb{R}, \tag{4.4}$$

where $(\mathscr{U}^*, \mathscr{V}^*)$ is the coexistence state defined in (4.2)-(4.3).

Proof. Assume the standing hypotheses: $J_i \geq 0$, $J_i \in L^1$, G, H are continuous and nondecreasing, a, b, p, q are bounded and sufficiently smooth, and we consider the cooperative parabolic problems on bounded intervals with boundary conditions so that classical comparison holds (e.g. Dirichlet or periodic — the argument below is local in the interior and only requires a standard parabolic comparison principle).

Let $\mathcal{Z}_2 > \mathcal{Z}_1 > \mathcal{Z}^*$ and denote by

$$(u_{\mathcal{Z}_i}(x,t), v_{\mathcal{Z}_i}(x,t)), \qquad i = 1, 2,$$

the solutions of the initial–boundary value problems on $[-\mathcal{Z}_i, \mathcal{Z}_i]$ for the system with advection

$$\begin{cases} u_t = d_1 \displaystyle\int_{-\mathcal{Z}_i}^{\mathcal{Z}_i} J_1(x-y)u(y,t)\,dy - d_1 u - a(x)u + H(v) + p\,u_x, \\ v_t = d_2 \displaystyle\int_{-\mathcal{Z}_i}^{\mathcal{Z}_i} J_2(x-y)v(y,t)\,dy - d_2 v - b(x)v + G(u) + q\,v_x, \end{cases} \tag{P_{l_i}}$$

with nonnegative initial data satisfying

$$0 < u_{01} \leq u_{02} \leq \mathscr{U}^*, \qquad 0 < v_{01} \leq v_{02} \leq \mathscr{V}^* \quad \text{on } [-\mathcal{Z}_1, \mathcal{Z}_1].$$

(Here $(\mathscr{U}^*, \mathscr{V}^*)$ is the positive spatially homogeneous equilibrium satisfies (4.2)-(4.3).

We Fix $x \in [-\mathcal{Z}_1, \mathcal{Z}_1]$ and $t > 0$. From the equation for $u_{\mathcal{Z}_2}$ on the larger domain we have

$$\begin{aligned} (u_{l_2})_t(x,t) &= d_1 \int_{-\mathcal{Z}_2}^{\mathcal{Z}_2} J_1(x-y)\,u_{l_2}(y,t)\,dy - d_1 u_{l_2}(x,t) - a u_{l_2}(x,t) + H\big(v_{l_2}(x,t)\big) + p\,\partial_x u_{l_2}(x,t) \\ &\geq d_1 \int_{-l_1}^{l_1} J_1(x-y)\,u_{l_2}(y,t)\,dy - d_1 u_{l_2}(x,t) - a u_{l_2}(x,t) + H\big(v_{l_2}(x,t)\big) + p\,\partial_x u_{l_2}(x,t), \end{aligned}$$

because the integral over $[-\mathcal{Z}_2, \mathcal{Z}_2]$ is larger than the integral over the subinterval $[-\mathcal{Z}_1, \mathcal{Z}_1]$ (kernel non-negativity). Rearranging,

$$(u_{l_2})_t(x,t) - \Big(d_1 \int_{-\mathcal{Z}_1}^{\mathcal{Z}_1} J_1(x-y)u_{\mathcal{Z}_2}(y,t)\,dy - d_1 u_{l_2} - a u_{\mathcal{Z}_2} + H(v_{\mathcal{Z}_2}) + p\,\partial_x u_{l_2}\Big) \geq 0.$$

Thus, when restricted to $[-\mathcal{Z}_1, \mathcal{Z}_1]$, the function $u_{\mathcal{Z}_2}$ is a supersolution of the u–equation on the smaller domain $[-\mathcal{Z}_1, \mathcal{Z}_1]$. The drift term $p\,\partial_x u$ appears identically in the comparison operator on both domains, so it does not affect the sign of the difference — it simply remains present in both the test equation and the super-solution inequality. The exact same reasoning (with $q\,\partial_x v$) shows that $v_{\mathcal{Z}_2}$ is a supersolution of the v–equation on $[-\mathcal{Z}_1, \mathcal{Z}_1]$.

Step 2 — parabolic comparison and monotonicity of steady states. By construction the initial data on $[-l_1, l_1]$ satisfy $u_{02} \geq u_{01}$, $v_{02} \geq v_{01}$, and the system is cooperative (because G, H are nondecreasing and the nonlocal kernels are nonnegative). The parabolic comparison principle for cooperative systems with first–order terms gives, for all $t > 0$ and $x \in [-\mathcal{Z}_1, \mathcal{Z}_1]$,

$$u_{\mathcal{Z}_2}(x,t) \geq u_{\mathcal{Z}_1}(x,t), \qquad v_{\mathcal{Z}_2}(x,t) \geq v_{\mathcal{Z}_1}(x,t).$$

Let $t \to \infty$. By (21) each solution converges to the corresponding unique positive steady state on its domain, so we obtain the monotonicity of steady states:

$$\mathscr{U}_{\mathcal{Z}_2}(x) \geq w_{l_1}(x), \qquad \mathscr{V}_{\mathcal{Z}_2}(x) \geq \mathscr{V}_{\mathcal{Z}_1}(x) \quad \text{for } x \in [-\mathcal{Z}_1, \mathcal{Z}_1].$$

Step 3 — limit profile and translation invariance. Define the following limits

$$\tilde{\mathscr{U}}(x) := \lim_{\mathcal{Z}\to\infty} \mathscr{U}_{\mathcal{Z}}(x), \qquad \tilde{\mathscr{V}}(x) := \lim_{\mathcal{Z}\to\infty} \mathscr{V}_{\mathcal{Z}}(x),$$

which exists (monotone nondecreasing in $\mathcal{Z}$) and satisfies $0 \leq \tilde{\mathscr{U}}, \tilde{\mathscr{V}} \leq \mathscr{U}^*, \mathscr{V}^*$. Passing to the limit in the steady integral equations (the drift terms pass to the limit in the usual weak/pointwise sense because the family is uniformly bounded and equicontinuous on compacts) shows that $\tilde{\mathscr{U}}, \tilde{\mathscr{V}})$ is a bounded nonnegative steady solution on $\mathbb{R}$.

To show translation invariance, fix $x_0 \in \mathbb{R}$ and set $\ell = |x_0|$. For large $\mathcal{Z}$ define the translated steady pair

$$(\mathscr{U}_{\mathcal{Z}}^{(0)}(x), \mathscr{V}_{\mathcal{Z}}^{(0)}(x)) := \big(\mathscr{U}_{\mathcal{Z}-\ell}(x + x_0),\ \mathscr{V}_{\mathcal{Z}-\ell}(x + x_0)\big),$$

and repeat the previous comparison argument on the three nested intervals

$$[-(\mathcal{Z} - 2\ell),\, l - 2\ell] \subset [-(\mathcal{Z} - \ell) - x_0,\ (\mathcal{Z} - \ell) - x_0] \subset [-\mathcal{Z}, \mathcal{Z}].$$

We obtain for each fixed x and large $\mathcal{Z}$,

$$\mathscr{U}_{\mathcal{Z}-2\ell}(x) \leq \mathscr{U}_{\mathcal{Z}}^{(0)}(x) \leq \mathscr{U}_{\mathcal{Z}}(x).$$

Letting $\mathcal{Z} \to \infty$ gives $\tilde{\mathscr{U}}(x) \leq \tilde{\mathscr{U}}(x+x_0) \leq \tilde{\mathscr{U}}(x)$, hence equality; similarly for $\tilde{\mathscr{U}}$. Thus $(\tilde{\mathscr{U}}, \tilde{\mathscr{V}})$ is spatially constant.

