On a sequence of singular ball quotient surfaces on the line $K^2 = 9\chi - 18$

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Abstract

Starting from computer experiments with the fundamental group of the Cartwright–Steger surface, we construct an infinite tower $(X_n)_{n\geq 1}$ of normal projective surfaces obtained by successive $\mathbb{Z}/3$-Galois covers $X_n \to X_{n-1}$. For $n > 1$, their minimal resolutions $\widetilde{X}_n$ lie on the line $K^2 = 9\chi - 18$ (equivalently $c_1^2 = 3c_2 - 72$), which is parallel to the Bogomolov–Miyaoka–Yau line $K^2 = 9\chi$ of ball quotients. We compute the fundamental groups for the first cases, showing that $\pi_1(\widetilde{X}_n) = 1$ for $n = 1, \ldots, 5$. Motivated by the geometry of the construction, we conjecture that all $\widetilde{X}_n$ are simply connected.

Keywords: surfaces of general type; ball quotient surfaces; fundamental groups; simply connected surfaces.

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1 Introduction

For smooth complex algebraic surfaces S of general type, the Bogomolov–Miyaoka–Yau inequality $K^2 \leq 9\chi$ (equivalently $c_1^2 \leq 3c_2$) holds, where K^2 is the self-intersection of a canonical divisor of S and χ is the holomorphic Euler characteristic of S. Surfaces on the line $K^2 = 9\chi$ are known to be quotients of the complex unit ball $\mathbb{B}$ by a lattice $\Lambda \subset PU(2,1)$. They are rigid surfaces with infinite fundamental group. A long-standing problem in the geography of surfaces of general type is to determine whether there exists a neighborhood of this line in which the fundamental group must be infinite and, in particular, how closely the line can be approached by simply connected surfaces. An important step has been taken by Roulleau and Urzúa [RU15], who constructed a sequence of simply connected surfaces Z_n with

$$\lim_n \frac{K^2_{Z_n}}{\chi(Z_n)} = 9.$$

However, this is not optimal, because $\lim_n \left(9\chi(Z_n) - K^2_{Z_n}\right) = +\infty$, thus a substantial region near the Bogomolov–Miyaoka–Yau line remains unexplored for the existence of simply connected examples.

Surfaces of general type with invariants $K^2 = 9\chi = 9$ and $p_g = 0$ (hence $q = 0$) are the so-called *fake projective planes*. There are 50 pairs of complex-conjugated such surfaces, according to the results from the work of Prasad and Yeung [PY07], [PY10], and Cartwright and Steger [CS10].

In their computational classification, Cartwright and Steger [CS10] also found the construction of a remarkable surface S_1 with invariants

$$K_{S_1}^2 = 9,\ p_g(S_1) = q(S_1) = 1.$$

This ball quotient surface is given by a lattice $\Lambda \subset PU(2,1)$ which is a sublattice of a maximal lattice $\bar{\Gamma}$ constructed by Mostow, which is a Deligne-Mostow group whose associated weights $(2,2,2,7,11)/12$ satisfy the condition $\Sigma(INT)$ (see [Mos86]). More recently, an explicit construction of this surface has been given by Borisov and Yeung [BY20].

In this paper, we start from the Cartwright–Steger surface and, guided by computer experiments with its fundamental group, we uncover an infinite sequence (X_n) of normal projective surfaces lying on the line $K^2 = 9\chi - 18$ for $n > 1$. Our computations show that the first five smooth minimal resolutions $\widetilde{X}_1, \ldots, \widetilde{X}_5$ are simply connected. More experiments then point to a simple geometric description: each X_n arises as a Galois triple covering of X_{n-1}, produced by a uniform procedure. These findings motivate the central conjecture of the paper that all $\widetilde{X}_n$ are simply connected.

All computations were performed in Magma [BCP97]. A reproducible script certifying our claims is available as an arXiv ancillary file.

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2 Experiments with the Cartwright–Steger fundamental group

Let $\mathbb{B} \subset \mathbb{C}^2$ be the unit ball. Consider the Cartwright–Steger ball quotient surface $S_1 = \mathbb{B}/H_1$ and its $\mathbb{Z}/3$-quotient $X_1 = \mathbb{B}/G_1$, where H_1 has index 3 in G_1, and both H_1, G_1 are subgroups of a maximal lattice $\bar{\Gamma}$. The covering map $S_1 \to X_1$ is ramified at isolated points corresponding to order-3 elliptic elements in G_1.

