

ON FEW-DISTANCE SETS IN THE PLANE

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ABSTRACT. Let $g(k)$ be the maximum size of a planar set that determines at most k distances. We prove

$$\frac{\pi}{3\mathcal{C}(\Lambda_{\text{hex}})} k\sqrt{\log k} (1 + o(1)) \leq g(k) \leq C k \log k,$$

so $g(k) \asymp k\sqrt{\log k}$ with an explicit hexagonal constant. For any arithmetic lattice Λ ,

$$g_{\Lambda}(k) \geq \frac{\pi}{4} \mathcal{S}^*(\Lambda) k\sqrt{\log k} (1 + o(1)).$$

We also give quantitative stability: unless X is line-heavy or has two popular nonparallel shifts, either almost all ordered pairs lie below a high quantile of the distance multiset (near-center localization), or a constant fraction of $X \cap W$ lies in one residue class modulo 2Λ .

1. INTRODUCTION

Let $m(n)$ be the minimum number of distinct distances determined by n planar points. Via the Elekes–Sharir reduction and incidence geometry, Guth–Katz proved $m(n) \gtrsim n/\log n$ [2, 3]. We study the inverse problem

$$g(k) := \max\{|X| : X \subset \mathbb{R}^2, |D(X)| \leq k\}.$$

The Guth–Katz bound gives $g(k) \lesssim k \log k$, while lattice windows already yield $g(k) \gtrsim k\sqrt{\log k}$ through Bernays–Landau asymptotics for represented norms.

For small k , Erdős–Fishburn determined $g(k)$ up to $k \leq 5$ and conjectured that extremizers for larger k come from triangular lattice subsets [1].

We determine the growth of $g(k)$ up to constants and make the constants explicit in the lattice setting. For each arithmetic lattice Λ we obtain a lower bound with the sharp $k\sqrt{\log k}$ scale and an explicit constant depending on covolume and the Bernays constant of the associated form; universally we retain the $g(k) \lesssim k \log k$ upper bound. We also prove a quantitative stability theorem: unless X is line-heavy or has two popular nonparallel shifts, either almost all ordered pairs lie below a high quantile of the distance multiset or a constant fraction of $X \cap W$ concentrates in a single residue class modulo 2Λ .

2. PRELIMINARIES AND NOTATION

Definition 2.1. Let $D(X) = \{|x - y| : x \neq y\}$. For each realized radius $t \in D(X)$ let

$$m_t := \#\{(p, q) \in X^2 : p \neq q, |p - q| = t\}$$

Define

$$Q_{\text{ord}}(X) := \sum_{t \in D(X)} m_t^2.$$

By Cauchy–Schwarz, writing $n := |X|$ and $k := |D(X)|$,

$$(2.1) \quad Q_{\text{ord}}(X) \geq \frac{(\sum_{t \in D(X)} m_t)^2}{k} = \frac{n^2(n-1)^2}{k}.$$

Definition 2.2. Let $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ be a rank-2 lattice. A Λ -rectangle is

$$W = \{a_0 + iv_1 + jv_2 : 0 \leq i < L_1, 0 \leq j < L_2\} \quad (L_1, L_2 \in \mathbb{N}).$$

It is *proper* if $L_1, L_2 \geq 2$.

Definition 2.3. Let $\Lambda \subset \mathbb{R}^2$ be a rank-2 lattice with a fixed $\mathbb{Z}$ -basis (v_1, v_2) . Identify $\Lambda \cong \mathbb{Z}^2$ by

$$u = (u_1, u_2) \in \mathbb{Z}^2 \longleftrightarrow \lambda(u) := u_1v_1 + u_2v_2 \in \Lambda.$$

Define the quadratic form $Q_\Lambda : \mathbb{Z}^2 \rightarrow \mathbb{R}_{\geq 0}$ by $Q_\Lambda(u) := |\lambda(u)|^2$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^2$.

We say that Λ is arithmetic if Q_Λ is proportional to a rational positive-definite binary quadratic form. Equivalently, Λ is commensurable with $\mathbb{Z}^2$. In this case, after scaling by a positive real and an $\text{SL}_2(\mathbb{Z})$ change of variables on u , we associate to Λ a primitive integral positive-definite binary quadratic form F_Λ and write $\mathcal{C}(\Lambda) := C(F_\Lambda)$ for its Bernays constant (Appendix A). We also define $s(\Lambda) > 0$ by

$$\forall u \in \mathbb{Z}^2 \quad Q_\Lambda(u) = s(\Lambda) F_\Lambda(u),$$

so $s(\Lambda)$ records the fixed proportionality between Euclidean squared norms on Λ and the integral model. For the unimodular hexagonal lattice one has $s(\Lambda_{\text{hex}}) = 2/\sqrt{3}$.

Definition 2.4. For an arithmetic lattice Λ , write $Q_\Lambda(\lambda) = s(\Lambda) F_\Lambda(u)$ with F_Λ primitive integral positive definite and $s(\Lambda) > 0$. Let $\mathcal{A}(\Lambda)$ be the covolume and $\mathcal{C}(\Lambda)$ the Bernays constant of F_Λ . Define

$$S^*(\Lambda) := \frac{s(\Lambda)}{\mathcal{A}(\Lambda) \mathcal{C}(\Lambda)}.$$

Definition 2.5. Let $\mathcal{L}(X)$ be the finite set of lines determined by unordered pairs of points of X . For $\ell \in \mathcal{L}(X)$ set $s_\ell(X) := |X \cap \ell|$. Then

$$\sum_{\ell \in \mathcal{L}(X)} \binom{s_\ell(X)}{2} = \binom{|X|}{2}.$$

Definition 2.6. For $\alpha \in (0, 1]$, a finite set $X \subset \mathbb{R}^2$ is α -line-heavy if

$$\max_{\ell \in \mathcal{L}(X)} s_\ell(X) \geq \alpha |X|.$$

When we say line-heavy without specifying α , we mean α is an absolute fixed constant.

Definition 2.7. For $x \in \mathbb{R}^2$ and $R > 0$, set $B(x, R) := \{y \in \mathbb{R}^2 : |y - x| \leq R\}$. For a lattice $\Lambda \subset \mathbb{R}^2$, the covering radius $\mu(\Lambda)$ is the least $\rho > 0$ with $B(0, \rho) + \Lambda = \mathbb{R}^2$. We write $\mathcal{A}(\Lambda)$ for the covolume.

Definition 2.8. For a rank-2 lattice $\Lambda \subset \mathbb{R}^2$, define the shortest vector length

$$\lambda_1(\Lambda) := \min\{|\lambda| : \lambda \in \Lambda \setminus \{0\}\}.$$

Definition 2.9. For finite $A, B \subset \mathbb{R}^2$ put $r_{A \rightarrow B}(v) := \#\{(a, b) \in A \times B : b - a = v\}$ and $r_A(v) := r_{A \rightarrow A}(v)$. The additive energy is $E_+(A) := \sum_v r_A(v)^2$.

