SIMPLE B_4 REPRESENTATIONS ASSOCIATED TO CYCLOTOMIC HECKE ALGEBRAS

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Abstract. We determine the structure of the cyclotomic Hecke algebra corresponding to the complex reflection group G_{25} also when it is not semisimple, as long as the generators are diagonalizable. In particular, we classify all simple representations of the braid group B_4 for which the generators are diagonalizable and satisfy a cubic polynomial. This will be used in the classification of braided tensor categories of type G_2.

It is a classical result by Coxeter that the relation $\sigma_i^p = 1$ defines a finite quotient of the braid group B_n for $p > 2$ and $n > 2$ if and only if $p = 3$ and $n \leq 5$ or $n = 3$ and $p \leq 5$. The corresponding quotient groups are special cases of complex reflection groups. Hecke algebra deformations have been defined for these groups which, for generic parameters, have the same representation theory as the group ring. The main work in this paper consists of studying the representations for the Hecke algebra $\mathcal{K}_4$ for $n = 4$ and $p = 3$. It can be defined as a quotient of the group algebra of B_4 with the additional relation $\prod_{i=1}^3(\sigma_j - \lambda_i 1)$ and corresponds to the complex reflection group with the label G_{25}. While this seems to be a worthwhile problem in its own right, our research was motivated by our work on classifying tensor categories with the tensor product rules of the simple Lie algebra of type G_2. We will only need representations for which the image of the braid generators are diagonalizable. So our results will mostly be about this case, which is also more accessible to the methods we use. Our main results in this paper are:

Theorem A (see Theorem 4.4). The algebra $\mathcal{K}_4$ is not semisimple if and only if the eigenvalues of σ_1 are zeros of one of the following polynomials: $\lambda_i - \lambda_j$, $\lambda_i + \lambda_j$, $\lambda_i + \theta\lambda_j$, $\lambda_i \pm \sqrt{-1}\lambda_j$, $\lambda_i^2 + \lambda_j\lambda_k$, $\lambda_i^2 - \theta\lambda_j\lambda_k$ or $\lambda_i^3 - \lambda_j^2\lambda_k$, where $\{i, j, k\} = \{1, 2, 3\}$ and θ is a primitive third root of unity.

This result would also follow from the calculation of Schur elements of a conjectured symmetrizing trace on $\mathcal{K}_4$ in [13], see the discussion in Section 5.1. There exists a complete set of mutually non-isomorphic $\mathcal{K}_4$ modules V_γ each of which is simple for a Zariski open set of eigenvalues; we call these regular modules. The canonical central element Δ_4^2 acts via a scalar λ_γ on V_γ which is a monomial in the eigenvalues (possibly multiplied with a primitive third root of unity, which we ignore for the moment). Let $\mathfrak{p}$ be the prime ideal generated by one of the polynomials $\neq \lambda_i - \lambda_j$ listed in Theorem A. We say $\gamma \sim_{\mathfrak{p}} \gamma'$ if $\lambda_\gamma - \lambda_{\gamma'} \in \mathfrak{p}$.

Theorem B (see Theorem 4.6) Let R be a domain in which $\lambda_i \neq \lambda_j$ for $i \neq j$. Then the blocks of $\mathcal{K}_4$, defined over the quotient field $Q(R/\mathfrak{p})$, are given by the equivalence classes with respect to $\sim_{\mathfrak{p}}$. Moreover, if we order the labels in such an equivalence class $\gamma_1 > \gamma_2 > \ ... \ \gamma_r$ via alphabetical order of the monomials λ_γ, we obtain an exact sequence

$$0 \ \to \ V_{\gamma_1} \ \to \ V_{\gamma_2} \ \to \ ... \ \to \ V_{\gamma_r} \ \to \ 0,$$

where the induced subquotient modules for each V_{γ_i} are simple.

For more details, see the table before Theorem 4.6. The simple modules in Theorem B may not stay simple if the eigenvalues satisfy more than one polynomial in Theorem A. We determine all possible simple subquotient modules also in these cases, see Table 4. It turns out that all of these modules are isomorphic to either regular modules or modules appearing in Theorem B. Hence we obtain

Theorem C (see Theorem 4.7) Table 1 and Table 3 give a complete classification of all simple modules of $\mathcal{K}_4$, subject to the condition that the braid generators are diagonalizable.

Unfortunately, even though we can give a fairly uniform description of our results, our proofs involve a lot of calculations and case-by-case studies. In order to study representations which are not already known from the study of Hecke algebras and BMW algebras, we use the explicit representations of $\mathcal{K}_4$ obtained in [14]. To determine their structure when they are not simple, we use different bases which we call weight and path bases. We then calculate diagonal matrix entries of eigenprojections of braid generators with respect to these bases, using a generalization of the quantum Jucys-Murphy elements. The main technical tool is Theorem 3.4, which first appeared in [26]. We can then read off these diagonal entries the existence and structure of subquotient modules for the representations in [14]; this is outlined at the beginning of Section 4. Here is our paper in more detail.

We review basic facts about the cubic Hecke algebras $\mathcal{K}_3$ and $\mathcal{K}_4$, regular modules and BMW algebras in Section 1. We also introduce path and weight bases. In Section 2 we find restrictions on the eigenvalues for which $\mathcal{K}_4$ is not semisimple, and show that path bases exist in all such cases. Section 3 is devoted to calculate the previously mentioned matrix entries for path bases. This is then used in Section 4 to determine all blocks and all possible simple representations of $\mathcal{K}_4$, assuming we have no repeated eigenvalues. We discuss connections of our results to other results and possible applications in the last section.

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1. Cyclotomic Hecke algebras for G_4 and G_{25}

1.1. **Definitions.** We first review a few basic facts about Artin's braid group, see e.g. [11]. The braid group B_n is defined by generators σ_i, $1 \leq i < n$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i < n-1$. We define elements Δ_k inductively by

$\Delta_1 = 1$ and

$$\Delta_{k+1} = \sigma_1\sigma_2 \ldots \sigma_k\Delta_k = \Delta_k\sigma_k\sigma_{k-1} \ldots \sigma_1.$$

Then Δ_n^2 is the generator of the center of B_n, see e.g. [11] Section 1.3.3. Let $\gamma_n = \Delta_n^2\Delta_{n-1}^{-2}$. Then $\gamma_n = \sigma_{n-1} \ldots \sigma_2\sigma_1^2\sigma_2 \ldots \sigma_{n-1}$, which can be fairly easily seen by using the usual description of braids via pictures, see e.g. [11]. We will also need the following well-known identity

$$\Delta_4\sigma_i\Delta_4^{-1} \; = \; \sigma_{4-i}, \quad 1 \leq i < 4. \tag{1.1}$$

It was shown by Coxeter [6] that B_n with the additional relation $\sigma_i^3 = 1$ is a finite group only if $n \leq 5$. These quotients correspond to the complex reflection groups denoted by G_4, G_{25} and G_{32} in [22]. This was generalized to the quotient $\mathcal{K}_n$ of the group algebra $\mathbb{C}B_n$ of the braid group B_n defined by the relations $(\sigma_i - \lambda_1)(\sigma_i - \lambda_2)(\sigma_i - \lambda_3) = 0$, $1 \leq i < n$ as follows.

Theorem 1.1. [3], [15] *The algebras $\mathcal{K}_n$ have dimensions 3, 24, 648 and 155,520 for $n = 2, 3, 4, 5$. In particular, they have bases with respect to which the standard generators act via matrices whose entries are Laurent polynomials in the eigenvalues λ_i over a finite extension of $\mathbb{Z}$. Their $\mathbb{C}$ span is isomorphic to the group ring of the corresponding complex reflection group for a Zariski open set of the eigenvalues.*

The algebras $\mathcal{K}_n$ are infinite-dimensional for $n > 5$.

We define a certain class of modules, called regular modules, which play the role of Specht modules (for Hecke algebras of type A) or cellular modules (for BMW algebras, see Section 1.9) for our algebras $\mathcal{K}_n$.

Definition 1.2. We call a $\mathcal{K}_n$ module (for $n \leq 5$) a *regular* module, if

(a) it has a basis with respect to which the braid generators are represented by matrices whose entries are in a finite extension of $\mathbb{Z}[\lambda_1, \lambda_2, \lambda_3]$,

(b) the representation is irreducible for a Zariski open subset of the eigenvalues (λ_i).

Remark 1.3. It is easy to see that two regular modules which are isomorphic for a Zariski dense subset of the eigenvalues may not be isomorphic when they are no longer simple. Indeed, as any braid group B_n is isomorphic to its opposite group, we obtain from any representation $\rho : B_n \to \mathrm{End}(V)$ a representation ρ^t defined by $\rho^t(\beta) = \rho(\beta)^t$, the transpose matrix of $\rho(\beta)$. So any quotient module of V corresponds to a submodule of V^t (which is V with the ρ^t action of B_n). If V is not semisimple, this entails that V^t is usually not isomorphic to V. More generally, it follows that any exact sequence $\ldots \to V_i \to V_{i+1} \to \ldots$ of B_n modules defines an exact sequence $\ldots \to V_{i+1}^t \to V_i^t \to \ldots$ of B_n modules.

1.2. **Conventions.** While we are primarily interested in representations over the complex numbers $\mathbb{C}$, it will be useful to deal with our algebras in a more general setting. In the following, we assume F_0 to be a field of characteristic 0 which contains a primitive third and fourth root of unity. We define $R = F_0[\lambda_i^{\pm 1}, (\lambda_i - \lambda_j)^{-1}]$, for $1 \leq i \neq j \leq 3$, where the λ_i are viewed as variables. F is assumed to be a field with $\lambda_i \in F$, $\lambda_i \neq \lambda_j$ for $i \neq j$. We will usually encounter F as the quotient field $Q(R/\mathfrak{p})$ for a prime ideal $\mathfrak{p} \subset R$. We may write $F\mathcal{K}_n$ if we want to stress the field over which we define the algebra; otherwise we just write $\mathcal{K}_n$.

Statements concerning regular modules usually mean that it holds for the given module V or its dual module V^t. The discerning reader should replace statements dealing with non-semisimple regular modules as: For the given regular module V or its dual module V^t the following holds... .

1.3. **The algebra $\mathcal{K}_3$.** We determine the structure of the algebra $\mathcal{K}_3$ in this section. These results are known to experts, see e.g. the review in [5], Chapter 2. They could also be derived e.g. from results in [23].

Theorem 1.4. [2], [23] *The algebra $\mathcal{K}_3$ is semisimple over a field F of characteristic 0 except if one of the following polynomials in the eigenvalues is equal to 0:*

$$\lambda_i - \lambda_j, \quad \lambda_i^2 - \lambda_i\lambda_j + \lambda_j^2 \quad \text{or} \quad \lambda_i^2 + \lambda_j\lambda_k,$$

where $\{i,j,k\} = \{1,2,3\}$. In the semisimple case, we have three 1-dimensional representations $\{\lambda_i\}$, three two-dimensional representations $\{\lambda_i\lambda_j\}$ and one three-dimensional representation $\{\lambda_1\lambda_2\lambda_3\}$. Here the expression between the brackets $\{\ \}$ is the determinant of σ_1 in the given representation.

We also expect the following result to be known.

Corollary 1.5. *Replacing a regular representation V by its conjugate V^t if necessary, see Remark 1.3, we obtain the following exact sequences, which do not split:*
(a) If $\lambda_i = -\theta\lambda_j$, we have

$$0 \to \{\lambda_i\} \to \{\lambda_i\lambda_j\} \to \{\lambda_j\} \to 0. \tag{1.2}$$

(b) If $\lambda_1^2 + \lambda_2\lambda_3 = 0$, we have

$$0 \to \{\lambda_2\lambda_3\} \to \{\lambda_1\lambda_2\lambda_3\} \to \{\lambda_1\} \to 0. \tag{1.3}$$

Proof. The first exact sequence is known from the study of Iwahori-Hecke algebras, see Section 1.7, (b) for more details. Statement (b) follows e.g. from Corollary 1.18,(a). It can also be shown directly using the formulas for braid matrices in Theorem 3.4 and Corollary 3.5, and the results in Section 3.6, Example (1).

Lemma 1.6. *Assume that the eigenvalues are mutually distinct, and two of the equations $\lambda_i^2 - \lambda_i\lambda_j + \lambda_j^2 = 0$ or $\lambda_i^2 + \lambda_j\lambda_k = 0$ in Theorem 1.4 are satisfied. Then we have $\lambda_j = -\theta\lambda_k$, $\lambda_k = -\theta\lambda_i$ for a primitive 3rd root of unity θ and indices $\{i,j,k\} = \{1,2,3\}$. In particular, all simple representations of $\mathcal{K}_3$ are 1-dimensional in this case, except for $\{\lambda_i\lambda_k\}$.*

Proof. If $\lambda_i^2 = -\lambda_j\lambda_k$ and $\lambda_j^2 = -\lambda_i\lambda_k$, we can solve for λ_k in both equations. It follows that $\lambda_i^2/\lambda_j = \lambda_j^2/\lambda_i$, which implies that $\lambda_i = \theta\lambda_j$ for some primitive third root of unity θ (as we assumed $\lambda_i \neq \lambda_j$). Substituting this into $\lambda_j^2 = -\lambda_i\lambda_k$, we obtain $\lambda_j = -\theta\lambda_k$ and $\lambda_k = -\theta\lambda_i$. Similar proofs work for any choice of the equations mentioned in the statement. We can rule out cases $\lambda_i = -\theta\lambda_j$ and $\lambda_j = -\theta^2\lambda_k$, as this would imply $\lambda_i = \lambda_k$. This is excluded in our setting.

Assume $\lambda_1 = -\theta\lambda_2$ and $\lambda_2 = -\theta\lambda_3$. Then, by Corollary 1.5, the representations $\{\lambda_1\lambda_2\}$ and $\{\lambda_2\lambda_3\}$ are not simple. Hence any subquotient module in these modules must be 1-dimensional. Moreover, these equations also imply $\lambda_1^2 + \lambda_2\lambda_3 = 0 = \lambda_3^2 + \lambda_1\lambda_2$. Hence $\{\lambda_1\lambda_2\lambda_3\}$ has a 1-dimensional submodule, and a 2-dimensional quotient $\{\lambda_2\lambda_3\}$ which is not simple. This shows that $\{\lambda_1\lambda_2\lambda_3\}$ has a composition series with one-dimensional factors.

Remark 1.7. Using the explicit results in [23], it is also possible to determine all simple modules of $\mathcal{K}_3$ when we have repeated eigenvalues. E.g. if $\lambda_1 = \lambda_2 = \lambda_3$, it follows from [23], Theorem 2.9 that we have exactly one simple module of dimensions 1, 2 and 3. We will not pursue these cases further in this paper.

1.4. **Regular modules of $\mathcal{K}_4$.** We want to carry out a similar analysis for the algebra $\mathcal{K}_4$. As a first step, we recall what structure it has when it is semisimple.

Theorem 1.8. [14] *Let F be a field of characteristic zero which contains a primitive third root of unity θ. The algebra $\mathcal{K}_4$, defined over F, has the same dimension and, for generic eigenvalues, the same decomposition into a direct sum of full matrix rings as the group algebra of the complex reflection group G_{25}. A full list of regular modules, up to permutation of the eigenvalues, is listed in Table 1.*

The regular modules of the algebra $\mathcal{K}_4$ are already determined by the determinant $\det(\sigma_1)$ of a generator of the braid group, except for the 9-dimensional representations, which also depend on the choice of a primitive third root of unity θ. We will refer to a representation by $\{\det(\sigma_1)\}$, where we add the suffix θ to the nine-dimensional representation $\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$. We list the scalar by which the central element Δ_4^2 acts in the third column below. The last column tells us the decomposition of the simple $\mathcal{K}_4$ module into a direct sum of $\mathcal{K}_3$ modules. Weights will be defined in Section 1.5. *Convention*: $V = X + Y$ means that V has a composition series with factors X and Y.

dim	$\det(\sigma_1)$	Δ_4^2	weights	$\mathcal{K}_3$ rep
9	$\lambda_1^3\lambda_2^3\lambda_3^3$	$\theta\lambda_1^4\lambda_2^4\lambda_3^4$	$(i,j),\ 1 \le i,j \le 3$	see $R9$
8	$\lambda_1^4\lambda_2^2\lambda_3^2$	$\lambda_1^6\lambda_2^3\lambda_3^3$	(1,1): 2, $(i,j), i \neq j$: 1	see $R8$
6	$\lambda_1^3\lambda_2^2\lambda_3$	$\lambda_1^6\lambda_2^4\lambda_3^2$	$(1,i),(i,1)$ and $(2,2)$	see $R6$
3	$\lambda_1\lambda_2\lambda_3$	$\lambda_1^4\lambda_2^4\lambda_3^4$	(i,i)	$\{\lambda_1\lambda_2\lambda_3\}$
3	$\lambda_1^2\lambda_2$	$\lambda_1^8\lambda_2^4$	$(1,1),(1,2),(2,1)$	$\{\lambda_1\lambda_2\} + \{\lambda_1\}$
2	$\lambda_1\lambda_2$	$\lambda_1^6\lambda_2^6$	$(1,1),(2,2)$	$\{\lambda_1\lambda_2\}$
1	λ_1	λ_1^{12}	$(1,1)$	$\{\lambda_1\}$

$R9$: $\{\lambda_1\lambda_2\lambda_3\} + \{\lambda_1\lambda_2\} + \{\lambda_1\lambda_3\} + \{\lambda_2\lambda_3\}$,
$R8$: $\{\lambda_1\lambda_2\lambda_3\} + \{\lambda_1\lambda_2\} + \{\lambda_1\lambda_3\} + \{\lambda_1\}$,
$R6$: $\{\lambda_1\lambda_2\lambda_3\} + \{\lambda_1\lambda_2\} + \{\lambda_1\}$.