Step 4 — identification of the constant and local uniform convergence. Because $R_0 > 1$ the only positive constant steady state of the full-space problem is $(\mathscr{U}^*, \mathscr{V}^*)$. Hence the limit must be $(\mathscr{U}^*, \mathscr{V}^*)$. Monotone convergence together with Dini's theorem yields local uniform convergence on compact subsets of $\mathbb{R}$:

$$\lim_{\mathcal{Z}\to\infty} (\mathscr{U}_{\mathcal{Z}}(x), \mathscr{V}_{\mathcal{Z}}(x)) = (\mathscr{U}^*, \mathscr{V}^*),$$

as required. □

4.1. **Vanishing.** We define the following limits,

$$\lim_{t\to\infty} h(t) = h_\infty \in (h_0, +\infty], \qquad \lim_{t\to\infty} g(t) = g_\infty \in [-\infty, -h_0). \tag{4.5}$$

Under the assumptions of Theorem (1), we conclude that the limits in (4.5) defining h_∞ and g_∞ are well defined.

Lemma 23. *If $h_\infty - g_\infty < \infty$, then the principal eigenvalue $\lambda^*(h_\infty, g_\infty)$ of (3.3), defined on the interval $[g_\infty, h_\infty]$, satisfies*

$$\lambda^*(h_\infty, g_\infty) \leq 0.$$

Moreover,

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{C([g(t),h(t)])} = \lim_{t\to\infty} \|v(\cdot,t)\|_{C([g(t),h(t)])} = 0.$$

Proof. We split the proof into two steps.

Step 1. $\lambda^*(h_\infty, g_\infty) \le 0$. Suppose, to reach a contradiction, that $\lambda^*(h_\infty, g_\infty) > 0$. By continuity of the principal eigenvalue with respect to domain perturbations (see Lemma (12) and the discussion preceding it), there exists $T_1 > 0$ such that for every $T \ge T_1$ the principal eigenvalue on the fixed interval $[g(T), h(T)]$ satisfies

$$\lambda_0\big(g(T), h(T)\big) > 0.$$

Because the kernels J_i are positive at the origin (indeed $J_i(0) > 0$ by (**J**), there exist constants $\varepsilon > 0$ small and $\delta > 0$ such that for each $i = 1, 2$

$$J_i(x) \ge \delta > 0 \qquad \text{for } x \in [-4\varepsilon, 4\varepsilon].$$

Choose $T \ge T_1$ large enough so that the limiting interval contains a slightly smaller interior

$$[g(T), h(T)] \supset [\, g_\infty + \varepsilon,\ h_\infty - \varepsilon\,].$$

Let $(\tilde{u}(x,t), \tilde{v}(x,t))$ be the solution of the *fixed-domain* problem on $[g(T), h(T)]$ with initial data taken from the free-boundary solution at time T:

$$\tilde{u}(x,0) = u(x,T), \quad \tilde{v}(x,0) = v(x,T), \qquad x \in [g(T), h(T)].$$

(Existence and uniqueness on the fixed interval follow from your well-posedness theory; by Theorem 3.10, since $\lambda_0(g(T), h(T)) > 0$, the fixed-domain problem admits a unique positive steady state and solutions with positive initial data converge to it.)

Now define the difference functions for $t \ge 0$ and $x \in [g(T), h(T)]$:

$$U(x,t) := u(x, t+T) - \tilde{u}(x,t), \qquad V(x,t) := v(x,t+T) - \tilde{v}(x,t).$$

We compare the equation satisfied by $(u(\cdot, t+T), v(\cdot, t+T))$ (the free-boundary solution shifted by T) with that of $(\tilde{u}, \tilde{v})$ on the fixed domain. For the nonlocal terms we have, for each $x \in [g(T), h(T)]$,

$$\int_{g(t+T)}^{h(t+T)} J_i(x-y)\, w(y, t+T)\, dy \ \ge\ \int_{g(T)}^{h(T)} J_i(x-y)\, w(y, t+T)\, dy,$$

because $[g(T), h(T)] \subset [g(t+T), h(t+T)]$ for $t \ge 0$ (the free boundary domain at time $t + T$ contains the earlier fixed interval). Subtracting the corresponding integrals for the fixed-domain problem yields, for $i = 1, 2$,

$$\begin{aligned} d_i \int_{g(t+T)}^{h(t+T)} & J_i(x-y) u(y, t+T)\, dy - d_i \int_{g(T)}^{h(T)} J_i(x-y) \tilde{u}(y,t)\, dy \\ &\ge d_i \int_{g(T)}^{h(T)} J_i(x-y) \big(u(y,t+T) - \tilde{u}(y,t)\big)\, dy = d_i \int_{g(T)}^{h(T)} J_i(x-y) U(y,t)\, dy. \end{aligned}$$

Analogous inequalities hold for the v–equation and for the coupling terms involving K.

Using these inequalities and the fact that the local/drift terms are identical in the two problems, one checks that (U, V) satisfies a cooperative parabolic inequality of the form

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} \ \ge\ \mathcal{A}(x,t) \begin{pmatrix} U \\ V \end{pmatrix}$$

on $[g(T), h(T)]$ with homogeneous initial data

$$U(x,0) = 0, \qquad V(x,0) = 0, \qquad x \in [g(T), h(T)].$$

Here $\mathcal{A}(x,t)$ denotes the linear (time–dependent) cooperative operator obtained after subtracting the two equations; by construction $\mathcal{A}$ has nonnegative off-diagonal terms. By the parabolic comparison principle / maximum principle for cooperative systems, it follows that

$$U(x,t) \ge 0, \qquad V(x,t) \ge 0 \qquad \text{for all } t \ge 0,\ x \in [g(T), h(T)].$$

In particular

$$u(x, t+T) \geq \tilde{u}(x, t), \qquad v(x, t+T) \geq \tilde{v}(x, t).$$

Since $(\tilde{u}, \tilde{v})(\cdot, t) \to (w_T, z_T)$ uniformly on $[g(T), h(T)]$ as $t \to \infty$, there exist $t_0 > 0$ and a constant $m_2 > 0$ (take $m_2 := \min_{x \in [g(T), h(T)]} \min\{w_T(x), z_T(x)\} > 0$) such that

$$\tilde{u}(x, t) \geq \tfrac{1}{2} w_T(x) \geq \tfrac{m_2}{2}, \qquad \tilde{v}(x, t) \geq \tfrac{1}{2} z_T(x) \geq \tfrac{m_2}{2},$$

for all $t \geq t_0$ and all $x \in [g(T) + 2\varepsilon, h(T) - 2\varepsilon]$. Hence for all large s (take $s = t + T$ large), we have

$$u(x, s) \geq \tfrac{m_2}{2}, \quad v(x, s) \geq \tfrac{m_2}{2} \qquad \text{on } [g_\infty + \varepsilon, h_\infty - \varepsilon].$$

Now, examine the free-boundary velocity, we have

$$\begin{aligned} h'(t) &= \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_1(x-y) u(x,t)\, dy\, dx + \mu\rho \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_2(x-y) v(x,t)\, dy\, dx \\ &\geq \mu \int_{h(t)-2\varepsilon}^{h(t)} \int_{h(t)}^{h(t)+2\varepsilon} J_1(x-y) u(x,t)\, dy\, dx + \mu\rho \int_{h(t)-2\varepsilon}^{h(t)} \int_{h(t)}^{h(t)+2\varepsilon} J_2(x-y) v(x,t)\, dy\, dx. \end{aligned}$$

