

A microscopic approach to nonlinear theory of spin-charge separation

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The fate of spin-charge separation beyond the low energy remains elusive up to now. Here we develop a microscopic theory of the correlation functions using the strong coupling expansion of the Hubbard model and demonstrate its validity down to the experimentally relevant $r_s > 1$. Evaluating the spectral function, we show the general stability of the nonlinear spin-charge modes in whole energy band and investigate all the nonlinear features systematically. We confirm the general prediction experimentally in semiconductor quantum wires. Furthermore, we observe a signal consistent with a continuum of the nonlinear excitations and with a final spectral density around the $3k_F$ point, indicating the robustness of the Hubbard model predictions for a finite range interaction.

Interactions restructure completely the many-body spectrum of electrons in one dimension (1D), resulting in the formation of the Luttinger liquid [1, 2] instead of the Fermi liquid already at low energy. Such a dramatic change manifests itself in the appearance of the pseudo-gap and in the separation of the spin and charge excitations [3, 4], both of which were confirmed experimentally in a variety of systems [5–15]. The fate of these effects beyond the linear regime remains unknown, with attempts via field theory blocked by many divergences [16] or being inconclusive [17–23]. On the other hand, the microscopic approach via the Hubbard model has been partially successful, as the exact Lieb-Wu solution [24] allowed the calculation of the full spectrum [25–27], but a direct attempt to bring the algebraic method [28, 29], developed for the correlation functions of spin chains [30, 31], to the Hubbard model still failed [32].

Here we choose a different path of constructing the t/U expansion for the correlation functions starting from the $U = \infty$ point, where the Lieb-Wu wave functions factorise into the spin and charge sectors [33] allowing to use the algebraic method with only some adjustments [34]. Evaluating the occupation numbers, we show the validity of such an expansion down to the interaction strengths $r_s > 1$, and from the spectral function we find the general stability of the spin-charge-separated modes in the whole energy band and systematically investigate the nonlinear features. Testing these predictions for the realistic screened-Coulomb interaction, we measure semiconductor quantum wires as in [10] using the magnetotransport spectroscopy technique [35–37] and find a signal consistent with a broad continuum of the nonlinear excitations around the charge mode in the particle sector as well as a finite spectral density around the $3k_F$ point, indicating experimentally the reliability of the Hubbard-model predictions for a finite-range interaction. This shows spin-charge splitting of the whole energy band, presenting a novel mechanism for band-structure engin-

eerings based solely on interactions in a simple crystal.

We analyse the 1D Hubbard model describing electrons with short-range interaction,

$$H = -t \sum_{j\alpha} \left(c_{j\alpha}^\dagger c_{j+1,\alpha} + c_{j\alpha}^\dagger c_{j-1,\alpha} \right) + U \sum_j n_{j\uparrow} n_{j\downarrow}, \quad (1)$$

where $c_{j\alpha}$ are the Fermi operators at site j for the spin-1/2 index $\alpha = \uparrow$ or $\downarrow$, $n_{j\alpha} = c_{j\alpha}^\dagger c_{j\alpha}$ is the local-density operator for the spin species α , t is the hopping amplitude, $U > 0$ is the repulsive on-site interaction energy, and we consider the periodic boundary condition, $c_{j+L} = c_j$, for a chain of length L . This model was diagonalised exactly in the N -particle sector by Lieb and Wu [24] via the solutions of a set of nonlinear equations for N charge quasimomenta $\mathbf{k} = (k_1, \dots, k_N)$ and M spin quasimomenta $\mathbf{q} = (q_1, \dots, q_M)$, which give the corresponding eigenenergies ($E = t \sum_j (k_j)^2/2$), total momenta ($P = \sum_j k_j$), and eigenstates as the Lieb-Wu wave functions $|\Psi\rangle$ [38]. Here, we consider only the low-particle densities $N/L \ll 1$ in the thermodynamic limit $N, L \gg 1$.

In the infinite-interaction limit, $U = \infty$, the Lieb-Wu wave functions factorise, $|\Psi^0\rangle = |\Psi_c^0\rangle \otimes |\Psi_s^0\rangle$, into product of a Slater determinant $|\Psi_c^0\rangle$ in the charge and a Bethe wave function $|\Psi_s^0\rangle$ in the spin sector [33]. The nonlinear equations for the spins also separate out into independent Bethe equations, $N q_m^0 - 2 \sum_{l \neq m} \varphi_{ml} = 2\pi J_m$, where $e^{i2\varphi_{lm}} = -(e^{iq_l^0 + iq_m^0} + 1 - 2e^{iq_l^0}) / (e^{iq_l^0 + iq_m^0} + 1 - 2e^{iq_m^0})$ are two-spinon scattering phases, simplifying the charge equations to single-particle quantisation conditions, $L k_j^0 - P_s = 2\pi I_j$, where $P_s = \sum_m q_m^0$ is the total spin momentum. For a large U/t these solutions depart slightly from the $U = \infty$ limit so that such deviations can be linearised in t/U as $\mathbf{k} = \mathbf{k}^0 + \mathbf{k}^1 t/U$ and $\mathbf{q} = \mathbf{q}^0 + \mathbf{q}^1 t/U$. In turn, linearisation of the nonlinear equations gives only a single-particle correction to the charge and a linear set of equations (that is solved via matrix inversion) for the spin quasimomenta in the first

t/U -order,

$$\mathbf{k}^1 = \mathbf{k}^0 \frac{4}{L} \sum_m (\cos q_m^0 - 1), \quad (2)$$

$$\mathbf{q}^1 = 4P\hat{Q}_0^{-1}(\mathbf{1} - \cos \mathbf{q}^0), \quad (3)$$

where the matrix elements of $\hat{Q}_0$ are $Q_{aa}^0 = N - \sum_{l \neq a} 4(1 - \cos q_l^0)/(e^{iq_l^0} + e^{-iq_a^0} - 2)/(e^{-iq_l^0} + e^{iq_a^0} - 2)$ and $Q_{ab}^0 = 4(1 - \cos q_b^0)/(e^{iq_b^0} + e^{-iq_a^0} - 2)/(e^{-iq_b^0} + e^{iq_a^0} - 2)$ for $a \neq b$.

Comparison of the zeroth with the first order for the charge quasimomenta in Eq. (2) gives a dimensionless parameter $\gamma = -Uk_j^0/(k_j^1 t)$ that is independent of j . Evaluation of the sum over m for the ground state in the thermodynamic limit, $\sum_m (\cos q_m^0 - 1)/N = -0.69(3)$, gives [39]

$$\gamma = 0.09\lambda_F \frac{U}{t}, \quad (4)$$

in which the Fermi wavelength $\lambda_F = 4L/N$ appears as an extra large factor in addition to U/t in the validity of this expansion. The same extra factor appears for the spin quasimomenta in Eq. (3), for which a typical $P \lesssim N/L$ and $[\hat{Q}_0^{-1}(\mathbf{1} - \cos \mathbf{q}^0)]_m \simeq 1$, making $1/\gamma$ the emergent small parameter controlling the large- U expansion of the nonlinear eigenvalue problem. The factor λ_F in Eq. (4) is similar to the generic parameter of the Coulomb interaction $r_s \propto \lambda_F$ [38], allowing us to interpret $\gamma \simeq r_s$ as the microscopic calculation of the phenomenologically introduced r_s in 1D [27].

The Lieb-Wu wave function can also be linearised in t/U as $|\Psi\rangle = |\Psi^0\rangle + t/U|\Psi^1\rangle$. This translates into the expansion matrix element $\langle f|c_{1\alpha}^\pm|0\rangle = \langle f|c_{1\alpha}^\pm|0\rangle^0 + t/U\langle f|c_{1\alpha}^\pm|0\rangle^1$ needed for the correlation functions, e.g., for the Green function $G_\alpha(k, E) = \sum_f [|\langle f|c_{k\alpha}^\pm|0\rangle|^2/(E - E_f + i\eta) + |\langle f|c_{k\alpha}|0\rangle|^2/(E + E_f - i\eta)]$, where $0 = (\mathbf{k}^0, \mathbf{q}^0)$ and $f = (\mathbf{k}^f, \mathbf{q}^f)$ are the ground state and an excited state, $c_k^\pm = \sum_j c_j^\pm e^{\pm ikj}/\sqrt{L}$ is the Fourier transform, $\langle f|c_{k\alpha}^\pm|0\rangle = \langle f|c_{1\alpha}^\pm|0\rangle\delta(k \pm P_f)$ due to the translational symmetry, η is an infinitesimally small real number, and without external magnetic field $G_\uparrow(k, E) = G_\downarrow(k, E)$ so we can consider only $\alpha = \uparrow$ here. The leading term factorises as $\langle f|c_{1\uparrow}|0\rangle^0 = \langle f|c_{1\uparrow}|0\rangle_c^0 \cdot \langle f|c_{1\uparrow}|0\rangle_s^0$, since the wave function factorises in the $U/t = \infty$ limit. The charge part was evaluated using the first quantisation directly in [33] giving, e.g., $\langle f|c_{1\uparrow}|0\rangle_c^0 = L^{-N+1/2} \det \hat{C}_0$ with the matrix elements $C_{a1}^0 = 1$ and $C_{ab}^0 = 2(k_a^{00} - k_{b-1}^{f0})^{-1} \sin(\frac{P_s - P_s^f}{2})$ for $b > 1$. The spin part was evaluated using the algebraic Bethe ansatz to deal with the Bethe wave function in [34] giving $\langle f|c_{1\uparrow}|0\rangle_s^0 = Z_0^{-1} Z_f^{-1} \prod_{lm} (e^{iq_l^{f0}} + e^{-iq_m^{00}} - 2) \prod_{l \neq m} (e^{iq_l^{f0}} + e^{-iq_m^{f0}} - 2)^{-\frac{1}{2}} \prod_{l \neq m} (e^{iq_l^{00}} + e^{-iq_m^{00}} - 2)^{-\frac{1}{2}} \det \hat{R}_{00}$ with $R_{Mb}^{00} = e^{ik_b^{00}} \prod_{l \neq b} (e^{iq_l^{00}} + e^{-iq_b^{00}} - 2)/\prod_l (e^{iq_l^{f0}} + e^{-iq_b^{00}} - 2)$ and

$R_{ab}^{00} = [e^{iq_b^{00}(N-1)} \prod_{l \neq a} (e^{iq_l^{f0} + iq_b^{00}} + 1 - 2e^{iq_l^{f0}})/(2e^{iq_b^{00}} - e^{iq_l^{f0} + iq_b^{00}} - 1) - 1]/(e^{-q_a^{f0}} - e^{-iq_b^{00}})/(e^{q_a^{f0}} - e^{-iq_b^{00}} - 2)$ for $a < M$. Here $Z_{0/f}^2 = \det \hat{Q}_0^{0/f}$ are the Gaudin normalisation factors [40].

The linear term in the wave function comes from three sources, $|\Psi^1\rangle = |\Psi_s^1\rangle + |\Psi_c^1\rangle + |\Psi_{sc}^1\rangle$, two are the expansions of $|\Psi^0\rangle$ in the linear terms of quasimomenta in Eqs. (2,3) that do not break the spin-charge factorisation of $|\Psi^0\rangle$ and one is the mixing term $|\Psi_{sc}^1\rangle$. This makes the linear term of the matrix element also a linear superposition of the same three contributions $\langle f|c_{1\uparrow}|0\rangle^1 = \langle f|c_{1\uparrow}|0\rangle_c^1 + \langle f|c_{1\uparrow}|0\rangle_{cs}^1 + \langle f|c_{1\uparrow}|0\rangle_s^1$. The charge part comes from the t/U expansion of the bra and ket states, $\langle f|c_{1\uparrow}|0\rangle_c^1 = (\langle f_c^0|c_{1\uparrow}|0_c^1\rangle + \langle f_c^1|c_{1\uparrow}|0_c^0\rangle)\langle f|c_{1\uparrow}|0\rangle_s^0$. We evaluate both contributions simultaneously by changing $\mathbf{k}_{0/f}^0 \rightarrow \mathbf{k}_{0/f}^0 + g\mathbf{k}_{0/f}^1$ in $\langle f|c_{1\uparrow}|0\rangle_c^0$, repeating the same calculation as for the $\langle f|c_{1\uparrow}|0\rangle_c^0$ as in [33], and taking a derivative and a limit of the resulting determinant expression as $\lim_{g \rightarrow 0} \partial_g \det(\hat{C}_0 + g\hat{C}_1) = \det \hat{C}_0 \text{Tr}(\hat{C}_0^{-1}\hat{C}_1)$ using generic matrix identities. For the whole charge part, we find that the $U/t = \infty$ matrix element appears as a factor $\langle f|c_{1\uparrow}|0\rangle_c^1 = \langle f|c_{1\uparrow}|0\rangle^0 T_c$ and

$$T_c = \text{Tr}(\hat{C}_0^{-1}\hat{C}_1), \quad (5)$$

where the derivatives of the entries of $\hat{C}_0$ under the shifts by $g\mathbf{k}_{0/f}^1$ are

$$C_{a1}^1 = -i \frac{k_a^{01}(L-1)}{2}, \quad (6)$$

$$C_{ab}^1 = 2 \left(\frac{L \cos \frac{P_s - P_s^f}{2}}{2} - \frac{\sin \frac{P_s - P_s^f}{2}}{k_a^{00} - k_{b-1}^{f0}} \right) \frac{k_a^{01} - k_{b-1}^{f1}}{k_a^{00} - k_{b-1}^{f0}}, \quad (7)$$

for $b > 1$.