Table 1

Remark 1.9. (a) Let $\Gamma = \{\gamma\}$ be a labeling set for the isomorphism classes of simple modules of $\mathcal{K}_4$, and let λ_γ be the scalar via which Δ_4^2 acts on the regular module V_γ. If $\lambda_\gamma \neq \lambda_{\gamma'}$ for all $\gamma' \neq \gamma$, the eigenprojection z_γ of Δ_4^2 for this scalar is a central idempotent of $\mathcal{K}_4$. It corresponds to the simple representation V_γ in the semisimple case. Unfortunately, we are not aware of a general result which would imply that the generic representation V_γ will be simple whenever z_γ is well-defined, even though we will be able to prove this later, with some effort.

(b) Assuming the result in (a), we could narrow down the values for the eigenvalues for which $\mathcal{K}_4$ is not semisimple. It suffices to just set equal the eigenvalues of Δ_4^2 for two generic representations. One could thus show that $\mathcal{K}_4$ is not semisimple only if one of the following expressions is equal to zero: $\lambda_i^{12} - \lambda_j^{12}$, $\lambda_i^6 \pm \lambda_j^3\lambda_k^3$ or $\lambda_i^3 - \lambda_j^2\lambda_k$, where $\{i, j, k\} = \{1, 2, 3\}$. Compare this with the precise result, see Theorem 4.4.

1.5. **Weights and paths.** Unless noted otherwise, we assume that we only deal with representations of $\mathcal{K}_4$ in which the images of the braid generators are diagonalizable. While regular representations are well-defined for any choice of eigenvalues, it is hard to see for which values of the eigenvalues they are not simple. As we are primarily interested in the case where the images of the braid generators are diagonalizable, we can construct different bases which are more useful in that aspect. Observe that the images of σ_1 and σ_3 form a commutative subalgebra of the image of B_4 in any representation of B_4. This motivates the following definition.

Definition 1.10. Let W be a representation of B_4. We call any common eigenvector of σ_1 and σ_3 a *weight vector*. If σ_1 has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$, we say that a vector $v \in W$ has weight (i, j) if σ_1 and σ_3 act on it via multiplication by λ_i and λ_j respectively. The dimension of the corresponding weight space $V(i, j)$ is called the multiplicity of the weight.

Remark 1.11. It follows from the definitions and 1.1 that Δ_4 maps the weight space for the weight (i, j) onto the weight space for the weight (j, i).

The weights and their multiplicities are listed in the second to last column in Table 1. We see that the weights have multiplicity at most 1 in all regular modules, except for $V = \{\lambda_r^4\lambda_s^2\lambda_t^2\}$, where the weight (r, r) has multiplicity 2. This will allow us to construct convenient bases for regular modules even in the case of multiplicity > 1, see Section 2.1.

We will also use another well-known type of bases, *path bases*. They can be defined for any sequence of finite-dimensional semisimple algebras $\mathcal{A}_n$ for which the restriction of a simple $\mathcal{A}_n$ module to $\mathcal{A}_{n-1}$ decomposes into a direct sum of mutually nonisomorphic $\mathcal{A}_{n-1}$ modules. A standard example would be $\mathcal{A}_n = \mathbb{C}S_n$, the group algebra of the symmetric group S_n. We will only give the definitions for the special case in this paper.

Let us assume that the algebras $\mathcal{K}_2 \subset \mathcal{K}_3 \subset \mathcal{K}_4$ are semisimple. It follows from the restriction rules stated in Table 1 that any simple $\mathcal{K}_n$ module V_n decomposes into a direct sum of mutually nonisomorphic $\mathcal{K}_{n-1}$ modules for $n = 3, 4$ in the generic case. Hence, if γ_i

labels a simple $\mathcal{K}_i$ module, and z_{γ_i} is the corresponding central idempotent in $\mathcal{K}_i$, it follows that dim $z_{\gamma_2}z_{\gamma_3}z_{\gamma_4}V \leq 1$ for any regular $\mathcal{K}_4$ module V.

Definition 1.12. If the product $z_{\gamma_2}z_{\gamma_3}z_{\gamma_4}$ is nonzero, we call $t = (\gamma_i)$ a *path*, the idempotent $p_t = z_{\gamma_2}z_{\gamma_3}z_{\gamma_4}$ a *path idempotent* and any nonzero $v_t \in p_tV$ a *path vector* for V. We also define $t(i) = \gamma_i$.

1.6. **Bases of path vectors or weight vectors.** We shall use bases of path or weight vectors to determine when a regular module is simple. We list a number of simple properties for such bases in the following general setting. Let $\mathcal{A}$ be an algebra over a field F, and let V be a finite-dimensional $\mathcal{A}$-module. We assume we have a family $(P_t)_{t\in I} \subset \mathrm{End}(V)$ of rank 1 idempotents in the image of $\mathcal{A}$ such that $P_tP_s = 0$ for $t \neq s$ and a basis (v_t) with $v_t \in P_tV$. We define a relation

$$s \leftarrow t \ \Leftrightarrow\ P_s\mathcal{A}v_t \neq 0, \tag{1.4}$$

i.e. there exists an element $a \in \mathcal{A}$ such that its st matrix coefficient with respect to our basis is not equal to zero. We also denote by $\leftarrow$ the transitive closure of this relation, and we write $s \leftrightarrow t$ if $s \leftarrow t$ and $t \leftarrow s$. We then have the following lemma, whose proof follows from the definitions and basic linear algebra.

Lemma 1.13. *Let $0 = V_0 \subset V_1 \subset\ \dots\ \subset V_m = V$ be a composition series with simple quotients V_i/V_{i-1}. Then we obtain bases for the factors V_i/V_{i-1} from the vectors v_t labeled by indices t in an equivalence class of the relation $\leftrightarrow$. In particular, V is simple if all indices are equivalent to each other.*

We collect a few basic facts about path bases. More details can e.g. be found in [18] for more general braid representations. An explicit example for the matrix blocks mentioned in the following lemma appears in Lemma 4.2.

Lemma 1.14. *Let V be a $\mathcal{K}_4$ module, and let S_i be the matrix representing σ_i for $i = 2, 3$ with respect to a path basis (v_t).*

(a) Let $t = (\gamma_j)$ be a path. Then S_iv_t is a linear combination of paths v_s with $s \in [t; i]$, the set of all paths $s = (\nu_j)$ such that $\nu_j = \gamma_j$ for all $j \neq i$. So, S_i is a direct sum of matrix blocks acting on the subspaces spanned by vectors v_s with $s \in [t; i]$.

(b) Assume one of the eigenprojections E of S_i has rank 1 in the block labeled by the paths in $[t; i]$. Then $s \leftrightarrow s'$ for $s, s' \in [t; i]$ if $d_sd_{s'} \neq 0$ for the diagonal entries d_s, $d_{s'}$ of E.

Proof. Statement (a) follows from the fact that S_i commutes with $z_{t(j)}$ for all $j \neq i$. For part (b), let P_s be the projection on the span of v_s. If $s, s' \in [t; i]$, we have $P_sEP_{s'}EP_s = d_sd_{s'}P_s \neq 0$, using rank $E = 1$. This implies $s \leftarrow t$ and $t \leftarrow s$.

1.7. **Regular modules.** Here we list explicit regular modules which we will use to study the representation theory of the algebras $\mathcal{K}_3$ and $\mathcal{K}_4$.

(a) For modules of $\mathcal{K}_n$, $n = 3, 4$, in which σ_i acts with only two distinct eigenvalues, we obtain a representation of the Iwahori-Hecke algebra $\mathcal{H}_n$ of type A_{n-1}. A class of special modules, called Specht modules, were defined in [7] for all Hecke algebras $\mathcal{H}_n$ which satisfy the conditions of regular modules. Composition series for Specht modules have been determined in general, see e.g. [20], [9]. We will use the Specht modules of $\mathcal{H}_3$ and $\mathcal{H}_4$ as regular modules for the representations labeled by $\{\lambda_i\}$, $\{\lambda_i\lambda_j\}$ (for both $\mathcal{K}_3$ and $\mathcal{K}_4$) and $\{\lambda_i^2\lambda_j\}$. We can deduce the results already mentioned in Section 1.3 from [7] and [8] or the aforementioned lecture notes, as well as the following exact sequence when $\lambda_i^2+\lambda_j^2=0$:

$$0 \to \{\lambda_i\} \ \to \ \{\lambda_i^2\lambda_j\} \ \to \ \{\lambda_j^2\lambda_i\} \to \{\lambda_j\} \ \to \ 0. \tag{1.5}$$

In particular, we obtain a simple two-dimensional representation $\{\lambda_i\lambda_j\}^*$. Unlike the $\mathcal{K}_4$ module $\{\lambda_i\lambda_j\}$, it has the weights (i,j) and (j,i).

(b) The $\mathcal{K}_4$ modules $\{\lambda_i\}$, $\{\lambda_i\lambda_j\}$ and $\{\lambda_1\lambda_2\lambda_3\}$ are obtained from the modules of $\mathcal{K}_3$ denoted by the same symbol by mapping σ_3 to the matrix representing σ_1. In particular, the results in Section 1.3 also apply for these $\mathcal{K}_4$ modules.

(c) By [14], Proposition 7.3, each irreducible representation of the Hecke algebra $\mathcal{K}_4$ does appear in some pre-W-graph. Explicit matrix representations for the 6, 8 and 9-dimensional representations were determined in [2], [14], based on this; see [16], Section 4.4, Table 4 for explicit matrices. In all these representations, the entries of the matrices representing the braid generators are in the ring $\mathbb{Z}[\theta][\lambda_1,\lambda_2,\lambda_3]$, where θ is a primitive third root of unity, i.e. they are regular modules in our definition. An illustration for the paths for semsimple $\{\lambda_i^3\lambda_j^2\lambda_k\}$ and $\{\lambda_i^4\lambda_j^2\lambda_k^2\}$ are given by the paths in the Bratteli diagram in [17], Section 2.1 from 0 to $3\Lambda_1$ and $\Lambda_1+\Lambda_2$ respectively. See the proof of Lemma 4.3 for paths for the 9-dimensional representation.

1.8. **Permutation automorphisms.** We consider the algebras $\mathcal{K}_n$, $n\le 5$, defined over the polynomials $R=F_0[\lambda_1,\lambda_2,\lambda_3]$. Obviously, any permutation $\pi\in S_3$ induces an automorphism of R via permuting the eigenvalues. We can extend this to an automorphism of $\mathcal{K}_n$, also denoted by π via the map

$$\pi : \sum_{\beta\in B_n} a_\beta\beta \ \mapsto \ \sum_{\beta\in B_n}\pi(a_\beta)\beta;$$

it is clear that this map factors over the defining ideal for $\mathcal{K}_n$. We list some simple properties of these permutation automorphisms.

Lemma 1.15. *Let $\pi\in S_3$ also denote the corresponding automorphism of $R=F[\lambda_1,\lambda_2,\lambda_3]$ and of $\mathcal{K}_n$. Then we have*

(a) π induces a permutation of the simple components of $\mathcal{K}_n$, $n\le 5$. If $\{\lambda_1^{e_1}\lambda_2^{e_2}\lambda_3^{e_3}\}$ denotes a simple representation of $\mathcal{K}_n$, $n\le 4$, then π maps it to the simple representation denoted by $\{\pi(\lambda_1^{e_1}\lambda_2^{e_2}\lambda_3^{e_3})\}$.

(b) π induces a permutation of path idempotents.

(c) If p_t is a path idempotent, and P_{ki} the eigenprojection of S_k for the eigenvalue λ_i, then

$$\pi(p_t)P_{k\pi(i)}\pi(p_t) \ = \pi(\alpha)\pi(p_t),$$

where $\alpha\in R$ is defined by $p_tP_{ki}p_t=\alpha p_t$.

1.9. *BMW* **algebras.** It has been independently shown in [1] and [21] that one obtains a finite dimensional quotient algebra $\mathcal{BM}_n$ of FB_n for *all* n if we add to the cubic relation for $\mathcal{K}_n$ one more relation, see the quoted papers for details. If $P_1(\lambda_3)$ denotes the eigenprojection of σ_1 for the eigenvalue λ_3, the relation could be expressed as

$$\lambda_3^{\pm 1}(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\ P_1(\lambda_3)\sigma_2^{\pm 1}P_1(\lambda_3)\ =\ \lambda_3(\lambda_1+\lambda_2)P_1(\lambda_3),$$

with matching signs in the exponents. It was shown in [1] and [21] that the simple components of $\mathcal{BM}_n$ are labeled by Young diagrams μ with $|\mu| = n-2k$ boxes, $0 \le k \le \lfloor n/2 \rfloor$, in the generic semisimple case. If $V_{n,\mu}$ denotes a simple $\mathcal{BM}_n$ module, it decomposes as a $\mathcal{BM}_{n-1}$ module into the direct sum of modules $V_{n-1,\gamma}$, where γ runs through the diagrams obtained by removing or, if $|\mu| \le n-2$, adding a box from/to μ. One deduces from this that $\mathcal{BM}_3 \cong \{\lambda_1\lambda_2\lambda_3\} \oplus \{\lambda_1\lambda_2\} \oplus \{\lambda_1\} \oplus \{\lambda_2\}$, and that

$$\mathcal{BM}_4 \cong \{\lambda_1^3\lambda_2^2\lambda_3\} \oplus \{\lambda_1^2\lambda_2^3\lambda_3\} \oplus \{\lambda_1\lambda_2\lambda_3\} \oplus \mathcal{H}_4(\lambda_1,\lambda_2), \tag{1.6}$$

where $\mathcal{H}_4(\lambda_1,\lambda_2)$ is the direct sum of all simple representations of $\mathcal{K}_4$ in which the eigenvalue λ_3 does not appear. Here the eigenvalue λ_3 corresponds to the representation of $\mathcal{BM}_2$ labeled by the empty diagram $\emptyset$, λ_1 corresponds to [2] and λ_2 corresponds to the diagram [1,1]. One checks that for $\mathcal{BM}_3$, [1] labels the representation $\{\lambda_1\lambda_2\lambda_3\}$, and for $\mathcal{BM}_4$, [2] labels $\{\lambda_1^3\lambda_2^2\lambda_3\}$, [1,1] labels $\{\lambda_1^2\lambda_2^3\lambda_3\}$ and $\emptyset$ labels $\{\lambda_1\lambda_2\lambda_3\}$.

The eigenvalues of the braid generators of the algebras $\mathcal{BM}_n = \mathcal{BM}_n(r,q)$ in the original papers were normalized such that $\lambda_1 = q$, $\lambda_2 = -q^{-1}$ and $\lambda_3 = r^{-1}$. If $r = \pm q^k$ for an integer k, the algebras $\mathcal{BM}_n$ may no longer be semisimple. We have the following result, see [25], Corollary 5.6.