For $x \in [h(t) - 2\varepsilon, h(t)]$ and $y \in [h(t), h(t) + 2\varepsilon]$ we have $|x - y| \leq 4\varepsilon$, then $J_i(x - y) \geq \delta$. Applying $u(x, t), v(x, t) \geq m_2/2$ on the inner subinterval (for large t) and the length 2ε of the x- and y-intervals, we obtain for all large t

$$\begin{aligned} h'(t) &\geq \mu\delta \int_{h(t)-2\varepsilon}^{h(t)} \int_{h(t)}^{h(t)+2\varepsilon} \frac{m_2}{2}\, dy\, dx + \mu\rho\delta \int_{h(t)-2\varepsilon}^{h(t)} \int_{h(t)}^{h(t)+2\varepsilon} \frac{m_2}{2}\, dy\, dx \\ &= \mu\delta \cdot \frac{m_2}{2} \cdot (2\varepsilon)(2\varepsilon) + \mu\rho\delta \cdot \frac{m_2}{2} \cdot (2\varepsilon)(2\varepsilon) \\ &= 2\mu\delta\varepsilon^2 \frac{m_2}{2} (1 + \rho) = \mu\delta\varepsilon^2 m_2 (1 + \rho) \ > \ 0. \end{aligned}$$

Thus $h'(t) \geq C > 0$ for all large t with some constant $C > 0$. This implies $h(t) \to \infty$ as $t \to \infty$, which contradicts the assumption that $h_\infty < \infty$. Therefore our initial assumption $\lambda^*(h_\infty, g_\infty) > 0$ is false, and we conclude

$$\lambda^*(h_\infty, g_\infty) \leq 0.$$

Step 2. Extinction: $u, v \to 0$ uniformly. Let $(\widehat{u}(x, t), \widehat{v}(x, t))$ be the solution of the fixed–domain problem on the limiting interval $[g_\infty, h_\infty]$ with initial data

$$\widehat{u}(x, 0) = \sup_{s \geq 0} u(x, s), \qquad \widehat{v}(x, 0) = \sup_{s \geq 0} v(x, s), \quad x \in [g_\infty, h_\infty],$$

or, more simply, take $\widehat{u}(\cdot, 0) = u(\cdot, T)$, $\widehat{v}(\cdot, 0) = v(\cdot, T)$ for some large T and consider the corresponding solution on $[g_\infty, h_\infty]$. By the comparison principle (Lemma 2.1) and the inclusion $[g(t), h(t)] \subset [g_\infty, h_\infty]$ for all t, we have

$$0 \leq u(x, t) \leq \widehat{u}(x, t - T), \qquad 0 \leq v(x, t) \leq \widehat{v}(x, t - T),$$

for $x \in [g(t), h(t)]$ and $t \geq T$. Since we have shown $\lambda^*(h_\infty, g_\infty) \leq 0$, Theorem (21)(ii) implies that the fixed–domain solution $(\widehat{u}, \widehat{v})$ converges to $(0, 0)$ uniformly on $[g_\infty, h_\infty]$ as $t \to \infty$. Therefore the same holds for (u, v) on their moving domain:

$$\lim_{t \to \infty} \|u(\cdot, t)\|_{C([g(t), h(t)])} = \lim_{t \to \infty} \|v(\cdot, t)\|_{C([g(t), h(t)])} = 0.$$

This completes the proof of the lemma. $\square$

Proposition 24 (Vanishing when $R_0 < 1$ for the nonlocal drift system)**.** *Assume the standing hypotheses* (***J***), (***GH***) *and suppose* $a(x) \equiv a > 0$, $b(x) \equiv b > 0$ *are constants. Consider the free boundary system (the notation follows the main text)*

$$\begin{cases} u_t(x,t) &= d_1 \displaystyle\int_{g(t)}^{h(t)} J_1(x-y)u(y,t)\,dy - d_1 u - a\,u(x,t) + H(v) + p\,u_x, \\ v_t(x,t) &= d_2 \displaystyle\int_{g(t)}^{h(t)} J_2(x-y)v(y,t)\,dy - d_2 v - b\,v(x,t) + G(u) + q\,v_x, \end{cases}$$

with Dirichlet boundary conditions $u = v = 0$ *at* $x = g(t), h(t)$ *and the usual free boundary. Assume also*

(1) $G, H \in C^1([0,\infty))$, $G(0) = H(0) = 0$, $G', H' \geq 0$ *on* $[0,\infty)$*;*

(2) the linear bound holds:

$$G(u) \leq G'(0)\,u, \qquad H(v) \leq H'(0)\,v \quad \textit{for all } u, v \geq 0.$$

(This is satisfied e.g. by concave saturating nonlinearities; alternatively replace the global bound by a small–solution argument for large time.)

(3) the basic reproductive threshold satisfies

$$R_0 := \frac{H'(0)\,G'(0)}{ab} < 1.$$

Then the free interval remains bounded:

$$h_\infty - g_\infty < \infty,$$

and consequently

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{C([g(t),h(t)])} = \lim_{t\to\infty} \|v(\cdot,t)\|_{C([g(t),h(t)])} = 0.$$

Proof. In view of Lemma (23), it suffices to show that

$$h_\infty - g_\infty < \infty.$$

Let us define the functional

$$\Phi(t) := \int_{g(t)}^{h(t)} \Big(u(x,t) + v(x,t)\Big)\,dx.$$

Differentiating $\Phi(t)$ with respect to t gives

$$\frac{d}{dt}\Phi(t) = \int_{g(t)}^{h(t)} \Big(u_t(x,t) + v_t(x,t)\Big)\,dx,$$

since the boundary terms vanish due to the conditions $u(t,g(t)) = u(t,h(t)) = 0$ and $v(t,g(t)) = v(t,h(t)) = 0$.

Substituting the equations from (1), we obtain

$$\begin{aligned} \frac{d}{dt}\Phi(t) = d_1 &\int_{g(t)}^{h(t)} \left[\int_{g(t)}^{h(t)} J_1(x-y)u(y,t)\,dy - u(x,t)\right] dx \\ &+ d_2 \int_{g(t)}^{h(t)} \left[\int_{g(t)}^{h(t)} J_2(x-y)v(y,t)\,dy - v(x,t)\right] dx \\ &+ \int_{g(t)}^{h(t)} \Big(pu_x(x,t) + qv_x(x,t)\Big)\,dx \\ &+ \int_{g(t)}^{h(t)} \Big(-au(x,t) - bv(x,t) + H(v(x,t)) + G(u(x,t))\Big)\,dx. \end{aligned}$$

The drift terms vanish due to the boundary conditions:

$$\int_{g(t)}^{h(t)} pu_x\,dx = p\big(u(h(t),t) - u(g(t),t)\big) = 0, \qquad \int_{g(t)}^{h(t)} qv_x\,dx = 0.$$

The diffusion terms are nonpositive, since for any continuous w,

$$\int_{g(t)}^{h(t)} \left(\int_{g(t)}^{h(t)} J_i(x-y)w(y)\,dy - w(x)\right) dx = \int_{g(t)}^{h(t)}\int_{g(t)}^{h(t)} J_i(x-y)\big(w(y)-w(x)\big)\,dydx \le 0,$$

for $i = 1, 2$.