We evaluate the spin part $\langle f|c_{k\uparrow}|0\rangle_s^1$ using the same trick with shifting $\mathbf{q}_{0/f}^0 \rightarrow \mathbf{q}_{0/f}^0 + g\mathbf{q}_{0/f}^1$ in $\langle f|c_{1\uparrow}|0\rangle_s^0$, repeating the same calculation as for the $\langle f|c_{1\uparrow}|0\rangle_s^0$ in [34], and taking $\lim_{g \rightarrow 0} \partial_g$ and obtain analogously $\langle f|c_{1\uparrow}|0\rangle_s^1 = \langle f|c_{1\uparrow}|0\rangle^0 T_s$ with

$$T_s = i \sum_{lm} \frac{q_l^{f1} e^{iq_l^{f0}} - q_m^{01} e^{-iq_m^{00}}}{e^{iq_l^{f0}} + e^{-iq_m^{00}} - 2} - \frac{i}{2} \sum_{l \neq m} \frac{q_l^{01} e^{iq_l^{00}} - q_m^{01} e^{-iq_m^{00}}}{e^{iq_l^{00}} + e^{-iq_m^{00}} - 2} + \text{Tr}(\hat{R}_{00}^{-1} \hat{R}_{01}) - \frac{i}{2} \sum_{l \neq m} \frac{q_l^{f1} e^{iq_l^{f0}} - q_m^{f1} e^{-iq_m^{f0}}}{e^{iq_l^{f0}} + e^{-iq_m^{f0}} - 2} + \text{Tr}(\hat{R}_{f0}^{-1} \hat{R}_{f1}), \quad (8)$$

where the extra sums appear due to a different normalisation of the Bethe wave functions in the algebraic representation, the second trace with the matrix $R_{aM}^{f0} = 1$ and $R_{ab}^{f0} = [e^{iq_b^{00}N} \prod_{l \neq a} (e^{iq_b^{f0} + iq_l^{00}} + 1 - 2e^{iq_l^{00}})/(2e^{iq_b^{f0}} - e^{iq_b^{f0} + iq_l^{00}} - 1) - 1]/(e^{-q_b^{f0}} - e^{-iq_a^{00}})/(e^{q_a^{00}} - e^{-iq_b^{f0}} - 2)$

for $b < M$ is due to extra mathematical complications in applying the Slavnov's formula [41] to linear expansion of the bra and ket states of the spin matrix element, and the derivatives of the entries of $\hat{R}_{0/f,0}$ shifted by $g\mathbf{q}_{0/f}^1$ are $R_{ab}^{0/f,1} = \sum_j q_j^{0/f,1} \partial_{q_{0/f,0}} R_{ab}^{0/f,0}$.

In the mixing part $\langle f|c_{1\uparrow}|0\rangle_{cs}^1$, the spin and charge coordinates mix in a linear way. After summation over the charge coordinates, the remaining spin dependence has a part proportional to $\langle f|c_{1\uparrow}|0\rangle^0$ that gives $\langle f|c_{1\uparrow}|0\rangle_{cs}^1 = \langle f|c_{1\uparrow}|0\rangle^0 T_{cs}$, with

$$T_{cs} = P_f \left[4 \frac{\sum_m \cos q_m^{f0} - \sum_m \cos q_m^{00} + \cos(P_s^f - P_s^0)}{\tan \frac{P_s^f - P_s^0}{2}} + 2i \frac{\sum_m e^{iq_m^{00}} - \sum_m e^{-iq_m^{f0}}}{N} + 2ie^{i(P_s^f - P_s^0)} \right]. \quad (9)$$

The other part of the spin dependence has the same structure as $\langle f|c_{1\uparrow}|0\rangle_s^1$ but with $\mathbf{q}_{0/f}^1 = 4P_f(\cos \mathbf{q}_{0/f}^0 - \mathbf{1})/N$ instead of the linear term of the spin quasimomenta in Eq. (3), giving the same result as in Eq. (8) but with different values of $\mathbf{q}_{0/f}^1$. Therefore, in the linear term of the whole matrix element the zeroth-order matrix element appears as a factor, $\langle f|c_{1\uparrow}|0\rangle^1 = \langle f|c_{1\uparrow}|0\rangle^0 T$ with $T = T_c + T_s + T_{cs}$, in which T_c and T_{cs} are given by Eqs. (5, 9) and T_s is given by Eq. (8), where the second mixing part is added to quasimomenta in Eq. (3) as $\mathbf{q}_{0/f}^1 \rightarrow \mathbf{q}_{0/f}^1 + 4P_f(\cos \mathbf{q}_{0/f}^0 - \mathbf{1})/N$ since this contribution to the matrix element is linear in $\mathbf{q}_{0/f}^1$.

The normalisation of the Lieb-Wu wave functions in this work was chosen to be unity in the $U = \infty$ limit, $\langle \Psi^0 | \Psi^0 \rangle = 1$, in the same way as in [34]. To the linear order that we calculate here, the normalisation changes as $\langle \Psi | \Psi \rangle = 1 + \delta Z t / U$, which we evaluate using the same linear expansion of the Lieb-Wu functions and the same methods as for the matrix element above. We find that the charge part does not contribute and the spin together with the mixing part give

$$\delta Z = 2\text{ReTr}(\hat{Q}_0^{-1} \hat{Q}_1) - \frac{4N}{L} \sum_m (\cos q_m^0 - 1), \quad (10)$$

where $\hat{Q}_1$ are the derivatives of entries of $\hat{Q}_0$ shifted by $g\mathbf{q}^1$, which are presented in [38]. Combining this result with the linear expansion above, we find the modulus squared needed for the observables as $|\langle f|c_{1\uparrow}|0\rangle|^2 = |\langle f|c_{1\uparrow}|0\rangle^0|^2 [1 + (2\text{Re}T - \delta Z_f - \delta Z_0)t/U]$. The full details on the derivations of Eqs. (5–10) are in [38]. The expressions for the matrix element $\langle f|c_{1\uparrow}^\dagger|0\rangle$ in the linear order are the same as in Eqs. (5–10), in which the quasimomenta are swapped as $\mathbf{k}^f, \mathbf{q}^f \leftrightarrow \mathbf{k}^0, \mathbf{q}^0$ and the particle and spin quantum numbers are increased by one, $N \rightarrow N + 1$ and $M \rightarrow M + 1$ as in the zeroth order [34].

Analysing correlation functions at a finite U , we start from the occupation numbers $n_k = \sum_f |\langle f|c_{k\uparrow}|0\rangle|^2$. Numerical evaluation of the sum over f using the leading

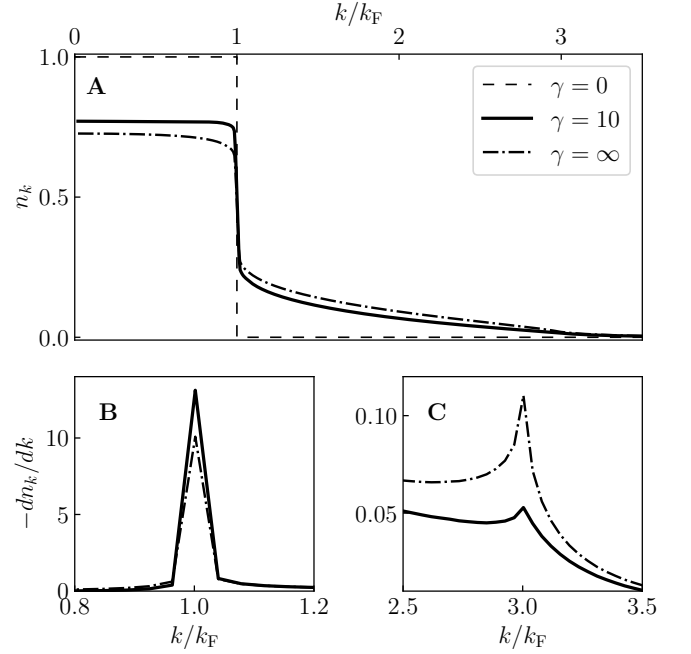


Figure 1. **A** Occupation numbers n_k evaluated using Eqs. (5–10) for $N = 100$ particles. The dash-dotted line is the infinite-interaction limit $\gamma = \infty$, the solid line is at finite interaction $\gamma = 10$, and the dashed line is the Fermi step of the free-particle limit $\gamma = 0$. **B** and **C** is the derivative of n_k in **A** with respect to k , exhibiting singularities around the k_F and $3k_F$ points, respectively.

two levels, $l \leq 1$, of the hierarchy of modes [42] is presented in Fig. 1 as a solid line. Above k_F , we find that a part of the spectral power, which was redistributed there at $U = \infty$ (dash-dotted line) to form the second Fermi point at $3k_F$ [34], moves back below k_F , recovering the Fermi function at $U = 0$ (the dashed line). The $3k_F$ Fermi point itself (which appears as a divergence in the first derivative $\partial_k n_k$ for Luttinger liquids [43, 44]) remains stable away from the $U = \infty$ limit, see Fig. 1C. Further, we analyse the part of the spectral weight above k_F per particle, $\mathcal{I} = 2 \int_{k_F}^{\infty} dk n_k / N$, to assess the validity of the t/U expansion for the correlation functions. Evaluating the integral over k numerically for different N and L , we find an additional N/L factor in the linear t/U -term that can be absorbed into γ defined in Eq. (4), and obtain $\mathcal{I} = 0.25 + 0.05/\gamma$. This demonstrates that the expansion for the correlation functions is controlled by the same γ as the expansion of the eigenvalues in Eqs. (2, 3).

Now, using the result in Eqs. (5–10), we evaluate another observable – the spectral function $A(k, E) = \sum_f |\langle f|c_{k\uparrow}^\dagger|0\rangle|^2 \delta(E - E_f + E_0) + \sum_f |\langle f|c_{k\uparrow}|0\rangle|^2 \delta(E + E_f - E_0)$ – in Fig. 2A. Around the $\pm k_F$ points, there are two singular peaks with different velocities and a “shadow band” described by the linear Tomonaga-Luttinger theory [3, 4]. Away from them, both peaks generally remain stable, see the dashed green and magenta lines, showing

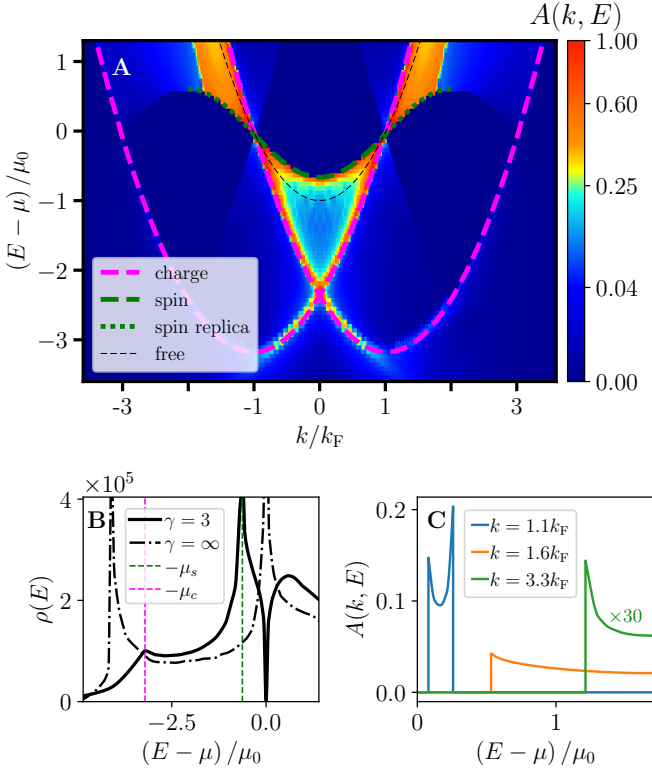


Figure 2. **A** Spectral function $A(k, E)$ evaluated for $N = 200$ particles and a finite interaction strength $\gamma = 3$. The magenta and green dashed lines are the dispersions of the pure charge and spin modes evaluated using the expansion in Eqs. (2,3). The green dotted line is the replica of the spin mode in the hole sector in the $E > \mu$ and $k > k_F$ region. The black dashed line is the free-particle dispersion for $\gamma = 0$. **B** Density of states $\rho(E)$ for $\gamma = 3$ (solid black line) and $\gamma = \infty$ (dash-dotted black line). The magenta and green dashed lines mark the charge ($-\mu_c$) and the spin ($-\mu_s$) chemical potentials for $\gamma = 3$ obtained as the minimum energy of the charge and spin dispersions w.r.t. the electron chemical potential μ in **A**. **C** Constant-momentum cuts of $A(k, E)$ in **A** around the Fermi point at $k = 1.1k_F$, of the nonlinear extension of the spin mode in the particle sector at $k = 1.6k_F$, and of the nonlinear charge mode above the $3k_F$ point at $k = 3.3k_F$.

the splitting of the whole single-particle band (the black dashed line) into two by interactions. On a more detailed level, the nonlinear parts of these modes are asymmetric w.r.t. the electronic chemical potential μ . In the hole sector $E < \mu$, the whole spin mode remains stable but the charge mode becomes unstable at the bottom of its dispersion. This instability is also apparent in the density of states $\rho(E) = L \int dk A(k, E)$, see the full black line in Fig. 2B. The van Hove singularity at the spin chemical potential $-\mu_s$ (defined as the distance from μ to the bottom of the green dispersion in Fig. 2A) remains stable but the van Hove singularity of the charge mode at $-\mu_c$ disappears at a finite U . The latter remains a singularity only in the $U = \infty$ limit, see the dash-dotted line in Fig. 2B, in which the spin singularity also contributes to the

low-energy behaviour around $E = \mu$ since the spin-mode dispersion is completely flat $\mu_s = 0$ [26, 34, 45]. At a finite U , μ_s becomes finite, revealing the power-law vanishing of $\rho(E)$, well-known from the linear theory [46].

In the particle sector $E > \mu$, the whole charge mode remains stable but the spin mode becomes a weaker singularity, only a jump instead of a singular peak, see the orange cut of $A(k, E)$ in Fig. 2C. The states forming the latter mode always have a pair of degenerate spin quasimomenta making $\langle f | c_{1\uparrow} | 0 \rangle_s = 0$ in the $l = 0$ level of the hierarchy. However, the states from the continuum of the $l = 1$ level do not have such a degeneracy and their squeezing from a wide $k_F < k < 3k_F$ region to the proximity of the black dashed line produces a finite jump at the replica of the main spin dispersion (marked by the green dotted line in Fig. 2A) the dispersion of this replica, however, is indistinguishable from the principal spin mode from the $l = 0$ level in the particle sector. Around the k_F point, the hierarchy breaks down so that all many-body excitations have comparable amplitudes, and the spin mode regains a singular peak obtained from the linear theory in [3, 4], see the blue cut of $A(k, E)$ in Fig. 2C.