Theorem 1.16. *If $r = \pm q^k$ for some integer k, and q is not a root of unity, the algebra $\mathcal{BM}_n$ has a semisimple quotient, whose simple components are labeled by Young diagrams μ with $n-2k$ boxes, $0 \le k \le \lfloor n/2 \rfloor$, where now μ can only be taken from a restricted subset $\Lambda(\pm,k)$ of Young diagrams. The dimensions of its simple representations can be calculated by the same restriction rule as before, but with only Young diagrams from $\Lambda(\pm,k)$ allowed. The set $\Lambda(+,N-1)$ consists of all Young diagrams with $\le N$ boxes in the first two columns, while the set $\Lambda(-,N+1)$ consists of all Young diagrams with $\le N$ boxes in the first column.*

Remark 1.17. The set $\Lambda(+,N-1)$ labels the irreducible representations of $O(N)$ which appear in tensor powers of its vector representation $\mathbb{C}^N$, while the set $\Lambda(-,N+1)$ labels the irreducible representations of $Sp(N)$ for N even. In particular, we can identify the eigenprojections for the eigenvalues λ_1, λ_2 and λ_3 in $\mathcal{K}_2$ with the irreducible representations of $O(N)$ and $Sp(N)$ labeled by the Young diagrams [2], [1,1] and $\emptyset$. It follows from the definitions that a representation of $\mathcal{BM}_4$ labeled by the Young diagram μ has weight (i,j) with multiplicity m just corresponds to the fact that the $O(N)$ or $Sp(N)$ representation labeled by μ appears with multiplicity m in the tensor product of the representations corresponding to λ_i and λ_j.

Corollary 1.18. *(a) If $\lambda_2^2 = -\lambda_1\lambda_3$, we obtain a two-dimensional quotient of the $\mathcal{BM}_3$ representation $\{\lambda_1\lambda_2\lambda_3\}$ which is isomorphic to $\{\lambda_1\lambda_3\}$.*

(b) We obtain nontrivial quotients of the 6-dimensional representation $\{\lambda_1^3\lambda_2^2\lambda_3\}$ *as follows:*

(1) If $\lambda_1 = -\lambda_3$*, we obtain a 3-dimensional quotient* $\{\lambda_2|\lambda_1\lambda_3\}^*$ *in* $\{\lambda_1^3\lambda_2^2\lambda_3\}$*. It has the weights* $(22), (13)$ *and* (31)*, and it is isomorphic to* $\{\lambda_1\lambda_2\lambda_3\}$ *as a* $\mathcal{K}_3$ *module. The submodule is isomorphic to the generic representation* $\{\lambda_1^2\lambda_2\}$*.*

(2) If $\lambda_2 = -\lambda_3$*, we obtain a 4-dimensional quotient* $\{\lambda_1^2\lambda_2\lambda_3\}$ *with weights* $(12), (21), (13)$ *and* (31)*. It is isomorphic to* $\{\lambda_1\lambda_2\lambda_3\} + \{\lambda_1\}$ *as a* $\mathcal{K}_3$ *module. The submodule is the generic* K_4 *module* $\{\lambda_1\lambda_2\}$*.*

(3) If $\lambda_2^2 = -\lambda_1\lambda_3$*, we have a quotient and a submodule which are isomorphic to the generic Hecke algebra representations* $\{\lambda_1^2\lambda_3\}$ *and* $\{\lambda_1\lambda_2^2\}$*.*

(4) If $\lambda_1^3 = \lambda_2^2\lambda_3$*, we obtain a 5-dimensional quotient* $\{\lambda_1^2\lambda_2^2\lambda_3\}$ *with the weights* $(22), (12),$ $(21), (13)$ *and* (31)*. It is isomorphic to* $\{\lambda_1\lambda_2\lambda_3\} + \{\lambda_1\lambda_2\}$ *as a* $\mathcal{K}_3$ *module.*

(c) All subquotient modules described in (a) and (b) are simple, as long as the eigenvalues do not satisfy an additional relation.

Proof. The statements do not change if we multiply the images of the braid generators by a constant. It is therefore enough to prove the statements with the eigenvalues $\lambda_1 = q$, $\lambda_2 = -q^{-1}$ and $\lambda_3 = r^{-1}$.

We start with proving the statements in part (b). Observe that $\lambda_2 = -\lambda_3$ implies $r^{-1} = q^{-1}$. Hence we are in the case $(+, 1)$ corresponding to $O(2)$, where only diagrams are allowed with ≤ 2 boxes in the first two columns. It follows that the $\mathcal{BM}_4$ representations labeled by $[2]$ and $[1, 1]$ have dimensions 4 and 3 respectively. One derives from the tensor product rules of $O(2)$ that the representation $V_{[2]}$ appears in the tensor products $V_\emptyset \otimes V_{[2]}$ and $V_{[1,1]} \otimes V_{[2]}$. The same applies if we change the order of the tensor factors. Hence the image of $\mathcal{BM}_4$ in the representation labeled by $[2]$ has the weights $(1, 2), (2, 1), (1, 3)$ and $(3, 1)$. This proves (2), using the fact that the kernel of this representation must have weights $(1, 1)$ and $(2, 2)$. Similarly, we deduce from the tensor product rules of $O(2)$ that the representation $V_{[1,1]}$ appears in $V_{[2]} \otimes V_{[2]}$, and its tensor product with the trivial representation $V_\emptyset$. This shows the 3-dimensional quotient of $\{\lambda_2^3\lambda_1^2\lambda_3\}$ has the weights $(1, 1)$, $(2, 3)$ and $(3, 2)$. Applying the permutation automorphism for the transposition (12), we obtain statement (1).

Observe that $r = -q^3$ implies that $\lambda_2^2 = -\lambda_1\lambda_3$ for our choice of eigenvalues. So this corresponds to the case of $Sp(2)$, where only diagrams with one row are allowed. It follows that only the representations corresponding to $\{\lambda_1\}$ and $\{\lambda_3\}$ are allowed in the image of $\mathcal{BM}_2$ and the representation $\{\lambda_1\lambda_2\lambda_3\}$ has a quotient isomorphic to $\{\lambda_1\lambda_3\}$. This proves statement (a). We leave it to the reader to check that we get a 3-dimensional quotient for the $\mathcal{BM}_4$ representation labeled by $[2]$ with weights $(1, 1)$, $(1, 3)$ and $(3, 1)$ which corresponds to the usual representation $\{\lambda_1^2\lambda_3\}$. The kernel has the same dimension and weights as the usual $\{\lambda_2^2\lambda_1\}$.

One similarly links the case $\lambda_2^3 = \lambda_1^2\lambda_3$ to the $\mathcal{BM}_4$ representation with $r = -q^5$, which corresponds to $Sp(4)$. As the diagram $[1, 1, 1]$ is no longer allowed, the $\mathcal{BM}_4$ representation labeled by $[1, 1]$ only has dimension 5. It follows that the $\mathcal{K}_4$ representation $\{\lambda_2^3\lambda_1^2\lambda_3\}$ has a 5-dimensional quotient with weights $(1,1)$, $(2,1),(1,2)$, $(2,3)$ and $(3,2)$. Applying the permutation automorphism for the transposition (12) gives us (4).

It was shown in the proof of [25], Corollary 5.6 that the quotient modules are simple unless q is a root of unity. One can check that this is equivalent with the statement about additional relations. This statement can also be directly proved with the methods in this paper.

2. Weight bases

We assume for the whole section that we only deal with representations of $\mathcal{K}_4$ in which the images of the braid generators are diagonalizable.

2.1. **Weight bases and semisimplicity.** Recall the definition of weights in Section 1.5. We see in Table 1 that the weights have multiplicity at most 1 in all regular modules, except for $V = \{\lambda_r^4\lambda_s^2\lambda_t^2\}$, where the weight (r,r) has multiplicity 2. We also want to define canonical weight vectors in the latter case. In the following, let S_i be the matrix representing σ_i in a fixed regular representation, and let

$$P_{kr} = \prod_{s\neq r}(S_k - \lambda_s 1) \quad \text{and} \quad B(i,j) = P_{1i}P_{3j}. \tag{2.1}$$

It should be clear that $P_{kr} = (\lambda_r - \lambda_s)(\lambda_r - \lambda_t)P_k(\lambda_r)$, where $P_k(\lambda_r)$ is the eigenprojection of S_k for its eigenvalue λ_r in V, and that the image of

$$B(i,j) = P_{1i}P_{3j}$$

is indeed the weight space of (i,j). We will first need the following result, where the second statement of (b) can be skipped for the first reading.

Lemma 2.1. *(a) The element Δ_4 maps the weight space $V(i,j)$ onto the weight space $V(j,i)$ for any regular $\mathcal{K}_4$ module V.*

(b) If $V = \{\lambda_1^4\lambda_2^2\lambda_3^2\}$, Δ_4 acts on the 2-dimensional weight space $V(1,1)$ with eigenvalues $\pm\lambda_1^3(\lambda_2^3\lambda_3^3)^{1/2}$.

Proof. Part (a) is a direct consequence of Eq 1.1. For the proof of (b), we use the explicit 8-dimensional representation in [16], with $a = \lambda_r$, $b = \lambda_s$, $c = \lambda_t$. One calculates that the matrix B which represents $\Delta_4 P_1(\lambda_1)P_3(\lambda_1)$ has nonzero entries only in the third and fifth row. The 2×2 submatrix for these rows and columns is given by

$$a^3bc\begin{bmatrix} c & c(b-c)\\ 1 & -c\end{bmatrix}.$$

This is all one needs to check that the characteristic polynomial of B is equal to $\lambda^6(\lambda^2 - a^6b^3c^3)$.

Definition 2.2. Let V be a regular $\mathcal{K}_4$ module. If (r,s) is a weight of V with multiplicity 1, we call $P_1(\lambda_r)P_3(\lambda_r)$ its canonical weight projection. If the weight (r,r) has multiplicity 2, its canonical weight projections are defined to be the eigenprojections of $\Delta_4P_1(\lambda_r)P_3(\lambda_r)$ for its non-zero eigenvalues, after possibly having to adjoin a square root of $a^6b^3c^3$ to the ground ring. A *weight basis* for V is obtained by picking a nonzero vector in the image of each weight projection.

2.2. **Conditions for simplicity of 6-dimensional representations $\{\lambda_i^3\lambda_j^2\lambda_k\}$.** We will use weight projections to construct a full system of matrix units for regular representations whenever the eigenvalues satisfy certain equations. We use the notations from 2.1.

Lemma 2.3. *The 6-dimensional module $\{\lambda_1^3\lambda_2^2\lambda_3\}$ is simple except possibly if $\lambda_1 = -\lambda_3$, $\lambda_2 = -\lambda_3$, $\lambda_1^3 = \lambda_2^2\lambda_3$ or $\lambda_2^2 = -\lambda_1\lambda_3$.*

Proof. It follows from Definition 2.2 that the module $\{\lambda_1^3\lambda_2^2\lambda_3\}$ has a basis of weight vectors whose weights are listed in Table 1. Using the explicit matrix representations in [16] and, preferably, some symbolic computing system, we calculate the quantity $d(i,j)$ from the identity

$$B(i,j)P_{23}B(i,j) = d(i,j)B(i,j).$$

The quantities $d(i,j)$ are the diagonal entries of P_{23}, up to a constant multiple involving factors of the form $\lambda_i - \lambda_j$. It follows from 1.1 that conjugation by Δ_4 fixes P_{23} and permutes $B(i,j)$ with $B(j,i)$; this implies $d(i,j) = d(j,i)$. We obtain the values

$$\begin{aligned} & & d(1,1) &= (\lambda_1-\lambda_3)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)(\lambda_1^3-\lambda_2^2\lambda_3),\\ & & d(2,2) &= \lambda_1(\lambda_2-\lambda_3)^2(\lambda_2+\lambda_3)(\lambda_2^2+\lambda_1\lambda_3),\\ d(1,2) &= & d(2,1) &= -\lambda_2(\lambda_1-\lambda_3)^2(\lambda_1+\lambda_3)(\lambda_2^2+\lambda_1\lambda_3),\\ d(1,3) &= & d(3,1) &= (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1+\lambda_2)^2\lambda_3^2. \end{aligned} \tag{2.2}$$

It follows from Lemma 1.13 that the module $V = \{\lambda_1^3\lambda_2^2\lambda_3\}$ is simple if all diagonal entries of the rank 1 matrix P_{23} are nonzero. This proves the claim, except that it could not be semisimple also if $\lambda_1 = -\lambda_2$. In this case, we still have a simple submodule U of dimension at least 4, containing the weights (11), (12), (21) and (2,2), as the corresponding diagonal entries of P_{23} are nonzero. As P_{12} equals the sum of the two weight projections for (2,1) and (2,2) in our representation, it follows that U must contain all $\mathcal{K}_3$ submodules in which P_{12} acts nonzero, i.e. $\{\lambda_1\lambda_2\lambda_3\}$ and $\{\lambda_1\lambda_2\}$. These are simple for $\lambda_1 = -\lambda_2$, so U must have dimension at least 5. It follows that it must have one of the weights of (13) or (31), and therefore, by Lemma 2.1(a), both weights. This shows that $U = V$.

2.3. **Conditions for simplicity of 8-dimensional and 9-dimensional representations.**

Lemma 2.4. *Assume that $\{r,s,t\} = \{1,2,3\}$ and that $\lambda_r \neq \lambda_s$ for $r \neq s$.*

(a) The representation $\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$ is simple except if $\lambda_r = -\theta\lambda_s$ or $\lambda_r\lambda_s = \theta\lambda_t^2$.

(b) The representation $\{\lambda_1^4\lambda_2^2\lambda_3^2\}$ is simple, except possibly if $\lambda_2 = -\lambda_3$, $\lambda_3^3 = \lambda_1^2\lambda_2$, $\lambda_2^3 = \lambda_1^2\lambda_3$ or $\lambda_1^6 = \lambda_2^3\lambda_3^3$.

Proof. The proof is similar to the one of Lemma 2.3, using Lemma 1.13. Let us do this first for the module $\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$. We define the scalar $\alpha_{ki,lj}$ by

$$B(k,i)S_2B(l,j)S_2B(k,i) = \alpha_{ki,lj}B(k,i).$$

It follows from an explicit calculation using the representation in [16] that

$$\alpha_{11,12} \;=\; -\theta\lambda_1\lambda_2^2\lambda_3^2(\lambda_3+\theta\lambda_2)(\lambda_2+\theta\lambda_1)(\lambda_1\lambda_2-\theta\lambda_3^2)(\lambda_1\lambda_3-\theta\lambda_2^2)/(\lambda_1-\lambda_2).$$

Applying the permutation automorphism for the transposition (23), see Lemma 1.15, we obtain $\alpha_{11,13} = (23)(\alpha_{11,12})$. It follows by inspection that $\alpha_{11,12}$ and $\alpha_{11,13}$ are not equal to zero if the eigenvalues satisfy the conditions in the statement. Hence the weights $(1,j)$, $1 \le j \le 3$ belong to the same equivalence class of indices, see 1.4. Applying an automorphism from Lemma 1.15 for a permutation π with $\pi(1) = i$, $i = 2,3$, we obtain the same statement for the weights (i,j), $1 \le j \le 3$. As (i,j) and (j,i) are equivalent by Remark 1.11 for $1 \le i,j \le 3$, it follows that all weights (i,j) are equivalent as long as $\pi(\alpha_{11,12}) \ne 0$ for all $\pi \in S_3$. The claim now follows from Lemma 1.13.

We can proceed similarly for the representations $\{\lambda_i^4\lambda_j^2\lambda_k^2\}$. By symmetry, it suffices to do this for $V = \{\lambda_1^4\lambda_2^2\lambda_3^2\}$ as follows. We define $\alpha_{ij,kl}$ as before. We first calculate by brute force

$$\alpha_{21,23} = \lambda_1^3\lambda_3(\lambda_3^3-\lambda_1^2\lambda_2)(\lambda_2+\lambda_3)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3).$$

As the permutation $(23) : \lambda_2 \leftrightarrow \lambda_3$ gives us an equivalent representation, we can argue as in the last paragraph that $\alpha_{31,32} = (23)(\alpha_{21,23})$ is nonzero whenever $\lambda_i^3 \ne \lambda_1^2\lambda_j$, where $\{i,j\} = \{2,3\}$, and $\lambda_2 \ne -\lambda_3$. Hence $(2,1) \leftrightarrow (2,3)$ and $(3,1) \leftrightarrow (3,2)$, with $\leftrightarrow$ as defined before Lemma 1.13. We can show $(i,j) \leftrightarrow (j,i)$ as in the previous paragraph. This already implies that our representation $\{\lambda_1^4\lambda_2^2\lambda_3^2\}$ has a simple subquotient module containing the span W of all vectors with weights (i,j), $i \ne j$. Assume W is contained in a proper submodule U of V. Then V/W and hence also V/U would only have the weight $(1,1)$. This entails that Δ_4^2 would act on V/U via the scalar $\lambda_1^{12} = \lambda_1^6\lambda_2^3\lambda_3^3$, which implies $\lambda_1^6 = \lambda_2^3\lambda_3^3$.