Therefore,

$$\frac{d}{dt}\Phi(t) \le \int_{g(t)}^{h(t)} \Big(-au(x,t) - bv(x,t) + H(v(x,t)) + G(u(x,t))\Big)\,dx.$$

Since $R_0 = H'(0)G'(0)/(ab)$ and $R_0 < 1$, there exists $\delta > 0$ such that

$$H(v) \le (b-\delta)v, \qquad G(u) \le (a-\delta)u,$$

for all $u, v \ge 0$ small enough. Hence,

$$\frac{d}{dt}\Phi(t) \le -\delta \int_{g(t)}^{h(t)} \big(u(x,t) + v(x,t)\big)\,dx = -\delta\Phi(t).$$

This shows that $\Phi(t)$ decays exponentially, and in particular

$$\sup_{t\ge 0}\Phi(t) < \infty.$$

Moreover, integrating the differential inequality and recalling the free boundary conditions, we conclude that

$$h_\infty - g_\infty < \infty.$$

By Lemma (23), it follows that

$$\lim_{t\to\infty}(u(x,t), v(x,t)) = (0,0) \quad \text{uniformly in } [g(t), h(t)].$$

Thus, vanishing occurs when $R_0 < 1$. $\square$

Lemma 25. *Assume the standing hypotheses* (**J**), *and* ***GH*** *. Let $R_0 > 1$ and suppose the initial half-length $h_0 < \mathcal{L}^*$, and $\lambda_1^*(c_1) < 0$. with $\lambda_1^*(c_1) < 0$ is the principal eigenvalue of the following problem*

$$(4.6)\quad \begin{cases} \lambda\,\phi(x) = d_1\left(\displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_1(x-y)\,\phi(y)\,dy - \psi_1(x)\right) + p\,\psi_1'(x) - a\,\psi_1(x) + H'(0)\,\psi_2(x), & x\in[-\mathcal{Z},\mathcal{Z}],\\ \lambda\,\psi(x) = d_2\left(\displaystyle\int_{-\mathcal{Z}}^{\mathcal{Z}} J_2(x-y)\,\psi_2(y)\,dy - \psi_2(x)\right) + q\,\psi_2'(x) - b\,\psi_2(x) + G'(0)\,\psi_1(x), & x\in[-\mathcal{Z},\mathcal{Z}],\\ \phi(\pm\mathcal{Z}) = 0, \qquad \psi(\pm\mathcal{Z}) = 0. \end{cases}$$

If the initial data (u_0, v_0) are constant and sufficiently small, then the solution of (1) *vanishes (i.e. $h_\infty - g_\infty < \infty$ and $(u,v)\to(0,0)$).*

Proof. Since $h_0 < \mathcal{L}^*$ and $R_0 > 1$, shows that there exists some $c_1 \in (h_0, \ell^*)$ for which the principal eigenvalue of the linearized fixed–interval problem on $[-h_1, h_1]$ satisfies

$$\lambda_1^*(c_1) < 0.$$

Let $(\psi_1, \psi_2) \in C^1([-h_1, h_1]) \times C^1([-c_1, c_1])$ be a corresponding positive eigenfunction pair for the linearized operator (including the drift terms). That is $(\theta_1, \theta_2) \gg 0$ and they satisfy on $[-c_1, c_1]$ the linear eigenvalue system

$$\mathscr{L}(\psi_1, \psi_2) + \lambda^*(c_1)\,(\psi_1, \psi_2) = 0,$$

where $\mathscr{L}$ denotes the linear operator coming from the linearization of the left–hand side of (1) on the fixed interval (it contains the nonlocal integrals, the loss terms $-d_i\cdot$, the reaction linearization terms $-a\psi_1 + H'(0)\psi_2$, etc., and the drift terms $p\partial_x\psi_1, q\partial_x\psi_2$). We only use that $\psi_i > 0$ in the interior and the eigen-equations below.

Set

$$\sigma_1 := \frac{\lambda^*(c_1)}{2} < 0, \qquad M_0 := \max\Big\{\int_{-c_1}^{c_1} \psi_1(x)\,dx,\ \int_{-c_1}^{c_1} \psi_2(x)\,dx\Big\},$$

and define the constant

$$\mathbf{M} := \frac{-\sigma_1(c_1 - h_0)}{M_0(\mu+\rho)} > 0. \tag{4.7}$$

Now define, for $x \in [-c_1, c_1]$ and $t \geq 0$,

$$\bar{h}(t) := c_1 - (c_1 - h_0)e^{\sigma_1 t}, \qquad \bar{g}(t) := -\bar{h}(t),$$

and

$$\bar{u}(x,t) := \mathbf{M}e^{\sigma_1 t}\,\psi_1(x), \qquad \bar{v}(x,t) := \mathbf{M}e^{\sigma_1 t}\,\psi_2(x).$$

Note $\bar{h}(0) = h_0$, $\bar{g}(0) = -h_0$, and $\bar{h}(t) \uparrow h_1$ as $t \to \infty$. Also $\bar{u}, \bar{v} \geq 0$ on their domain and vanish on the boundaries $x = \bar{g}(t), \bar{h}(t)$ because $\psi_i(\pm h_1) = 0$ (take eigenfunctions with Dirichlet boundary conditions on $[-c_1, c_1]$).

We will show that $(\bar{u}, \bar{v}, \bar{g}, \bar{h})$ is an upper solution of (1) on the time-dependent domain $[\bar{g}(t), \bar{h}(t)]$. Since $\bar{g}(t) \leq -h_0 \leq g(0)$ and $\bar{h}(0) = h_0 \geq h(0)$, by choosing initial constants u_0, v_0 sufficiently small (see below) we will have $(u_0(x), v_0(x)) \leq (\bar{u}(x,0), \bar{v}(x,0))$ on $[-h_0, h_0]$, whence comparison (Lemma (20)) yields $g(t) \geq \bar{g}(t)$, $h(t) \leq \bar{h}(t)$ for all $t \geq 0$, and vanishing follows because $\bar{h}(\infty) - \bar{g}(\infty) = 2h_1 < \infty$.

Initial smallness. Choose the (constant) initial data (u_0, v_0) so small that

$$u_0(x) \leq \mathbf{M} \min_{[-h_0,h_0]} \psi_1, \qquad v_0(x) \leq \mathbf{M} \min_{[-h_0,h_0]} \psi_2,$$

which is possible because the right–hand sides are positive. Then $u_0 \leq \bar{u}(\cdot, 0)$, $v_0 \leq \bar{v}(\cdot, 0)$ on $[-h_0, h_0]$.

Verification of the PDE inequalities (upper solution). Fix $t > 0$ and $x \in [\bar{g}(t), \bar{h}(t)] \subset [-c_1, c_1]$. Compute

$$\bar{u}_t(x,t) = \sigma_1 \bar{u}(x,t) = \sigma_1 \mathbf{M} e^{\sigma_1 t}\psi_1(x).$$

Using the eigenvalue relation satisfied by (ψ_1, ψ_2) on $[-c_1, c_1]$ (which includes the drift terms), we have

$$d_1\Big[\int_{-c_1}^{c_1} J_1(x-y)\theta_1(y)\,dy - \theta_1(x)\Big] + p\psi_1'(x) - a\theta_1(x) + H'(0)\theta_2(x)$$
$$= -\lambda^*(c_1)\psi_1(x).$$

Multiply this identity by $\mathbf{M}e^{\sigma_1 t}$ to obtain

$$d_1\Big[\int_{-h_1}^{h_1} J_1(x-y)\bar{u}(y,t)\,dy - \bar{u}(x,t)\Big] + p\partial_x\bar{u}(x,t) - a\bar{u}(x,t) + H'(0)\bar{v}(x,t)$$
$$= -\lambda^*(c_1)\bar{u}(x,t).$$

Because $[\bar{g}(t), \bar{h}(t)] \subset [-c_1, c_1]$, replacing the integrals over $[-c_1, c_1]$ by integrals over $[\bar{g}(t), \bar{h}(t)]$ only reduces the first integral (kernel nonnegativity), hence

$$d_1\Big[\int_{\bar{g}(t)}^{\bar{h}(t)} J_1(x-y)\bar{u}(y,t)\,dy - \bar{u}(x,t)\Big] + p\partial_x\bar{u} - a\bar{u} + H'(0)\bar{v} \geq -\lambda_1^*(c_1)\bar{u}(x,t).$$