Here we turn to an experiment on tunneling spectroscopy in a 1D geometry (quantum wire) to search for the nonlinear features predicted by the Hubbard model in a real system with a screened Coulomb interaction. The design of our device [10, 47, 48] is based on a GaAs/Al_{0.33}Ga_{0.67}As double-quantum-well heterostructure. To amplify the 1D signal, the electrons in the upper well are confined to an array of 300 highly homogeneous wires of length $L = 18 \mu\text{m}$ by applying a negative voltage V_{FG} on the finger gates, see the inset in Fig. 3A. The relative position of the Fermi energies of the wires and of the 2DEG in the lower well is shifted by applying a bias V_{dc} between the wells and the electron k -vector along the wire is shifted by $\Delta k = eBd/\hbar$ in the tunneling process by the Lorentz force from the in-plane magnetic field B applied perpendicular to the wires, where e is the electronic charge and $d = 32 \text{ nm}$ is the center-to-center distance between the two wells, providing together both energy and momentum resolution. The inter-well current I measured for different B and V_{dc} probes the spectral function of 1D electrons via its convolution with the 2D spectral function, see more details in [38]. An area of the upper well not covered by the finger gates always contributes parasitically to I , which we remove by repeating the measurement at $V_{FG} = -750 \text{ mV}$, when the wires are completely pinched off, and subtracting this signal from the 1D data at less negative V_{FG} .

The conductance $G = dI/dV_{dc}$ for $V_{FG} = -664 \text{ mV}$ in the single 1D-subband regime with large density $n_{1D} \approx 45 \mu\text{m}^{-1}$ is presented in Fig. 3A. The data is visualized as the dG/dB derivative to show the positions of the peaks as white lines between red and blue regions. The peak marked by the black dashed line is the 2D disper-

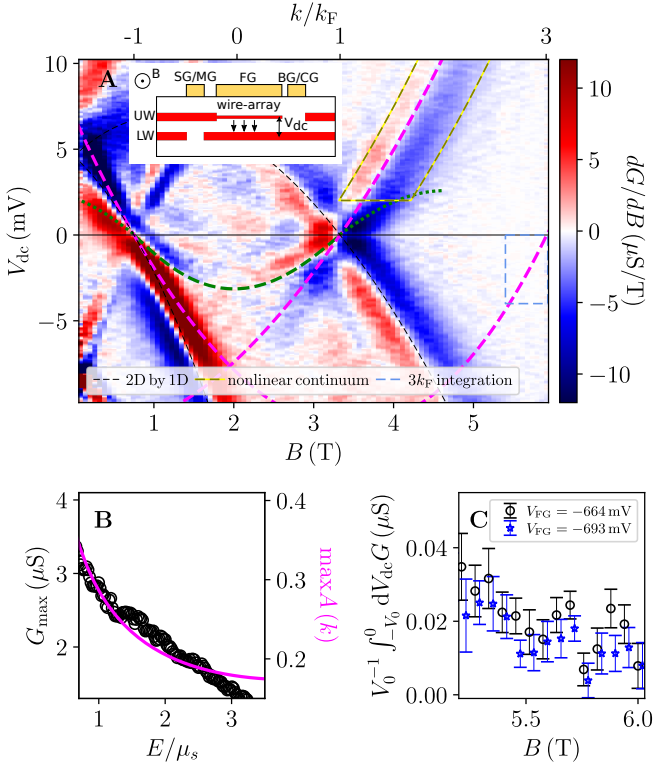


Figure 3. **A** The conductance $G(B, V_{dc})$ measured for $V_{FG} = -664$ mV and presented as the dG/dB derivative. The green and magenta dashed lines are the dispersions of spin and charge modes obtained from the full Lieb-Wu equations for $\gamma = 1.25$, and corrected for capacitance using $c_{UL} = 5.6$ mFm $^{-2}$ and $c_{UW} = 4.7$ mFm $^{-2}$, see details in [27]. The upper horizontal axis is the linear transformation of B using the two crossing points with $V_{dc} = 0$ line as $B_{lo} = 0.75$ T is $-k_F$ and $B_{hi} = 3.33$ T is k_F . The inset is a schematic of the cross-section of our device. **B** Open black circles are the conductance along the charge mode in the region marked by the olive-yellow dashed line in **A**, for which E/μ_s is obtained as V_{dc} divided by the voltage of the minimum of the dashed-green parabola in **A**. The magenta line is the maximum of $A(k, E)$ in Fig. 2A along the charge mode in the particle sector. **C** Integration over V_{dc} of the conductance G within the light-blue dashed rectangle of height $V_0 = 4$ mV in **A** as a function of B for $V_{FG} = -664$ mV (black circles) and $V_{FG} = -693$ mV (blue stars).

sion of the electrons in the lower well measured by the wires in the upper well. The pair of peaks marked by the green and magenta dashed lines have the same pattern as the maxima of the spectral function calculated in Fig. 2A. Fitting their dispersions by using the solution of the full Lieb-Wu equations [24] and $m_0 = 0.0525 m_e$ [49], where m_e is the free electron mass, we obtain a moderate value of the interaction parameter $\gamma = 1.25$. Here the charge peak in $A(k, E)$ in the hole sector manifests as a minimum in dG/dB due to peculiarities of the transport theory, which were already understood in the linear regime in [50].

Furthermore, we find a signal consistent with the broad continuum of the nonlinear excitations predicted around the charge mode in the particle sector. In Fig. 3A it can be seen as a large asymmetry of this line (on the momentum scale of $\simeq k_F$), see the area enclosed by the olive-yellow dashed line. The observed amplitude of this mode also decreases significantly in accordance with the Hubbard-model prediction, see the comparison in Fig. 3B and more analysis on the asymmetry in [38].

We also look for the many-body excitations around the $3k_F$ point in the signal. The predicted amplitude of $A(k, E)$ around this point is ~ 100 times smaller than that around k_F , bringing the expected amplitude of G around $3k_F$ just below the observed noise $|G| \sim 0.05 \mu S$. However, motivated by the relation $n_k = \int_{-\infty}^0 dE A(k, E)$, we integrate G over V_{dc} in the light-blue dashed rectangle in Fig. 3A at each field and find a finite signal, the black circles in Fig. 3C. Repeating this measurement at a more negative $V_{FG} = -693$ mV (and a smaller $n_{1D} \approx 42 \mu m^{-1}$), we find a very similar pattern, the blue stars in Fig. 3C. The non-monotonic dependence of the integrated G cannot be explained by a still possible contribution from the far tail of the 2D signal in this region but, on the other hand, does not match the shape of n_k around $3k_F$ in Fig. 1A. The mismatch could be due to contribution of the next ($l = 1, 2$) levels of the hierarchy [42], which were previously observed in spectroscopy [27, 47], or due to a peculiarity of transport theory, which is still lacking for nonlinear Luttinger liquids. The latter may also be a reason for the absence of a quantitative match in the comparison in Fig. 3B.

In conclusion, we have developed a microscopic theory for the correlation function of the Hubbard model for $r_s > 1$ and have used it to show the stability of the nonlinear spin-charge separated modes and to investigate systematically their features. We have confirmed some of these predictions experimentally in a semiconductor quantum wires, indicating the reliability of the Hubbard-model prediction for a finite-range interaction.

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Data availability—Experimental data presented in this work are available at the University of Cambridge data repository (<http://...>).

Author Contributions—O.T. planned the project and performed all the calculations. Y.J., M.M. and W.K.T. fabricated the device and with C.J.B.F. performed the transport measurements shown. O.T. and C.J.B.F. analyzed the experimental data. O.T. wrote the manuscript with help from C.J.B.F..

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Supplementary material for A microscopic approach to nonlinear theory of spin-charge separation

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I. LIEB-WU SOLUTION

The N -particle eigenstates of the 1D Hubbard model in Eq. (1) of the main text were constructed by Lieb and Wu in [1]. These eigenstate have the form of a superposition of plain waves according to Bethe's hypothesis [2], which in second quantisation $|\Psi\rangle = \sum_{\mathbf{j}, \boldsymbol{\alpha}} \psi_{\mathbf{j}\boldsymbol{\alpha}} \prod_{i=1}^N c_{j_i \alpha_i}^\dagger |0\rangle$, reads as

$$\psi_{\mathbf{j}\boldsymbol{\alpha}} = \sum_Q (-1)^{QO} A_{QO\boldsymbol{\alpha}} e^{i\mathbf{Q}\mathbf{k} \cdot O\mathbf{j}}, \quad (1)$$

where $j_1 \dots j_N = \mathbf{j}$ and $\alpha_1 \dots \alpha_N = \boldsymbol{\alpha}$ are the coordinates and spin configurations of N Fermi particles with spin-1/2 on a chain of length L , O is the permutation that orders all N coordinates so that

$$Oj_1 < \dots < Oj_N, \quad (2)$$

the charge quasimomenta associated with the orbital degrees of freedom of every particle are $\mathbf{k} = k_1 \dots k_N$, and $\sum_Q$ is the sum over all permutations Q of N quasimomenta k_j . The amplitudes $A_{QO\boldsymbol{\alpha}}$ in this superposition depend additionally on the spin configuration $O\boldsymbol{\alpha}$. A way of constructing them using the Bethe hypothesis was proposed in [3, 4], producing another “nested” Bethe-ansatz wave function for the spin degrees of freedom as

$$A_{QO\boldsymbol{\alpha}} = \sum_R \prod_{1 \leq l < m \leq M} \sqrt{-\frac{e^{i(Rq_m + Rq_l)} + 1 - 2e^{iRq_l}}{e^{i(Rq_m + Rq_l)} + 1 - 2e^{iRq_m}}} \prod_{l=1}^M \frac{i\frac{U}{2t} e^{iRq_l}}{(e^{iRq_l} - 1) Qk_{x_l} + i\frac{U}{2t}} \prod_{j=1}^{x_l-1} \frac{(e^{iRq_l} - 1) Qk_j + i\frac{U}{2t} e^{iRq_l}}{(e^{iRq_l} - 1) Qk_j + i\frac{U}{2t}}, \quad (3)$$

where $x_1 \dots x_M = \mathbf{x}$ are the coordinates of M spins $\uparrow$ in the configuration $O\boldsymbol{\alpha}$ of all spins of N particles, $q_1 \dots q_M = \mathbf{q}$ are the spin quasimomenta associated with these M spins $\uparrow$, and $\sum_R$ is the sum over all permutations R of M quasimomenta q_j . Note that this spin wave function is written in the coordinate representation, in which the normalisation factors are defined in this work. The expression in Eq. (3), used as the starting point in this work, is obtained by substitution of the algebraic-to-coordinate mapping for the spin quasimomenta as

$$\lambda_l = -\frac{iU}{4t} \frac{e^{iq_l} + 1}{e^{iq_l} - 1} \quad (4)$$

in the original expression in the algebraic representation in [3, 4]. The low-particle-density limit, $N \ll L$, is already taken in Eq. (3). For a large lattice filling, the charge quasimomenta have to be changed as $Qk_j \rightarrow \sin Qk_j$ in this expression.

Application of the periodic boundary condition to the wave function in Eq. (1) solves the eigenvalue problem by a set of coupled nonlinear equations, the Lieb-Wu equations [1],

$$Lk_j = \frac{1}{i} \sum_{l=1}^M \log \left(1 + \frac{1}{\frac{1}{e^{iq_l} - 1} - \frac{2it}{U} k_j} \right) + 2\pi I_j, \quad (5)$$

$$\frac{1}{i} \sum_{j=1}^N \log \left(1 + \frac{1}{\frac{1}{e^{iq_m} - 1} - \frac{2it}{U} k_j} \right) = \frac{1}{i} \sum_{l=1 \neq m}^M \log \left(-\frac{e^{i(q_l + q_m)} + 1 - 2e^{iq_m}}{e^{i(q_l + q_m)} + 1 - 2e^{iq_l}} \right) + 2\pi J_m, \quad (6)$$

where N non-equal integers I_j and M non-equal integers J_m define the solution for the corresponding charge and spin quasimomenta k_j and q_j for a given value of the interaction strength U/t . This solution gives the eigenenergy of the corresponding N -particle state as $E = t \sum_j k_j^2/2$ and its total momentum as $P = \sum_j k_j$. The low-particle-density limit, $N \ll L$, is already taken in these Lieb-Wu equations. The expressions for arbitrary density are obtained via the $k_j \rightarrow \sin k_j$ substitution in the r.h.s. of Eq. (5) and in the whole of Eq. (6).