2.4. **Semisimplicity.** We can now almost prove one direction of one of the main results of this paper.

Proposition 2.5. *Assume $\lambda_i \ne \lambda_j$ for $i \ne j$. A regular module V is simple unless the eigenvalues satisfy one of the polynomials listed next to it in Table 2 below, or, possibly, if they satisfy $\lambda_j+\lambda_k = 0$ or $\lambda_i^2 = \lambda_j\lambda_k$ for the module $\{\lambda_i^4\lambda_j^2\lambda_k^2\}$. In particular, $\mathcal{K}_4$ is semisimple if the eigenvalues do not satisfy any of the polynomials listed here.*

Proof. It suffices to prove that all regular modules are simple. Such modules with different labels also have different weights, see Table 1. Hence they are not isomorphic, which will imply semisimplicity of $\mathcal{K}_4$. For the Iwahori-Hecke algebra modules in $\mathcal{K}_4$, i.e. modules in which σ_1 satisfies a quadratic equation, it is well-known that they are not simple if and only if $(-\lambda_i/\lambda_j)$ is an n-th root of unity, $n = 2,3,4$, see the discussion in Sections 1.3 and 1.7. The module $\{\lambda_1\lambda_2\lambda_3\}$ is not simple if and only if $\lambda_i^2+\lambda_j\lambda_k = 0$, by Theorem 1.4 and Corollary 1.5. It follows from Lemmas 2.3 and 2.4 that also the 6-, 8- and 9-dimensional modules have to be simple if the eigenvalues do not satisfy one of the equations in the statement. This is summarized in Table 2. It will be shown in Theorem 4.4 that the module V is indeed not semisimple if the eigenvalues are in the locus of one of the ideals listed in Table 2.

regular module V	not semisimple
$\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$	$(\lambda_i+\theta\lambda_j)$, $(\lambda_i^2-\theta\lambda_j\lambda_k)$
$\{\lambda_1^4\lambda_2^2\lambda_3^2\}$	$(\lambda_2^3-\lambda_1^2\lambda_3)$, $(\lambda_3^3-\lambda_1^2\lambda_2)$, $(\lambda_1^2-\theta^{\pm1}\lambda_2\lambda_3)$
$\{\lambda_1^3\lambda_2^2\lambda_3\}$	$(\lambda_1+\lambda_3)$, $(\lambda_2+\lambda_3)$, $(\lambda_2^2+\lambda_1\lambda_3)$, $(\lambda_1^3-\lambda_2^2\lambda_3)$
$\{\lambda_1\lambda_2\lambda_3\}$	$(\lambda_i^2+\lambda_j\lambda_k)$
$\{\lambda_1^2\lambda_2\}$	$(\lambda_1\pm\sqrt{-1}\lambda_2)$
$\{\lambda_1\lambda_2\}$	$(\lambda_1+\theta\lambda_2)$

Table 2

2.5. **Existence of path bases.** Path bases do not exist in general. The following lemma will show that we can use them when $\mathcal{K}_4$ is not semisimple.

Lemma 2.6. *Let V be a regular module, and let $\mathfrak{p}$ be one of the prime ideals listed next to it in Table 2. Then V has a path basis over the quotient field $Q(R/\mathfrak{p})$.*

Proof. As we assumed $\sigma_1=\Delta_2$ to be diagonalizable, it is sufficient to check that Δ_3^2 acts via different eigenvalues on the $\mathcal{K}_3$ submodules of V. This is fairly straightforward to check, using Table 1. E.g if $V=\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$, such eigenvalues would only coincide if $\lambda_i^3=\lambda_j^3$ or $\lambda_i^2=-\lambda_j\lambda_k$, where $\{i,j,k\}=\{1,2,3\}$. This is not possible in $R/\mathfrak{p}$ for $\mathfrak{p}=(\lambda_i+\theta\lambda_j)$ or $\mathfrak{p}=(\lambda_i^2-\theta\lambda_j\lambda_k)$, as none of the former polynomials is in the later ideals. The same strategy works for $V=\{\lambda_1^4\lambda_2^2\lambda_3^2\}$.

One similarly observes that the eigenvalues of Δ_3^2 for the $\mathcal{K}_3$ summands in $V=\{\lambda_1^3\lambda_2^2\lambda_3\}$ coincide only if $\lambda_1^4=\lambda_2^2\lambda_3^2$, $\lambda_3^2=-\lambda_1\lambda_2$ or $\lambda_1^3=-\lambda_2^3$. This again is not possible in $R/\mathfrak{p}$ for $\mathfrak{p}$ as in Table 2 for our given V.

3. Path representations

The main result of this section is the calculation of matrix entries for the braid generators and their eigenprojections for a path basis of a regular module. This will allow us to determine possible subquotient modules, in view of Lemma 1.14.

3.1. **Affine braid group AB_2.** The affine braid group AB_2 is defined via generators τ and σ and relation $\sigma\tau\sigma\tau=\tau\sigma\tau\sigma$. We will review how any path representation of braid groups will give us representations of the affine braid group AB_2 on a finite dimensional vector space W. More precisely, we will obtain matrices A and T which satisfy the relation

$$ATAT=TATA \tag{3.1}$$

$$ATAT=\delta 1_W, \tag{3.2}$$

acting on a vector space W with a basis $\{v_r\}$ such that

$$Tv_r = x_r v_r, \quad x_i \neq x_j \text{ for } i \neq j. \tag{3.3}$$

It follows from 3.2 and 3.3 that

$$(A^{-1})_{rs} = \frac{x_r x_s}{\delta} a_{rs}. \tag{3.4}$$

3.2. **Generalized quantum Jucys-Murphy elements.** We recall how one can obtain representations of AB_2 from representations of ordinary braid groups using (quantum) Jucys-Murphy elements. This was first done for Hecke algebras in [4].

Lemma 3.1. *Let V be a B_{n+1} module with a path basis, labeled by ν. Let λ be a label for a simple B_{n-1} module which appears if V is viewed as a B_{n-1} module, and let $W(\lambda,\nu)$ be the vector space with a basis labeled by all paths of length 2 from λ to ν. Then the assignment*

$$\tau \mapsto T = (\Delta_n^2)_{|W(\lambda,\nu)}, \qquad \sigma \mapsto A = (\sigma_n)_{|W(\lambda,\nu)},$$

makes $W(\lambda,\nu)$ into an AB_2 module satisfying 3.1, 3.2, with $\delta = \Delta_{n-1}^2(\lambda)\Delta_{n+1}^2(\nu)$, and 3.3, except possibly that some of the x_rs coincide; here x_r is given by the scalar $\Delta_n^2(\mu_r)$ for the path $r : \lambda \to \mu_r \to \nu$, and $\Delta_k^2(\gamma)$ is the scalar via which Δ_k^2 acts on a simple B_k module labeled by γ.

Proof. This is well-known. We give some details to the readers who want to prove it for themselves. It is probably easiest shown using pictures as in e.g. [11] that $\gamma_n = \Delta_n^2\Delta_{n-1}^{-2} = \sigma_{n-1} \dots \sigma_2\sigma_1^2\sigma_2 \dots \sigma_{n-1}$. We then have

$$\sigma_n\gamma_n\sigma_n\gamma_n = \gamma_{n+1}\gamma_n = \gamma_n\gamma_{n+1} = \gamma_n\sigma_n\gamma_n\sigma_n. \tag{3.5}$$

Hence

$$\tau \mapsto \gamma_n, \qquad \sigma \mapsto \sigma_n,$$

defines a group homomorphism from AB_2 into B_{n+1}. As Δ_{n-1}^2 commutes with both γ_n and with σ_n, this remains a group homomorphism if we replace γ_n by $\gamma_n\Delta_{n-1}^2 = \Delta_n^2$. To prove the claimed value of δ for 3.2, observe that $\gamma_{n+1}\gamma_n = \Delta_{n+1}^2\Delta_{n-1}^{-2}$, which acts via the scalar $\alpha\alpha'^{-1}$ on W. Hence $\sigma_n\Delta_n^2\sigma_n\Delta_n^2 = \gamma_{n+1}\gamma_n\Delta_{n-1}^4$ acts via the scalar $\alpha\alpha'^{-1}\alpha'^2$, as claimed.

3.3. **Representations with A having at most 2 eigenvalues.** We consider representations of AB_2 on a vector space W satisfying 3.1, 3.2, 3.3 and $(A - \lambda_1 1)(A - \lambda_2 1) = 0$. The following is a version of the well-known quantum Jucys-Murphy construction, see [4]

Lemma 3.2. *(a) If A acts via the scalar λ on a basis vector v_r, then $\delta = x_r^2\lambda^2$. Hence we only have two possible values $x_r = \pm\sqrt{\delta}/\lambda$ of T for given δ.*

(b) If A only has two eigenvalues λ_1 and λ_2, with λ_1 having multiplicity 1, then $\dim W \leq 4$. *We have at most one irreducible 2-dimensional subrepresentation span$\{v_r, v_s\}$ of AB_2. The diagonal entries $d_r(\lambda_1)$ and $d_s(\lambda_1)$ of the eigenprojection of A for its eigenvalue λ_1 with respect to this basis are given by*

$$d_r(\lambda_1) = -\frac{\lambda_1 x_s + \lambda_2 x_r}{(x_r - x_s)(\lambda_1 - \lambda_2)}, \qquad d_s(\lambda_1) = \frac{\lambda_1 x_r + \lambda_2 x_s}{(x_r - x_s)(\lambda_1 - \lambda_2)}. \tag{3.6}$$

Proof. Part (a) is obvious. For part (b), it follows from 3.4 and $A+\lambda_1\lambda_2A^{-1}=(\lambda_1+\lambda_2)I$ that

$$(\delta+\lambda_1\lambda_2x_rx_s)a_{rs}=\delta(\lambda_1+\lambda_2)\delta_{rs}.$$

Hence $a_{rs}=0$ or $\delta+\lambda_1\lambda_2x_rx_s=0$ for $r\neq s$. As $x_s\neq x_{s'}$ for $s\neq s'$ by assumption, there is at most one s for given r for which this is the case. As $a_{rs}\neq 0$ for these r,s, we deduce from this that

$$\delta=-\lambda_1\lambda_2x_rx_s.$$

and

$$a_{rr}=\frac{\delta(\lambda_1+\lambda_2)}{\delta+\lambda_1\lambda_2x_r^2},\qquad a_{ss}=\frac{\delta(\lambda_1+\lambda_2)}{\delta+\lambda_1\lambda_2x_s^2}.$$

By definition, $d_r(\lambda_1)$ is the r-th diagonal entry of the eigenprojection $P=(A-\lambda_2I)/(\lambda_1-\lambda_2)$ of A for the eigenvalue λ_1. We obtain the claimed result from this via a straightforward calculation. The restriction of A to span $\{v_r,v_s\}$ has determinant $\lambda_1\lambda_2$ from which we calculate

$$a_{rs}a_{sr}=\frac{-(\lambda_2x_r+\lambda_1x_s)(\lambda_2x_s+\lambda_1x_r)}{(x_r-x_s)^2}.$$

Hence we obtain a 2-dimensional irreducible subrepresentation of W spanned by v_r and v_s. As λ_1 appears with multiplicity 1 in W, the complement of this subrepresentation is contained in the eigenspace of A with eigenvalue λ_2. By (a), we only have two possibilities for the eigenvalue x_r for the basis vector of a one-dimensional representation of AB_2.

3.4. **Examples.** We will need the following examples of two-dimensional representations of AB_2. It will be convenient in one case to call the second eigenvalue λ_3 in stead of λ_2, so we also list the eigenvalues of A.

paths	x_1	x_2	e.values	$d_1(\lambda_1)$	$d_2(\lambda_1)$
$\{1\}\to\{\lambda_1\lambda_2\}$	λ_1^2	λ_2^2	λ_1,λ_2	$\frac{-\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}$	$\frac{\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2}{(\lambda_1-\lambda_2)^2}$
$\{\lambda_2\}\to\{\lambda_1^3\lambda_2^2\lambda_3\}$	$(\lambda_1\lambda_2\lambda_3)^2$	$-(\lambda_1\lambda_2)^3$	λ_1,λ_2	$\frac{\lambda_2(\lambda_1^2-\lambda_3^2)}{(\lambda_1-\lambda_2)(\lambda_3^2+\lambda_1\lambda_2)}$	$\frac{\lambda_1(\lambda_3^2-\lambda_2^2)}{(\lambda_1-\lambda_2)(\lambda_3^2+\lambda_1\lambda_2)}$
$\{\lambda_3\}\to\{\lambda_1^4\lambda_2^2\lambda_3^2\}$	$(\lambda_1\lambda_2\lambda_3)^2$	$-(\lambda_1\lambda_3)^3$	λ_1,λ_2	$\frac{\lambda_1^2\lambda_3-\lambda_2^3}{(\lambda_1-\lambda_2)(\lambda_2^2+\lambda_1\lambda_3)}$	$\frac{\lambda_1\lambda_2(\lambda_2-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_2^2+\lambda_1\lambda_3)}$
$\{\lambda_2\}\to\{\lambda_1^4\lambda_2^2\lambda_3^2\}$	$(\lambda_1\lambda_2\lambda_3)^2$	$-(\lambda_1\lambda_2)^3$	λ_1,λ_3	$\frac{\lambda_1^2\lambda_2-\lambda_3^3}{(\lambda_1-\lambda_3)(\lambda_3^2+\lambda_1\lambda_2)}$	$\frac{\lambda_1\lambda_3(\lambda_3-\lambda_2)}{(\lambda_1-\lambda_3)(\lambda_3^2+\lambda_1\lambda_2)}$

Here $\{1\}\to\{\lambda_1\lambda_2\}$ stands for the two paths for the two-dimensional representation $\{\lambda_1\lambda_2\}$ of K_3, and $\gamma\to\nu$ stands for the two paths of length 2 from the K_2 representation λ to the K_4 representation ν in lines 2 to 4.

3.5. **More complicated representations of AB_2.** We now consider representations of AB_2 where A satisfies a cubic equation, and, moreover, one of its eigenvalues, say λ_3, has multiplicity 1. So if P is the corresponding eigenprojection for λ_3, the matrices A and T satisfy, besides relations 3.1, 3.2 and 3.3, also the relation

$$(A-\lambda_1 I)(A-\lambda_2 I)(A-\lambda_3 I)=0 \quad \text{and} \quad \text{rank } P=1. \tag{3.7}$$

It follows from these conditions that

$$A+\lambda_1\lambda_2 A^{-1}=(\lambda_1+\lambda_2)I+\lambda_3^{-1}(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)P. \tag{3.8}$$

We deduce from the previous equation and 3.4 that

$$\frac{\delta+\lambda_1\lambda_2 x_i x_j}{\delta}a_{ij}=(\lambda_1+\lambda_2)\delta_{ij}+\lambda_3^{-1}(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)p_{ij}. \tag{3.9}$$

The following lemma and theorem are slight improvements of [26], Proposition 5.7, given in notation more suitable for our purposes.