Definition 2.10. Let Λ be a rank-2 lattice and fix an aspect-ratio bound $\mathcal{A}_0 \geq 1$. A set $W \subset z + \Lambda$ is *inner-regular with parameters* $(c, R; \mathcal{A}_0)$ if

$$B(z, (1 - c)R) \cap (z + \Lambda) \subseteq W \subseteq B(z, R) \cap (z + \Lambda),$$

and every minimal Λ -aligned rectangle containing W has side lengths within a factor $\mathcal{A}_0$. Implicit constants may depend on Λ and $\mathcal{A}_0$.

Lemma 2.11. *Let $\Lambda \subset \mathbb{R}^2$ be a lattice with Euclidean covering radius $\mu(\Lambda)$, and fix $\tau \in \mathbb{R}^2$. For every $R > \mu(\Lambda)$,*

$$\{\lambda \in \Lambda : |\lambda| \leq 2R - 2\mu(\Lambda)\} \subseteq \{x - y : x, y \in (\tau + \Lambda) \cap B(0, R)\}.$$

Proof. Let $\lambda \in \Lambda$ with $|\lambda| \leq 2R - 2\mu(\Lambda)$. Then the two Euclidean disks $B(0, R - \mu(\Lambda))$ and $B(-\lambda, R - \mu(\Lambda))$ intersect, since

$$\text{dist}(0, -\lambda) = |\lambda| \leq 2(R - \mu(\Lambda)).$$

Pick any t in the intersection, so $|t| \leq R - \mu(\Lambda)$ and $|t + \lambda| \leq R - \mu(\Lambda)$. By the definition of the covering radius there exists $z \in \tau + \Lambda$ with $|z - t| \leq \mu(\Lambda)$. Set $y := z$ and $x := z + \lambda$. Then $x, y \in \tau + \Lambda$, $x - y = \lambda$, and

$$|y| \leq |z - t| + |t| \leq \mu(\Lambda) + (R - \mu(\Lambda)) = R,$$

$$|x| = |y + \lambda| \leq |(z - t) + (t + \lambda)| \leq \mu(\Lambda) + (R - \mu(\Lambda)) = R.$$

Thus $x, y \in (\tau + \Lambda) \cap B(0, R)$ and $x - y = \lambda$, proving the inclusion. $\square$

3. INCIDENCE BOUNDS

Definition 3.1. Fix a finite set $X \subset \mathbb{R}^2$. For $g \in \text{SE}(2)$ write $r_g := |\{x \in X : g(x) \in X\}|$. Let

$$\mathcal{G} = \mathcal{G}(X) := \{g \in \text{SE}(2) : r_g \geq 2\}$$

be the finite set of direct isometries that map at least two points of X into X . Write $\mathbb{T} := \{g \in \mathcal{G} : g \text{ is a translation}\}$ and $\mathbb{N} := \mathcal{G} \setminus \mathbb{T}$.

Lemma 3.2. *Let $X \subset \mathbb{R}^2$ with $|X| = n$ and let $T \subset D(X)$ with $|T| = L$. Then*

$$\sum_{t \in T} m_t \leq \sqrt{Q_{\text{ord}}(X)} \sqrt{L}.$$

Moreover,

$$Q_{\text{ord}}(X) = \sum_{t \in D(X)} m_t^2 \leq E_+(X) + C n^3 \log n + O(n^2).$$

Proof. For the first inequality, apply Cauchy–Schwarz to $\sum_{t \in T} m_t$.

For the second, by Lemma 4.1,

$$\sum_{g \in \mathcal{G}} r_g^2 = Q_{\text{ord}}(X) + O(n^2),$$

where $\mathcal{G} = \{g \in \text{SE}(2) : r_g \geq 2\}$. Decompose $\mathcal{G}$ into translations $\mathbb{T}$ and non-translations $\mathbb{N}$. For translations,

$$\sum_{g \in \mathbb{T}} r_g^2 = E_+(X) + O(n^2).$$

For nontranslations, by the Elekes–Sharir incidence bound and Guth–Katz,

$$\sum_{g \in \mathbb{N}} r_g^2 \ll n^3 \log n,$$

see Elekes–Sharir [2] and Guth–Katz [3]. Combine to get

$$Q_{\text{ord}}(X) \leq E_+(X) + C n^3 \log n + O(n^2).$$

$\square$

Proposition 3.3. *Let $X \subset \mathbb{R}^2$ with $|X| = n$ and $|D(X)| = k$. List the distinct radii as $t_1 < \dots < t_k$. For $\theta \in (0, 1)$ set $t_\star := t_{\lfloor (1-\theta)k \rfloor}$. If*

$$\sum_{\substack{t \in D(X) \\ t \leq t_\star}} m_t \geq (1 - \eta)n(n - 1) \quad \text{for some } \eta \in (0, 1/2),$$

then there exists $z \in X$ such that

$$|X \cap B(z, t_\star)| \geq (1 - \eta)n.$$

Proof. Form the directed graph $\vec{G}$ on vertex set X by placing an arc $p \rightarrow q$ between distinct $p, q \in X$ if and only if $|p - q| \leq t_\star$. The hypothesis states that the number of ordered edges (arcs) in $\vec{G}$ is at least $(1 - \eta)n(n - 1)$. Hence the average out-degree satisfies

$$\bar{d} = \frac{1}{n} \cdot [\text{ordered edges}] \geq (1 - \eta)(n - 1).$$

Choose $z \in X$ with $\deg_{\vec{G}}^+(z) \geq \bar{d}$. Then the number of points of X at distance $\leq t_\star$ from z equals $1 + \deg_{\vec{G}}^+(z)$, so

$$|X \cap B(z, t_\star)| = 1 + \deg_{\vec{G}}^+(z) \geq 1 + (1 - \eta)(n - 1) \geq (1 - \eta)n.$$

as required. $\square$

Theorem 3.4. *Let Λ be an arithmetic rank-2 lattice, normalized by similarity so that $\lambda_1(\Lambda) = 1$. There exists $k_0(\Lambda) \in \mathbb{N}$ such that for all $k \geq k_0(\Lambda)$,*

$$\frac{\pi}{4} S^*(\Lambda) k \sqrt{\log k} (1 + o_\Lambda(1)) \leq g_\Lambda(k) \leq C k \log k,$$

where $C > 0$ is an absolute constant.