A. t/U expansion of eigenvalue equations

The Lieb-Wu equations (5,6) can be solved explicitly for large U by expanding them in the Taylor series in t/U . The leading term in such an expansion is obtained by taking the $U \rightarrow \infty$ limit of Eqs. (5, 6), which gives [5]

$$Lk_j^0 = P_s + 2\pi I_j, \quad (7)$$

$$Nq_m^0 = 2 \sum_{l \neq m} \varphi_{ml} + 2\pi J_m, \quad (8)$$

where

$$e^{i2\varphi_{ml}} = -\frac{e^{i(q_l^0 + q_m^0)} + 1 - 2e^{iq_m^0}}{e^{i(q_l^0 + q_m^0)} + 1 - 2e^{iq_l^0}}, \quad (9)$$

are the two-spinon scattering phases, the superscript 0 was added to both charge k_j^0 and spin q_j^0 quasimomenta to mark the zeroth-order term in the t/U Taylor series for them. The total spin momentum

$$P_s = \sum_l q_l^0 \quad (10)$$

becomes a quantum number in this limit since the spin part in Eq. (8) decouples completely, becoming an independent set of M nonlinear equation for q_j^0 only, which are exactly the same as the Bethe equations for the antiferromagnetic Heisenberg chain [2]. Once the solution for q_m^0 is found from Eq. (8) for a set of J_m , each of the remaining N equations for k_j in Eq. (8) becomes just an independent single-particle quantisation condition, which is solved immediately as

$$k_j^0 = \frac{P_s + 2\pi J_m}{L}. \quad (11)$$

To find the next term in the t/U expansion, the Lieb-Wu equations (5, 6) have to be expanded in a Taylor series in t/U around the $U = \infty$ point. In order to evaluate the linear term for the spin and charge quasimomenta, they are expanded up to linear order in t/U around the solutions of Eqs. (7, 8),

$$k_j = k_j^0 + \frac{t}{U} k_j^1, \quad (12)$$

$$q_m = q_m^0 + \frac{t}{U} q_m^1. \quad (13)$$

Then, these expansions are substituted into Eqs. (5, 6) and they are, in turn, expanded up to linear order in t/U , in which the equations for the quasimomenta become linear in k_j^1 and q_j^1 ,

$$Lk_j^1 = \sum_l q_l^1 + 4k_j^0 \sum_l (\cos q_l^0 - 1), \quad (14)$$

$$Nq_m^1 = 4 \sum_{l \neq m} \frac{q_m^1 (1 - \cos q_l^0) - q_l^1 (1 - \cos q_m^0)}{(e^{iq_m^0} + e^{-iq_l^0} - 2)(e^{iq_l^0} + e^{-iq_m^0} - 2)} - 4(\cos q_m^0 - 1) \sum_j k_j^0. \quad (15)$$

Summation of Eq. (15) over m makes the first term on its r.h.s. vanish, giving

$$\sum_m q_m^1 = -\frac{4}{N} \sum_l (\cos q_l^0 - 1) \sum_j k_j^0, \quad (16)$$

since the summand is asymmetric w.r.t. exchange of the summation indices l and m . Substitution of the last expression in Eq. (14) gives the first order for the charge quasimomenta k_j^1 explicitly in terms of the solutions of Eqs. (7, 8) only,

$$k_j^1 = \frac{4}{L} \left(k_j^0 - \frac{P^0}{N} \right) \sum_l (\cos q_l^0 - 1), \quad (17)$$

where we note that the quantum number $P^0 = \sum_j k_j^0 \equiv P$ is independent of U due to the translational invariance.

The first order for the spin quasimomenta q_m^1 can be found from Eq. (15) via matrix inversion as

$$q_m^1 = \left[\hat{Q}_0^{-1} \mathbf{v} \right]_m, \quad (18)$$

where $\mathbf{v}$ is the second term on the r.h.s. of Eq. (15) written in the vectorial form

$$v_m = (1 - \cos q_m^0) P, \quad (19)$$

and the matrix

$$Q_{ml}^0 = \begin{cases} N - \sum_{k=1 \neq m}^M \frac{4(1 - \cos q_k^0)}{(e^{iq_m^0} + e^{-iq_k^0} - 2)(e^{iq_k^0} + e^{-iq_m^0} - 2)}, & m = l, \\ \frac{4(1 - \cos q_m^0)}{(e^{iq_m^0} + e^{-iq_l^0} - 2)(e^{iq_l^0} + e^{-iq_m^0} - 2)}, & m \neq l, \end{cases} \quad (20)$$

is the same matrix whose determinant gives the normalisation of the Bethe wave functions, $Z^2 = \det \hat{Q}_0$ [6]. Since the normalisation factor is always finite for the physically meaningful states, $\det \hat{Q}_0$ is also finite, making the matrix $\hat{Q}_0$ invertible for all eigenstates of the Heisenberg chain.

In the thermodynamic limit, $N, L \gg 1$, the contribution of P^0/N in Eq. (17) becomes subleading for physically relevant momenta within a few-Fermi-momenta range so it can be neglected,

$$k_j^1 = \frac{4}{L} k_j^0 \sum_l (\cos q_l^0 - 1). \quad (21)$$

The result in the last equation is presented in Eq. (2) of the main paper. The result in Eq. (18) is presented in Eq. (3) of the main paper.

B. t/U expansion of wave function

The Lieb-Wu wave function (1) simplifies for large U , which manifests in its expansion in the Taylor series in t/U . In the $U = \infty$ limit, the amplitude in Eq. (3) becomes independent of the charge quasimomenta k_j^0 and their permutation Q ,

$$A_{QO\alpha} = \sum_R e^{i \sum_{l < m} \varphi_{R_l, R_m} + i \sum_l q_l^0 x_l}, \quad (22)$$

where the two-spinon scattering phase φ_{lm} is given by Eq. (9). Then, the ordering permutation O of spin configurations α is absorbed in the relabelling of α under the sum over all α , this amplitude can be taken out of the sum over the permutation Q in Eq. (1), the ordering permutation O of the charge coordinates $\mathbf{j}$ is absorbed in the relabelling of the permutation Q under the sum over all Q , and the whole Lieb-Wu wave function factorises in this limit as [5]

$$\psi_{\mathbf{j}\alpha}^0 = \psi_{\mathbf{j}}^{0c} \cdot \psi_{\alpha}^{0s}, \quad (23)$$

where

$$\psi_{\mathbf{j}}^{0c} = L^{-N} \sum_Q (-1)^Q e^{iQ\mathbf{k}^0 \cdot \mathbf{j}} \quad (24)$$

is a Slater determinant that depends only on the charge coordinates $\mathbf{j}$,

$$\psi_{\alpha}^{0s} = Z^{-1} \sum_R e^{i \sum_{l < m} \varphi_{Rl, Rm} + i \sum_l R q_l^0 x_l} \quad (25)$$

is the Bethe wave function for the antiferromagnetic spin chain [2] that depends only on the spin configurations α , and $Z = \sqrt{\det \hat{Q}_0}$ is the Gaudin normalisation factor [6]. Here each of the factorised charge and spin wave functions are already normalised to unity so that the whole Lieb-Wu wave function (23) is also normalised to unity.

In the representation of second quantisation, such a wave function factorised into the spin and charge sectors in Eq. (23) can be written as a direct product,

$$|\Psi^0\rangle = |\Psi_c^0\rangle \otimes |\Psi_s^0\rangle, \quad (26)$$

in which the charge part is

$$|\Psi_c^0\rangle = \frac{1}{L^{\frac{N}{2}} N!^{\frac{1}{2}}} \sum_{Q, \mathbf{j}} (-1)^Q e^{iQ\mathbf{k}^0 \cdot O\mathbf{j}} a_{j_1}^\dagger \dots a_{j_N}^\dagger |0\rangle \quad (27)$$

and the spin part is

$$|\Psi_s^0\rangle = \frac{1}{Z} \sum_{R, x_1 < \dots < x_M} e^{i \sum_{l < m} \varphi_{Rl, Rm} + i R \mathbf{q} \cdot \mathbf{x}} S_{x_1}^+ \dots S_{x_M}^+ |\downarrow\rangle, \quad (28)$$

where $a_j^\pm$ are the spinless ladder operators obeying Fermi statistics on the same 1D lattice of length L and $S_j^\pm$ are the spin-1/2 flip operators on the spin chain of length N formed by the spin degree of freedom of N particles. These operators can be recombined into the original electron operators $c_{j\alpha}^\pm$ of the Hubbard model in Eq. (1) of the main text by introducing an insertion (deletion) operator of a site in the spin-down state at a given position x on the spin chain I_x (D_x) as

$$c_{j\uparrow}^\dagger = a_j^\dagger S_x^+ I_x, \quad (29)$$

$$c_{j\downarrow}^\dagger = a_j^\dagger I_x, \quad (30)$$

$$c_{j\uparrow} = a_j D_x S_x^-, \quad (31)$$

$$c_{j\downarrow} = a_j D_x S_x^-. \quad (32)$$

In the r.h.s. of these equations, the $a_j^\pm$ and $S_x^\pm$ operators obey the Fermi and spin commutation rules but the I_x and D_x operators do not. In general, these insertion and deletion operators also do not commute with the spin operators.

Like the Lieb-Wu equations (5, 6) in the previous subsection, the Lieb-Wu wave function (1) can also be expanded in a Taylor series in t/U around the $U = \infty$ point. Substitution of the linear expansion for the charge and spin quasimomenta around the $U = \infty$ point from Eqs. (12, 13) into Eq. (1) and expansion of the resulting expression up to linear order in t/U gives

$$\psi_{\mathbf{j}\alpha} = \psi_{\mathbf{j}\alpha}^0 + t/U \psi_{\mathbf{j}\alpha}^1, \quad (33)$$

where the zeroth term $\psi_{\mathbf{j}\alpha}^0$ is factorised into the spin and charge sectors as given by Eq. (23) and the linear term has three contributions,

$$\psi_{\mathbf{j}\alpha}^1 = \psi_{\mathbf{j}\alpha}^{1c} + \psi_{\mathbf{j}\alpha}^{1sc} + \psi_{\mathbf{j}\alpha}^{1s}. \quad (34)$$

The first and the third terms in $\psi_{\mathbf{j}\alpha}^1$ come from the linear terms in charge and spin quasimomenta in Eqs. (12, 13), respectively, when the amplitude in Eq. (3) has the form of the $U = \infty$ limit in Eq. (22). These terms do not contain any mixing between the charge and spin variables so that both $\psi_{\mathbf{j}\alpha}^{1s}$ and $\psi_{\mathbf{j}\alpha}^{1c}$ are still factorised in the spin and charge sectors

$$\psi_{\mathbf{j}\alpha}^{1s} = \psi_{\mathbf{j}}^{0c} \cdot \frac{1}{Z} \sum_R e^{i \sum_{l < m} \varphi_{R_l R_m} + i R \mathbf{q}^0 \cdot \mathbf{x}} \left[2 \sum_{l < m} \frac{R q_l^1 (1 - \cos R q_m^0) - R q_m^1 (1 - \cos R q_l^0)}{(e^{i R q_l^0} + e^{-i R q_m^0} - 2) (e^{-i R q_l^0} + e^{i R q_m^0} - 2)} + i R \mathbf{q}^1 \cdot \mathbf{x} \right], \quad (35)$$

$$\psi_{\mathbf{j}\alpha}^{1c} = \left[\frac{i}{L^{\frac{N}{2}} N!^{\frac{1}{2}}} \sum_Q (-1)^Q e^{i Q \mathbf{k}^0 \cdot O \mathbf{j}} Q \mathbf{k}^1 \cdot O \mathbf{j} \right] \cdot \psi_{\alpha}^{0s}, \quad (36)$$

where $\psi_{\mathbf{j}}^{0c}$ and ψ_{α}^{0s} are given in Eqs. (23) and (24) respectively. The second term in Eq. (34) comes from the linear expansion of Eq. (3) representing the linearised version of the original mixing between the spin and charge degrees of freedom in the Lieb-Wu wave function,

$$\psi_{\mathbf{j}\alpha}^{1sc} = -\frac{4}{L^{\frac{N}{2}} N!^{\frac{1}{2}} Z} \sum_{Q,R} (-1)^Q e^{i Q \mathbf{k}^0 \cdot O \mathbf{j} + i \sum_{l < m} \varphi_{R_l R_m} + i R \mathbf{q} \cdot \mathbf{x}} \sum_{m'} \left[\frac{1 - e^{i R q_{m'}^0}}{2} Q k_{x_{m'}}^0 + (1 - \cos R q_{m'}^0) \sum_{j'=1}^{x_{m'}} Q k_{j'}^0 \right]. \quad (37)$$

In the representation of second quantisation, the expansion of the Lieb-Wu wave function up to linear order in t/U reads as

$$|\Psi\rangle = |\Psi^0\rangle + t/U |\Psi^1\rangle. \quad (38)$$

The zeroth term is already given by Eq. (26). The three contributions to the linear term of the wave function in Eqs. (34) can be written in this representation as

$$|\Psi^1\rangle = |\Psi_c^1\rangle + |\Psi_{sc}^1\rangle + |\Psi_s^1\rangle, \quad (39)$$

where

$$|\Psi_c^1\rangle = \frac{i}{L^{\frac{N}{2}} N!^{\frac{1}{2}}} \sum_{Q,\mathbf{j}} (-1)^Q e^{i Q \mathbf{k}^0 \cdot O \mathbf{j}} Q \mathbf{k}^1 \cdot O \mathbf{j} a_{j_1}^\dagger \cdots a_{j_N}^\dagger |0\rangle \otimes |\Psi_s^0\rangle, \quad (40)$$

$$|\Psi_{sc}^1\rangle = -\frac{4i}{L^{\frac{N}{2}} N!^{\frac{1}{2}} Z} \sum_{Q,R,\mathbf{j},\mathbf{x}} (-1)^Q e^{i Q \mathbf{k}^0 \cdot O \mathbf{j} + i \sum_{l < m} \varphi_{R_l R_m} + i R \mathbf{q} \cdot \mathbf{x}} \sum_{m'} \left[\frac{1 - e^{i R q_{m'}^0}}{2} Q k_{x_{m'}}^0 + (1 - \cos R q_{m'}^0) \sum_{j'=1}^{x_{m'}} Q k_{j'}^0 \right] a_{j_1}^\dagger \cdots a_{j_N}^\dagger |0\rangle \otimes S_{x_1}^+ \cdots S_{x_M}^+ |\Downarrow\rangle, \quad (41)$$

$$|\Psi_s^1\rangle = |\Psi_c^0\rangle \otimes \frac{1}{Z} \sum_{R,\mathbf{x}} e^{i \sum_{l < m} \varphi_{R_l R_m} + i R \mathbf{q}^0 \cdot \mathbf{x}} \left[i 2 \sum_{l < m} \frac{R q_l^1 (1 - \cos R q_m^0) - R q_m^1 (1 - \cos R q_l^0)}{(e^{i R q_l^0} + e^{-i R q_m^0} - 2) (e^{-i R q_l^0} + e^{i R q_m^0} - 2)} + i R \mathbf{q}^1 \cdot \mathbf{x} \right] S_{x_1}^+ \cdots S_{x_M}^+ |\Downarrow\rangle. \quad (42)$$

Here $|\Psi_c^0\rangle$ and $|\Psi_s^0\rangle$ are the factorised charge and spin parts of the Lieb-Wu wave function in the $U = \infty$ limit given by Eqs. (27, 28) and the sum over the spin coordinates $\mathbf{x}$ always runs only over the ordered set of coordinates, $x_1 < \cdots < x_M$, which is not specified explicitly in what follows for brevity.