Lemma 3.3. *Assume W is an AB_2 module satisfying conditions 3.1, 3.2 and 3.3, with $n\geq 2$. Moreover, we assume A has at most three eigenvalues, one of which, say μ, has multiplicity 1. If $\delta\neq\lambda_k^2x_r^2$ for all eigenvalues $\lambda_k\neq\mu$, and $1\leq r\leq n$, then A and T generate the full $n\times n$ matrix algebra.*

Proof. Let us first do the more complicated case with A having three eigenvalues, with $\mu=\lambda_3$. As P has rank 1, we can write it as $P=vw^T$ for two vectors $v,w\in F^n$. Assume $v_i=0$. Then it follows from 3.9 and our assumption $\delta+\lambda_1\lambda_2x_ix_j\neq 0$ that $a_{ij}=0$ for $j\neq i$. Hence a_{ii} must be an eigenvalue of A. It can not be equal to λ_3 as the i-th column of its eigenprojection P would be equal to 0. Hence $a_{ii}\in\{\lambda_1,\lambda_2\}$. If $a_{ii}=\lambda_1$, it follows from 3.9 that $\lambda_1(1+\delta^{-1}\lambda_1\lambda_2x_i^2)=\lambda_1+\lambda_2$, from which one deduces $\lambda_1^2x_i^2=\delta$. One similarly shows that $a_{ii}=\lambda_2$ implies $\lambda_2^2x_i^2=\delta$. One proves that $w_i=0$ implies $\lambda_k^2x_i^2=0$ for $k\in\{1,2\}$ the same way. We can therefore assume that $v_rw_r\neq 0$ for $1\leq r\leq n$. Hence all entries of P are nonzero. Moreover, as all the x_is are distinct, we can also obtain the diagonal matrix entries as eigenprojections of T. This proves that A and T generate the full $n\times n$ matrix algebra.

If A only has two eigenvalues, the claim follows from Lemma 3.2.

Theorem 3.4. *(generalized q-Jucys-Murphy, see [26]) We assume a representation of AB_2 satisfying conditions 3.1, 3.2, 3.3 and 3.7 with $(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\neq 0$. If $\delta+\lambda_1\lambda_2x_rx_s\neq 0$ for $1\leq r,s\leq n$ and $\delta\notin\{\lambda_k^2x_r^2,\ k=1,2,\ 1\leq r\leq n\}$, then*

(a) the diagonal entries $d_r(\lambda_3)$ of P are nonzero and

(b) they are given by

$$d_r(\lambda_3) \;=\; \frac{\lambda_3\ b(r,3)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\ \prod_{s\neq r}\frac{\delta+\lambda_1\lambda_2x_rx_s}{\delta(x_r-x_s)},$$

where

$$b(r,3) \;=\; (-1)^{n-1}\lambda_3\frac{\lambda_1\lambda_2x_r+\delta x_r^{-1}}{\delta}\ (\prod_{t=1}^n x_t)-(\lambda_1+\lambda_2)(\frac{-\delta}{\lambda_1\lambda_2})^{(n-1-\varepsilon)/2}x_r^{\varepsilon},$$

and $\varepsilon = 0$ for n odd and $\varepsilon = 1$ for n even. If all $d_r(\lambda_3) \neq 0$, our AB_2 representation is irreducible, and $A = (a_{rt})$ is given uniquely, up to conjugation by a diagonal matrix by

$$a_{rt} = \frac{b(r,3)}{x_r - x_t} \prod_{s \neq r,t} \frac{\delta + \lambda_1\lambda_2 x_r x_s}{\delta(x_r - x_s)}.$$

Proof. This is essentially just a version of the proof of [26], Proposition 5.7 in our more general notation. We outline the main steps for the readers' convenience. Let $P = vw^T$ be as in the proof of Lemma 3.3. As $v_i \neq 0$, it follows from $Av = \lambda_3 v$, after solving for a_{ij} in Eq 3.9, that

$$\sum_j \frac{\delta}{\delta + \lambda_1\lambda_2 x_i x_j}((\lambda_1+\lambda_2)\delta_{ij} \ + \ \frac{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}{\lambda_3} v_i w_j)v_j \ = \ \lambda_3 v_i.$$

Dividing by v_i and setting $d_j = v_j w_j$, we obtain the linear system

$$\sum_j \frac{\delta(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{\lambda_3(\delta + \lambda_1\lambda_2 x_i x_j)} \, d_j \ = \ \lambda_3 - \frac{\delta(\lambda_1+\lambda_2)}{\delta + \lambda_1\lambda_2 x_i^2}.$$

This can be transformed to

$$\sum_j \frac{1}{1 + \lambda_1\lambda_2 x_i x_j/\delta} \, d_j \ = \ \frac{\lambda_3^2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} \ - \ \frac{\lambda_3(\lambda_1+\lambda_2)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(1+\lambda_1\lambda_2 x_i^2/\delta)}.$$

This system can now be solved after substituting $\tilde{x}_i = (-\lambda_1\lambda_2/\delta)^{1/2} x_i$, using Lemmas 5.5 and 5.6 from [26]. This yields the expressions for $d_r(\lambda_3)$ as stated.

If all $d_r(\lambda_3) \neq 0$, the rank 1 matrix P is uniquely determined by its diagonal entries, up to conjugation by a diagonal matrix. Setting $P = (d_r(\lambda_3))_{rs}$, we obtain the matrix A from 3.9 as claimed.

3.6. **Examples.** We now give some examples of AB_2 representations of dimension 3, which will be needed later. In this case, the formula for the r-th diagonal entry $d_r(\lambda_3)$ of the eigenprojection P of A for the eigenvalue λ_3 becomes

$$d_r(\lambda_3) \ = \ \frac{\lambda_3}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} [\lambda_3 \frac{\lambda_1\lambda_2 x_r + \delta x_r^{-1}}{\delta} (\prod_{t=1}^{3} x_t) + (\lambda_1+\lambda_2)\frac{\delta}{\lambda_1\lambda_2})] \prod_{s\neq r} \frac{\delta + \lambda_1\lambda_2 x_r x_s}{\delta(x_r - x_s)}.$$

(1) We first consider the 3-dimensional representation of $\mathcal{K}_3$, with three paths $\{1\} \to \{\lambda_1\lambda_2\lambda_3\}$ of length 2. We have eigenvalues $x_i = \lambda_i^2$ of T for $1 \leq i \leq 3$, and $\delta = (\lambda_1\lambda_2\lambda_3)^2$. We then calculate the i-th diagonal entry $d_i(\lambda_j)$ of the eigenprojection of A for the eigenvalue λ_j as

$$d_i(\lambda_j) = \frac{(\lambda_i^2 + \lambda_j\lambda_k)(\lambda_j^2 + \lambda_i\lambda_k)}{(\lambda_i-\lambda_j)^2(\lambda_i-\lambda_k)(\lambda_k-\lambda_j)}, \quad i \neq j,$$

where $\{i,j,k\} = \{1,2,3\}$, and

$$d_i(\lambda_i) = \frac{(\lambda_j+\lambda_k)^2\lambda_i^2}{(\lambda_j-\lambda_i)^2(\lambda_i-\lambda_k)^2}.$$

This can be checked explicitly for $d_i(\lambda_3)$ using Theorem 3.4. The other cases are obtained from this using the permutation automorphisms in Lemma 1.15.

(2) We next consider the case for the three paths $\{\lambda_1\} \to \{\lambda_1^3\lambda_2^2\lambda_3\}$, with $x_1 = (\lambda_1\lambda_2\lambda_3)^2$, $x_2 = -(\lambda_1\lambda_2)^3$, $x_3 = \lambda_1^6$ and $\delta = \lambda_1^8\lambda_2^4\lambda_3^2$. Let $d_r(\lambda_i)$ be the r-th diagonal entry of the eigenprojection of A for the eigenvalue λ_i. Moreover, our indices satisfy $\{i,j,k\} = \{r,s,t\} = \{1,2,3\}$. Then the formula for the diagonal entries simplifies to

$$d_r(\lambda_i) \;=\; \frac{\lambda_i\lambda_1^4\lambda_2^2\lambda_3}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\,[(\lambda_j^{-1}+\lambda_k^{-1})\lambda_1^4\lambda_2^2\lambda_3 - x_r - \lambda_1^7\lambda_2^3\lambda_3\lambda_i x_r^{-1})]\prod_{s\neq r}\frac{1+\delta^{-1}\lambda_j\lambda_k x_r x_s}{x_s-x_r}.$$

We calculate the expression in straight brackets above for all values of r and i below. E.g. for $i=2$ the formula above would simplify to $(\lambda_1+\lambda_3)\lambda_1^3\lambda_2^2 - x_r - \lambda_1^7\lambda_2^4\lambda_3 x_r^{-1}$.

$r\backslash i$	1	2	3
1	$\lambda_1^2\lambda_2\lambda_3^{-1}(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2\lambda_3)$	$-\lambda_1^2\lambda_2^2\lambda_3^{-1}(\lambda_1-\lambda_3)^2(\lambda_1+\lambda_3)$	$-\lambda_1^2\lambda_2(\lambda_1^2-\lambda_2\lambda_3)(\lambda_1-\lambda_3)$
2	$\lambda_1^3(\lambda_2^2+\lambda_1\lambda_3)(\lambda_1+\lambda_2)$	$\lambda_1^3\lambda_2(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)$	$\lambda_1^3(\lambda_2^2+\lambda_1\lambda_3)(\lambda_2+\lambda_3)$
3	$-\lambda_1^2(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_2\lambda_3)$	$-\lambda_1(\lambda_1^2-\lambda_2^2)(\lambda_1^3-\lambda_2^2\lambda_3)$	$-\lambda_1(\lambda_1^3-\lambda_2^2\lambda_3)(\lambda_1^2-\lambda_2\lambda_3)$

We also have

$$x_1-x_2 = \lambda_1^2\lambda_2^2(\lambda_3^2+\lambda_1\lambda_2), \quad x_2-x_3 = -\lambda_1^3(\lambda_2^3+\lambda_1^3), \quad x_3-x_1 = \lambda_1^2(\lambda_1^4-\lambda_2^2\lambda_3^2).$$

In the following table we calculate the quantity $1+\lambda_j\lambda_k x_r x_s/\delta = 1-\lambda_1^4\lambda_2^2\lambda_3/\lambda_i x_t$, where $\{r,s,t\} = \{1,2,3\}$.

$i\backslash t$	1	2	3
1	$1-\lambda_1\lambda_3^{-1}$	$1+\lambda_3\lambda_2^{-1}$	$1-\lambda_1^{-3}\lambda_2^2\lambda_3$
2	$1-\lambda_1^2\lambda_2^{-1}\lambda_3^{-1}$	$1+\lambda_1\lambda_2^{-2}\lambda_3$	$1-\lambda_1^{-2}\lambda_2\lambda_3$
3	$1-\lambda_1^2\lambda_3^{-2}$	$1+\lambda_1\lambda_2^{-1}$	$1-\lambda_1^{-2}\lambda_2^2$

For $i=1$ we obtain the values

$$\begin{aligned} d_1(\lambda_1) &= \frac{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)(\lambda_1^3-\lambda_2^2\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1^2+\lambda_2\lambda_3)(\lambda_3^2+\lambda_1\lambda_2)} \\ d_2(\lambda_1) &= \frac{-(\lambda_2^2+\lambda_1\lambda_3)(\lambda_1^3-\lambda_2^2\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_3^2+\lambda_1\lambda_2)(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)}, \\ d_3(\lambda_1) &= \frac{\lambda_1^2\lambda_2(\lambda_2+\lambda_3)}{(\lambda_1^2+\lambda_2\lambda_3)(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)}. \end{aligned} \tag{3.10}$$

For $i=2$ we obtain the values

$$\begin{aligned}
d_1(\lambda_2) &= \frac{-\lambda_2(\lambda_1+\lambda_3)(\lambda_1-\lambda_3)^2(\lambda_2^2+\lambda_1\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1^2+\lambda_2\lambda_3)(\lambda_3^2+\lambda_1\lambda_2)},\\
d_2(\lambda_2) &= \frac{\lambda_2(\lambda_2+\lambda_3)(\lambda_1^2-\lambda_2\lambda_3)^2}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3^2+\lambda_1\lambda_2)(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)},\\
d_3(\lambda_2) &= \frac{-(\lambda_1^3-\lambda_2^2\lambda_3)(\lambda_2^2+\lambda_1\lambda_3)}{(\lambda_2-\lambda_3)(\lambda_1^2+\lambda_2\lambda_3)(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)}.
\end{aligned}\tag{3.11}$$

For $i=3$ we obtain the values

$$\begin{aligned}
d_1(\lambda_3) &= \frac{\lambda_3^2(\lambda_1+\lambda_2)^2(\lambda_1-\lambda_2)}{(\lambda_2-\lambda_3)(\lambda_1^2+\lambda_2\lambda_3)(\lambda_3^2+\lambda_1\lambda_2)},\\
d_2(\lambda_3) &= \frac{-(\lambda_2+\lambda_3)(\lambda_1-\lambda_2)(\lambda_1+\lambda_3)(\lambda_2^2+\lambda_1\lambda_3)}{(\lambda_3^2+\lambda_1\lambda_2)(\lambda_2-\lambda_3)(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)},\\
d_3(\lambda_3) &= \frac{\lambda_2(\lambda_1+\lambda_3)(\lambda_1^3-\lambda_2^2\lambda_3)}{(\lambda_2-\lambda_3)(\lambda_1^2+\lambda_2\lambda_3)(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)}.
\end{aligned}\tag{3.12}$$

(3) We consider the case with three paths $\{\lambda_1\}\to\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$, where we have $x_1=-(\lambda_1\lambda_2)^3$, $x_2=-(\lambda_1\lambda_3)^3$, $x_3=(\lambda_1\lambda_2\lambda_3)^2$ and $\delta=\theta\lambda_1^6\lambda_2^4\lambda_3^4$. We then obtain the diagonal entries (for the eigenprojection of the eigenvalue λ_3)

$$\begin{aligned}
d_1(\lambda_3) &= \frac{\theta(\lambda_1+\theta\lambda_3)(\lambda_2+\theta\lambda_3)(\lambda_3+\theta\lambda_2)(\lambda_3^2-\theta\lambda_1\lambda_2)}{(\lambda_2-\lambda_3)^2(\lambda_3-\lambda_1)(\lambda_3^2+\lambda_1\lambda_2)},\\
d_2(\lambda_3) &= \frac{\theta^{-1}\lambda_2\lambda_3(\lambda_1+\theta\lambda_3)(\lambda_2^2-\theta\lambda_1\lambda_3)}{(\lambda_2-\lambda_3)^2(\lambda_1-\lambda_3)(\lambda_2^2+\lambda_1\lambda_3)},\\
d_3(\lambda_3) &= \frac{\theta\lambda_1\lambda_3(\lambda_1+\theta\lambda_2)(\lambda_3-\theta\lambda_2)^2(\lambda_3+\theta\lambda_2)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_2^2+\lambda_1\lambda_3)(\lambda_3^2+\lambda_1\lambda_2)}.
\end{aligned}\tag{3.13}$$

The diagonal entries for the eigenprojection of λ_2 are obtained from the ones just calculated by permuting λ_2 with λ_3 and then by permuting d_1 with d_2. The diagonal entries for the eigenprojection of λ_1 are given by

$$\begin{aligned}
d_1(\lambda_1) &= \frac{\theta^{-1}\lambda_1\lambda_3(\lambda_2^2-\theta\lambda_1\lambda_3)(\lambda_2+\theta\lambda_3)}{(\lambda_2-\lambda_3)(\lambda_3^2+\lambda_1\lambda_2)(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)},\\
d_2(\lambda_1) &= \frac{\theta^{-1}\lambda_1\lambda_2(\lambda_3^2-\theta\lambda_1\lambda_2)(\lambda_3+\theta\lambda_2)}{(\lambda_2-\lambda_3)(\lambda_2^2+\lambda_1\lambda_3)(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)},\\
d_3(\lambda_1) &= \frac{-\theta(\lambda_2^2-\theta\lambda_1\lambda_3)(\lambda_3^2-\theta\lambda_1\lambda_2)(\lambda_1+\theta\lambda_2)(\lambda_1+\theta\lambda_3)}{(\lambda_3^2+\lambda_1\lambda_2)(\lambda_2^2+\lambda_1\lambda_3)(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)}.
\end{aligned}\tag{3.14}$$

3.7. **Example.** We consider the case with four paths $\{\lambda_1\}\to\{\lambda_1^4\lambda_2^2\lambda_3^2\}$ of length 2. We then have $x_1=\lambda_1^6$, $x_2=-(\lambda_1\lambda_2)^3$, $x_3=(\lambda_1\lambda_2\lambda_3)^2$, $x_4=-(\lambda_1\lambda_3)^3$ and $\delta=\lambda_1^8\lambda_2^3\lambda_3^3$. Plugging these values into the formula for d_i in Theorem 3.4, we obtain

$$d_r(\lambda_3) = \frac{x_r - \lambda_1^6\lambda_2^3\lambda_3^3 x_r^{-1}}{\lambda_1^{13}\lambda_2^4\lambda_3^5(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \prod_{s\neq r} \frac{\lambda_1^7\lambda_2^2\lambda_3^3 + x_r x_s}{(x_r - x_s)}.$$

Plugging the values of x_i into these formulas, we obtain

$$\begin{aligned}
d_1(\lambda_3) &= \frac{(\lambda_1^2\lambda_2-\lambda_3^3)(\lambda_1^4+\lambda_1^2\lambda_2\lambda_3+\lambda_2^2\lambda_3^2)(\lambda_1-\lambda_2)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_1^2+\lambda_2\lambda_3)(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)(\lambda_1^2-\lambda_1\lambda_3+\lambda_3^2)},\\
d_2(\lambda_3) &= \frac{-(\lambda_2^3-\lambda_1^2\lambda_3)(\lambda_3^3-\lambda_1^2\lambda_2)}{(\lambda_1\lambda_2+\lambda_3^2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)},\\
d_3(\lambda_3) &= \frac{-\lambda_2\lambda_3(\lambda_1+\lambda_3)^2(\lambda_2^3-\lambda_1^2\lambda_3)}{(\lambda_2-\lambda_3)(\lambda_1^2+\lambda_2\lambda_3)(\lambda_2^2+\lambda_1\lambda_3)(\lambda_3^2+\lambda_1\lambda_2)},\\
d_4(\lambda_3) &= \frac{-\lambda_3^2(\lambda_1+\lambda_2)^2(\lambda_1-\lambda_2)}{(\lambda_2^2+\lambda_1\lambda_3)(\lambda_1^2-\lambda_1\lambda_3+\lambda_3^2)(\lambda_2-\lambda_3)}. \qquad (3.15)
\end{aligned}$$

3.8. **Determining matrix blocks for the braid generators.** We will encounter non-semisimple representations of AB_2 in the following context. Assume that δ and the eigenvalues x_r of T are monomials in the eigenvalues λ_i. Let $R = F_0[\lambda_1, \lambda_2, \lambda_3]$ and let $\mathfrak{p}$ be a prime ideal in R. We assume as before W to be an AB_2 module with a basis of eigenvectors of T with mutually distinct eigenvalues. Moreover, we assume that the matrix coefficients of A are in the localization $R_\mathfrak{p}$. This implies that we can also get a representation over $Q(R/\mathfrak{p})$.