Proof. For the lower bound, let $R > \mu(\Lambda)$ and consider disk windows $W_R = (\tau + \Lambda) \cap B(z, R)$ for some $z \in \mathbb{R}^2$. By Proposition 5.1 and Theorem A.1,

$$|D(W_R)| = \frac{\mathcal{C}(\Lambda)}{s(\Lambda)} \frac{4R^2}{\sqrt{\log(\frac{4R^2}{s(\Lambda)})}} (1 + o_\Lambda(1)).$$

Put $T := \frac{4R^2}{s(\Lambda)}$. Then

$$k = \mathcal{C}(\Lambda) \frac{T}{\sqrt{\log T}} (1 + o_\Lambda(1)) \quad (T \rightarrow \infty).$$

We invert this asymptotically. Rearranging gives

$$T = \frac{k}{\mathcal{C}(\Lambda)} \sqrt{\log T} (1 + o_\Lambda(1)).$$

Let $U := \log T$. Taking logs yields

$$U = \log T = \log k - \log \mathcal{C}(\Lambda) + \frac{1}{2} \log U + o_\Lambda(1).$$

Since $U \rightarrow \infty$, this implies $U = \log k + O(\log \log k)$, hence

$$\sqrt{\log T} = \sqrt{U} = \sqrt{\log k} (1 + o_\Lambda(1)).$$

Substituting back gives

$$T = \frac{k}{\mathcal{C}(\Lambda)} \sqrt{\log k} (1 + o_\Lambda(1)),$$

and therefore

$$R^2 = \frac{s(\Lambda)}{4} T = \frac{s(\Lambda)}{4\mathcal{C}(\Lambda)} k \sqrt{\log k} (1 + o_\Lambda(1)).$$

Hence

$$\begin{aligned} |W_R| &= \frac{\pi}{\mathcal{A}(\Lambda)} R^2 + O_\Lambda(R) \\ &= \frac{\pi}{\mathcal{A}(\Lambda)} \cdot \frac{s(\Lambda)}{4\mathcal{C}(\Lambda)} k \sqrt{\log k} (1 + o_\Lambda(1)) \\ &= \frac{\pi}{4} \mathcal{S}^*(\Lambda) k \sqrt{\log k} (1 + o_\Lambda(1)), \end{aligned}$$

and so $g_\Lambda(k) \geq |W_R|$ gives the claimed lower bound.

For the upper bound, since $g_\Lambda(k) \leq g(k)$ for every fixed Λ and $g(k) \ll k \log k$ by Guth–Katz [3], we obtain $g_\Lambda(k) \ll k \log k$ with an absolute implied constant. $\square$

4. ADDITIVE STRUCTURE AT POSITIVE ENERGY

Lemma 4.1. *Let $X \subset \mathbb{R}^2$, $|X| = n$. Put*

$$Q^* = \#\{(p, q, p', q') \in X^4 : p \neq q, p' \neq q', |p - q| = |p' - q'| > 0\}.$$

Then

$$Q^* = \sum_{t \in D(X)} m_t^2 = Q_{\text{ord}}(X).$$

Let $\mathcal{G} := \{g \in \text{SE}(2) : r_g \geq 2\}$. Then

$$Q^* = \sum_{g \in \mathcal{G}} r_g (r_g - 1), \quad \sum_{g \in \mathcal{G}} r_g^2 = Q^* + O(n^2).$$

In particular, writing T for the translations in $\mathcal{G}$ and $\mathsf{N} := \mathcal{G} \setminus \mathsf{T}$,

$$\sum_{g \in \mathsf{T}} r_g^2 = E_+(X) + O(n^2), \quad \sum_{g \in \mathsf{N}} r_g^2 = Q^* - E_+(X) + O(n^2).$$

Proof deferred to Appendix B. $\square$

4.1. From large additive energy to inner-regular lattice windows.

Lemma 4.2. *Let $W = \{a_0 + iv_1 + jv_2 : 0 \leq i < L_1, 0 \leq j < L_2\}$ with $L_1, L_2 \geq 2$.*

(i) $E_+(W) = \Theta(|W|^3)$. Moreover, if $A \subseteq W$ and $|A| \geq (1 - \varepsilon)|W|$, then

$$E_+(A) \geq E_+(W) - 4\varepsilon|W|^3.$$

(ii) If $A \subset W$ has density $\beta = |A|/|W|$, there exist $s \in \{1, \dots, L_1 - 1\}$, $t \in \{1, \dots, L_2 - 1\}$ and $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that

$$|A \cap (A + \varepsilon_1 s v_1)| \geq \max\left\{0, \frac{\beta L_1 - 1}{2(L_1 - 1)}\right\} |A|, \quad |A \cap (A + \varepsilon_2 t v_2)| \geq \max\left\{0, \frac{\beta L_2 - 1}{2(L_2 - 1)}\right\} |A|.$$

(iii) If P is a proper Λ -rectangle and $T \subset \mathbb{Z}v_1 + \mathbb{Z}v_2$ is finite, there is a proper GAP P^ containing $\bigcup_{t \in T} (t + P)$ with side lengths enlarged by the spans of the T -coefficients and*

$$|P^*| = |P| + \Delta_\alpha L_2 + \Delta_\gamma L_1 + \Delta_\alpha \Delta_\gamma.$$

Proof. (i) Write each difference as $u = u_1 v_1 + u_2 v_2$ with $u_i \in \mathbb{Z}$. A pair $(x, y) \in W^2$ contributes to $r_W(u)$ iff the coordinates differ by u_1, u_2 , hence $r_W(u) = \max\{0, L_1 - |u_1|\} \cdot \max\{0, L_2 - |u_2|\}$ and

$$E_+(W) = \left(\sum_{d=-(L_1-1)}^{L_1-1} (L_1 - |d|)^2 \right) \left(\sum_{e=-(L_2-1)}^{L_2-1} (L_2 - |e|)^2 \right).$$

For a single side, $\sum_{d=-(L_1-1)}^{L_1-1} (L_1 - |d|)^2 = \frac{2}{3}L^3 + \frac{1}{3}L$, giving $E_+(W) = \Theta(|W|^3)$. For the deletion bound, removing one point from W destroys at most two ordered pairs for any fixed u , hence

$$r_A(u) \geq r_W(u) - 2|W \setminus A| = r_W(u) - 2\varepsilon|W|.$$

Therefore

$$E_+(A) = \sum_u r_A(u)^2 \geq \sum_u r_W(u)^2 - 4|W \setminus A| \sum_u r_W(u).$$