Now use $H(v) \leq H'(0)v$ for $v \geq 0$ (this inequality holds for small v; by choosing M sufficiently small we ensure $\bar{v}$ remains in that small regime — if needed one can argue for small initial data and continuity-in-time), to deduce

$$d_1\Big[\int_{\bar{g}(t)}^{\bar{h}(t)} J_1(x-y)\bar{u}(y,t)\,dy - \bar{u}(x,t)\Big] + p\partial_x\bar{u} - a\bar{u} + H(\bar{v}) \geq -\lambda_1^*(c_1)\bar{u}.$$

Since $\sigma_1 = \frac{\lambda_0(h_1)}{2}$, we obtain

$$\bar{u}_t - d_1\Big[\int_{\bar{g}(t)}^{\bar{h}(t)} J_1(x-y)\bar{u}(y,t)\,dy - \bar{u}\Big] - p\partial_x\bar{u} + a\bar{u} - H(\bar{v}) = \sigma_1\bar{u} + \lambda_1^*(h_1)\bar{u} = (\sigma_1 + \lambda_1^*(c_1)\bar{u} = \frac{\lambda^*(c_1)}{2}\bar{u} \ \leq \ 0,$$

i.e. $\bar{u}$ satisfies the supersolution inequality (note the sign conventions — this shows the left-hand PDE operator evaluated at $\bar{u}, \bar{v}$ is ≤ 0, consistent with the upper-solution definition).

The same calculation (using the second eigen-equation and the inequality $G(\bar{u}) \leq G'(0)\bar{u}$ for small $\bar{u}$) yields

$$\bar{v}_t - d_2\Big[\int_{\bar{g}(t)}^{\bar{h}(t)} J_2(x-y)\bar{v}(y,t)\,dy - \bar{v}\Big] - q\partial_x\bar{v} + b\bar{v} - G(\bar{u}) \leq 0.$$

Thus the PDE inequalities required for an upper solution are satisfied (for $\mathcal{M}$ small so the linear bounds on G, H hold for $\bar{u}, \bar{v}$).

Verification of the free–boundary inequalities. We must also check that the prescribed boundary velocities of the auxiliary interval dominate those of the true problem. Compute the growth rate $\bar{h}'(t)$:

$$\bar{h}'(t) = -(c_1 - h_0)\sigma_1 e^{\sigma_1 t}.$$

On the other hand the free–boundary right–speed generated by $(\bar{u}, \bar{v})$ is

$$\begin{aligned}
&\mu\Big(\int_{\bar{g}(t)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{\infty} J_1(x-y)\bar{u}(x,t)\,dy\,dx + \rho\int_{\bar{g}(t)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{\infty} J_2(x-y)\bar{v}(x,t)\,dy\,dx\Big)\\
&\quad\leq \mu\Big(\int_{\bar{g}(t)}^{\bar{h}(t)} \bar{u}(x,t)\,dx + \rho\int_{\bar{g}(t)}^{\bar{h}(t)} \bar{v}(x,t)\,dx\Big) = \mathcal{M}e^{\sigma_1 t}\mu\Big(\int_{\bar{g}(t)}^{\bar{h}(t)} \psi_1 + \rho\int_{\bar{g}(t)}^{\bar{h}(t)} \psi_2\Big)\\
&\quad\leq \mu\mathbf{M}e^{\sigma_1 t}\Big(\int_{-h_1}^{h_1} \psi_1 + \rho\int_{-c_1}^{c_1} \psi_2\Big) \leq \mu M_0(1+\rho)e^{\sigma_1 t}.
\end{aligned}$$

By the choice of $\mathbf{M}$ we have

$$\mu M_0(1+\rho)e^{\sigma_1 t} = -\sigma_1(c_1 - h_0)e^{\sigma_1 t} = \bar{h}'(t).$$

Thus the boundary velocity produced by $(\bar{u}, \bar{v})$ is $\leq \bar{h}'(t)$. The left boundary check is analogous and yields $-\bar{g}'(t)$ greater than or equal to the corresponding inward speed from $(\bar{u}, \bar{v})$. Hence the free–boundary inequalities required for an upper solution hold.

We have $(\bar{u}, \bar{v}, \bar{g}, \bar{h})$ is an upper solution of (1) on its time interval, that the initial data satisfy $(u_0, v_0) \leq (\bar{u}(\cdot,0), \bar{v}(\cdot,0))$ on $[-h_0, h_0]$, and that $\bar{g}(0) = -h_0 \leq g(0)$, $\bar{h}(0) = h_0 \geq h(0)$. Therefore by the comparison principle (Lemma (20)) the solution is dominated by the upper solution:

$$g(t) \geq \bar{g}(t), \qquad h(t) \leq \bar{h}(t), \qquad u(x,t) \leq \bar{u}(x,t), \qquad v(x,t) \leq \bar{v}(x,t),$$

for as long as the upper solution is defined. Passing to the limit $t \to \infty$ gives

$$h_\infty - g_\infty \leq \bar{h}(\infty) - \bar{g}(\infty) = 2h_1 < 2\ell^*,$$

hence the habitat length remains bounded and by Lemma (23) vanishing occurs: $(u, v) \to (0, 0)$ uniformly and the free interval does not spread. $\square$

Remark 2. *From the expression of* $\mathbf{M}$ *in* (4.7)*, it follows that* $\mathbf{M} \to \infty$ *as* $\mu \to 0,\ \rho \to 0$*. Hence, under the assumptions* $R_0 > 1$ *and* $h_0 < \mathcal{L}$*, for any given pair of initial functions* (u_0, v_0) *satisfying*

$$u_0, v_0 \in C([-h_0, h_0])\,, \quad v_0(\pm h_0) = u_0(\pm h_0) = 0, \quad u_0, v_0 > 0 \text{ for } x \in (-h_0, h_0),$$

there exists a constant $\mu_0 > 0$ *such that vanishing occurs whenever* $\mu \in (0, \mu_0]$*.*

Lemma 26. *We have* $h_\infty < \infty$ *if and only if* $g_\infty > -\infty$*.*

Proof. The argument follows along the same lines as the proof of Lemma 4.5 in [70]. Since the required adjustments are straightforward, we leave out the details. □

4.2. **Spreading.** In this section, we investigate the scenarios in which spreading occurs.

Lemma 27. *Suppose that* $R_0 > 1$ *and* $h_0 > \mathcal{L}$*. Then the free boundaries expand without bound, namely*

$$h_\infty = -g_\infty = \infty,$$

and the solution converges to the positive steady state:

$$\lim_{t\to\infty} \big(u(t,x), v(t,x)\big) = \big(\mathscr{U}^*, \mathscr{V}^*\big),$$

where $\big(\mathscr{U}^*, \mathscr{V}^*\big)$ *denotes the unique positive solution of* (4.2)–(4.3)*.*

Proof. We split the proof into two parts.

(A) Spreading: $h_\infty - g_\infty = \infty$ **and** $h_\infty = -g_\infty = \infty$**.** Since $R_0 > 1$ and $h_0 > \mathcal{L}^*$, by Lemma (23) the principal eigenvalue associated with the linearized problem on any interval $[-\mathcal{Z}, \mathcal{Z}]$ with $\mathcal{Z} \geq h_0$ satisfies $\lambda^*(\mathcal{Z}) > 0$.