II. ALGEBRAIC BETHE ANSATZ

The spin part of the wave function in the $U = \infty$ limit in Eq. (25) is the same as the Bethe wave function for the antiferromagnetic Heisenberg model in 1D [2]. The Bethe wave function in the coordinate representation in Eq. (25) is not factorised in terms of the single-spin states, making calculation of the matrix elements impossible. However, an algebraic representation of the Bethe wave function was invented in [7] to factorise it in terms of operators with specific commutation relations, which can be used for the analytical calculations of matrix elements. Here, we briefly introduce this algebraic representation, which will be used later for dealing with the spin part of matrix elements of the Hubbard model. It is more convenient to do this algebraic construction for a more general model, the XXZ spin model,

$$H = \sum_{j=1}^N \left(\frac{S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+}{2} + \Delta S_j^z S_{j+1}^z \right), \quad (43)$$

for which the eigenstates are the same as in Eq. (25) where the spin quasimomenta q_j^0 satisfy almost the same set of equations (8). The only difference is in the two-magnon scattering phase,

$$e^{i2\varphi_{ml}} = - \frac{e^{i(q_l^0 + q_m^0)} + 1 - 2\Delta e^{iq_m^0}}{e^{i(q_l^0 + q_m^0)} + 1 - 2\Delta e^{iq_l^0}} \quad (44)$$

that needs to be used for the model in Eq. (43) instead of the scattering phase in Eq. (9). For $\Delta = 1$ the XXZ model in Eq. (43) becomes the antiferromagnetic Heisenberg model $H = \sum_j \mathbf{S}_i \cdot \mathbf{S}_{j+1}$ and the two-spinon scattering phase in Eq. (44) becomes that of the antiferromagnetic Heisenberg model in Eq. (9).

In this work we follow the notations of the book in [8]. The M -spinon eigenfunction of the XXZ model can be represented by the algebraic Bethe operators generated by the Yang-Baxter equation [3, 9] as

$$|\mathbf{u}\rangle = \prod_{m=1}^M C(u_j) |\Downarrow\rangle, \quad (45)$$

where u_j are M complex parameters corresponding to the M spin quasimomenta q_j , $|\Downarrow\rangle$ is the “vacuum state” of the spin chain, and $C(u)$ is a matrix element of the monodromy matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \quad (46)$$

This matrix $T(u)$ is defined in an auxiliary 2×2 space, it is a function of a complex parameter u , and its four entries are operators that act in the space of N spins forming the chain. When this matrix is a solution of the Yang-Baxter equation,

$$R(u-v) (T(u) \otimes T(v)) = (T(v) \otimes T(u)) R(u-v), \quad (47)$$

the M -body scattering matrices factorise into products of only two-body scattering matrices. This equation is defined by the R -matrix that acts on a 4×4 tensor product space $V_1 \otimes V_2$, where V_1 and V_2 are two-element subspaces. For the XXZ model in Eq. (43) the R -matrix is [8]

$$R(u) = \begin{pmatrix} 1 & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & 1 \end{pmatrix}, \quad (48)$$

where

$$b(u) = \frac{\sinh(u)}{\sinh(u+2\eta)}, \quad c(u) = \frac{\sinh(2\eta)}{\sinh(u+2\eta)}, \quad (49)$$

and η is a real parameter corresponding to Δ in the XXZ model. Note that this R -matrix also satisfies the Yang-Baxter equation defined by itself,

$$R_{12}(u_1 - u_2) R_{13}(u_1) R_{23}(u_2) = R_{23}(u_2) R_{13}(u_1) R_{12}(u_1 - u_2). \quad (50)$$

For a chain consisting of only one spin, the solution of Eq. (47) can be constructed by identifying one two-element subspace of the R -matrix in Eq. (48) with the two-state spin-1/2 space of a single lattice spin on site j . Then, this R -matrix becomes the quantum version of the Lax matrix [10] for such a single-site chain, $L_j = R_{1j}$, in which the other two-element subspace plays the role of the auxiliary 2×2 space of the T -matrix in Eq. (46). In this auxiliary subspace the single-site Lax matrix reads as

$$L_j(u) = \begin{pmatrix} \frac{\cosh(u+\eta 2S_j^z)}{\cosh(u-\eta)} & -i \frac{\sinh(2\eta) 2S_j^-}{\cosh(u-\eta)} \\ -i \frac{\sinh(2\eta) 2S_j^+}{\cosh(u-\eta)} & \frac{\cosh(u-\eta 2S_j^z)}{\cosh(u-\eta)} \end{pmatrix}. \quad (51)$$

Further, the monodromy matrix for the whole chain consisting of N spins is constructed as

$$T(u) = \prod_{j=1}^N L_j(u), \quad (52)$$

providing a definition of the algebraic Bethe ansatz operators in terms of the physical spin operators of the model in Eq. (43). By construction, the T -matrix in Eq. (52) satisfies the Yang-Baxter equation (47) with the R -matrix in Eq. (48), see details in [8].

The entries of the Yang-Baxter equation (47) in the 4×4 space with the R -matrix of the XXZ model in Eq. (48) give the explicit form of the commutations relation between all four Bethe ansatz operators $A(u)$, $B(u)$, $C(u)$, and $D(u)$ defined in Eq. (46). The commutation relations that we will need later are

$$[B_u, C_v] = \frac{c(u-v)}{b(u-v)} (A_u D_v - A_v D_u), \quad (53)$$

$$A_u C_v = \frac{1}{b(u-v)} C_v A_u - \frac{c(u-v)}{b(u-v)} C_u A_v, \quad (54)$$

$$D_u C_v = \frac{1}{b(u-v)} C_v D_u - \frac{c(v-u)}{b(v-u)} C_u D_v, \quad (55)$$

$$[A_u, D_v] = \frac{c(u-v)}{b(u-v)} (C_v B_u - C_u B_v), \quad (56)$$

where the subscripts u and v were introduced for brevity, *e.g.*, $A_u \equiv A(u)$.

The transition matrix $\tau(u)$ is given by the trace of the T -matrix in the algebraic approach,

$$\tau(u) = A(u) + D(u). \quad (57)$$

This operator also commutes with itself for different values of u , *i.e.*, $[\tau(u), \tau(v)] = 0$ for $u \neq v$, being a linear superposition of all the conserved quantities of the problem, including the XXZ model. Therefore, if a state $|\mathbf{u}\rangle$, parametrised by a set of u_j , is an eigenstates of $\tau(u)$ it is also an eigenstates of the Hamiltonian in Eq. (43),

$$\tau(u) |\mathbf{u}\rangle = \mathcal{T}_u |\mathbf{u}\rangle, \quad (58)$$

where $\mathcal{T}_u$ is the corresponding eigenvalue of the transition matrix. Solution of this eigenvalue problem imposes a constraint on the M parameters u_j as [8]

$$\frac{a_m}{d_m} = \prod_{l=1 \neq m}^M \frac{b_{ml}}{b_{lm}}, \quad (59)$$

where a_m and d_m are the vacuum eigenvalues of the A_u and D_u operators, $A_u |\Downarrow\rangle = a_u |\Downarrow\rangle$ and $D_u |\Downarrow\rangle = d_u |\Downarrow\rangle$, and the subscripts were introduced further as $a_m \equiv a(u_m)$, $d_m \equiv d(u_m)$, $b_{ml} \equiv b(u_m - u_l)$, and $b_{mu} \equiv b(u_m - u)$ for brevity. When this constraint is obeyed, *i.e.*, a particular set of u_j gives an eigenstate of $\tau(u)$, the corresponding eigenvalue of $\tau(u)$ is

$$\mathcal{T}_u = a_u \prod_{m=1}^M \frac{1}{b_{um}} + d_u \prod_{m=1}^M \frac{1}{b_{mu}}. \quad (60)$$

The vacuum eigenvalues a_m and d_m are evaluated straightforwardly using the expressions constructed in Eqs. (51, 52) and the properties of Pauli matrices as

$$a_u = \frac{\cosh^N(u - \eta)}{\cosh^N(u + \eta)} \quad \text{and} \quad d_u = 1. \quad (61)$$

Under substitution of these expressions and of the expression for $b(u)$ in Eq. (49), the constraint in Eq. (59) reads as

$$\frac{\cosh^N(u_m - \eta)}{\cosh^N(u_m + \eta)} = \prod_{l=1 \neq m}^M \frac{\sinh(u_m - u_l - 2\eta)}{\sinh(u_m - u_l + 2\eta)} \quad (62)$$

and the eigenvalue $\mathcal{T}_u$ in Eq. (60) as

$$\mathcal{T}_u = \frac{\cosh^N(u - \eta)}{\cosh^N(u + \eta)} \prod_{m=1}^M \frac{\sinh(u - u_m + 2\eta)}{\sinh(u - u_m)} + \prod_{m=1}^M \frac{\sinh(u_m - u + 2\eta)}{\sinh(u_m - u)}. \quad (63)$$

The set of equations (62) are the Bethe equations for the XXZ model in Orbach parametrisation [11]. Substitution of the inverse mapping from Orbach (which is also known as algebraic) to the coordinate parametrisation,

$$u_m = \frac{1}{2} \ln \left(\frac{1 - e^{iq_m - 2\eta}}{1 - e^{-iq_m - 2\eta}} \right) - \frac{iq_m}{2}, \quad (64)$$

$$\eta = \frac{1}{2} \text{acosh} \Delta, \quad (65)$$

into Eq. (62) recovers the Bethe equations in the coordinate representation in Eq. (8) with the two-magnon scattering phase in Eq. (44).

The advantage of the algebraic over the coordinate representation of the Bethe wave function can already be seen in calculation of the scalar product of two Bethe states. In the algebraic representation of the scalar product of two Bethe states $\langle \mathbf{v} |$ and $|\mathbf{u} \rangle$ in Eq. (45), each $B(u_j)$ operator in the bra state can be commuted through the product of all $C(u_j)$ operators in the ket state using the commutation relations in Eqs. (53–55). At the end of the commutation procedure, the $B(u_j)$ operators acting upon the vacuum states $|\downarrow\rangle$ give zero and the $A(u_j)$ and $D(u_j)$ operators generated by the $B(u_j)$ and $C(u_j)$ commutation relation (53) give their vacuum eigenvalues a_j and d_j . When the ket state $|\mathbf{u}\rangle$ is parametrised by u_j that satisfy the Bethe equations (62), the result of such commutation for the scalar product of two Bethe states can be written in a compact form as a determinant of an $M \times M$ matrix, which is also known as the Slavnov formula [12],

$$\langle \mathbf{v} | \mathbf{u} \rangle = \frac{\prod_{lm} \sinh(v_l - u_m) \det \hat{G}}{\prod_{l < m} \sinh(v_l - v_m) \prod_{l < m} \sinh(u_l - u_m)}, \quad (66)$$

where the $M \times M$ matrix $\hat{G}$ is given by the derivatives of the eigenvalue of the transition matrix $\mathcal{T}_u$ as $G_{ab} = \partial_{u_a} \mathcal{T}(v_b)$. The explicit form of these derivatives for the $\mathcal{T}_u$ in Eq. (63) is

$$G_{ab} = \frac{\cosh^N(v_b - \eta)}{\cosh^N(v_b + \eta)} \frac{\sinh(2\eta)}{\sinh^2(v_b - u_a)} \prod_{l=1 \neq a}^M \frac{\sinh(v_b - u_l + 2\eta)}{\sinh(v_b - u_l)} - \frac{\sinh(2\eta)}{\sinh^2(u_a - v_b)} \prod_{l=1 \neq a}^M \frac{\sinh(u_l - v_b + 2\eta)}{\sinh(u_l - v_b)}. \quad (67)$$

When v_j satisfy the Bethe equations (62) instead of u_j , the result is almost the same in Eq. (66), with the only difference that the derivatives of the eigenvalue of the transition matrix taken over the v_j instead of u_j as $G_{ab} = \partial_{v_a} \mathcal{T}(u_b)$.

A. Normalisation factor

The normalisation factor of the Bethe states in the algebraic representation in Eq. (45) can be evaluated by applying the formula for the scalar product in Eq. (66) on the pair of the same states, $Z^2 = \langle \mathbf{u} | \mathbf{u} \rangle$. Since some of the matrix elements in Eq. (67) are divergent under the direct substitution of $\mathbf{v} = \mathbf{u}$, which is, however, regularised by the zeros in the prefactor in Eq. (66), this substitution needs to be evaluated by taking the $\mathbf{v} \rightarrow \mathbf{u}$ limit, which gives [13]

$$Z^2 = \sinh^M(2\eta) \prod_{l \neq m} \frac{\sinh(u_l - u_m + 2\eta)}{\sinh(u_l - u_m)} \det \hat{F}, \quad (68)$$

where the matrix $\hat{F}$ is

$$F_{ab} = \begin{cases} -N \frac{\sinh(2\eta)}{\cosh(u_a + \eta) \cosh(u_a - \eta)} - \sum_{l=1 \neq a}^M \frac{\sinh(4\eta)}{\sinh(u_a - u_l - 2\eta) \sinh(u_a - u_l + 2\eta)}, & a = b, \\ \frac{\sinh(4\eta)}{\sinh(u_b - u_a - 2\eta) \sinh(u_b - u_a + 2\eta)}, & a \neq b. \end{cases} \quad (69)$$

The mapping of the last result back into the coordinate representation, in which we calculate the spin matrix element in this work, is done by substituting the relations in Eqs. (64, 65) in Eq. (68). In the Heisenberg limit, which we also need for the Hubbard model in this work, this substitution becomes degenerate since $\eta = 0$ and $u_m = i\pi/2$ for $\Delta = 1$. Therefore, the $\eta \rightarrow 0$ limit needs to be taken in doing this inverse mapping. Expanding Eq. (64) upto the linear order in η ,

$$u_m = \frac{i\pi}{2} + i\eta \cot \frac{q_m}{2}, \quad (70)$$

substituting the last expansion in Eq. (68), and taking the $\eta \rightarrow 0$ limit

$$\lim_{\eta \rightarrow 0} Z^2 \Big|_{u_m \rightarrow \frac{i\pi}{2} + i\eta \cot \frac{q_m}{2}}, \quad (71)$$

gives

$$Z^2 = (-4)^M \prod_m \sin^2 \frac{q_m}{2} \prod_{l \neq m} \frac{\cot \frac{q_m}{2} - \cot \frac{q_l}{2} - 2i}{\cot \frac{q_m}{2} - \cot \frac{q_l}{2}} \det \hat{Q}_0. \quad (72)$$

Here the determinant of the matrix $\hat{Q}_0$ given by Eq. (20) is the same as in the Gaudin normalisation factor in the coordinate representation [6] but the prefactor in the algebraic representation is not unity but a function of spin quasimomenta. Since we calculate the spin matrix element in the coordinate representation using the algebraic method in this work, the non-unity prefactor in Eq. (72) will be taken into account where needed in what follows.