Since $\sum_u r_W(u) = |W|^2$ and $|W \setminus A| = \varepsilon|W|$, we get $E_+(A) \geq E_+(W) - 4\varepsilon|W|^3$. (ii) Index W by (i, j) with $0 \leq i < L_1$, $0 \leq j < L_2$. For a fixed column j let $b_j := |\{i : (i, j) \in A\}|$. For $s \in \{1, \dots, L_1 - 1\}$ put $N_j(s) := \sum_{i=0}^{L_1-1-s} \mathbf{1}_A(i, j) \mathbf{1}_A(i+s, j)$. Then $\sum_{s=1}^{L_1-1} N_j(s) = \binom{b_j}{2}$. Averaging over s and summing over j gives

$$\frac{1}{L_1 - 1} \sum_{s=1}^{L_1-1} |A \cap (A + s v_1)| = \frac{1}{L_1 - 1} \sum_{j=0}^{L_2-1} \binom{b_j}{2}.$$

By Cauchy-Schwarz, $\sum_{j=0}^{L_2-1} \binom{b_j}{2} \geq \frac{|A|^2}{2L_2} - \frac{|A|}{2}$, so for some s

$$|A \cap (A + s v_1)| \geq \left(\frac{\beta L_1 - 1}{2(L_1 - 1)} \right) |A|.$$

If the RHS is negative, use the trivial 0 bound. Replacing s by $-s$ if needed gives ε_1 . The v_2 case is identical with rows/columns swapped, giving t and ε_2 .

(iii) Write each $t \in T$ as $t = \alpha_t v_1 + \gamma_t v_2$ with $\alpha_t, \gamma_t \in \mathbb{Z}$ and set

$$\alpha_{\min} = \min_t \alpha_t, \quad \gamma_{\min} = \min_t \gamma_t, \quad \Delta_\alpha = \max_t \alpha_t - \alpha_{\min}, \quad \Delta_\gamma = \max_t \gamma_t - \gamma_{\min}.$$

Let $a_0^* := a_0 + \alpha_{\min} v_1 + \gamma_{\min} v_2$ and $P^* := \{a_0^* + i v_1 + j v_2 : 0 \leq i < L_1 + \Delta_\alpha, 0 \leq j < L_2 + \Delta_\gamma\}$. Then $\bigcup_{t \in T} (t + P) \subset P^*$ and $|P^*| = (L_1 + \Delta_\alpha)(L_2 + \Delta_\gamma) = |P| + \Delta_\alpha L_2 + \Delta_\gamma L_1 + \Delta_\alpha \Delta_\gamma$. $\square$

Proposition 4.3. *Let v_1, v_2 be nonparallel, $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, and*

$$P = \{a_0 + i v_1 + j v_2 : 0 \leq i < L_1, 0 \leq j < L_2\}$$

with $L_1 \geq L_2 \geq 2$. Let $A \subseteq P$ and write $\beta := |A|/|P| \in [0, 1]$. Then there exists a Λ -rectangle $W \subseteq P$ of side lengths $L_2 \times L_2$ with exactly $4L_2 - 4$ lattice boundary points (hence $\asymp |W|^{1/2}$) such that

$$\frac{|A \cap W|}{|W|} \geq \frac{\beta}{2}.$$

Proof. Write the column sums $b_i := |\{j \in \{0, \dots, L_2 - 1\} : (i, j) \in A\}|$ for $i = 0, \dots, L_1 - 1$, so that $\sum_{i=0}^{L_1-1} b_i = |A| = \beta L_1 L_2$.

Consider the L_1 cyclic length- L_2 column windows

$$\mathcal{W}_s^{\text{cyc}} := \{s, s+1, \dots, s+L_2-1\} \pmod{L_1} \quad (s = 0, 1, \dots, L_1-1).$$

Each point of A lies in exactly L_2 of these cyclic windows, hence

$$\frac{1}{L_1} \sum_{s=0}^{L_1-1} \sum_{i \in \mathcal{W}_s^{\text{cyc}}} b_i = \frac{L_2}{L_1} \sum_{i=0}^{L_1-1} b_i = \frac{L_2}{L_1} |A|.$$

Therefore there exists s^* with

$$\sum_{i \in \mathcal{W}_{s^*}^{\text{cyc}}} b_i \geq \frac{L_2}{L_1} |A|.$$

If the window $\mathcal{W}_{s^*}^{\text{cyc}}$ is nonwrapping (i.e. $s^* \leq L_1 - L_2$), put

$$W := \{a_0 + iv_1 + jv_2 : s^* \leq i \leq s^* + L_2 - 1, 0 \leq j < L_2\} \subset P.$$

Then $|A \cap W| = \sum_{i \in \mathcal{W}_{s^*}^{\text{cyc}}} b_i \geq (L_2/L_1)|A|$, hence

$$\frac{|A \cap W|}{|W|} \geq \frac{(L_2/L_1)|A|}{L_2^2} = \frac{|A|}{L_1 L_2} = \beta \geq \frac{\beta}{2}.$$

If $\mathcal{W}_{s^*}^{\text{cyc}}$ wraps (so $s^* > L_1 - L_2$), it decomposes as a disjoint union of two contiguous nonwrapping parts

$$J_1 = [s^*, L_1 - 1], \quad J_2 = [0, s^* + L_2 - 1 - L_1],$$

with $|J_1| + |J_2| = L_2$, hence $\max\{|J_1|, |J_2|\} \geq \lceil L_2/2 \rceil$. One of these parts, call it J , satisfies

$$\sum_{i \in J} b_i \geq \frac{1}{2} \sum_{i \in \mathcal{W}_{s^*}^{\text{cyc}}} b_i \geq \frac{1}{2} \cdot \frac{L_2}{L_1} |A|.$$

Since J is a prefix (resp. suffix) of $[0, L_1 - 1]$ and $|J| \geq \lceil L_2/2 \rceil$, the block $[0, L_2 - 1]$ (resp. $[L_1 - L_2, L_1 - 1]$) is a contiguous nonwrapping interval of length L_2 that contains J . We take that block as our window.

Let W be the nonwrapping L_2 -column block in P obtained by extending J on one side to length L_2 (this is always possible since J is a prefix or suffix of $[0, L_1 - 1]$). Then $W \subset P$ and

$$|A \cap W| \geq \sum_{i \in J} b_i \geq \frac{1}{2} \cdot \frac{L_2}{L_1} |A|.$$

Dividing by $|W| = L_2^2$ gives

$$\frac{|A \cap W|}{|W|} \geq \frac{1}{2} \cdot \frac{L_2}{L_1} \cdot \frac{|A|}{L_2^2} = \frac{1}{2} \cdot \beta.$$

In both cases W is a Λ -rectangle of side lengths $L_2 \times L_2$, and W has exactly $4L_2 - 4$ boundary points, as claimed. $\square$

Lemma 4.4. *Let $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ and let $W \subset z + \Lambda$ be inner-regular with parameter $c \in [0, 1)$ and radius R , with $(1 - c)R > \mu(\Lambda)$ and bounded aspect ratio.*

(i) *We have*

$$B(z, (1 - c)R) \cap (z + \Lambda) \subseteq W \subseteq B(z, R) \cap (z + \Lambda),$$

and $R \asymp_\Lambda |W|^{1/2}$. There exists a subset $W_{\text{in}} \subset W$ with

$$|W \setminus W_{\text{in}}| \ll_\Lambda |W|^{1/2}$$

such that $W_{\text{in}} + t \subset W$ for every $t \in \Lambda$ whose (v_1, v_2) -coordinates lie in $\{0, 1\}^2$.