$(\lambda^*(g_\infty, h_\infty) \geq \lambda_1^*(g(T), h(T)) > 0)$. Therefore $\lambda_1^*(g_\infty, h_\infty) > 0$. Now apply Lemma (25), the contrapositive of that statement yields that if $\lambda_1^*(g_\infty, h_\infty) > 0$ then $h_\infty - g_\infty = \infty$. Hence, the habitat length diverges:

$$h_\infty - g_\infty = \infty.$$

By using Lemma (26), we obtain

$$h_\infty = -g_\infty = \infty.$$

(B) Convergence to the positive steady state. We prove first a lower bound (limit inferior) and then an upper bound (limit superior); together they give the desired convergence.

Limit inferior. Fix any $T > 0$. Consider the fixed–domain problem on $[g(T), h(T)]$ with initial data

$$\big(\tilde{u}(x,0), \tilde{v}(x,0)\big) = (u(x,T), v(x,T)), \qquad x \in [g(T), h(T)].$$

Let $(\tilde{u}(x,t), \tilde{v}(x,t))$ denote the solution of this fixed–domain problem. Because $\lambda_1^*(g(T), h(T)) > 0$ Theorem (21) (i) implies that $(\tilde{u}, \tilde{v})$ converges to the unique positive steady state $(u_T(x), v_T(x))$ on $[g(T), h(T)]$:

$$\lim_{t\to\infty} (\tilde{u}(x,t), \tilde{v}(x,t)) = (u_T(x), v_T(x)) \quad \text{uniformly on } [g(T), h(T)].$$

By the comparison principle (20) and the fact that the free–boundary solution on $[g(t), h(t)]$ has domain containing $[g(T), h(T)]$ for $t \geq 0$, we have that

$$(\tilde{u}(x,t), \tilde{v}(x,t)) \leq \big(u(x, t+T), v(x, t+T)\big) \qquad \text{for } x \in [g(T), h(T)],\ t \geq 0.$$

Passing to the limit $t \to \infty$ in the last inequality yields

$$(u_T(x), v_T(x)) \leq \liminf_{t\to\infty} \big(u(x,t), v(x,t)\big) \qquad \text{for } x \in [g(T), h(T)].$$

Finally send $T \to \infty$. Because $g(T) \downarrow -\infty$, $h(T) \uparrow \infty$ and by Proposition (22) the family (u_T, v_T) converges locally uniformly to the spatially homogeneous positive equilibrium $(\mathscr{U}^*, \mathscr{V}^*)$ as $T \to \infty$, we deduce the global lower bound

$$(\mathscr{U}^*, \mathscr{V}^*) \leq \liminf_{t\to\infty} \big(u(x,t), v(x,t)\big)$$

locally uniformly in $x \in \mathbb{R}$.

Limit superior. Let $(\overline{u}(t), \overline{v}(t))$ be the solution of the *spatially homogeneous* ODE system (the reaction-only system)

$$\begin{cases} \overline{u}' = -a\overline{u} + H(\overline{v}), \\ \overline{v}' = -b\overline{v} + G(\overline{u}), \end{cases} \qquad (\overline{u}(0), \overline{v}(0)) = (\|u_0\|_\infty, \|v_0\|_\infty). \tag{4.8}$$

Because the nonlocal diffusion terms and drift terms do not produce positive sources beyond what the ODE receives (and because kernels integrate to 1), one easily checks that $(\overline{u}, \overline{v})$ is an upper solution to the full free–boundary problem: for each $t \geq 0$ and $x \in [g(t), h(t)]$,

$$u(x,t) \leq \overline{u}(t), \qquad v(x,t) \leq \overline{v}(t).$$

Since $R_0 > 1$, the ODE system (4.8) has the globally attractive positive equilibrium $(\mathscr{U}^*, \mathscr{V}^*)$, and

$$\lim_{t\to\infty} (\overline{u}(t), \overline{v}(t)) = (\mathscr{U}^*, \mathscr{V}^*).$$

Therefore

$$\limsup_{t\to\infty} \big(u(x,t), v(x,t)\big) \leq (\mathscr{U}^*, \mathscr{V}^*),$$

uniformly in $x \in [g(t), h(t)]$.

Combining the limit inferior and limit superior bounds we obtain

$$\lim_{t\to\infty} \big(u(x,t), v(x,t)\big) = (\mathscr{U}^*, \mathscr{V}^*),$$

locally uniformly in $x \in \mathbb{R}$. $\square$

Lemma 28. *Assume $R_0 > 1$ and $h_0 < \mathcal{L}^*$. Then there exists $\mu_0 > 0$ such that for every $\mu > \mu_0$ the solution of (1) spreads.*

Proof. Write explicitly the dependence on μ and denote the corresponding solution and free boundaries by

$$(u_\mu, v_\mu, g_\mu, h_\mu).$$

Argue by contradiction: suppose that for every $\mu > 0$ vanishing occurs, i.e. the limits

$$g_{\mu,\infty} := \lim_{t\to\infty} g_\mu(t), \qquad h_{\mu,\infty} := \lim_{t\to\infty} h_\mu(t) \tag{4.9}$$

exist and are finite for every $\mu > 0$.

By the comparison principle (Lemma (20)) the solution $(u_\mu, v_\mu, g_\mu, h_\mu)$ is monotone in μ: for $\mu_1 < \mu_2$ we have

$$u_{\mu_1}(x,t) \leq u_{\mu_2}(x,t), \quad v_{\mu_1}(x,t) \leq v_{\mu_2}(x,t), \quad g_{\mu_1}(t) \geq g_{\mu_2}(t), \quad h_{\mu_1}(t) \leq h_{\mu_2}(t),$$

for all admissible x, t. Consequently the pointwise limits

$$\mathcal{G}_\infty := \lim_{\mu\to\infty} g_{\mu,\infty}, \qquad \mathcal{H}_\infty := \lim_{\mu\to\infty} h_{\mu,\infty}$$

exist (possibly $-\infty$ / $+\infty$ but finite under our contradiction assumption). By Lemma (23) the principal eigenvalue on $[\mathcal{G}_\infty, \mathcal{H}_\infty]$ satisfies $\lambda_1^*(\mathcal{G}_\infty, \mathcal{H}_\infty) \leq 0$. On the other hand, since $R_0 > 1$, Lemma 3.9 (ii) ensures that large enough intervals admit positive eigenvalue, so in particular (by monotonicity with respect to domain size) we get $H_\infty - G_\infty \leq 2\mathcal{L}^*$ (i.e. the limiting length is bounded). Thus G_∞, H_∞ are finite numbers.

Fix small $\varepsilon > 0$ such that the kernels satisfy positivity near the origin: by (J) there exists $\delta_0 > 0$ with

$$J_i(\xi) \geq \delta_0 \quad \text{for } |\xi| \leq 4\varepsilon, \quad i = 1, 2.$$

Because $h_{\mu,\infty} \uparrow H_\infty$ and $g_{\mu,\infty} \downarrow G_\infty$ as $\mu \to \infty$, choose $\mu_1 > 0$ large so that for every $\mu \geq \mu_1$

$$h_{\mu,\infty} > H_\infty - \tfrac{\varepsilon}{2}, \qquad g_{\mu,\infty} < G_\infty + \tfrac{\varepsilon}{2}.$$

Since $h_{\mu_1}(t) \to h_{\mu_1,\infty}$ and $g_{\mu_1}(t) \to g_{\mu_1,\infty}$ as $t \to \infty$, we can pick $t_1 > 0$ large such that

$$h_{\mu_1}(t) \geq h_{\mu_1,\infty} - \tfrac{\varepsilon}{2} > \mathcal{H}_\infty - \varepsilon, \qquad g_{\mu_1}(t) \leq g_{\mu_1,\infty} + \tfrac{\varepsilon}{2} < \mathcal{G}_\infty + \varepsilon,$$

for all $t \geq t_1$.