Corollary 3.5. *Assume that the diagonal entries d_r of P are not in $\mathfrak{p}R_\mathfrak{p}$ except for one entry d_t, which is not in $\mathfrak{p}^2R_\mathfrak{p}$. Also assume that the entries $\delta + \lambda_1\lambda_2 x_r x_s \notin \mathfrak{p}$ for all indices r, s. Then there exist two non-equivalent non-semisimple representations of AB_2 over the quotient field $Q(R/\mathfrak{p})$, in which the basis vector v_t either spans a 1-dimensional sub module or quotient module.*

Proof. By assumption, we have $d_t d_s \in \mathfrak{p}R_\mathfrak{p}\backslash\mathfrak{p}^2R_\mathfrak{p}$ for all $s \neq t$. As P is a rank 1 matrix, this implies $p_{ts}p_{st} = d_t d_s \in \mathfrak{p}R_\mathfrak{p}\backslash\mathfrak{p}^2R_\mathfrak{p}$ for all $s \neq t$. Hence, for given s, exactly one of the two entries p_{ts} or p_{st} is in $\mathfrak{p}R_\mathfrak{p}$, say p_{st}. This implies that $\bar{p}_{st} = 0$ and $\bar{p}_{ts} \neq 0$ in $Q(R/\mathfrak{p})$. But as P has rank 1, we have $\bar{p}_{\tilde{s}t} = 0$ for all $\tilde{s} \neq t$. It follows that v_t spans a 1-dimensional AB_2 submodule in the representation over $Q(R/\mathfrak{p})$. If $p_{ts} \in \mathfrak{p}R_\mathfrak{p}$, one similarly shows that v_t spans a 1-dimensional quotient module.

Remark 3.6. Assume we have eigenvalues λ_i such that the matrix entries $d_r(\lambda_3)$, $1 \le r \le 4$ above are well-defined, and define the projection P via the matrix $(d_r(\lambda_3))_{rs}$. Then it is straightforward, if moderately tedious to determine the entries of the braid matrix A using the last formula in Theorem 3.4. One can thus check that the entry a_{rs} is zero mod $(\lambda_2^3 - \lambda_1^2\lambda_3)$ if and only if $r \in \{2, 3\}$ and $s \in \{1, 4\}$. One similarly checks that the entries a_{rs} are zero mod $\lambda_1^4 + \lambda_1^2\lambda_2\lambda_3 + \lambda_2^2\lambda_3^2$ if and only if $r = 1$ and $s > 1$.

4. Structure results for $\mathcal{K}_4$

4.1. **Strategy.** Let V be a regular module. We already know that it has to be simple unless its eigenvalues satisfy one of the polynomials listed in Proposition 2.5. We fix such a polynomial and the corresponding prime ideal $\mathfrak{p}$. It follows from Lemma 2.6 that our path representations are well-defined, if we consider V as a $Q(R/\mathfrak{p})$ module for $\mathfrak{p}$ one of the corresponding prime ideals; here $Q(R/\mathfrak{p})$ is the quotient field of $R/\mathfrak{p}$. We now describe how knowledge of diagonal entries of eigenprojections of the braid generators in these representations will be used to determine the structure of these modules. It may perhaps be more instructive following one of the examples worked out in the proofs of Lemmas 4.2 or 4.3.

(1) *Bases for simple subquotients* Observe that we have at least two rank 1 eigenprojections in each block of the matrix S_i for a braid generator σ_i in the path representation. We determine the equivalence classes of the relation defined as the transitive closure of $s \leftrightarrow t$ if $d_s d_t \notin \mathfrak{p}$ for the diagonal entries of at least one of these eigenprojections, see Lemma 1.14. As we shall see, there will be at most two such equivalence classes. If we only have one such equivalence class, V will be simple as a $Q(R/\mathfrak{p})$ module, see Lemma 1.13.

(2) *Composition series* If we have two equivalence classes, we are going to show that they correspond to a sub and a quotient module of V as follows:

(a) Pick a matrix block for a braid generator (see Lemma 1.14) of maximum size with paths in both equivalence classes. Calculate the braid matrix using Theorem 3.4, Corollary 3.5 or Remark 3.6. This depends on a choice about the off-diagonal entries of the eigenprojection P. We can choose the matrices $(d_r(\lambda_i))_{rs}$ or $(d_s(\lambda_i))_{rs}$. Replacing one choice by the other one will result in interchanging sub and quotient modules of V. In the language of Remark 1.3, this would correspond to interchanging the module V by V^t.

(b) If there are other matrix blocks with non-equivalent paths, put question marks in the corresponding off-diagonal entries.

(c) Use braid relations to determine which of the question mark off-diagonal entries have to be zero mod $\mathfrak{p}$.

This determines the structure of V as a $Q(R/\mathfrak{p})$ module. We will deal with the structure of V as a $Q(R/\mathfrak{q})$ module for ideals $\mathfrak{q}$ larger than the ones treated so far in Section 4.7.

4.2. **Subquotients for the 6-, 8- and 9-dimensional representations of B_4.** We are going to determine the structure of these regular modules, viewed as $Q(R/\mathfrak{p})$ modules, as discussed in the previous subsection.

Lemma 4.1. *Let $\mathfrak{p}$ be one of the ideals listed in Table 2 for the module $V = \{\lambda_1^3\lambda_2^2\lambda_3\}$. Then the $Q(R/\mathfrak{p})$ module V or V^t has simple quotient and submodules as listed in Corollary 1.18.*

Proof. The existence of the claimed sub and quotient modules has already been shown in Corollary 1.18. To prove simplicity of these modules, we can apply Lemma 1.13, either by using the matrix coefficients in Lemma 2.3, or the diagonal entries of $P_3(\lambda_3)$ in Example (2) in Section 3.6, corresponding to the paths from $\{\lambda_1\}$ to $\{\lambda_1^3\lambda_2^2\lambda_3\}$. Also observe that if we have a nontrivial quotient module, there exists at least one matrix block for S_i, $i \in \{2,3\}$ where some basis paths belong to the quotient, and some to the submodule. This is possible only if one of the diagonal entries d_t for a rank 1 eigenprojection of S_i is in $\mathfrak{p}R_\mathfrak{p}$. One can check from the results in Section 3.6, (2), that these diagonal entries are not in $\mathfrak{p}^2R_\mathfrak{p}$. But then the corresponding AB_2 representation does not split, see Corollary 3.5. Hence the $\mathcal{K}_3$ quotient can not split either.

We will have to work harder to prove nonsemisimplicity of the 8- and 9-dimensional regular modules, as we are not aware of a similar connection to quantum groups as in the previous lemma.

Lemma 4.2. *Let* $V = \{\lambda_1^4\lambda_2^2\lambda_3^2\}$.

(1) If $\mathfrak{p} = (\lambda_2+\lambda_3)$ *or* $\mathfrak{p} = (\lambda_1^2 - \lambda_2\lambda_3)$, *the* $Q(R/\mathfrak{p})$ *module* V *is simple.*

(2) If $\mathfrak{p} = (\lambda_2^3 - \lambda_1^2\lambda_3)$, *the* $Q(R/\mathfrak{p})$ *module* V *has a 5-dimensional quotient* $\{\lambda_1^2|\lambda_2^2\lambda_3\}$ *with weights* $(11), (12), (21)$, (23) *and* (32), *and a submodule isomorphic to the generic representation* $\{\lambda_1^2\lambda_3\}$. *The case* $\mathfrak{p} = (\lambda_3^3 - \lambda_1^2\lambda_2)$ *is similar.*

(3) If $\mathfrak{p} = (\lambda_1^2 - \theta\lambda_2\lambda_3)$, *the* $Q(R/\mathfrak{p})$ *module* V *has a 7-dimensional quotient representation* $\{\lambda_1^3\lambda_2^2\lambda_3^2\}$ *with the same weights as* $\{\lambda_1^4\lambda_2^2\lambda_3^2\}$, *but with the weight* $(1,1)$ *only appearing once.*

Proof. Observe that $\lambda_1^6 - \lambda_2^3\lambda_3^3 = (\lambda_1^2-\lambda_2\lambda_3)(\lambda_1^2-\theta\lambda_2\lambda_3)(\lambda_1^2-\theta^{-1}\lambda_2\lambda_3)$. Hence, by Lemma 2.4, the representation $\{\lambda_1^4\lambda_2^2\lambda_3^2\}$ is simple except possibly if $\lambda_2 = -\lambda_3$, $\lambda_1^2 = \lambda_2\lambda_3$ or one of the cases in (2) or (3) here apply. Let $\mathfrak{p}$ be the prime ideal generated by one of these irreducible polynomials. It follows from Table 1 and Theorem 1.4 that in all of these cases we have the isomorphism of $Q(R/\mathfrak{p})\mathcal{K}_3$ modules

$$\{\lambda_1^4\lambda_2^2\lambda_3^2\} \;\cong\; \{\lambda_1\} \oplus \{\lambda_1\lambda_2\} \oplus \{\lambda_1\lambda_2\lambda_3\} \oplus \{\lambda_1\lambda_3\}, \tag{4.1}$$

where each summand on the right hand side is simple. Now observe that all diagonal entries $d_r(\lambda_3)$ of the eigenprojection P in the 4×4 block of S_3 are not in $\mathfrak{p}$ if $\mathfrak{p} = (\lambda_2+\lambda_3)$ or $\mathfrak{p} = (\lambda_1^2 - \lambda_2\lambda_3)$, see Section 3.7. It follows that all paths for our module are equivalent, i.e. the $Q(R/\mathfrak{p})$-module $\{\lambda_1^4\lambda_2^2\lambda_3^2\}$ is simple in these cases. This proves (1).

Let now $\mathfrak{p} = (\lambda_2^3 - \lambda_1^2\lambda_3)$. We order the paths for $\{\lambda_1^4\lambda_2^2\lambda_3^2\}$ by starting with the left most $\mathcal{K}_3$ module in 4.1, and ordering the paths in a $\mathcal{K}_3$-module with the ordering of the eigenvalues λ_i. E.g the path corresponding to eigenvalue λ_2 for σ_1 in the $\mathcal{K}_3$ module $\{\lambda_1\lambda_2\lambda_3\}$ would be the fifth path. See also the paths into $\Lambda_1+\Lambda_2$ in the figure below [16], Remark 2.2.

As $\mathcal{K}_3$ is semisimple over $Q(R/\mathfrak{p})$ in this case, paths belonging to the same $\mathcal{K}_3$ module are equivalent. The 4 by 4 block for the matrix S_3 for σ_3 can be calculated from the example in Section 3.7. We will use the notation in that section, where the base vectors v_i, $1\le i\le 4$ correspond to the paths 1, 2, 4 and 7 here. We indicate by 0 those entries of S_4 which are 0 mod $\mathfrak{p}R_\mathfrak{p}$, see Remark 3.6. We similarly calculate the 2 by 2 blocks for S_3 for the paths from $\{\lambda_2\}$ resp $\{\lambda_3\}$ to $\{\lambda_1^4\lambda_2^2\lambda_3^2\}$, using the results in Section 3.4. Again, one of the diagonal entries for the path from $\{\lambda_3\}$ is zero mod $\mathfrak{p}R_\mathfrak{p}$. This implies that one of the two entries with

? has to be equal to 0 mod $\mathfrak{p}R_\mathfrak{p}$, see Corollary 3.5.

$$S_2 = \begin{bmatrix} x &&&&&&& \\ & x & x &&&&& \\ & x & x &&&&& \\ &&& x & x & x && \\ &&& x & x & x && \\ &&& x & x & x && \\ &&&&&& x & x \\ &&&&&& x & x \end{bmatrix} \qquad S_3 = \begin{bmatrix} x & x && x &&& x & \\ 0 & x && x &&& 0 & \\ && x && x &&& \\ 0 & x && x &&& 0 & \\ && x && x &&& \\ &&&&& x && ? \\ x & x && x &&& x & \\ &&&&& ? && x \end{bmatrix}$$

Now observe that $S_2S_3S_2v_1 \in \text{span}\{v_1, v_7, v_8\}$. But if the (6,8) entry $a^{(3)}_{68}$ of S_3 was nonzero, $S_3S_2S_3v_1$ would also have a nonzero coefficient for v_6. Hence $a^{(3)}_{68} = 0 \bmod \mathfrak{p}R_\mathfrak{p}$. It follows that the image of $\mathcal{K}_4$ has a submodule spanned by $\{v_1, v_7, v_8\}$ and a quotient spanned by the remaining basis vectors.

Part (3) with $\mathfrak{p} = (\lambda_1^2 - \theta\lambda_2\lambda_3)$ can be shown similarly, and is much easier. Again, calculating the matrix S_3 as in the previous case, it follows from Remark 3.6 that all non-diagonal entries of S_3 in the first row are equal to 0 mod $\mathfrak{p}R_\mathfrak{p}$. Hence the vectors v_2 to v_7 span a submodule. This submodule is irreducible, as it is semisimple as a $\mathcal{K}_3$ module, and all other diagonal entries of $P_3(\lambda_3)$ in the 4 by 4 block are nonzero mod $\mathfrak{p}R_\mathfrak{p}$.

Lemma 4.3. *We only get nontrivial quotients of the 9-dimensional representation $\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$ over $Q(R/\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ in the following cases, where $\{i, j, k\} = \{1, 2, 3\}$:*

(1) If $\mathfrak{p} = (\lambda_i^2 - \theta\lambda_j\lambda_k)$, we get a 7-dimensional quotient with the same weights as the subquotient appearing in the 8-dimensional representation.

(2) If $\mathfrak{p} = (\lambda_i + \theta\lambda_j)$, we have 6- and 3-dimensional subquotients with the same weights as the generic modules $\{\lambda_j^3\lambda_k^2\lambda_i\}$ and $\{\lambda_i^2\lambda_k\}$.

Proof. We have shown in Lemma 2.4 that the 9-dimensional representation $V = \{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$ is simple, except possibly if $\lambda_i = -\theta\lambda_j$ or $\lambda_i^2 = \theta\lambda_j\lambda_k$. We will again use path representations to study these cases, as indicated in figure 4.1. Here a vertex at level k indicates a representatioon of $\mathcal{K}_k$, with the scalar via which Δ_k^2 acts on it. So we have $x_1 = -(\lambda_1\lambda_2)^3$, $x_2 = -(\lambda_1\lambda_3)^3$, $x_3 = (\lambda_1\lambda_2\lambda_3)^2$, and $x_4 = -(\lambda_2\lambda_3)^3$.