(ii) *Fix $\varepsilon \in (0, 1)$ and $\delta \in (0, 1]$. Put*

$$\rho_\varepsilon := (1 - c - \varepsilon)R - \mu(\Lambda) > 0.$$

If $\lambda \in \Lambda$ satisfies $|\lambda| \leq (2 - \delta)\rho_\varepsilon$, then

$$r_W(\lambda) \geq c_0(\Lambda, c, \varepsilon, \delta) R^2.$$

Consequently, if $X \subset W$ with $|W \setminus X| = o_\Lambda(R^2)$, then for all sufficiently large R (depending on $\Lambda, c, \varepsilon, \delta$) every such λ lies in $D(X)$. Moreover, all differences $x - y$ with $x, y \in W$ satisfy $|x - y| \leq 2R$, and for every $\lambda \in \Lambda$ with

$$|\lambda| \leq 2(1 - c)R - 2\mu(\Lambda)$$

there exist $x, y \in W$ with $x - y = \lambda$.

Proof. (i) The containment and the estimate $R \asymp_\Lambda |W|^{1/2}$ follow from bounded aspect ratio and norm equivalence on Λ . For the residue-stable core W_{in} take

$$W_{\text{in}} := B(z, (1 - c)R - \Delta) \cap (z + \Lambda), \quad \Delta := \max\{|v_1|, |v_2|, |v_1 + v_2|\}.$$

Then $W_{\text{in}} + t \subset B(z, (1 - c)R) \subset W$ for $t \in \{0, v_1, v_2, v_1 + v_2\}$, and the removal bound $|W \setminus W_{\text{in}}| \ll_\Lambda R \asymp_\Lambda |W|^{1/2}$ follows from lattice-point counting in a belt of fixed thickness (via Lemma B.1).

(ii) Put $\rho := (1 - c)R - \mu(\Lambda)$ and fix $\varepsilon \in (0, 1)$. Let $\rho_\varepsilon := (1 - c - \varepsilon)R - \mu(\Lambda) > 0$. Fix any $\delta \in (0, 1]$. If $|\lambda| \leq (2 - \delta)\rho_\varepsilon$, then the two disks $B(z, \rho_\varepsilon)$ and $B(z, \rho_\varepsilon) - \lambda$ have a lens L whose area satisfies $\text{area}(L) \gg_\delta \rho_\varepsilon^2 \asymp R^2$. By Corollary B.2 the lens contains $\gg_{\Lambda, \varepsilon, \delta} R^2$ points of the translate $z + \Lambda$, each giving an ordered pair $(x, y) \in W \times W$ with $y - x = \lambda$.

If $X \subset W$ with $|W \setminus X| = \Delta$, then for every fixed λ ,

$$r_X(\lambda) \geq r_W(\lambda) - 2\Delta,$$

since removing a single point deletes at most two ordered λ -pairs.

Hence

$$r_W(\lambda) \gg_{\Lambda, c, \varepsilon, \delta} R^2 \quad \text{for all } |\lambda| \leq (2 - \delta)\rho_\varepsilon.$$

Consequently, if $X \subset W$ with $|W \setminus X| = o_\Lambda(R^2)$, then for every fixed $\delta \in (0, 1]$ and all sufficiently large R , every λ with $|\lambda| \leq (2 - \delta)\rho_\varepsilon$ lies in $X - X$, hence $|\lambda| \in D(X)$.

All differences $x - y$ with $x, y \in W$ satisfy $|x - y| \leq 2R$. Moreover, for every $\lambda \in \Lambda$ with $|\lambda| \leq 2(1 - c)R - 2\mu(\Lambda)$ there exist $x, y \in W$ with $x - y = \lambda$, by Lemma 2.11 applied in $z + \Lambda$ with radius $(1 - c)R$. $\square$

5. COUNTING REALIZED DISTANCES IN LATTICE WINDOWS

Proposition 5.1. *Let Λ be a rank-2 arithmetic lattice and let W_R be inner-regular:*

$$B(z, (1 - c)R) \cap (z + \Lambda) \subseteq W_R \subseteq B(z, R) \cap (z + \Lambda)$$

for some fixed $c \in [0, 1)$ with $(1 - c)R > \mu(\Lambda)$. Then, as $R \rightarrow \infty$,

$$\frac{\mathcal{C}(\Lambda)}{s(\Lambda)} \frac{4(1 - c)^2 R^2}{\sqrt{\log(\frac{4R^2}{s(\Lambda)})}} (1 + o_{\Lambda, c}(1)) \leq |D(W_R)| \leq \frac{\mathcal{C}(\Lambda)}{s(\Lambda)} \frac{4R^2}{\sqrt{\log(\frac{4R^2}{s(\Lambda)})}} (1 + o_\Lambda(1)).$$

Here $o_{\Lambda, c}(1)$ and $o_\Lambda(1)$ are uniform for fixed Λ (and fixed c in the lower bound).

Proof. By Lemma 2.11 applied in $z + \Lambda$ with radius $(1 - c)R$, every $\lambda \in \Lambda$ with $|\lambda| \leq 2(1 - c)R - 2\mu(\Lambda)$ occurs as a difference $x - y$ with $x, y \in W_R$, while trivially all differences satisfy $|x - y| \leq 2R$. Since $Q_\Lambda(\lambda) = s(\Lambda) F_\Lambda(u)$ for a primitive integral positive-definite binary quadratic form F_Λ ,

$$\mathcal{R}_{F_\Lambda}\left(\frac{(2(1 - c)R - 2\mu(\Lambda))^2}{s(\Lambda)}\right) \leq |D(W_R)| \leq \mathcal{R}_{F_\Lambda}\left(\frac{(2R)^2}{s(\Lambda)}\right).$$

Bernays–Landau for fixed F_Λ gives $\mathcal{R}_{F_\Lambda}(U) = \mathcal{C}(\Lambda) U / \sqrt{\log U} (1 + o(1))$. Since

$$\frac{(2(1 - c)R - 2\mu(\Lambda))^2}{s(\Lambda)} = \frac{4(1 - c)^2 R^2}{s(\Lambda)} (1 + O_\Lambda(R^{-1})),$$

the lower main term carries $(1 - c)^2$; the $\mu(\Lambda)$ correction is absorbed by $o(1)$. Replacing U by a fixed multiplicative constant changes $\sqrt{\log U}$ by $1 + o(1)$, so both denominators may be written as $\sqrt{\log(\frac{4R^2}{s(\Lambda)})(1 + o(1))}$. $\square$