Starting from the finite presentation of $\bar{\Gamma}$ given in [CKY17], we searched, using Magma, for two elements that generate $\bar{\Gamma}$ and yield short defining relators after simplification. This led to the concise presentation

$$\bar{\Gamma} \ \cong\ \left\langle u, w \;\middle|\; u^3,\ w^3,\ (u,\ wuw^{-1}uw),\ (uw)^8 \right\rangle,$$

where $(x,y) = x^{-1}y^{-1}xy$.

We then searched for short words in u, w that generate an index 288 subgroup isomorphic to the group $G_1 \leq \bar{\Gamma}$ corresponding to X_1. We found that G_1 is generated by the three conjugates

$$G_1 \cong \ \langle\, w^{uw},\ \ w^{w^u},\ \ w^{(u,w^{-1})} \,\rangle \ \leq\ \bar{\Gamma},$$

where $x^g = g^{-1}xg$.

The invariants of the abelianization of G_1 are $[3,3]$, i.e.

$$\mathrm{Abel}(G_1) := G_1/[G_1, G_1] \cong (\mathbb{Z}/3)^2.$$

To find the group H_1 we compute all index-3 normal subgroups of G_1. We get four subgroups. Their abelianizations have invariants $[0,0], [3,21], [3,3], [3,3]$. The first one is H_1.

Let G_2 be one of the subgroups with invariants $[3,3]$, say the one containing the generator w^{uw}. The computations show that G_2 also has four index-3 normal subgroups, whose abelianization invariants are

$$[7,0,0],\ [3,3],\ [3,3],\ [3,3]. \tag{1}$$

We check computationally that the first one is an index-3 subgroup of H_1, while the other three are conjugate subgroups.

Iterating this procedure we get a chain of nine successive index-3 subgroups

$$G_1 \geq G_2 \geq \cdots \geq G_{10},$$

at each step exhibiting the same subgroup pattern as in (1). For each G_i there is an index-9 normal subgroup H_{i+1} such that

$$G_i/H_{i+1} \cong (\mathbb{Z}/3)^2.$$

We check that

$$H_{i+1} = [G_i, G_i], \quad i = 1, \ldots 9.$$

Due to computer limitations, our computations do not go further, and we ask whether this index-3 subgroup pattern persists indefinitely.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H_2 \arrow[r] \arrow[d] & G_2 \arrow[d] \\
H_1 \arrow[r] & G_1
\end{tikzcd}$$

To answer this, we looked for a much simpler group that still exhibits the same index-3 subgroup pattern. A short randomized search among two–generator, small–relator groups pointed to the Euclidean triangle group

$$T \ = \ \langle\, x, y \mid x^3,\ y^3,\ (xy)^3 \,\rangle.$$

The group T contains an index 3 subgroup isomorphic to T, thus it has an infinite structure of index-3 subgroups similar to the one above, with abelianizations

$$[0,0],\ [3,3],\ [3,3],\ [3,3].$$

We then used Magma to find a surjective homomorphism

$$h : G_1 \twoheadrightarrow T,$$

which then implies that G_1 also has an infinite structure of index-3 subgroups as above.

From the map h we extracted order-3 generators a, b, c, d of G_1 such that

$$h(a) = h(b) = h(c) = x \in T, \quad h(d) = y \in T.$$

In particular, ab^{-1} and bc^{-1} are in the kernel of h.

We wonder if $\mathrm{Ker}(h)$ is normally generated by these two elements. Accordingly, set

$$Q \;:=\; G_1 \;/\; \langle\!\langle\, ab^{-1},\, bc^{-1} \,\rangle\!\rangle^{G_1}.$$

A Magma computation shows that

$$Q \;\cong\; T' \;:=\; \langle\, x, y \mid x^3,\; y^3,\; (xy)^3(yx)^3 \,\rangle,$$

which still exhibits the same index-3 normal-subgroup pattern as T.

So, we have seen that there is an infinite commutative diagram

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
S_2 & \longrightarrow & X_2 \\
\downarrow & & \downarrow \\
S_1 & \longrightarrow & X_1
\end{array}$$

where the arrows denote $\mathbb{Z}/3$-coverings of surfaces.

The group G_1 is generated by elements of order 3, hence by Armstrong theorem [Arm68], the smooth minimal model $\tilde{X}_1$ of the surface X_1 is simply connected. What can be said about $X_2, X_3, \ldots$?

Using Magma, we show that

$$G_{i+1} \;:=\; \langle\!\langle\, a, b, c \,\rangle\!\rangle^{G_i}, \quad i = 1, 2, 3$$

(the normal closure of $\langle\, a, b, c \,\rangle$ in G_i). Beyond this point, the presentations grow rapidly in length and complexity, and our computations stall.