Let us consider the case $\mathfrak{p} = (\lambda_1 + \theta\lambda_2)$. It follows from Section 3.6, Example (3) and Lemma 1.15 that all matrix entries of the eigenprojections $P_3(\lambda_i)$ are well-defined. Hence we can use the same method as in Lemma 4.2 to determine subquotient modules. We label the paths as in Lemma 4.2 based on the isomorphism of $\mathcal{K}_3$ modules

$$\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta \;\cong\; \{\lambda_1\lambda_2\} \oplus \{\lambda_1\lambda_3\} \oplus \{\lambda_1\lambda_2\lambda_3\} \oplus \{\lambda_2\lambda_3\}.$$

This determines the block structure for the matrix S_2. The matrix S_3 consists of three 3 by 3 blocks, which live on span $\{v_1, v_3, v_5\}$, span $\{v_2, v_6, v_8\}$ and span $\{v_4, v_7, v_9\}$. The diagonal entries of the eigenprojections $P_3(\lambda_3)$ and $P_3(\lambda_1)$ for the first block are calculated in Section

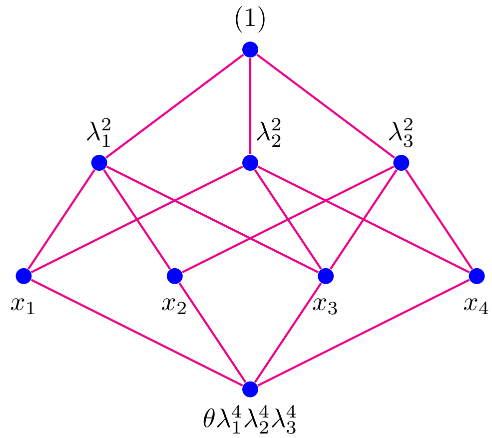

FIGURE 4.1. Paths for the 9-dimensional representation $\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$.

3.6, (3), and the ones for $P_3(\lambda_2)$ are obtained by applying the permutation (23). It follows that the diagonal entries of $P_3(\lambda_1)$ and $P_3(\lambda_3)$ for the vector v_5 are equal to zero mod $\mathfrak{p}R_\mathfrak{p}$.

One similarly gets the diagonal entries of the eigenprojections for the block acting on span $\{v_4, v_7, v_9\}$ by applying the permutation (13), see Lemma 1.15. One obtains that the diagonal entries of the projections $P_3(\lambda_2)$ and $P_3(\lambda_3)$ for the vector v_4 are equal to 0. Finally, we obtain from Section 3.4 that at least one of the two off-diagonal entries for the first 2 by 2 block of S_2 has to be equal to 0.

We now proceed as in the proof of Lemma 4.2 as follows: We can set the non-diagonal entries on the 5-th column of S_3 equal to 0. We then put question marks into the nondiagonal entries in the fourth row and fourth column of S_3 in the $(4,7,8)$ block and into the non-diagonal entries for the (12) block of S_2. It is then straightforward to check that $S_3S_2S_3v_5$ is in the span of all basis vectors except v_1, while this is true for $S_2S_3S_2v_5$ only if $(S_2)_{12} = 0$ mod $\mathfrak{p}R_\mathfrak{p}$. If the non-diagonal entries of the fourth column of S_3 were 0 mod $\mathfrak{p}R_\mathfrak{p}$, then $S_3S_2S_3v_4$ would be in the span of $\{v_1, v_3, v_4, v_5\}$, while $S_2S_3S_2v_4$ is in the span of the basis vectors v_i with $i \in \{2,5,6,7,8,9\}$. As this would contradict the braid relation, the non-diagonal entries of S_3 in its fourth row must be equal to 0 mod $\mathfrak{p}R_\mathfrak{p}$. This proves that we get a $\mathcal{K}_4$ submodule spanned by the vectors v_i with $i \in \{2,5,6,7,8,9\}$ and a quotient module spanned by the remaining basis vectors. As $\lambda_1^2 + \lambda_3^2 \neq 0$ mod $\mathfrak{p}R_\mathfrak{p}$, the 3-dimensional subquotient is isomorphic to $\{\lambda_1^2\lambda_3\}$, which has the weights $(11), (13)$ and (31). This also determines the weights of the 6-dimensional subquotient. The general case $\mathfrak{p} = (\lambda_i + \theta\lambda_j)$ follows from Lemma 1.15 by symmetry.

If $\mathfrak{p} = (\lambda_3^2 - \theta\lambda_1\lambda_2)$, we can assume $\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$ to be a semisimple $\mathcal{K}_3$ module. We leave it to the reader to check that one can set the nondiagonal entries of S_3 in the first row equal to 0, and that this will also force that its nondiagonal entries in the second row will have to be equal to zero mod $\mathfrak{p}R_{\mathfrak{p}}$. This implies that we have a quotient module spanned by the vectors v_1 and v_2, and a submodule spanned by the remaining basis vectors.

4.3. **Nonsemisimplicity of $\mathcal{K}_4$.** We can now prove the converse of Proposition 2.5.

Theorem 4.4. *The regular module V is not semisimple if and only if the eigenvalues of σ_1 are in the locus of one of the ideals listed next to it in Table 2, or if two of the eigenvalues coincide. In particular, the algebra $\mathcal{K}_4$ is not semisimple if and only if either σ_1 is not diagonalizable, or if its eigenvalues are in the locus of one of the ideals listed in Table 2.*

Proof. It remains to prove that the algebras $\mathcal{K}_4$ are not semisimple if one of the polynomials in Table 2 is zero at the eigenvalues of σ_1. For this, it is enough to find a representation which is not semisimple. This is trivially true if two of the eigenvalues coincide, by mapping all σ_i to the corresponding Jordan form. For the polynomials $\lambda_i + \theta\lambda_j$ and $\lambda_i \pm \sqrt{-1}\lambda_j$, the result is known from the study of Iwahori Hecke algebras. For the other polynomials, we have shown in Lemmas 4.1, 4.2 and 4.3 that our regular modules have nontrivial quotients which do not split if the eigenvalues satisfy one of the polynomials listed in Table 2.

Lemma 4.5. *Assume that the eigenvalues of σ_i satisfy at most one of the polynomials in Table 2. If two simple $\mathcal{K}_4$ representations are isomorphic as $\mathcal{K}_3$ modules, and they have the same weights, then they are isomorphic, except for the two non-isomorphic 9-dimensional modules.*

Proof. This follows for the generic modules by inspection of Table 1. However, it might be conceivable that e.g. the 7-dimensional modules appearing as subquotients of the 8- and 9-dimensional generic representations might not be isomorphic. To rule that out, observe that they have isomorphic path bases; for the subquotient of the nine-dimensional representation one obtains it by removing the paths going through $\{\lambda_2\lambda_3\}$ in the picture in the proof of Lemma 4.3. We will denote the path going through the representation $\{\lambda_i\}$ and the $\mathcal{K}_3$ representation on which Δ_3^2 acts via x_r by (λ_i, x_r). Then we can assume that in the path representations for both representations the 3 by 3 blocks of S_3, as well as the diagonal entries of the 2 by 2 blocks coincide, by Theorem 3.4. One can now use the braid relations to show that the remaining entries for S_3 are determined by this. For example, one can calculate the $((\lambda_3, x_1), (\lambda_3, x_1))$ entry of S_3 by comparing the $((\lambda_1, x_3), (\lambda_3, x_3))$ entries of $S_2S_3S_2$ and $S_3S_2S_3$. The other entries of S_3 can be calculated similarly.

One can similarly prove the uniqueness of the 5- and 4-dimensional simple representations. The four 3-dimensional representations $\{\lambda_1\lambda_2\lambda_3\}$ and $\{\lambda_i|\lambda_j\lambda_k\}$ are isomorphic as $\mathcal{K}_3$-modules. But their path bases are also bases of weight vectors, which determines the diagonal matrix S_3.

4.4. **Exceptional B_4 representations.** We can now list all new simple representations of $\mathcal{K}_4$ if the eigenvalues of σ_i satisfy exactly one irreducible polynomial in Table 2. Here $\mathfrak{p}$ is the prime ideal generated by one of those polynomials, and $Q(R/\mathfrak{p})$ is the corresponding quotient field.

$\mathfrak{p}$	dim	notation	Δ_4^2	weights	$\mathcal{K}_3$ module
$(\lambda_1 \pm i\lambda_2)$	2	$\{\lambda_1\lambda_2\}^*$	$-\lambda_1^6\lambda_2^6$	$(1,2)$	$\{\lambda_1\lambda_2\}$
$(\lambda_1+\lambda_3)$	3	$\{\lambda_2\|\lambda_1\lambda_3\}$	$\lambda_1^4\lambda_2^4\lambda_3^4$	(2,2), (1,3),	$\{\lambda_1\lambda_2\lambda_3\}$
$(\lambda_2+\lambda_3)$	4	$\{\lambda_1^2\lambda_2\lambda_3\}$	$-\lambda_1^6\lambda_2^3\lambda_3^3$	(1,2), (1,3)	$\{\lambda_1\lambda_2\lambda_3\}+\{\lambda_1\}$
$(\lambda_1^3-\lambda_2^2\lambda_3)$	5	$\{\lambda_2^2\|\lambda_1^2\lambda_3\}$	$\lambda_1^6\lambda_2^4\lambda_3^2$	(2,2), (1,2), (1,3)	$\{\lambda_1\lambda_2\lambda_3\}+\{\lambda_1\lambda_2\}$
$(\lambda_1^2-\theta\lambda_2\lambda_3)$	7	$\{\lambda_1^3\lambda_2^2\lambda_3^2\}$	$\theta\lambda_1^4\lambda_2^4\lambda_3^4$	$(1,1),(1,2),(1,3),(2,3)$	see 4.2

$$\{\lambda_1^3\lambda_2^2\lambda_3^2\} \;\cong\; \{\lambda_1\lambda_2\lambda_3\}+\{\lambda_1\lambda_2\}+\{\lambda_1\lambda_3\}. \tag{4.2}$$

Table 3

As in Table 1, we refer to the representation by the determinant of one of the standard generators σ_i, where we have additional symbols if the same determinant has appeared before for a generic representation. We do not list quotients that can be obtained by permuting the eigenvalues, and we only list one of the two weights (i,j) and (j,i) for $i \neq j$.

4.5. **Blocks.** We now give an explicit description of the blocks of $\mathcal{K}_4$ if the eigenvalues λ_i are mutually distinct. We will list the corresponding exact sequences in the next section. This will be proved in Theorem 4.6.

(1) $\lambda_i=-\lambda_j$: $[\{\lambda_i^2\lambda_k\},\{\lambda_i^3\lambda_k^2\lambda_j\},\{\lambda_j^3\lambda_k^2\lambda_i\},\{\lambda_j^2\lambda_k\}]$, $[\{\lambda_i\lambda_k\},\{\lambda_k^3\lambda_i^2\lambda_j\},\{\lambda_k^3\lambda_j^2\lambda_i\},\{\lambda_j\lambda_k\}]$,

(2) $\lambda_i=-\theta\lambda_j$: $[\{\lambda_i\},\{\lambda_i\lambda_j\},\{\lambda_j\}]$, $[\{\lambda_i^2\lambda_k\},\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta,\{\lambda_j^3\lambda_k^2\lambda_i\}]$ and $[\{\lambda_j^2\lambda_k\},\{\lambda_1^3\lambda_2^3\lambda_3^3\}_{\theta^2},\{\lambda_i^3\lambda_k^2\lambda_j\}]$,

(3) $\lambda_i^2=-\lambda_j^2$: $[\{\lambda_i^2\lambda_j\},\{\lambda_j^2\lambda_i\},\{\lambda_i\},\{\lambda_j\}]$

(4) $\lambda_i^2=-\lambda_j\lambda_k$: $[\{\lambda_i\},\{\lambda_i\lambda_j\lambda_k\},\{\lambda_j\lambda_k\}]$, $[\{\lambda_j^2\lambda_k\},\{\lambda_j^3\lambda_i^2\lambda_k\},\{\lambda_i^2\lambda_j\}]$,

(5) $\lambda_i^2=\theta\lambda_j\lambda_k$: $[\{\lambda_j\lambda_k\},\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta,\{\lambda_i^4\lambda_j^2\lambda_k^2\},\{\lambda_i\}]$,

(6) $\lambda_i^3=\lambda_j^2\lambda_k$: $[\{\lambda_i\},\{\lambda_i^3\lambda_j^2\lambda_k\},\{\lambda_j^4\lambda_i^2\lambda_k^2\},\{\lambda_j^2\lambda_k\}]$

4.6. **Exact sequences.** We list exact sequences corresponding to the blocks in the previous section. Here the notation is somewhat ambiguous in case the listed representation is not simple: we may have to replace it by its transpose representation, see Remark 1.3. For the explicit regular representations referred to before, either the given sequence or the sequence with the arrows reversed for the transposed representations will hold. The first blocks in (2) and (4) appear for representations where σ_1 and σ_3 have the same image; the corresponding exact sequences have already appeared for $\mathcal{K}_3$, see Section 1.3, and are not listed here again. Also, we have only listed one exact sequence for the second and third block listed in (2), as they are very similar.

$\mathfrak{p}$	exact sequence for $Q(R/\mathfrak{p})$
$(\lambda_1+\lambda_3)$	$0\to\{\lambda_1^2\lambda_2\}\to\{\lambda_1^3\lambda_2^2\lambda_3\}\to\{\lambda_1\lambda_2^2\lambda_3^3\}\to\{\lambda_2\lambda_3^2\}\to 0$
$(\lambda_2+\lambda_3)$	$0\to\{\lambda_1\lambda_2\}\to\{\lambda_1^3\lambda_2^2\lambda_3\}\to\{\lambda_1^3\lambda_2\lambda_3^2\}\to\{\lambda_1\lambda_3\}\to 0$
$(\lambda_1+\theta\lambda_2)$	$0\to\{\lambda_1^2\lambda_3\}\to\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta\to\{\lambda_1\lambda_2^3\lambda_3^2\}\to 0$
$(\lambda_1\pm i\lambda_2)$	$0\to\{\lambda_1\}\to\{\lambda_1^2\lambda_2\}\to\{\lambda_1\lambda_2^2\}\to\{\lambda_2\}\to 0$
$(\lambda_1^2+\lambda_2\lambda_3)$	$0\to\{\lambda_2^2\lambda_3\}\to\{\lambda_1^2\lambda_2^3\lambda_3\}\to\{\lambda_1^2\lambda_2\}\to 0$
$(\lambda_1^2-\theta\lambda_2\lambda_3)$	$0\to\{\lambda_2\lambda_3\}\to\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta\to\{\lambda_1^4\lambda_2^2\lambda_3^2\}\to\{\lambda_1\}\to 0$
$(\lambda_1^3-\lambda_2^2\lambda_3)$	$0\to\{\lambda_2^2\lambda_3\}\to\{\lambda_1^2\lambda_2^4\lambda_3^2\}\to\{\lambda_1^3\lambda_2^2\lambda_3\}\to\{\lambda_1\}\to 0$

The following theorem gives proofs and concise statements of the results listed in the previous subsections.

Theorem 4.6. *Let $\mathfrak{p}$ be a prime ideal generated by one of the polynomials listed in Table 2, and let $F = Q(R/\mathfrak{p})$ be the quotient field for $R/\mathfrak{p}$. Then we have*

(a) The blocks of $\mathcal{K}_4$, defined over F, are given by equivalence classes of regular modules, where $V_\gamma \sim V_{\gamma'}$ if and only if $\lambda_\gamma = \lambda_{\gamma'}$ mod $\mathfrak{p}$ for the eigenvalues $\lambda_\gamma, \lambda_{\gamma'}$ of Δ_4^2. See Section 4.5 for explicit lists.

(b) Order the labels γ in a block by alphabetical order for the eigenvalues λ_γ. After possibly replacing V_γ by V_γ^t if necessary, we obtain exact sequences

$$0 \;\to\; V_{\gamma_1} \;\to\; V_{\gamma_2} \;\to\; \dots \;\to\; V_{\gamma_r} \;\to\; 0,$$

where the induced subquotient modules for each V_{γ_i} are simple. See the table above for explicit exact sequences.