Definition 5.2. For a rank-2 lattice Λ and $k \in \mathbb{N}$ put

$$g_\Lambda(k) := \max\{|X| : \exists \tau \in \mathbb{R}^2 \text{ with } X \subset \tau + \Lambda, |D(X)| \leq k\}.$$

6. RESIDUE CLASSES AND CONCENTRATION

6.1. Rigidity of near-optimizers.

Proposition 6.1. *Let $A_0 \subset \mathbb{R}^2$ be finite. Suppose there exists $\mathcal{U} \subset A_0 - A_0$ with $|\mathcal{U}| \geq \beta |A_0|$ and*

$$|A_0 \cap (A_0 + u)| \geq \rho |A_0| \quad (\forall u \in \mathcal{U}),$$

for some $\beta, \rho \in (0, 1]$. Then there exist nonparallel vectors u_1, u_2 , a full-rank lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, a Λ -rectangle W , and a set $A \subseteq A_0 \cap W$ such that

$$|W| \geq c(\beta, \rho) |A_0|, \quad |A| \geq c(\beta, \rho) |A_0|, \quad |A \cap (A + u_i)| \geq c(\beta, \rho) |A| \quad (i = 1, 2).$$

Proof. By Proposition 6.2, $E_+(A_0) \geq \beta \rho^2 |A_0|^3$. Apply Proposition B.3(i) to obtain $A' \subseteq A_0$ with $|A'| \geq c_1(\beta, \rho) |A_0|$ and $|A' - A'| \leq K |A'|$ where $K \leq C_1(\beta, \rho)$. By Proposition B.3(ii), A' lies in a proper rank-2 GAP P with $|P| \leq C_2(\beta, \rho) |A'|$. The two GAP steps give nonparallel v_1, v_2 and the lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$. Apply Proposition 4.3 to $A' \subset P$ to obtain a Λ -rectangle $W \subset P$ with $|W| \geq c_2(\beta, \rho) |A'|$ and $|A' \cap W| \geq c_3(\beta, \rho) |W|$. Set $A := A' \cap W$. Then $|A| \geq c(\beta, \rho) |W|$, so $\beta_W := |A|/|W| \geq c(\beta, \rho)$. Apply Lemma 4.2(ii) to $A \subset W$: there exist $s, t \in \{1, \dots, L_2 - 1\}$ and signs $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that

$$|A \cap (A + \varepsilon_1 s v_1)| \geq c'(\beta, \rho) |A|, \quad |A \cap (A + \varepsilon_2 t v_2)| \geq c'(\beta, \rho) |A|.$$

Since $s, t \leq L_2 - 1$, these overlaps occur entirely inside W , so no boundary loss arises. This gives the desired two nonparallel heavy shifts inside a single Λ -rectangle W , with $|W| \geq c(\beta, \rho) |A_0|$ and $|A| \geq c(\beta, \rho) |A_0|$. $\square$

Proposition 6.2. *Let $A_0 \subset \mathbb{R}^2$ and suppose there exists $\mathcal{U} \subset A_0 - A_0$ with $|\mathcal{U}| \geq \beta |A_0|$ and $|A_0 \cap (A_0 + u)| \geq \rho |A_0|$ for all $u \in \mathcal{U}$. Then*

$$E_+(A_0) \geq \sum_{u \in \mathcal{U}} r_{A_0}(u)^2 \geq \beta \rho^2 |A_0|^3.$$

Consequently, by Proposition B.3, there exists $A' \subseteq A_0$ with $|A'| \geq c(\beta, \rho) |A_0|$ and $|A' - A'| \leq K(\beta, \rho) |A'|$, and A' lies in a proper rank-2 GAP P with $|P| \leq C K(\beta, \rho)^C |A'|$.

Proof. Immediate from $E_+(A_0) = \sum_v r_{A_0}(v)^2$, the hypothesis, and the standard BSG and Freiman statements (Prop. B.3). $\square$

Lemma 6.3. *Let $A \subset \mathbb{R}^2$ with $|A - A| \leq K |A|$. Then*

$$\left| \left\{ u \in A - A : r_A(u) \geq \frac{|A|}{2K} \right\} \right| \geq \frac{|A|}{2}.$$

Proof. Write $D := |A - A|$ and $M := |\{u : r_A(u) \geq T\}|$. For any $T \in (0, |A|)$,

$$|A|^2 = \sum_u r_A(u) \leq M |A| + (D - M) T.$$

With $T = |A|/(2K)$ and $D \leq K|A|$ this gives

$$|A|^2 \leq \frac{|A|^2}{2} + M|A|\left(1 - \frac{1}{2K}\right),$$

hence $M \geq |A|/(2 - 1/K) \geq |A|/2$. $\square$

Lemma 6.4. *Let $A \subset \mathbb{R}^2$ be finite and fix a direction u . For each line $L \parallel u$, write $s_L := |A \cap L|$. Then*

$$\sum_{\substack{v \parallel u \\ v \neq 0}} r_A(v) = \sum_{L \parallel u} s_L (s_L - 1).$$

Proof. Partition ordered pairs $(x, y) \in A^2$ with $x \neq y$ by the line L parallel to u that contains them. Pairs from L contribute exactly $s_L(s_L - 1)$, and each such pair has difference $y - x$ parallel to u . Summing over L gives the identity. $\square$

6.2. Period lattices and residue decomposition.

Proposition 6.5. *Let $L \subset \mathbb{R}^2$ be a rank-2 lattice and let $X' \subset \mathbb{R}^2$ be finite. Write its canonical decomposition modulo $2L$ as $X' = \bigsqcup_{j=1}^m X_j$ with $1 \leq m \leq 4$ and $X_j \subset c_j + 2L$. Put $N := |X'|$ and $m_j := |X_j|$. Then*

$$E_+(X') \leq 4N^2 \max_{1 \leq j \leq m} m_j.$$

In particular, if $\delta := 1 - \max_j(m_j)/N \in [0, 3/4]$, then $E_+(X') \leq 4(1 - \delta)N^3$.