III. t/U EXPANSION OF THE MATRIX ELEMENT $\langle f | c_{1\uparrow}^\pm | 0 \rangle$

The matrix elements of the ladder operators $c_{1\alpha}^\pm$ are needed for calculation of the Green function and of a range of observables relevant in practice. Since the Hubbard model without a magnetic field in Eq. (1) of the main text has the symmetry w.r.t. $\uparrow \leftrightarrow \downarrow$, it is sufficient to consider only $\alpha = \uparrow$. Let us start from the annihilation operators $c_{k\uparrow}$. Its matrix element that is required for the Green function at zero temperature is evaluated between the ground state 0 and an excited state f , $\langle f | c_{1\uparrow} | 0 \rangle$. For large U , the expansion of the wave function in Eq. (38) gives the t/U expansion of the matrix element as

$$\langle f | c_{1\uparrow} | 0 \rangle = \langle f | c_{1\uparrow} | 0 \rangle^0 + \frac{t}{U} \langle f | c_{1\uparrow} | 0 \rangle^1. \quad (73)$$

The zeroth term for the matrix element here factorises into the charge and spin parts,

$$\langle f | c_{1\uparrow} | 0 \rangle^0 = \langle f | c_{1\uparrow} | 0 \rangle_c^0 \cdot \langle f | c_{1\uparrow} | 0 \rangle_s^0, \quad (74)$$

since the zeroth order of the wave function in Eq. (26) is factorised into these two sectors. The charge part is evaluated as an N -fold sum over the charge coordinates $\mathbf{j}$ using Slater determinants in Eq. (24), producing a determinant of the Vandermonde type as a result [5],

$$\langle f | c_{1\uparrow} | 0 \rangle_c^0 = \frac{1}{L^{-N+\frac{1}{2}}} \det \hat{C}_0, \quad (75)$$

where the matrix elements of the $N \times N$ matrix $\hat{C}_0$ are

$$C_{ab}^0 = \begin{cases} e^{-i \frac{k_a^{00}(L-1)}{2}}, & b = 1, \\ \frac{\sin\left(\frac{k_a^{00} - k_{b-1}^{f0}}{2} L\right)}{\sin\left(\frac{k_a^{00} - k_{b-1}^{f0}}{2}\right)}, & b > 1. \end{cases} \quad (76)$$

Substitution of the explicit solutions for the charge quasimomenta in the $U = \infty$ limit from Eq. (11) in the above gives

$$C_{ab}^0 = \begin{cases} 1, & b = 1, \\ 2 \frac{\sin\left(\frac{P_s^0 - P_s^f}{2}\right)}{k_a^{00} - k_{b-1}^{f0}}, & b > 1, \end{cases} \quad (77)$$

where the limit of low-particle density, $N/L \ll 1$, was also taken and the overall real phase was neglected since it does not affect the observables. The last expression is presented in the main text after Eq. (4).

The spin part of the matrix element in Eq. (74) is evaluated by means of the algebraic Bethe ansatz [7]. This technique was successfully applied to calculation of the correlation function of the antiferromagnetic 1D-Heisenberg model [14, 15]. However, this result cannot be used for the Hubbard model directly, since the ladder operator $c_{1\uparrow}$ here changes the length of the spin chain by one site, making the construction of [7] for the bra $\langle f_s^0 |$ and ket $|0_s^0\rangle$ states in the spin part of Eq. (74) incompatible with each other. This problem was resolved by developing a representation of the algebra for the longer spin chain through the other algebra for the shorter one in [16], where the spin part of the matrix element for the Hubbard model was obtained as

$$\langle f | c_{1\uparrow} | 0 \rangle_s^0 = \frac{1}{Z_0 Z_f} \frac{\prod_{lm} (e^{iq_l^{f0}} + e^{-iq_m^{00}} - 2) \det \hat{R}_{00}}{\prod_{l \neq m} (e^{iq_l^{f0}} + e^{-iq_m^{f0}} - 2)^{\frac{1}{2}} \prod_{l \neq m} (e^{iq_l^{00}} + e^{-iq_m^{00}} - 2)^{\frac{1}{2}}}, \quad (78)$$

where the matrix elements of the $M \times M$ matrix $\hat{R}_{00}$ are

$$R_{ab}^{00} = \begin{cases} \frac{e^{i(N-1)q_b^{00}} \prod_{l \neq a} \frac{e^{iq_l^{f0} + iq_b^{00}} + 1 - 2e^{iq_l^{f0}}}{2e^{iq_b^{00}} - e^{iq_l^{f0} + iq_b^{00}} - 1}}{(e^{-q_a^{f0}} - e^{-iq_b^{00}})(e^{q_a^{f0}} - e^{-iq_b^{00}} - 2)}}, & a < M, \\ e^{ik_b^{00}} \frac{\prod_{l \neq b} (e^{iq_l^{00}} + e^{-iq_b^{00}} - 2)}{\prod_l (e^{iq_l^{f0}} + e^{-iq_b^{00}} - 2)}, & a = M, \end{cases} \quad (79)$$

and the normalisation factors $Z_{0/f} = \sqrt{\det \hat{Q}_0^{f/0}}$ are given by the determinant of the matrix $\hat{Q}_0^{f/0}$ in Eq. (20) with the spin quasimomenta in the $U = \infty$ limit for the bra $\langle f_s^0 |$ and the ket $|0_s^0\rangle$ states, respectively.

The linear term in Eq. (73) separates into three independent terms as

$$\langle f | c_{1\uparrow} | 0 \rangle^1 = \langle f | c_{1\uparrow} | 0 \rangle_c^1 + \langle f | c_{1\uparrow} | 0 \rangle_{cs}^1 + \langle f | c_{1\uparrow} | 0 \rangle_s^1 \quad (80)$$

following separation of linear term of the wave function in Eq. (39) into the same three terms. We will evaluate each of these contributions to the matrix element in the linear order in Eq. (80) independently below.

A. Charge part

The first term in Eq. (80) has two contributions originating from the linear terms in the bra and ket wave functions of the charge type in Eq. (40),

$$\langle f | c_{1\uparrow} | 0 \rangle_c^1 = (\langle f_c^0 | c_{1\uparrow} | 0_c^1 \rangle + \langle f_c^1 | c_{1\uparrow} | 0_c^0 \rangle) \langle f | c_{1\uparrow} | 0 \rangle_s^0, \quad (81)$$

where the spin part in the zeroth order in t/U is already given by Eq. (78). We simplify evaluation of the N -fold sum over the coordinates in the charge part by noting that the linear term of the wave function $|\Psi_c^1\rangle$ in Eq. (40) can be obtained from the zeroth order term $|\Psi_c^0\rangle$ in Eq. (27) by substituting $\mathbf{k}^0 \rightarrow \mathbf{k}^0 + g\mathbf{k}^1$ in the latter, taking the derivative of the result w.r.t. g , and by taking the $g \rightarrow 0$ limit at the end,

$$|\Psi_c^1\rangle = \lim_{g \rightarrow 0} d_g \left(|\Psi_c^0\rangle_{\mathbf{k}^0 \rightarrow \mathbf{k}^0 + g\mathbf{k}^1} \right), \quad (82)$$

where the charge quasimomenta in the first t/U order $\mathbf{k}^1$ are given by Eq. (21).

Using this trick for both terms in Eq. (81) as $\mathbf{k}_{0/f}^0 \rightarrow \mathbf{k}_{0/f}^0 + g\mathbf{k}_{0/f}^1$ and commuting the $\lim_{g \rightarrow 0} d_g$ with the N -fold sum over the charge coordinates $\mathbf{j}$, we express the whole charge as a single term,

$$\langle f | c_{1\uparrow} | 0 \rangle_c^1 = \langle f | c_{1\uparrow} | 0 \rangle_s^0 \lim_{g \rightarrow 0} d_g \langle f_c^0 + g f_c^1 | c_{1\uparrow} | 0_c^0 + g 0_c^1 \rangle^0, \quad (83)$$

where application of the chain rule in calculating the derivative d_g recovers the two contributions in Eq. (81). Calculation of the charge matrix element for the zeroth-order wave functions $|\Psi_c^0\rangle$ in the last expression was already done in [5]. Additional caution needs to be used to reuse this calculation here, since the $g \rightarrow 0$ limit and the d_g derivative in the first order in Eq. (83) do not commute with each other and also do not commute with the low-density limit taken in Eq. (77). Therefore, we use the result in Eq. (76), substitute $\mathbf{k}_{0/f}^0 \rightarrow \mathbf{k}_{0/f}^0 + g\mathbf{k}_{0/f}^1$ into it, and expand the resulting matrix under the determinant up to linear order in g as

$$\hat{C}_0 + g\hat{C}_1, \quad (84)$$

where the matrix elements in the first order in g are

$$C_{ab}^1 = \begin{cases} -i \frac{k_a^{01}(L-1)}{2}, & b = 1, \\ 2 \left(\frac{L \cos \frac{P_s^0 - P_s^f}{2}}{2} - \frac{\sin \frac{P_s^0 - P_s^f}{2}}{k_a^{00} - k_{b-1}^{f0}} \right) \frac{k_a^{01} - k_{b-1}^{f1}}{k_a^{00} - k_{b-1}^{f0}}, & b > 1, \end{cases} \quad (85)$$

in which the low-density limit was taken at the last step and the matrix element of $\hat{C}_0$ in the low-density limit are given in Eq. (77). Then, we use the Jacobi formula for evaluating the derivative w.r.t. g of the determinant from Eq. (75) with the matrix (84) under it as

$$d_g \det(\hat{C}_0 + g\hat{C}_1) = \det(\hat{C}_0 + g\hat{C}_1) \text{tr} \left[(\hat{C}_0 + g\hat{C}_1)^{-1} \hat{C}_1 \right]. \quad (86)$$

Lastly, substitution of Eq. (75) with the last expression into Eq. (83) gives

$$\langle f | c_{1\uparrow} | 0 \rangle_c^1 = \langle f | c_{1\uparrow} | 0 \rangle_s^0 \frac{1}{L^{-N+\frac{1}{2}}} \det \hat{C}_0 \text{tr} \left(\hat{C}_0^{-1} \hat{C}_1 \right), \quad (87)$$

where the $g \rightarrow 0$ limit also insures that all the higher-than-linear terms in the g -expansion in Eq. (84) do not contribute to the charge part of the matrix element. Identification of the charge part of the matrix element in the zeroth order $\langle f | c_{1\uparrow} | 0 \rangle_c^0$ in Eq. (87) by means of Eq. (75) allows to factor out the whole zeroth-order matrix element in Eq. (87) as

$$\langle f | c_{1\uparrow} | 0 \rangle_c^1 = \langle f | c_{1\uparrow} | 0 \rangle^0 T_c, \quad (88)$$

where

$$T_c = \text{tr} \left(\hat{C}_0^{-1} \hat{C}_1 \right). \quad (89)$$

The results in Eqs. (85, 89) are presented in Eqs. (5–7) of the main paper.

B. Spin part

The third term in Eq. (80), the spin contribution to the linear term, can be evaluated using the same trick with the derivative and the limit, as for the charge term in the first order in Eq. (83), since the linear term of the wave function of the spin type $|\Psi_s^1\rangle$ in Eq. (39) can also be expressed using the zeroth-order term $|\Psi_s^0\rangle$ in Eq. (28) as

$$|\Psi_s^1\rangle = \lim_{g \rightarrow 0} d_g \left(|\Psi_s^0\rangle_{\mathbf{q}^0 \rightarrow \mathbf{q}^0 + g\mathbf{q}^1} \right), \quad (90)$$

where the spin quasimomenta in the first t/U order $\mathbf{q}^1$ are given by Eq. (18). However, the application of this trick to the spin part of matrix element is somewhat less straightforward. In the charge part in Eq. (81), the N -fold sum over the charge coordinates is performed in the first-quantisation formalism, which imposes no additional requirements on the bra and ket states so that we can use the trick for both of them simultaneously in Eq. (83). For the spin part, the M -fold sum over the spin coordinates is performed indirectly using the algebraic Bethe ansatz by means of the Slavnov formula [12] in Eq. (66). Application of this formula requires one of the bra or ket states to be an eigenstate, which is not satisfied when both of the states are shifted by $g\mathbf{q}_{0/f}^1$ simultaneously. Thus, we have to apply the trick for the bra and ket states in the spin part of the matrix element separately.

Moreover, the spin part of the matrix element in the zeroth order in Eq. (78) was derived under the specific condition of the bra state being an eigenstate in [16], so we have to derive additionally the explicit expression for the spin matrix

element in the zeroth order under the condition of ket state being an eigenstate here. We will do so by repeating the steps from [16], but with the different assumption of the ket instead of the bra state being an eigenstate. Our starting point is the representation of the spin part of the matrix element in the zeroth order in the representation of the algebraic Bethe ansatz,

$$\langle f | c_{1\uparrow} | 0 \rangle_s^0 = e^{-i(P_s^0 - P_s^f)(N-1)} \frac{\langle \mathbf{v} | D_N S_N^- | \mathbf{u} \rangle}{Z_f Z_0}, \quad (91)$$

where the decomposition of the electron ladder operator into the spin and charge parts in Eq. (31) was used, $\mathbf{v}$ are the spin quasimomenta $\mathbf{q}_0^f$ and $\mathbf{u}$ are the spin quasimomenta $\mathbf{q}_0^0$ in the algebraic representation Eq. (64), $Z_{f/0}$ are the normalisation factors of the Bethe functions in the algebraic representation in Eq. (68), and the spin chain was shifted $N - 1$ times to the right in order to simplify the algebraic manipulations in what follows. Here we assume that $|\mathbf{u}\rangle$ is an eigenstate instead of $\langle \mathbf{v} |$.