Proof. We have determined composition series with simple factors for all regular modules in the previous lemmas. Isomorphisms between the various subquotient modules have been established in Lemma 4.5. Replacing a module V by V^t, we can interchange sub and quotient modules, if necessary. So we obtain the claimed exact sequences.

4.7. **No more exceptional representations.** We have listed new simple representations of $\mathcal{K}_4$ when the eigenvalues satisfy one of the relations in Proposition 2.5. These representations will in general not stay simple if they also satisfy a second equation. Nevertheless, we have the following result.

Theorem 4.7. *Table 1 and Table 2 list all possible simple representations of $\mathcal{K}_4$ for which $\lambda_i \neq \lambda_j$, $1 \leq i \neq j \leq 3$. The simple modules in a composition series of a regular module in case when the eigenvalues satisfy two of the relations in Proposition 2.5 are listed in Table 4.*

module	ideal	simple components
$\{\lambda_1^3\lambda_2^2\lambda_3\}$	$(\lambda_1+\lambda_3, \lambda_1^3-\lambda_2^2\lambda_3)$	$\{\lambda_2\mid\lambda_1\lambda_3\}+\{\lambda_1\lambda_2\}^*+\{\lambda_1\}$
$\{\lambda_1^3\lambda_2^2\lambda_3\}$	$(\lambda_2+\lambda_3, \lambda_1^3-\lambda_2^2\lambda_3)$	$\{\lambda_1^2\lambda_2\lambda_3\}+\{\lambda_1\}+\{\lambda_2\}$
$\{\lambda_1^3\lambda_2^2\lambda_3\}$	$(\lambda_2^2+\lambda_1\lambda_3, \lambda_1^3-\lambda_2^2\lambda_3)$	$\{\lambda_2^2\lambda_1\}+\{\lambda_1\lambda_3\}^*+\{\lambda_1\}$
$\{\lambda_1^4\lambda_2^2\lambda_3^2\}$	$(\lambda_2^3-\lambda_1^2\lambda_3, \lambda_3^3-\lambda_1^2\lambda_2, \lambda_2^2+\lambda_3^2)$	$\{\lambda_1^2\lambda_2\}+\{\lambda_1^2\lambda_3\}+\{\lambda_2\lambda_3\}^*$
$\{\lambda_1^4\lambda_2^2\lambda_3^2\}$	$(\lambda_2^3-\lambda_1^2\lambda_3, \lambda_3^3-\lambda_1^2\lambda_2, \lambda_2+\lambda_3)$	$\{\lambda_1\mid\lambda_2\lambda_3\}+\{\lambda_1\lambda_2\}^*+\{\lambda_1\lambda_3\}^*+\{\lambda_1\}$
$\{\lambda_1^4\lambda_2^2\lambda_3^2\}$	$(\lambda_2^3-\lambda_1^2\lambda_3, \lambda_1^2-\theta\lambda_2\lambda_3, \lambda_1^2+\lambda_3^2)$	$\{\lambda_1^2\mid\lambda_2^2\lambda_3\}+\{\lambda_1\lambda_3\}^*+\{\lambda_1\}$
$\{\lambda_1^4\lambda_2^2\lambda_3^2\}$	$(\lambda_2^3-\lambda_1^2\lambda_3, \lambda_1^2-\theta\lambda_2\lambda_3, \lambda_1+\lambda_3)$	$\{\lambda_2^2\lambda_1\lambda_3\}+\{\lambda_1^2\lambda_3\}+\{\lambda_1\}$
$\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$	$(\lambda_1+\theta\lambda_2, \lambda_1^2-\theta\lambda_2\lambda_3)$	$\{\lambda_1^2\lambda_3\}+\{\lambda_2\lambda_3\}+\{\lambda_2^2\lambda_1\lambda_3\}$
$\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$	$(\lambda_2+\theta\lambda_3, \lambda_1^2-\theta\lambda_2\lambda_3)$	$\{\lambda_1^2\mid\lambda_3^2\lambda_2\}+\{\lambda_1\lambda_2\}^*+\{\lambda_2\}+\{\lambda_3\}$
$\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$	$(\lambda_1+\theta\lambda_2, \lambda_2+\theta\lambda_3)$	$\{\lambda_1^2\lambda_3\}+\{\lambda_2^2\lambda_1\}+\{\lambda_3^2\lambda_2\}$

Table 4

Proof. Let V be a regular module. Then we have a canonical basis of weight vectors. It follows from Theorem 4.4 that V is simple over the quotient field $Q(R/\mathfrak{q})$ for any prime ideal $\mathfrak{q}$ of R which does not contain an ideal listed in Table 2. If it only contains one of these ideals, we have a composition series whose factors can be determined in the table in Section 4.6. We now apply the same strategy to these modules for prime ideals $\mathfrak{q}$ which contain more than one of the ideals. Picking two such ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$, we observe the following scenarios:

(a) There is no proper ideal which contains both $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

(b) If $\mathfrak{p}_1$ and $\mathfrak{p}_2$ generate a proper ideal, determine all maximal ideals $\mathfrak{m}$ which contain both $\mathfrak{p}_1$ and $\mathfrak{p}_2$. For determining all possible subquotient modules, we can already use the information obtained for both $R/\mathfrak{p}_1$ and $R/\mathfrak{p}_2$, see examples below.

Case (a) can happen when a polynomial $\lambda_i-\lambda_j$ is in the ideal generated by $\mathfrak{p}_1$ and $\mathfrak{p}_2$; recall that we assume these polynomials to be invertible in R. These two ideals likely would still be contained in a nontrivial ideal without our assumption $\lambda_i \neq \lambda_j$. As an example, consider the two ideals $\mathfrak{p}_1 = (\lambda_1^2-\theta\lambda_2\lambda_3)$ and $\mathfrak{p}_2 = (\lambda_2^2-\theta\lambda_1\lambda_3)$ for $V = \{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$. Then we obtain

$$\lambda_1^3-\lambda_2^3 = \lambda_1(\lambda_1^2-\theta\lambda_2\lambda_3)-\lambda_2(\lambda_2^2-\theta\lambda_1\lambda_3) \ \in \ (\mathfrak{p}_1,\mathfrak{p}_2).$$

As $(\lambda_1-\lambda_2)$ is invertible, one of the two factors of $(\lambda_1-\theta\lambda_2)(\lambda_1-\theta^2\lambda_2) = (\lambda_1^3-\lambda_2^3)/(\lambda_1-\lambda_2)$ has to be in in the prime ideal $(\mathfrak{p}_1,\mathfrak{p}_2)$. If $(\lambda_1-\theta\lambda_2)\in(\mathfrak{p}_1,\mathfrak{p}_2)$, then so is $\lambda_1-\lambda_3 = \lambda_2^{-1}\theta^{-1}(\lambda_1^2-\theta\lambda_2\lambda_3-\lambda_1(\lambda_1-\theta\lambda_2))$. One similarly shows that $\lambda_2-\lambda_3\in(\mathfrak{p}_1,\mathfrak{p}_2)$ if $(\lambda_1-\theta^2\lambda_2)\in(\mathfrak{p}_1,\mathfrak{p}_2)$.

For $V=\{\lambda_1^3\lambda_2^2\lambda_3\}$, the ideal generated by $(\lambda_1+\lambda_3)$ and $(\lambda_2+\lambda_3)$ obviously contains the invertible polynomial $\lambda_1-\lambda_2$. As another example for case (a), one similarly shows that the ideal generated by $(\lambda_2^2+\lambda_1\lambda_3)$ and $(\lambda_2+\lambda_3)$ contains $\lambda_1-\lambda_2$ from

$$\lambda_1-\lambda_2 = \lambda_3^{-1}[(\lambda_2^2+\lambda_1\lambda_3)-\lambda_2(\lambda_2+\lambda_3)].$$

For case (b), observe that both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ define a partition of the weight basis. If all weights have multiplicity 1, the intersections of these subsets give us a refined partition. It only remains to check that the path vectors labeled by the elements of any of its components do indeed span a simple subquotient of V. As an example for this case, consider the ideal generated by $(\lambda_1^2 - \theta\lambda_2\lambda_3)$ and $(\lambda_2 + \theta\lambda_3)$ for the 9-dimensional generic representation $\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$. Then it follows from Tables 1 and 3 that the equation $\lambda_2 = -\theta\lambda_3$ induces a partition of its weights into $\{(2,2),(1,2),(2,1)\}$ and the remaining weights. Similarly, the equation $\lambda_1^2 = \theta\lambda_2\lambda_3$ induces a partition of the weights of $\{\lambda_1^3\lambda_2^3\lambda_3^3\}_\theta$ into $\{(2,2),(3,3)\}$ and its complement. Hence, if both equations are satisfied, we obtain the refined partition

$$\{(2,2)\} \cup \{(1,2),(2,1)\} \cup \{(3,3)\} \cup \{(1,1),(1,3),(3,1),(2,3),(3,2)\}.$$

We deduce from this and Table 3 the result stated in the second to last line of Table 4. Observe that the corresponding simple subrepresentations have already appeared before. E.g. one can check directly that our two defining equations also imply $\lambda_1^2 = -\lambda_2^2$, a necessary condition for the $\mathcal{K}_4$ representation $\{\lambda_1\lambda_2\}^*$. We similarly obtain the results listed in Table 4 for the other cases concerning the 6- and 8-dimensional regular modules.

The situation is slightly more complicated for the 8-dimensional representation, say, $\{\lambda_1^4\lambda_2^2\lambda_3^2\}$. As the weight (1,1) has multiplicity 2, there is more than one refinement of our path basis compatible with the partitions coming from the ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$. They can be associated to different ideals $\mathfrak{q}_1$ and $\mathfrak{q}_2$, both of which contain $\mathfrak{p}_1$ and $\mathfrak{p}_2$. As an example, consider the two ideals $\mathfrak{p}_1 = (\lambda_2^3 - \lambda_1^2\lambda_3)$ and $\mathfrak{p}_2 = (\lambda_3^3 - \lambda_1^2\lambda_2)$. First, observe that we have subquotients isomorphic to $\{\lambda_1^2\lambda_2\}$ and $\{\lambda_1^2\lambda_3\}$, by Lemma 4.2. Moreover, we obtain

$$(\lambda_2 + \lambda_3)(\lambda_2^2 + \lambda_3^2) = (\lambda_2 - \lambda_3)^{-1}[\lambda_2(\lambda_2^3 - \lambda_1^2\lambda_3) - \lambda_3(\lambda_3^3 - \lambda_1^2\lambda_2)] \in (\mathfrak{p}_1, \mathfrak{p}_2).$$

We consider the two cases of ideals $\mathfrak{q}_1 = (\lambda_2^3 - \lambda_1^2\lambda_3, \lambda_3^3 - \lambda_1^2\lambda_2, \lambda_2 + \lambda_3)$ and $\mathfrak{q}_2 = (\lambda_2^3 - \lambda_1^2\lambda_3, \lambda_3^3 - \lambda_1^2\lambda_2, \lambda_2^2 + \lambda_3^2)$. They are both nontrivial, with $(i, 1, -1)$ and $(\sqrt{i}, 1, -i)$ elements in their loci, respectively.

In the first case, we derive from $\lambda_2 = -\lambda_3$ and the relations we started out with that $\lambda_1^2 = -\lambda_2^2 = -\lambda_3^2$. This splits the subquotients $\{\lambda_1^2\lambda_i\} = \{\lambda_1\lambda_i\}^* + \{\lambda_1\}$ for $i = 2, 3$. The remaining weights are (23), (32) and, possibly (1,1) if the $\{\lambda_1\}$ summands in the split of $\{\lambda_1^2\lambda_i\}$ coincide for $i = 2, 3$. As $\lambda_2^2 \neq -\lambda_3^2$, there is no 2-dimensional representation with weights (2,3) and (3,2). Hence, the remaining summand is $\{\lambda_1|\lambda_2\lambda_3\}$, which gives us line 5 in Table 4 for the representation $\{\lambda_1^4\lambda_2^2\lambda_3^2\}$.

In the second case, we deduce from $\lambda_2^2 = -\lambda_3^2$ and $\lambda_2^3 = \lambda_1^2\lambda_3$ that $\lambda_1^2 = -\lambda_2\lambda_3 \neq -\lambda_3^2$, and similarly $\lambda_1^2 \neq -\lambda_2^2$. Hence, the subquotients $\{\lambda_1^2\lambda_2\}$ and $\{\lambda_1^2\lambda_3\}$ remain irreducible, and the remaining summand necessarily has to be $\{\lambda_2\lambda_3\}^*$. This gives us line 4 in Table 4 for the representation $\{\lambda_1^4\lambda_2^2\lambda_3^2\}$.

If $\mathfrak{p}_1 = (\lambda_2^3 - \lambda_1^2\lambda_3)$ and $\mathfrak{p}_2 = (\lambda_1^2 - \theta\lambda_2\lambda_3)$, we similarly determine ideals $\mathfrak{q}_1 = (\mathfrak{p}_1, \mathfrak{p}_2, \lambda_1^2 + \lambda_3^2)$ and $\mathfrak{q}_2 = (\mathfrak{p}_1, \mathfrak{p}_2, \lambda_1 + \lambda_3)$, which lead to subquotients listed in lines 6 and 7 of Table 4.

5. Related results and applications

5.1. **Schur elements and semisimplicity.** Our Theorem A proves a special case of a general conjecture, which would determine the exact values of parameters for which a cyclotomic Hecke algebra is semisimple. We sketch this here, following the review in [5], Chapter 2. A symmetrizing trace for an algebra $\mathcal{A}$ over a ground ring R is a functional $tr : \mathcal{A} \to R$ such that $(a, b) = tr(ab)$ defines a symmetric non-degenerate bilinear form. A canonical symmetrizing trace for a cyclotomic Hecke algebra has been conjectured by Broué, Malle and Michel for all cyclotomic Hecke algebras. This has been verified for a large class of such Hecke algebras, see the review in [5], but not for G_{25}. The trace can be characterized by so-called Schur elements, which have been calculated for all cyclotomic Hecke algebras in [13], assuming the existence of the trace. It is well known that an algebra $\mathcal{A}$ is semisimple if and only if all Schur elements are nonzero. Comparing the Schur elements for G_{25} in [13] with the polynomials in Theorem A shows that we have confirmed the conjecture for this particular Hecke algebra.

5.2. **Reconstructing categories of type G_2.** Our original motivation for this research comes from our project of classifying semisimple braided rigid tensor categories $\mathcal{C}$ whose tensor product rules are the ones of the representations of the Lie algebra of type G_2 or its associated fusion tensor categories. Such classifications have been obtained for classical Lie types in [12] and [24], but are not known for exceptional Lie types. In [18] it was shown that if the image of the braid group B_3 in $\mathrm{End}(V^{\otimes 3})$ generates this algebra, then also $\mathrm{End}(V^{\otimes n})$ is generated by the image of B_n; here V is the object in $\mathcal{C}$ corresponding to the simple 7-dimensional representation of the Lie algebra $\mathfrak{g}(G_2)$. From this we can conclude that the given tensor category $\mathcal{C}$ must be equivalent to Rep $U_q\mathfrak{g}(G_2)$, the Drinfeld-Jimbo qantum group of type G_2, with q not a root of unity.

In order to prove the missing step, we need to examine all possible representations of the braid group B_4 into $\mathrm{End}(V^{\otimes 4})$ to eventually prove that the image of B_3 must generate $\mathrm{End}(V^{\otimes 3})$. This will be done in [19], also for the associated finite fusion categories of type G_2, using the results of this paper. It is expected that the results of our paper would also be useful in proving similar classification results for braided tensor categories of other exceptional Lie types.

5.3. **Scope of our method.** A priori, our method of using path bases would only work if both $\mathcal{K}_2$ and $\mathcal{K}_3$ are semisimple. We managed to make it work even in the case where $\mathcal{K}_3$ is not semisimple. It might be possible to extend this even to the general case with also $\mathcal{K}_2$ not semisimple, i.e. the braid generators are not diagonalizable. Similar situations were studied in [10] in the context of Iwahori-Hecke algebras of type A. But more work would be needed.

We expect that our methods will also be useful for determining the structure of other cyclotomic Hecke algebras corresponding to complex reflection groups where some of the generators have order 3, such as e.g. G_{32}.

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