Proof. Write $r_{X_i \rightarrow X_j}(v) := |\{(x, y) \in X_i \times X_j : y - x = v\}|$. Then $r_{X'}(v) = \sum_{j=1}^m r_{X' \rightarrow X_j}(v)$ and by Cauchy–Schwarz,

$$E_+(X') = \sum_v r_{X'}(v)^2 \leq 4 \sum_{j=1}^m \sum_v r_{X' \rightarrow X_j}(v)^2.$$

For fixed j , $\sum_v r_{X' \rightarrow X_j}(v) = Nm_j$ and $r_{X' \rightarrow X_j}(v) \leq m_j$, hence $\sum_v r_{X' \rightarrow X_j}(v)^2 \leq Nm_j^2$. Summing over j gives $E_+(X') \leq 4N \sum_j m_j^2 \leq 4N(\max_j m_j) \sum_j m_j = 4N^2 \max_j m_j$. The final inequality follows by substituting $m_{\max} = (1 - \delta)N$. $\square$

Corollary 6.6. *Let $X' \subset P$ decompose as $X' = \bigsqcup_{j=1}^m X_j$ into residue classes modulo $2L$, with $m \leq 4$ and $N := |X'|$. If $E_+(X') \geq \alpha N^3$ for some $\alpha \in (0, 1]$, then*

$$\max_{1 \leq j \leq m} |X_j| \geq \frac{\alpha}{4} N.$$

Proof. By Proposition 6.5, $E_+(X') \leq 4N^2 \max_j |X_j|$. Thus $4N^2 \max_j |X_j| \geq \alpha N^3$, so $\max_j |X_j| \geq (\alpha/4)N$. $\square$

7. HEX CONSTRUCTION AND GLOBAL BOUNDS

7.1. Arithmetic reduction.

Theorem 7.1. *As $k \rightarrow \infty$,*

$$\frac{\pi}{3\mathcal{C}(\Lambda_{\text{hex}})} k \sqrt{\log k} (1 + o(1)) \leq g(k) \leq C k \log k.$$

for some absolute constant $C > 0$.

Proof. The lower bound follows from Theorem 3.4 applied to Λ_{hex} . For the upper bound, by Guth–Katz [3], any n -point planar set determines at least $cn/\log n$ distinct distances. Hence if $|D(X)| \leq k$ then

$$n \leq C_1 k \log n.$$

Define $f(x) := x - C_1 k \log x$. Then $f'(x) = 1 - \frac{C_1 k}{x}$, so f' is $\geq \frac{1}{2}$ on $[2C_1 k, \infty)$. Set $M := 2C_1 k \log k$. For all sufficiently large k we have $M \geq 2C_1 k$ and

$$C_1 k \log M < M.$$

For k sufficiently large we have $\log \log k \leq \frac{1}{2} \log k$, hence $\log M = \log(2C_1 k \log k) \leq \log k + \log(2C_1) + \log \log k \leq \frac{3}{2} \log k + O(1)$, which yields $C_1 k \log M < M$.

If $n \geq M$, then $f(n) \geq f(M) > 0$, contradicting $n \leq C_1 k \log n$. Hence $n < M = 2C_1 k \log k$, and taking the supremum yields $g(k) \ll k \log k$. $\square$

7.2. Quantitative stability.

Lemma 7.2. *Let $X \subset \mathbb{R}^2$ with $|X| = n$ and $|D(X)| \leq k$, and assume $k \leq Cn/\log n$. Fix $\sigma \in (0, 1/4]$ and set $\theta_k = (\log k)^{-1/2-\sigma}$. Let $t_\star$ be the $(1 - \theta_k)$ -quantile of $D(X)$. Then for all sufficiently large k , at least one holds:*

- (i) *some line ℓ satisfies $|X \cap \ell| \geq cn$;*
- (ii) *there exist two nonparallel vectors v_1, v_2 with $r_X(v_i) \geq cn$ ($i = 1, 2$);*
- (iii) *there exists $z \in X$ with $|X \cap B(z, t_\star)| \geq (1 - o(1))n$.*

Here $c, C > 0$ are absolute constants.

Proof. By (2.1) and $k \leq Cn/\log n$, $Q_{\text{ord}}(X) \geq c_0 n^3 \log n$. By Lemma 3.2, $Q_{\text{ord}}(X) \leq E_+(X) + C_1 n^3 \log n + O(n^2)$. If $E_+(X) \geq c_2 n^3$, apply Proposition B.3(i) to get $A \subseteq X$ with $|A| \geq cn$ and $|A - A| \leq K|A|$ where $K \leq C$. By Lemma 6.3, either two nonparallel u in $A - A$ satisfy $r_A(u) \gg n$ (yielding (ii) for X), or all popular u are parallel. Then Lemma 6.4 gives

$$\sum_{L \parallel u} s_L(s_L - 1) = \sum_{v \parallel u} r_A(v) \geq \frac{|A|}{2} \cdot \frac{|A|}{2K} \gg |A|^2,$$

so $\max_{L \parallel u} s_L \gg |A| \gg n$, yielding (i) with an absolute constant. If $E_+(X) < c_2 n^3$, then

$$\sum_{\text{top } L} m_t \leq \sqrt{Q_{\text{ord}}(X)} \sqrt{L} \leq C_3 n^2 \theta_k^{1/2}$$

with $L = \lfloor \theta_k k \rfloor$. Since $\theta_k^{1/2} = o(1)$, the bottom $(1 - \theta_k)k$ radii support $(1 - o(1))n(n - 1)$ ordered pairs. Proposition 3.3 yields (iii). $\square$

Theorem 7.3. *There exist absolute constants $c, C > 0$ such that the following holds. Let $X \subset \mathbb{R}^2$ with $|X| = n$ and $|D(X)| \leq k$, and assume $k \leq Cn/\log n$. Then either*

- (i) *some line ℓ contains at least cn points of X ; or*
- (ii) *there exist nonparallel vectors v_1, v_2 , a rank-2 lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, a Λ -rectangle W , and a set $A \subseteq X \cap W$ with $|W| \geq cn$ and $|A| \geq cn$ such that $|A \cap (A + v_i)| \geq c|A|$ for $i = 1, 2$. Moreover, with $N := |X \cap W|$ and the residue decomposition modulo 2Λ ,*

$$E_+(X \cap W) \leq 4N^2 \max_j |X_j|, \quad \text{so if } E_+(X \cap W) \geq \alpha N^3 \text{ then } \max_j |X_j| \geq (\alpha/4)N;$$

- (iii) *There exists $z \in X$ such that $|X \cap B(z, t_\star(X, \theta_k))| \geq (1 - o(1))n$.*

Proof. As above, $Q_{\text{ord}}(X) \gg n^3 \log n$ and $Q_{\text{ord}}(X) \leq E_+(X) + Cn^3 \log n + O(n^2)$. If $E_+(X) \gg n^3$, apply Proposition 6.1 to obtain (ii) and then Proposition 6.5 for the residue estimate. Otherwise use Lemma 7.2 to obtain (i) or (iii). $\square$

APPENDIX A. BERNAYS–LANDAU ASYMPTOTIC

Theorem A.1 (Bernays–Landau). *Let F be a primitive positive definite integral binary quadratic form and $\mathcal{R}_F(T) := \#\{n \leq T : n \text{ is represented by } F\}$. Then*

$$\mathcal{R}_F(T) = C(F) \frac{T}{\sqrt{\log T}} (1 + o(1)) \quad (T \rightarrow \infty),$$

where $C(F) > 0$ depends only on F .