To handle G_5 we use a workaround: we compute the quotient

$$J \;:=\; G_5 \;/\; \langle\!\langle\, a, b, c \,\rangle\!\rangle^{G_4}$$

and use the Magma function `SearchForIsomorphism` to check if it is isomorphic to $\mathbb{Z}/3$. In the process, Magma reports (twice) that a generator of J is trivial. We append this relation and the algorithm confirms that $J \;\cong\; \mathbb{Z}/3$.

Summing up, the groups $G_1, \ldots, G_5$ are generated by elements of order 3, which imply that the smooth minimal surfaces $\tilde{X}_1, \ldots, \tilde{X}_5$ are simply connected.

3 The fibration on the surface X_1

Our references for this section are [CS10] and [CKY17].

The Cartwright-Steger surface is $S_1 := \mathbb{B}/H_1$, with invariants $K^2 = 9\chi = 9$, $p_g = q = 1$. The group G_1 is the normalizer of H_1 in $\bar{\Gamma}$. The automorphism group $\sigma := \mathrm{Aut}(S_1)$ is $\mathbb{Z}/3 \cong G_1/H_1$. The quotient surface $X_1 := S_1/\sigma$ has 9 singularities: 3 points O'_1, O'_2, O'_3 of type $\frac{1}{3}(1,1)$, and 6 points $Q'_1, \ldots, Q'_6$ of type $\frac{1}{3}(1,2)$ (A_2 singularities, or ordinary *cusps*).

Denote by $O_i, Q_j \in S_1$ the fixed points of σ corresponding to O'_i, Q'_j, respectively. The Albanese fibration $\alpha : S_1 \to E$ has three fibers $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_3$ such that $O_1, O_2, O_3 \in \bar{F}_1$, $Q_1, Q_2, Q_3 \in \bar{F}_2$ and $Q_4, Q_5, Q_6 \in \bar{F}_3$. Moreover, $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_3$ are reduced. The invariants of the smooth minimal resolution $\tilde{X}_1$ of X_1 are

$$K^2_{\tilde{X}_1} = \chi(\tilde{X}_1) = 2,\ p_g(\tilde{X}_1) = 1,\ q(\tilde{X}_1) = 0.$$

By the functorial properties of the Albanese map, the automorphism σ acts on the elliptic curve E through an automorphism σ_E, which has 3 fixed points, and α induces a fibration

$$f : \tilde{X}_1 \longrightarrow \mathbb{P}^1 = E/\sigma_E.$$

Let F_i and D_i be the total and strict transform in $\tilde{X}_1$, respectively, of the image of $\bar{F}_i$ on $X_1 := S_1/\sigma$.

Proposition 1. *The fibers F_1, F_2, F_3 have the following configuration:*

- $F_1 = 3D_1 + B_1 + B_2 + B_3$,
- $F_2 = 3D_2 + 2(A_1 + A_2 + A_3) + A'_1 + A'_2 + A'_3$,
- $F_3 = 3D_3 + 2(A_4 + A_5 + A_6) + A'_4 + A'_5 + A'_6$,

where B_i is the (-3)-curve which resolves the singularity O'_i, $i = 1, 2, 3$, and A_i, A'_i are the (-2)-curves which resolve the singularity Q'_i, $i = 1, \dots, 6$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \tilde{S}_1 & \xrightarrow{\pi} & S_1 \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\sigma} \\ \tilde{X}_1 & \xrightarrow{\varphi} & X_1 \end{array}$$

where φ is the minimal resolution of the singularities of X_1, the map ψ is a $\mathbb{Z}/3$-covering ramified over the exceptional divisors from φ, and π is a sequence of blowups centered at the fixed points of σ. Let $\tilde{F}_i := \pi^*(\bar{F}_i)$, $i = 1, 2, 3$. The pullback of a fiber in $\tilde{X}_1$ is a union of three fibers in $\tilde{S}_1$, except for $\psi(F_i) = 3\tilde{F}_i$, $i = 1, 2, 3$. This implies that every component of a fiber F_i which is not contained in the branch locus of ψ must be of multiplicity 3. We have then

$$\begin{aligned} F_1 &= 3D_1 + \textstyle\sum_1^3 a_{1i} B_i, \\ F_2 &= 3D_2 + \textstyle\sum_1^3 (a_{2i} A_i + b_{2i} A'_i), \\ F_3 &= 3D_3 + \textstyle\sum_4^6 (a_{3i} A_i + b_{3i} A'_i), \end{aligned}$$

for some effective divisors D_1, D_2, D_3, and $a_{ji}, b_{ji} \in \mathbb{N}$.