First, we express the local spin operator S_N^- in Eq. (91) in terms of the algebraic Bethe-ansatz operators in Eq. (46) by means of the Drinfeld twist [17] as

$$S_N^- = B_\xi \tau_\xi^{N-1}, \quad (92)$$

where $\xi = -i\pi/2 + \eta$ and the transition matrix τ_ξ is given by Eq. (63). Substitution of the last expression into the matrix element in the r.h.s. of Eq. (91) gives

$$\langle \mathbf{v} | D_N S_N^- | \mathbf{u} \rangle = e^{iP_s^0(N-1)} \langle \downarrow | \prod_{m=1}^{M-1} B^{N-1}(v_m) B_\xi^N \prod_{m=1}^M C^N(u_m) | \downarrow \rangle, \quad (93)$$

where $C^N(u)$ and $B^N(u)$ are the Bethe-ansatz operators constructed for the chain of N spins and $B^{N-1}(u)$ are the Bethe-ansatz operators for the chain of $N - 1$ spins due to the D_N operator in Eq. (91).

The last expression is almost a scalar product of two Bethe states, but the algebraic Bethe ansatz operators in the bra and ket states are constructed for the chains of different lengths, making the Slavnov formula in Eq. (66) inapplicable. In order to restore its applicability, we need to express the operators in the bra state using the same algebra as in the ket state by using the construction of the Bethe-ansatz operators in Eq. (52). Singling out the operators for the chain of $N - 1$ spins in this construction for N spins we obtain

$$\begin{pmatrix} A_u^N & B_u^N \\ C_u^N & D_u^N \end{pmatrix} = \begin{pmatrix} A_u^{N-1} & B_u^{N-1} \\ C_u^{N-1} & D_u^{N-1} \end{pmatrix} \begin{pmatrix} \frac{\cosh(u+2\eta S_N^z)}{\cosh(u+\eta)} & -i \frac{\sinh 2\eta S_N^-}{\cosh(u+\eta)} \\ -i \frac{\sinh 2\eta S_N^+}{\cosh(u+\eta)} & \frac{\cosh(u-2\eta S_N^z)}{\cosh(u+\eta)} \end{pmatrix}. \quad (94)$$

Multiplying this equation by the inverse of the matrix in the second factor in the r.h.s., and picking the top-right element of the resulting matrix equation, we get

$$B_u^{N-1} = \frac{\cosh(u+\eta)}{\cosh(u-2\eta S_N^z)} B_u^N + i \frac{\sinh 2\eta}{\cosh(u-2\eta S_N^z)} S_N^- A_u^{N-1}. \quad (95)$$

Lastly, we substitute this expression for B_u^{N-1} into Eq. (93) and obtain

$$\langle \mathbf{v} | D_N S_N^- | \mathbf{u} \rangle = e^{iP_s^0(N-1)} \langle \downarrow | \prod_{m=1}^{M-1} B^N(v_j) B^N(\xi) \prod_{m=1}^M C^N(u_j) | \downarrow \rangle, \quad (96)$$

where the second term in Eq. (95) does not contribute since $S_N^- B_\xi^N \sim S_N^- S_N^- = 0$ and

$$\frac{\cosh(v_m + \eta)}{\cosh(v_m - 2\eta S_N^z)} B_\xi^N = B_\xi^N, \quad (97)$$

since the N^{th} spin to the left of the B_ξ^N operator is always in the $\downarrow$ state.

The expression in Eq. (96) is a scalar product of two Bethe states constructed for the same chain consisting of N spins. Applying the Slavnov formula in Eq. (66) to it, performing the inverse mapping to the coordinate representation in Eq. (70), and taking the $\eta \rightarrow 0$ limit of the result, as in Eq. (71), we obtain

$$\langle \mathbf{v} | D_N S_N^- | \mathbf{u} \rangle = e^{iP_s^0(N-1)} \frac{2^M i^M \prod_l \left(1 + i \cot \frac{q_l^{00}}{2}\right) \prod_{lm} \left(\cot \frac{q_m^{00}}{2} - \cot \frac{q_l^{f0}}{2}\right) \det \hat{G}_f}{\prod_l \left(1 - i \cot \frac{q_l^{f0}}{2}\right) \prod_{l < m} \left(\cot \frac{q_l^{f0}}{2} - \cot \frac{q_m^{f0}}{2}\right) \prod_{l < m} \left(\cot \frac{q_l^{00}}{2} - \cot \frac{q_m^{00}}{2}\right)}, \quad (98)$$

where the matrix $\hat{G}_f$ is

$$G_{ab}^f = \begin{cases} \frac{e^{iq_b^{f0}N} \prod_{l=1 \neq a}^M \frac{\cot \frac{q_b^{f0}}{2} - \cot \frac{q_l^{00}}{2} - 2i}{\cot \frac{q_b^{f0}}{2} - \cot \frac{q_l^{00}}{2}} - \prod_{l=1 \neq a}^M \frac{\cot \frac{q_b^{f0}}{2} - \cot \frac{q_l^{00}}{2} + 2i}{\cot \frac{q_b^{f0}}{2} - \cot \frac{q_l^{00}}{2}}}{\left(\cot \frac{q_b^{f0}}{2} - \cot \frac{q_a^{00}}{2} \right)^2}, & b < M, \\ \frac{1}{\left(1 - i \cot \frac{q_a^{00}}{2} \right) \left(1 + i \cot \frac{q_a^{00}}{2} \right)}, & b = M. \end{cases} \quad (99)$$

Substituting the result in Eq. (98) and the normalisation of the algebraic Bethe states in Eq. (72) into Eq. (91) and rearranging terms, we obtain the same expression for the spin matrix element as in Eq. (78),

$$\langle f | c_{1\uparrow} | 0 \rangle_s^0 = \frac{1}{Z_0 Z_f} \frac{\prod_{lm} \left(e^{iq_l^{f0}} + e^{-iq_m^{00}} - 2 \right) \det \hat{R}_{f0}}{\prod_{l \neq m} \left(e^{iq_l^{f0}} + e^{-iq_m^{f0}} - 2 \right)^{\frac{1}{2}} \prod_{l \neq m} \left(e^{iq_l^{00}} + e^{-iq_m^{00}} - 2 \right)^{\frac{1}{2}}}, \quad (100)$$

but with a different matrix $\hat{R}_{f0}$ under the determinant, the elements of which are

$$R_{ab}^{f0} = \begin{cases} \frac{e^{iq_b^{f0}N} \prod_{l=1 \neq a}^M \left(-\frac{e^{iq_b^{f0}+iq_l^{00}+1-2e^{iq_l^{00}}}}{e^{iq_b^{f0}+iq_l^{00}+1-2e^{iq_b^{f0}}}} \right) - 1}{\left(e^{-iq_b^{f0}} - e^{-iq_a^{00}} \right) \left(e^{iq_a^{00}} + e^{-iq_b^{f0}} - 2 \right)}, & b < M, \\ 1, & b = M, \end{cases} \quad (101)$$

instead of the matrix $\hat{R}_{00}$ in Eq. (79). In the result in Eq. (100), the normalisation factors $Z_{0/f} = \sqrt{\det \hat{Q}_0^{f/0}}$ are in the coordinate representation and the property $P_s^f(N-1) = 2\pi \times \text{integer number}$ so that $\exp(iP_s^f(N-1)) = 1$ was used. In the $U = \infty$ limit, only one spin matrix in Eq. (78) was needed for the observables in [16] since $|\det \hat{R}_{f0}|^2 = |\det \hat{R}_{00}|^2$, but in the first t/U -order the relative phase between $\det \hat{R}_{00}$ and $\det \hat{R}_{f0}$ affects the result through derivatives, so both have to be introduced and evaluated.

Turning back to the spin part of the matrix element in the first order, we are now ready to evaluate the third term in Eq. (80), which has two contributions originating from the linear terms in the bra and ket wave functions of the spin type in Eq. (42),

$$\langle f | c_{1\uparrow} | 0 \rangle_s^1 = \langle f | c_{1\uparrow} | 0 \rangle_c^0 \left(\langle f_s^0 | c_{1\uparrow} | 0_s^1 \rangle + \langle f_s^1 | c_{1\uparrow} | 0_s^0 \rangle \right), \quad (102)$$

where the charge part in the zeroth t/U order $\langle f | c_{1\uparrow} | 0 \rangle_c^0$ is already given by Eq. (75). Using the expression for the wave function of the spin type in the first t/U order in Eq. (90) and applying the same trick as for the charge part of the matrix element in the linear order in Eq. (83) but only to the ket state in first term in Eq. (102), we obtain

$$\langle f_s^0 | c_{1\uparrow} | 0_s^1 \rangle = \lim_{g \rightarrow 0} d_g \langle f_s^0 | c_{1\uparrow} | 0_s^0 + g 0_s^1 \rangle^0. \quad (103)$$

In the matrix element on the r.h.s. of the above expression, the spin quasimomenta $\mathbf{q}_0^f$ in the bra state satisfy the Bethe equations so evaluation of this spin part of the matrix element in the zeroth t/U order by means of the algebraic Bethe ansatz gives Eq. (78), in which the spin quasimomenta of the ground state are shifted as $\mathbf{q}_0^0 \rightarrow \mathbf{q}_0^0 + g\mathbf{q}_1^0$. Expanding the matrix under the determinant in Eq. (78) after the shift up to linear order in g , similarly to Eq. (84), we obtain

$$\hat{R}_{00} + g\hat{R}_{01}, \quad (104)$$

where the matrix in the first order in g is $R_{ab}^{01} = \sum_m q_m^{01} \partial_{q_m^{00}} R_{ab}^{00}$. Then, using the Jacobi formula for the derivative of a determinant as in Eq. (87), we obtain

$$\begin{aligned} \langle f_s^0 | c_{1\uparrow} | 0_s^1 \rangle &= \frac{1}{Z_0 Z_f} \frac{\prod_{lm} \left(e^{iq_l^{f0}} + e^{-iq_m^{00}} - 2 \right)}{\prod_{l \neq m} \left(e^{iq_l^{f0}} + e^{-iq_m^{f0}} - 2 \right)^{\frac{1}{2}} \prod_{l \neq m} \left(e^{iq_l^{00}} + e^{-iq_m^{00}} - 2 \right)^{\frac{1}{2}}} \\ &\times \left[\det \hat{R}_{00} \text{tr} \left(\hat{R}_{00}^{-1} \hat{R}_{01} \right) - \frac{i}{2} \sum_{l \neq m} \frac{q_l^{01} e^{iq_l^{f0}} - q_m^{01} e^{-iq_m^{00}}}{e^{iq_l^{00}} + e^{-iq_m^{00}} - 2} - i \sum_{lm} \frac{q_m^{01} e^{-iq_m^{00}}}{e^{iq_l^{f0}} + e^{-iq_m^{00}} - 2} \right], \end{aligned} \quad (105)$$

where the last two terms containing the sums over the spin quasimomenta in the first t/U order q_m^{01} originate from the derivatives w.r.t. g of the prefactor in front of the determinant in Eq. (78) under the $\mathbf{q}_0^0 \rightarrow \mathbf{q}_0^0 + g\mathbf{q}_1^0$ shift, and the contribution from the $\mathbf{q}_0^0 \rightarrow \mathbf{q}_0^0 + g\mathbf{q}_1^0$ shift in the normalisation factor Z_0 is not taken into account here. The latter involves additional algebraic complications and will be considered separately as a part of Taylor expansion in t/U for the normalisation factor of the whole Lieb-Wu wave function in a separate section below.

The second term in Eq. (102) can be expressed in the same way, using the tricks in Eqs. (90) and (83) as

$$\langle f_s^0 | c_{1\uparrow} | 0_s^0 \rangle = \lim_{g \rightarrow 0} d_g \langle f_s^0 + g f_s^1 | c_{1\uparrow} | 0_s^0 \rangle. \quad (106)$$

Then, we repeat the same steps as after Eq. (103), but using the result for the sum over the spin coordinates in Eq. (100) instead of Eq. (78) and obtain

$$\begin{aligned} \langle f_s^1 | c_{1\uparrow} | 0_s^0 \rangle = & \frac{1}{Z_0 Z_f} \frac{\prod_{lm} (e^{iq_l^{f0}} + e^{-iq_m^{00}} - 2)}{\prod_{l \neq m} (e^{iq_l^{f0}} + e^{-iq_m^{f0}} - 2)^{\frac{1}{2}} \prod_{l \neq m} (e^{iq_l^{00}} + e^{-iq_m^{00}} - 2)^{\frac{1}{2}}} \\ & \times \left[\det \hat{R}_{f0} \text{tr} (\hat{R}_{f0}^{-1} \hat{R}_{f1}) - \frac{i}{2} \sum_{l \neq m} \frac{q_l^{f1} e^{iq_l^{f0}} - q_m^{f1} e^{-iq_m^{f0}}}{e^{iq_l^{f0}} + e^{-iq_m^{f0}} - 2} + i \sum_{lm} \frac{q_l^{f1} e^{iq_l^{f0}}}{e^{iq_l^{f0}} + e^{-iq_m^{00}} - 2} \right], \quad (107) \end{aligned}$$

where $R_{ab}^{f1} = \sum_m q_m^{f1} \partial_{q_m^{f0}} R_{ab}^{f0}$ is the linear term in the Taylor expansion of the matrix $\hat{R}_{f0}$ in g under the $\mathbf{q}_0^f \rightarrow \mathbf{q}_0^f + g\mathbf{q}_1^f$ shift of the spin quasimomenta and the last two terms in the above expression containing the sums over the spin quasimomenta in the first t/U order q_m^{f1} originate from the derivatives w.r.t. g of the prefactor under the same $\mathbf{q}_0^f \rightarrow \mathbf{q}_0^f + g\mathbf{q}_1^f$ shift.