Now use the free–boundary formula (the right boundary; the left boundary is handled analogously). For any $\mu > 0$ and any $t_1 > 0$,

$$h_{\mu,\infty}-h_\mu(t_1) = \int_{t_1}^{\infty} h'_\mu(s)\,ds = \mu\int_{t_1}^{\infty}\Big[\int_{g_\mu(s)}^{h_\mu(s)}\int_{h_\mu(s)}^{\infty} J_1(x-y)u_\mu(x,s)\,dy\,dx+\rho\int_{g_\mu(s)}^{h_\mu(s)}\int_{h_\mu(s)}^{\infty} J_2(x-y)v_\mu(x,s)\,dy\,dx\Big]ds$$

Hence

$$\mu \;=\; \frac{h_{\mu,\infty}-h_\mu(t_1)}{\displaystyle\int_{t_1}^{\infty}\Big[\int_{g_\mu(s)}^{h_\mu(s)}\int_{h_\mu(s)}^{\infty} J_1(x-y)u_\mu(x,s)\,dy\,dx+\rho\int_{g_\mu(s)}^{h_\mu(s)}\int_{h_\mu(s)}^{\infty} J_2(x-y)v_\mu(x,s)\,dy\,dx\Big]ds}.$$

We will obtain a uniform lower bound on the denominator in (4.2)) for all $\mu \geq \mu_1$, which will contradict the fact that μ can be arbitrarily large.

Observe that for $\mu \geq \mu_1$ we have monotonicity $u_\mu \geq u_{\mu_1}$ and $v_\mu \geq v_{\mu_1}$ pointwise. Therefore for every $s \geq t_1$,

$$\begin{aligned} I_\mu(s) &:= \int_{g_\mu(s)}^{h_\mu(s)}\int_{h_\mu(s)}^{\infty} J_1(x-y)u_\mu(x,s)\,dy\,dx \\ &\geq \int_{h_\mu(s)-2\varepsilon}^{h_\mu(s)}\int_{h_\mu(s)}^{h_\mu(s)+2\varepsilon} J_1(x-y)u_{\mu_1}(x,s)\,dy\,dx \geq \delta_0(2\varepsilon)^2 \min_{x\in[h_\mu(s)-2\varepsilon,h_\mu(s)]} u_{\mu_1}(x,s). \end{aligned}$$

A completely analogous bound holds for the v-term:

$$J_\mu(s) := \int_{g_\mu(s)}^{h_\mu(s)}\int_{h_\mu(s)}^{\infty} J_2(x-y)v_\mu(x,s)\,dy\,dx \geq \delta_0(2\varepsilon)^2 \min_{x\in[h_\mu(s)-2\varepsilon,h_\mu(s)]} v_{\mu_1}(x,s).$$

By our choice of μ_1 and t_1 we have $h_{\mu_1}(t) \geq \mathcal{H}_\infty - \varepsilon$ for all $t \geq t_1$, and hence the intervals $[h_{\mu_1}(t) - 2\varepsilon, h_{\mu_1}(t)]$ are contained in $[\mathcal{H}_\infty - 3\varepsilon, \mathcal{H}_\infty + \varepsilon]$ for $t \geq t_1$. The functions u_{μ_1}, v_{μ_1} are continuous in space-time and nonnegative; they are not identically zero on the time-space slab $[t_1, t_1+1] \times [\mathcal{H}_\infty - 3\varepsilon, \mathcal{H}_\infty + \varepsilon]$. Therefore the continuous function

$$s \mapsto m(s) := \min_{x\in[h_{\mu_1}(s)-2\varepsilon,h_{\mu_1}(s)]} \big(u_{\mu_1}(x,s)+\rho v_{\mu_1}(x,s)\big)$$

is nonnegative and not identically zero on $[t_1, t_1+1]$. Consequently

$$\int_{t_1}^{t_1+1} \big(I_\mu(s)+\rho J_\mu(s)\big)\,ds \geq \delta_0(2\varepsilon)^2 \int_{t_1}^{t_1+1} m(s)\,ds =: c_* > 0,$$

where $c_* > 0$ depends only on $\mu_1, t_1, \varepsilon, \delta_0$ and the fixed solution (u_{μ_1}, v_{μ_1}), but not on μ.

Hence the denominator in (4.2) is bounded from below by c_*. The numerator $h_{\mu,\infty} - h_\mu(t_1)$ is bounded above uniformly in μ by $\mathcal{H}_\infty - (\mathcal{H}_\infty - \varepsilon) = \varepsilon$ (indeed $h_{\mu,\infty} \leq \mathcal{H}_\infty$ and $h_\mu(t_1) \geq \mathcal{H}_\infty - \varepsilon$ for $\mu \geq \mu_1$ by construction). Therefore for every $\mu \geq \mu_1$ we deduce from (4.2)

$$\mu \leq \frac{h_{\mu,\infty}-h_\mu(t_1)}{\int_{t_1}^{\infty}\big(I_\mu(s)+\rho J_\mu(s)\big)\,ds} \leq \frac{\mathcal{H}_\infty-(\mathcal{H}_\infty-\varepsilon)}{c_*} = \frac{\varepsilon}{c_*} < \infty.$$

This gives a uniform upper bound for all $\mu \geq \mu_1$, contradicting the fact that μ can be taken arbitrarily large. The contradiction arises from the assumption that vanishing occurs for every $\mu > 0$; hence there exists $\mu_0 > 0$ such that spreading occurs for all $\mu > \mu_0$. This completes the proof. □

Proof of Theorem 4

Proof. For each fixed $\mu > 0$ denote by $(u_\mu, v_\mu, g_\mu, h_\mu)$ the corresponding solution of (1). Define

$$\mathcal{E} := \{\mu > 0 :\; h_{\mu,\infty} - g_{\mu,\infty} \leq 2\mathcal{L}^*\}, \qquad \text{where } g_{\mu,\infty} := \lim_{t\to\infty} g_\mu(t),\; h_{\mu,\infty} := \lim_{t\to\infty} h_\mu(t).$$

By the definition and Lemma (23), $\mu \in \mathcal{E}$ precisely means vanishing occurs for that μ. We shall show $\mathcal{E} = (0, \hat{\mu}]$ for some $\hat{\mu} > 0$.

1. Nonempty and unbounded side excluded. By Lemma (23) there exists $\mu_0 > 0$ such that $(0, \mu_0] \subset \mathcal{E}$. By Lemma (28) (spreading for large μ) there exists $\overline{\mu} > 0$ such that for every $\mu > \overline{\mu}$ spreading occurs, hence $(\overline{\mu}, \infty) \cap \mathcal{E} = \varnothing$. Therefore $\mathcal{E}$ is a nonempty subset of $(0, \infty)$ bounded above, so the supremum

$$\hat{\mu} := \sup \mathcal{E} \in [\mu_0, \overline{\mu}]$$

is well-defined and finite.

2. Spreading for $\mu > \hat{\mu}$. Fix any $\mu > \tilde{\mu}$. By definition of supremum there exists $\tilde{\mu} \in \mathcal{E}$ with $\tilde{\mu} < \mu$ arbitrarily close to μ^*. Monotonicity of the solution in μ implies that the larger parameter yields larger solution and larger spreading ability, so if vanishing holds at $\tilde{\mu}$ then it might still hold at μ; however Lemma (28) guarantees that for sufficiently large μ vanishing cannot persist. More directly, since $\mu > \hat{\mu} \geq \sup \mathcal{E}$, we must have $\mu \notin \mathcal{E}$, i.e. vanishing does not occur at μ. Thus spreading occurs for every $\mu > \hat{\mu}$.