According to [Rit22, Section 5], the fibers of the Albanese map of S_1 are smooth. This implies $B_i D_1 = 1$, $i = 1, 2, 3$, and $A_i D_2 = 1$, $A'_i D_2 = 0$, $i = 1, 2, 3$ (possibly relabeling $A_i \leftrightarrow A'_i$). Now $F_1 B_i = F_2 A_i = F_2 A'_i = 0$, for $i = 1, 2, 3$, gives $a_{1i} = 1$, $a_{2i} = 2$, $b_{2i} = 1$, for $i = 1, 2, 3$. The configuration of F_3 is analogous to the one of F_2.

□

4 Geometric construction of the tower

4.1 Basics on Galois triple covers

Our reference here is [Tan92].

Let $\tilde{X}$ be a smooth surface. A Galois triple cover $\pi : \tilde{Y} \to \tilde{X}$ is determined by divisors L, M, B, C on $\tilde{X}$ such that $B \in |2L - M|$ and $C \in |2M - L|$. The branch locus of π is $B+C$ and $3L \equiv 2B+C$, $3M \equiv B+2C$ (we say that $2B+C$ and $B+2C$ are 3-*divisible*). The surface $\tilde{Y}$ is normal iff $B+C$ is reduced. The singularities of $\tilde{Y}$ lie over the singularities of $B+C$.

Now suppose that $\tau : \tilde{X} \to X$ is the minimal resolution of a normal surface X with a set $\{s_1, \dots, s_{\bar{n}}\}$ of ordinary cusps, a set $\{q_1, \dots, q_{\bar{m}}\}$ of singularities of type $\frac{1}{3}(1,1)$, and no other singularities. Let $B_i := \tau^{-1}(q_i)$ be the (-3)-curve which resolves q_i, $i = 1, \dots, \bar{m}$. If the (-2)-curves A_i, A_i' satisfying $\tau^{-1}(s_i) = A_i + A_i'$ can be relabeled such that

$$\sum_1^{\bar{n}} (2A_i + A_i') + \sum_1^{\bar{m}} B_i \equiv 3J,$$

for some divisor J, then we say that the singular set of X is 3-divisible. We have a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & X \end{array}$$

where the vertical arrows over X and $\tilde{X}$ are Galois triple covers ramified over the singular set of X and over its resolution, respectively. The surface Y is smooth and $\tilde{Y} \to Y$ is a sequence of blowups.

Proposition 2. *If* $\bar{n} = 3n$ *and* $\bar{m} = 3m$, *with* $n, m \in \mathbb{N}$, *then*

$$\chi(Y) = 3\chi(\tilde{X}) - 2n - m, \quad K_Y^2 = 3K_{\tilde{X}}^2 + 3m.$$

Proof. If $m = 0$, this is just [Tan92, Lemma 2.2.4]. The contribution of $m \neq 0$ follows easily from [Tan92, Section 1.3]. $\square$

4.2 The construction

Recall that the resolution $\tilde{X}_1$ of the surface $X_1 := S_1/\sigma$ has a fibration onto $\mathbb{P}^1$ with singular fibers F_1, F_2, F_3. The 3-divisibility of the divisors

$$F_1 + 2F_2, \; F_1 + 2F_3, \; F_2 + 2F_3, \; F_1 + F_2 + F_3$$

implies that the following sets of singularities of X_1 are 3-divisible:

$$\begin{aligned} \Delta_1 &= \{O_1', O_2', O_3', Q_1', Q_2', Q_3'\}, \\ \Delta_2 &= \{O_1', O_2', O_3', Q_4', Q_5', Q_6'\}, \\ \Delta_3 &= \{Q_1', \dots, Q_6'\}, \\ \Delta_4 &= \{O_1', O_2', O_3', Q_1', \dots, Q_6'\}. \end{aligned}$$

To each Δ_i corresponds a $\mathbb{Z}/3$-covering $X_1(i) \to X_1$. One has $X_1(4) = S_1$, the Cartwright-Steger surface. The singular set of each $X_1(1), X_1(2)$ is a union of 9 ordinary cusps, and the singular set of $X_1(3)$ is a union of 9 singularities of type $\frac{1}{3}(1,1)$.

We now fix $X_2 := X_1(2)$ (or $X_1(1)$, the resulting surfaces have the same invariants). The resolution $\tilde{X}_2$ of X_2 has three singular fibers F_1', F_2', F_3' which are copies of F_2. Equivalently X_2 has three multiple fibers, each containing three cusp singularities of X_2. As above, we can construct three $\mathbb{Z}/3$-coverings of X_2, each ramified on 6 cusps, using the 3-divisible divisors

$$F_1' + 2F_2',\ F_1' + 2F_3',\ F_2' + 2F_3'.$$

Fixing any of these, we obtain a new surface X_3 which has the same special fibers as X_2. Repeating this process, we construct a sequence (X_n) of surfaces each containing three special fibers with the same configuration as above.