Proof. Bernays [5]; see also Tenenbaum [4, Ch. III.4]. $\square$

APPENDIX B. COMBINATORIAL AND GEOMETRIC FACTS

Proof of Lemma 4.1. The first identity $Q^* = \sum_t m_t^2$ is immediate from the definition of m_t , which counts ordered pairs (p, q) with $p \neq q$ and $|p - q| = t$.

For the isometry identity, each ordered pair of ordered pairs $((p, q), (p', q'))$ with $p \neq q, p' \neq q', |p - q| = |p' - q'| > 0$ determines a unique direct isometry $g \in \text{SE}(2)$ with $g(p) = p'$ and $g(q) = q'$. Moreover such a g satisfies $r_g \geq 2$, so $g \in \mathcal{G}$. Conversely, any $g \in \mathcal{G}$ and any ordered distinct pair (x, y) with $g(x), g(y) \in X$ produce one such quadruple. Thus $Q^* = \sum_{g \in \mathcal{G}} r_g(r_g - 1)$.

Summing gives $\sum_{g \in \mathcal{G}} r_g^2 = Q^* + \sum_{g \in \mathcal{G}} r_g$. Each ordered pair $(x, y) \in X^2$ with $x \neq y$ contributes to at most $O(1)$ isometries in $\mathcal{G}$ (those determined by (x, y) together with a second matched ordered pair), hence $\sum_{g \in \mathcal{G}} r_g = O(n^2)$ and $\sum_{g \in \mathcal{G}} r_g^2 = Q^* + O(n^2)$.

For translations, $\sum_{g \in \mathcal{T}} r_g^2 = \sum_v r_X(v)^2 = E_+(X)$, with the $O(n^2)$ adjustment if one includes $r_g = 1$ maps, which we do not since $g \in \mathcal{G}$. Subtracting yields the non-translation identity. $\square$

Lemma B.1. *Let $\Lambda \subset \mathbb{R}^2$ be a lattice with covolume $\mathcal{A}(\Lambda)$. There exist constants $r_\Lambda, R_\Lambda \asymp_\Lambda 1$ and $C_\Lambda > 0$ such that for every translate $\tau + \Lambda$ and every bounded convex set $K \subset \mathbb{R}^2$ with piecewise C^1 boundary,*

$$|(\tau + \Lambda) \cap K| = \frac{\text{area}(K)}{\mathcal{A}(\Lambda)} + O_\Lambda(1 + \text{perim}(K)),$$

uniformly in τ .

Proof. Fix a fundamental domain $\mathcal{F}$ for Λ with $B(0, r_\Lambda) \subset \mathcal{F} \subset B(0, R_\Lambda)$. Let $\mathcal{F}_\lambda := \lambda + \mathcal{F}$ for $\lambda \in \tau + \Lambda$. Cells with $\mathcal{F}_\lambda \subset K$ contribute exactly $\text{area}(K)/\mathcal{A}(\Lambda)$ up to an error bounded by the number of boundary cells. A cell intersects ∂K only if $\lambda \in \partial K + B(0, R_\Lambda)$. Hence the number of boundary cells is at most

$$\frac{\text{area}(\partial K + B(0, R_\Lambda))}{\text{area}(\mathcal{F})} \ll_\Lambda 1 + \text{perim}(K),$$

since $\text{area}(\partial K + B(0, R_\Lambda)) \leq c_\Lambda (R_\Lambda^2 + R_\Lambda \text{perim}(K))$ for convex K with piecewise C^1 boundary. Each boundary cell changes the count by at most 1, giving the stated error uniformly in τ . $\square$

Corollary B.2. *Let $\rho > 0$ and $u \in \mathbb{R}^2$ with $|u| \leq 2\rho$. For any $z \in \mathbb{R}^2$, the lens $L := B(z, \rho) \cap (B(z, \rho) - u)$ satisfies*

$$|(\tau + \Lambda) \cap L| = \frac{\text{area}(L)}{\mathcal{A}(\Lambda)} + O_\Lambda(\rho),$$

uniformly in τ, z, u . The perimeter obeys $\text{perim}(L) \leq 4\pi\rho$. In addition,

$$\text{area}(L) = 2\rho^2 \arccos\left(\frac{|u|}{2\rho}\right) - \frac{|u|}{2} \sqrt{4\rho^2 - |u|^2},$$

so if $|u| \leq (2 - \delta)\rho$ with fixed $\delta \in (0, 1]$ then $\text{area}(L) \gg_\delta \rho^2$. Hence $|(\tau + \Lambda) \cap L| \gg_{\Lambda, \delta} \rho^2$.

Proof. Apply Lemma B.1 with $K = L$. The boundary of L consists of two circular arcs of radius ρ , so $\text{perim}(L) \leq 4\pi\rho$. The explicit area formula is standard and yields the stated lower bound when $|u| \leq (2 - \delta)\rho$. $\square$

Proposition B.3. *There exist absolute constants $C, c > 0$ such that:*

- (i) (BSG) *If $A \subset \mathbb{R}^2$ satisfies $E_+(A) \geq \kappa |A|^3$ with $0 < \kappa \leq 1$, then there exists $A' \subseteq A$ with*

$$|A'| \geq c \kappa^C |A| \quad \text{and} \quad |A' - A'| \leq C \kappa^{-C} |A'|.$$
- (ii) (Freiman in $\mathbb{R}^2$) *If $A' \subset \mathbb{R}^2$ has $|A' - A'| \leq K |A'|$, then A' lies in a proper rank-2 GAP P with $|P| \leq C K^C |A'|$.*

Proof. (i) This is the Balog–Szemerédi–Gowers theorem. See, e.g., [6] or TaoVu. Applied to the additive energy $E_+(A) \geq \kappa |A|^3$ it yields $A' \subseteq A$ with $|A'| \geq c \kappa^C |A|$ and $|A' - A'| \leq C \kappa^{-C} |A'|$, for absolute constants $c, C > 0$.

(ii) This is Freiman’s theorem in $\mathbb{R}^2$. If $|A' - A'| \leq K |A'|$, then A' is contained in a proper rank-2 generalized arithmetic progression P with $|P| \leq C K^C |A'|$. See [7]. $\square$

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