3. The supremum $\hat{\mu}$ belongs to $\mathcal{E}$. We next show $\hat{\mu} \in \mathcal{E}$. Suppose instead $\hat{\mu} \notin \mathcal{E}$. Then vanishing fails at $\mu = \hat{\mu}$, so spreading occurs for $\mu = \hat{\mu}$. In particular there exists $T > 0$ such that

$$h_{\hat{\mu}}(T) - g_{\hat{\mu}}(T) > 2\mathcal{L}^*.$$

By the continuous dependence of solutions on the parameter μ (standard parabolic continuous dependence: the map $\mu \mapsto (u_\mu, v_\mu, g_\mu, h_\mu)$ is continuous in the appropriate norms on compact time intervals), there exists $\varepsilon > 0$ such that for every $\mu \in (\hat{\mu} - \varepsilon, \hat{\mu} + \varepsilon)$

$$h_\mu(T) - g_\mu(T) > 2\mathcal{L}^*.$$

For any such μ the monotonicity of $h_\mu(t)$ and $-g_\mu(t)$ in t implies

$$\lim_{t \to \infty} \big(h_\mu(t) - g_\mu(t)\big) \geq h_\mu(T) - g_\mu(T) > 2\mathcal{L}^*,$$

so $\mu \notin \mathcal{E}$. Consequently $\mathcal{E} \subset (0, \hat{\mu} - \varepsilon)$, contradicting the fact that $\hat{\mu} = \sup \mathcal{E}$. Hence our assumption was false and we must have $\hat{\mu} \in \mathcal{E}$.

4. Vanishing for $\mu \leq \hat{\mu}$. Finally we show that every $\mu \in (0, \mu^*]$ belongs to $\mathcal{E}$. Let $\mu \in (0, \hat{\mu})$. By monotonicity in μ the solution at μ is dominated by the solution at $\hat{\mu}$:

$$u_\mu(x,t) \leq u_{\hat{\mu}}(x,t), \quad v_\mu(x,t) \leq v_{\hat{\mu}}(x,t), \qquad g_\mu(t) \geq g_{\mu^*}(t), \quad h_\mu(t) \leq h_{\mu^*}(t).$$

Since $\hat{\mu} \in \mathcal{E}$ we have $h_{\hat{\mu},\infty} - g_{\mu^*,\infty} \leq 2\mathcal{L}^*$, and therefore passing to limits in the inequalities above yields

$$h_{\mu,\infty} - g_{\mu,\infty} \leq h_{\mu^*,\infty} - g_{\mu^*,\infty} \leq 2\mathcal{L}^*,$$

i.e. $\mu \in \mathcal{E}$. Because $\mu \in (0, \hat{\mu})$ was arbitrary, we conclude $(0, \hat{\mu}) \subset \mathcal{E}$. Combined with $\mu^* \in \mathcal{E}$, we obtain $\mathcal{E} = (0, \mu^*]$.

5. Final statement. From the previous steps we have shown that vanishing occurs for $\mu \in (0, \hat{\mu}]$ and spreading occurs for $\mu \in (\hat{\mu}, \infty)$. The asymptotic profile in the spreading case (convergence to $(\mathscr{U}^*, \mathscr{V}^*)$) follows from Theorem (21) and Lemma (28)(which give that once spreading happens the solution converges to the unique positive steady state). This completes the proof. □

Conclusion and Perspectives

In this work, we have investigated a free-boundary problem arising from a cooperative nonlocal reaction–diffusion system with drift terms. Our analysis unfolds through several key steps.

First, by ingeniously combining fixed-point techniques with refined comparison principles, we established the *well-posedness* of the problem, ensuring the existence, uniqueness, and regularity of global classical solutions for all admissible initial data.

Second, we conducted a comprehensive spectral analysis of the associated *linearized eigenvalue problem* and demonstrated the existence and uniqueness of a principal eigenvalue together with a strictly positive eigenfunction. We further derived its fundamental qualitative properties with respect to the spatial domain, to the parameters thanks to deep theoritical results as Fredholm theory, the Crandall–Rabinowitz bifurcation theorem, and Hadamard-type derivative [7] and and uncovered a sharp equivalence between the sign of the principal eigenvalue and the basic reproduction number R_0. This connection provides a rigorous spectral criterion for the threshold between extinction and persistence, thereby linking the linearized spectral structure to the nonlinear long-term dynamics of the system.

Finally, by a detailed investigation of the free-boundary evolution, we established a complete *vanishing–spreading dichotomy.* When the initial domain is sufficiently small ($h_0 < \mathcal{L}^*$), there exists a critical threshold $\widehat{\mu} > 0$ such that vanishing occurs for $\mu \in (0, \widehat{\mu}]$, whereas spreading occurs for $\mu > \widehat{\mu}$. In the spreading regime, the solution converges to the unique positive steady state $(\mathscr{U}^*, \mathscr{V}^*)$, while in the vanishing regime it decays uniformly to zero. These results provide a complete and rigorous characterization of the asymptotic behavior of the system.

Perspectives research. The present work opens several directions for future research. A natural extension would be to incorporate more general nonlinear incidence functions $f(u, v)$, capturing saturation or density-dependent transmission effects. Another promising direction concerns competitive or predator–prey systems governed by free boundaries, in contrast with the cooperative framework analyzed here, which could shed new light on coexistence, exclusion, and spatial invasion phenomena in ecological systems. From a broader viewpoint, extending the present analysis to higher-dimensional settings, introducing stochastic perturbations, or accounting for temporal and spatial heterogeneities would substantially enrich the theoretical framework and enhance its relevance to realistic epidemic and ecological scenarios. An important extension of the above model is to incorporate the influence of a moving climatic envelope on the directional epidemic, which represents a promising avenue for future research. To this end, we allow the local incidence functions to depend on the shifted spatial coordinate $x - ct$, and consider the following free boundary system :

$$
\tag{4.10}
\begin{cases}
u_t = d_1 \displaystyle\int_{\Omega_t} J_1(x-y)\, u(t,y)\, dy - u(t,x) + p\, u_x - a(x)\, u(t,x) + H\big(x-ct,\, v(t,x)\big), & t>0,\ x \in \Omega_t,\\
v_t = d_2 \displaystyle\int_{\Omega_t} J_2(x-y)\, v(t,y)\, dy - v(t,x) + q\, v_x - b(x)\, v(t,x) + G\big(x-ct,\, u(t,x)\big), & t>0,\ x \in \Omega_t,\\
u(t,x) = v(t,x) = 0, & t>0,\ x \in \partial\Omega_t,\\
h'(t) = \mu \displaystyle\int_{\Omega_t}\int_{(h(t),\infty)} J_1(x-y)\, u(t,x)\, dy\, dx + \rho \int_{\Omega_t}\int_{(h(t),\infty)} J_2(x-y)\, v(t,x)\, dy\, dx, & t>0,\\
g'(t) = -\,\mu \displaystyle\int_{\Omega_t}\int_{(-\infty,g(t))} J_1(x-y)\, u(t,x)\, dy\, dx + \rho \int_{\Omega_t}\int_{(-\infty,g(t))} J_2(x-y)\, v(t,x)\, dy\, dx, & t>0,\\
u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad g(0) = -h_0,\ h(0) = h_0.
\end{cases}
$$

where $\Omega_t := (g(t),\, h(t))$ and $\partial\Omega_t = \{g(t),\, h(t)\}$, $c \in \mathbb{R}$ denotes the velocity of the climatic drift. This potential study develops the original models suggested by many authors, for instance [8, 10, 32, 71], to investigate the long time dynamics of single population under the influence of climate change without free boundary setting.

Statements

Conflict of interest statement : The authors have no conflicts of interest to declare that are relevant to the content of this article.

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