Let G_{n+1} be the subgroup of $PU(2,1)$ such that $X_{n+1} = \mathbb{B}/G_{n+1}$. We define

$$H_{n+1} := H_n \cap G_{n+1} \quad \text{and} \quad S_{n+1} := \mathbb{B}/H_{n+1}.$$

(Notice that it is well defined for all $n \geq 1$, the group H_1 is the fundamental group of the Cartwright-Steger surface S_1.) Since H_n and G_{n+1} are index 3 normal subgroups of G_n, then H_{n+1} is an index 9 normal subgroup of G_n. We are therefore considering the following diagram

$$\begin{array}{ccc} S_{n+1} & \longrightarrow & X_{n+1} \\ \downarrow & & \downarrow \\ S_n & \longrightarrow & X_n \end{array} \tag{2}$$

such that $S_{n+1} \to X_n$ is a $(\mathbb{Z}/3)^2$-Galois covering.

We note that each group H_n is torsion free, because H_1 is torsion free, hence S_n is smooth and then the map $S_n \to X_n$ is ramified over the 9 cusps of X_n, for $n > 0$. Since $(\mathbb{Z}/3)^2$ has four $\mathbb{Z}/3$ subgroups, the covering $S_{n+1} \to X_n$ has four intermediate surfaces

$$X_{n+1},\ X'_{n+1},\ X''_{n+1},\ S_n,$$

that correspond to index 3 subgroups

$$G_{n+1},\ G'_{n+1},\ G''_{n+1},\ H_n$$

of G_n. Clearly the intersection of any two of these groups is H_{n+1}. We have inclusions of normal index 3 groups:

$$\begin{array}{ccc} H_{n+1} & \longrightarrow & G_{n+1} \\ \downarrow & & \downarrow \\ H_n & \longrightarrow & G_n \end{array}.$$

Let $\tilde{X}_n \to X_n$ be the smooth minimal resolution of X_n. One has:

Proposition 3. *The surface $\tilde{X}_n$ is minimal, and for $n > 1$:*

a) $K^2_{\tilde{X}_n} = 3^n,\ \chi(\tilde{X}_n) = 3^{n-2} + 2;$

b) $K^2_{S_n} = 3^{n+1},\ \chi(S_n) = 3^{n-1}.$

Proof. We have $K^2_{\tilde{X}_1} = \chi(\tilde{X}_1) = 2$. Then Proposition 2 with $n = m = 1$ gives that $K^2_{\tilde{X}_2} = 9$, $\chi(\tilde{X}_2) = 3$. Now Proposition 2 with $n = 2$, $m = 0$ gives a). Applying Proposition 2 to the covering $S_n \to X_n$ ($n = 3$, $m = 0$), we get b). $\square$

Corollary 4. *The surfaces S_n are smooth ball quotients. The surfaces $\tilde{X}_n$ are on the line $K^2 = 9\chi - 18$ for $n > 1$. In particular,* $\lim K^2_{\tilde{X}_n}/\chi(\tilde{X}_n) = 9$.

5 Conclusion

Starting from the Cartwright-Steger surface S_1 and its $\mathbb{Z}/3$-quotient X_1, we have constructed two sequences of surfaces (S_n), (X_n) such that the following diagram

$$\begin{array}{ccc} S_{n+1} & \longrightarrow & X_{n+1} \\ \downarrow & & \downarrow \\ S_n & \longrightarrow & X_n \end{array}$$

fits in a $(\mathbb{Z}/3)^2$-Galois covering $S_{n+1} \to X_n$. Moreover, for $n > 1$ the invariants of the smooth surfaces $\tilde{X}_n$ are on the line

$$K^2 = 9\chi - 18,$$

and the surfaces $\tilde{X}_1, \dots \tilde{X}_5$ are simply connected.

Motivated by the simplicity of the geometric construction of the $\mathbb{Z}/3$-covers $X_n \to X_{n-1}$, we formulate:

Conjecture 5. *All surfaces $\tilde{X}_n$ are simply connected.*

Remark 6. *In Section* 4.2*, we have chosen to study the tower of surfaces obtained by branching over cusps. If instead we choose singularities $\frac{1}{3}(1,1)$, we obtain a sequence of surfaces (W_n) with invariants on the line $K^2 = 9\chi - 12$. Moreover, one can compute that the fundamental group of the first few surfaces W_i is $\mathbb{Z}/7$.*

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