Under the substitution of both terms in Eqs. (105) and (107) into Eq. (102), the whole zeroth-order matrix element can be identified as a factor using Eqs. (78, 100), and we find the spin contribution to the linear term as

$$\langle f | c_{1\uparrow} | 0 \rangle_s^1 = \langle f | c_{1\uparrow} | 0 \rangle^0 T_s, \quad (108)$$

where

$$\begin{aligned} T_s = & i \sum_{lm} \frac{q_l^{f1} e^{iq_l^{f0}} - q_m^{01} e^{-iq_m^{00}}}{e^{iq_l^{f0}} + e^{-iq_m^{00}} - 2} - \frac{i}{2} \sum_{l \neq m} \frac{q_l^{01} e^{iq_l^{00}} - q_m^{01} e^{-iq_m^{00}}}{e^{iq_l^{00}} + e^{-iq_m^{00}} - 2} + \text{Tr}(\hat{R}_{00}^{-1} \hat{R}_{01}) \\ & - \frac{i}{2} \sum_{l \neq m} \frac{q_l^{f1} e^{iq_l^{f0}} - q_m^{f1} e^{-iq_m^{f0}}}{e^{iq_l^{f0}} + e^{-iq_m^{f0}} - 2} + \text{Tr}(\hat{R}_{f0}^{-1} \hat{R}_{f1}). \quad (109) \end{aligned}$$

The last result is presented in Eq. (8) of the main text.

C. Mixing part

In the second term in Eq. (80), the sums over the spin and charge coordinates are mixed with each other. Substitution of the linear terms in the bra and ket wave functions of mixing type in Eq. (41) gives two terms for this contribution as

$$\langle f | c_{1\uparrow} | 0 \rangle_{cs}^1 = \langle \Psi_{cs,f}^1 | a_1 D_1 S_1^- | \Psi_0^0 \rangle + \langle \Psi_f^0 | a_1 D_1 S_1^- | \Psi_{cs,0}^1 \rangle. \quad (110)$$

We will deal with the first term in the r.h.s. of the above expression first. Since $|\Psi_{cs}^1\rangle$ in Eq. (41) is linear in the charge coordinates and in the charge quasimomenta, the sum over the charge coordinates can be evaluated independently of the spin coordinates giving the “average” values of the charge quasimomenta of the bra state that depend on the coordinate on the spin chain,

$$\bar{k}_l^{f0} = \frac{1}{L^{N-1} (N-1)!} \sum_{j_2, \dots, j_N, Q, Q'} (-1)^{Q+Q'} e^{iQ\mathbf{k}_0^0 \cdot \mathcal{O}(1, j_2, \dots, j_N) - iQ'\mathbf{k}_f^0 \cdot \mathcal{O}(j_2, \dots, j_N)} Q' k_l^{f0}, \quad (111)$$

which in the thermodynamic limit gives

$$\bar{k}_l^{f0} = \langle f|c_{1\uparrow}|0\rangle_c^0 P_f \left[N^{-1} + i \cot \left(\frac{P_s^f - P_s^0}{2} \right) (\delta_{l,1} - \delta_{l,N-1}) \right]. \quad (112)$$

Substitution of the latter in the remaining sum over the spin coordinates gives the first term in Eq. (110) as

$$\begin{aligned} \langle \Psi_{cs,f}^1 | a_1 D_1 S_1^- | \Psi_0^0 \rangle &= \langle f|c_{1\uparrow}|0\rangle_c^0 \frac{4P_f}{Z_f Z_0} \sum_{R,R',x_2,\dots,x_M} e^{i \sum_{l < m} \varphi_{R_l R_m}^0 - i \sum_{l < m} \varphi_{R'_l R'_m}^f + i R \mathbf{q}_0^0 \cdot (1, x_2, \dots, x_M) - i R' \mathbf{q}_0^f \cdot (x_2, \dots, x_M)} \\ &\times i \sum_{m'} \left[\frac{1 - e^{-i R' q_{m'}^{f0}}}{2N} + \frac{1 - \cos R' q_{m'}^{f0}}{N} x_{m'} + i \cot \left(\frac{P_s^f - P_s^0}{2} \right) (1 - \cos R' q_{m'}^{f0}) \right], \end{aligned} \quad (113)$$

where the δ -functions originating from Eq. (112) are resolved by the sums over m' in Eq. (41) and the thermodynamic limit is also taken. The first and third terms in the second line of the last expression can be taken outside of the sum over the permutations R' and R since under the sum over m' they do not depend on R' or R , and, then, the remaining sum over the spin coordinates for them in the first line of the above expression is $\langle f|c_{1\uparrow}|0\rangle_s^0$ that was already evaluated in Eq. (78). Together, the first and third terms in Eq. (113) give

$$\langle \Psi_{cs,f}^1 | a_1 D_1 S_1^- | \Psi_0^0 \rangle = \langle f|c_{1\uparrow}|0\rangle^0 \sum_m \left[2i \frac{1 - e^{-i q_m^{f0}}}{N} - 4 \cot \left(\frac{P_s^f - P_s^0}{2} \right) (1 - \cos q_m^{f0}) \right]. \quad (114)$$

The second term in the second line in Eq. (113) depends explicitly on the spin coordinates. However, this dependence has the same form as the linear term of the bra wave function of the spin type in Eq. (42) with $\mathbf{q}_f^1 = 4P_f (\cos \mathbf{q}_f^0 - 1)/N$ when we realise that the derivative of the two-magnon scattering phase in Eq. (42) is zero for $\mathbf{q}_f^1 = 4P_f (\cos \mathbf{q}_f^0 - 1)/N$, since

$$\frac{(1 - \cos R' q_l^{f0})(1 - \cos R' q_m^{f0}) - (1 - \cos R' q_m^{f0})(1 - \cos R' q_l^{f0})}{(e^{i R' q_l^{f0}} + e^{-i R' q_m^{f0}} - 2)(e^{-i R' q_l^{f0}} + e^{i R' q_m^{f0}} - 2)} = 0. \quad (115)$$

Then, the sum over the spin coordinates in the second term in Eq. (113) can be evaluated using the same trick as for the spin contribution to the linear term in Eq. (106), giving the same result as in Eq. (107) but with $\mathbf{q}_f^1 = 4P_f (\cos \mathbf{q}_f^0 - 1)/N$.

Now we deal with the second term in the r.h.s. of Eq. (110). Similarly to the first term, it is linear in the charge coordinates and in the quasimomenta. Therefore, the sum over the charge coordinates can be evaluated independently of the spin coordinates, giving the “average” values of the charge quasimomenta of the ket state that depend on the coordinate on the spin chain,

$$\bar{k}_l^{00} = \frac{1}{L^{N-1} (N-1)!} \sum_{j_2, \dots, j_N, Q, Q'} (-1)^{Q+Q'} e^{i Q \mathbf{k}_0^0 \cdot O(1, j_2, \dots, j_N) - i Q' \mathbf{k}_f^0 \cdot O(j_2, \dots, j_N)} Q k_l^{00}, \quad (116)$$

which in the thermodynamic limit gives

$$\bar{k}_l^{00} = \langle f|c_{1\uparrow}|0\rangle_c^0 P_f \left[N^{-1} - \delta_{l,1} + i \cot \left(\frac{P_s^f - P_s^0}{2} \right) (\delta_{l,2} - \delta_{l,N}) \right]. \quad (117)$$

Substitution of the latter into the remaining sum over the spin coordinates gives the second term in Eq. (110) as

$$\begin{aligned} \langle \Psi_f^0 | a_1 D_1 S_1^- | \Psi_{cs,0}^1 \rangle &= \langle f|c_{1\uparrow}|0\rangle_c^0 \frac{4P_f}{Z_f Z_0} \sum_{R,R',x_2,\dots,x_M} e^{i \sum_{l < m} \varphi_{R_l R_m}^0 - i \sum_{l < m} \varphi_{R'_l R'_m}^f + i R \mathbf{q}_0^0 \cdot (1, x_2, \dots, x_M) - i R' \mathbf{q}_0^f \cdot (x_2, \dots, x_M)} \\ &\times (-i) \sum_{m'} \left[\frac{1 - e^{i R q_{m'}^{00}}}{2N} - \frac{1 - e^{i R q_{m'}^{00}}}{2} \delta_{x_{m'},1} + \frac{1 - \cos R q_{m'}^{00}}{N} x_{m'} + (1 - \cos R q_{m'}^{00}) \right. \\ &\quad \left. + i (1 - \cos R q_{m'}^{00}) \cot \left(\frac{P_s^f - P_s^0}{2} \right) - i (1 - \cos R q_{m'}^{00}) \cot \left(\frac{P_s^f - P_s^0}{2} \right) \delta_{x_{m'},1} \right], \end{aligned} \quad (118)$$

where the δ -functions originating from Eq. (117) are resolved by the sums in Eq. (41) and the thermodynamic limit is also taken. Similarly to the contributions in Eq. (114), the first and the fourth terms in the second line and the first term in the third line of the above expression are independent of the spin coordinates and the sum over the permutation R and R' , so they can be taken outside of them, and the remaining sums give $\langle f|c_{1\uparrow}|0\rangle_s^0$, which was already evaluated in Eq. (78). The third term in the second line in Eq. (118) has the same dependence on the spin coordinates as the linear term of the ket wave function of the spin type in Eq. (42) and can be evaluated in the same way as the corresponding term in Eq. (113), giving the spin contribution to the linear term in Eq. (105) but with $\mathbf{q}_0^1 = 4P_f (\cos \mathbf{q}_0^0 - 1) / N$.

The remaining two terms with $\delta_{x_{m'},1}$ in Eq. (118) do not have analogs in the first term in Eq. (113). The second term in the second line in Eq. (118) has the first part that is independent of $Rq_{m'}^{00}$ and gives $\langle f|c_{1\uparrow}|0\rangle_s^0$ as

$$\frac{1}{2} \frac{1}{Z_f Z_0} \sum_{R, R', x_2, \dots, x_M} e^{i \sum_{l < m} \varphi_{R_l R_m}^0 - i \sum_{l < m} \varphi_{R'_l R'_m}^f + i R \mathbf{q}_0^0 \cdot (1, x_2, \dots, x_M) - i R' \mathbf{q}_0^f \cdot (x_2, \dots, x_M)} = \frac{1}{2} \langle f|c_{1\uparrow}|0\rangle_s^0. \quad (119)$$

The other contribution to this term with $Rq_{m'}^{00}$ can be represented in the thermodynamic limit as a shift of the spin chain to the right by one site, evaluated as a spin part of the matrix element in the zeroth order, and reverse shifted back by one site, giving

$$\begin{aligned} -\frac{1}{2} \frac{1}{Z_f Z_0} \sum_{R, x_2, \dots, x_M} e^{i \sum_{l < m} \varphi_{R_l R_m}^0 - i \sum_{l < m} \varphi_{R'_l R'_m}^f + i R \mathbf{q}_0^0 \cdot (1, x_2, \dots, x_M) - i R' \mathbf{q}_0^f \cdot (x_2, \dots, x_M)} e^{i R q_1^{00}} \\ = -\frac{1}{2} e^{-i(P_s^0 - P_s^f)} \langle f|c_{1\uparrow}|0\rangle_s^0. \end{aligned} \quad (120)$$

The second term in the third line in Eq. (118) has the first part that is independent of $Rq_{m'}^{00}$ and gives $\langle f|c_{1\uparrow}|0\rangle_s^0$ as

$$\begin{aligned} -i \cot \left(\frac{P_s^f - P_s^0}{2} \right) \frac{1}{Z_f Z_0} \sum_{R, R', x_2, \dots, x_M} e^{i \sum_{l < m} \varphi_{R_l R_m}^0 - i \sum_{l < m} \varphi_{R'_l R'_m}^f + i R \mathbf{q}_0^0 \cdot (1, x_2, \dots, x_M) - i R' \mathbf{q}_0^f \cdot (x_2, \dots, x_M)} \\ = -i \cot \left(\frac{P_s^f - P_s^0}{2} \right) \langle f|c_{1\uparrow}|0\rangle_s^0. \end{aligned} \quad (121)$$

In the other contribution $\cos Rq_{m'}^{00}$ can be represented as a sum of two exponential functions,

$$\cos Rq_{m'}^{00} = \frac{e^{i R q_{m'}^{00}} + e^{-i R q_{m'}^{00}}}{2}, \quad (122)$$

and the same trick with the back-and-forth shift of the spin chain as in Eq. (120) applied to each of the two terms above gives

$$\begin{aligned} i \cot \left(\frac{P_s^f - P_s^0}{2} \right) \frac{1}{Z_f Z_0} \sum_{R, R', x_2, \dots, x_M} e^{i \sum_{l < m} \varphi_{R_l R_m}^0 - i \sum_{l < m} \varphi_{R'_l R'_m}^f + i R \mathbf{q}_0^0 \cdot (1, x_2, \dots, x_M) - i R' \mathbf{q}_0^f \cdot (x_2, \dots, x_M)} \\ \times \sum_{m'} \cos Rq_{m'}^{00} \delta_{x_{m'}, 1} = i \cot \left(\frac{P_s^f - P_s^0}{2} \right) \cos (P_s^0 - P_s^f) \langle f|c_{1\uparrow}|0\rangle_s^0. \end{aligned} \quad (123)$$

Together, all the terms without $x_{m'}$ in Eq. (118) give

$$\begin{aligned} \langle \Psi_f^0 | a_1 D_1 S_1^- | \Psi_{cs,0}^1 \rangle = \langle f|c_{1\uparrow}|0\rangle_s^0 P_f \left[\sum_m \left(-2i \frac{1 - e^{i q_m^{00}}}{N} + 4(1 - \cos q_m^{00}) \cot \left(\frac{P_s^f - P_s^0}{2} \right) \right. \right. \\ \left. \left. - 4i(1 - \cos q_m^{00}) \right) - 2i \left(1 - e^{-i(P_s^0 - P_s^f)} \right) + 4 \cot \left(\frac{P_s^f - P_s^0}{2} \right) (\cos (P_s^0 - P_s^f) - 1) \right] \end{aligned} \quad (124)$$

and the term with $x_{m'}$ gives the result in Eq. (105) with $\mathbf{q}_0^1 = 4P_f (\cos \mathbf{q}_0^0 - 1) / N$.

Under substitution of both contributions in Eqs. (114) and (124) into the spin-charge mixing term in linear order in Eq. (110), the whole matrix element in the zeroth order factorises as

$$\langle f|c_{1\uparrow}|0\rangle_{cs}^1 = \langle f|c_{1\uparrow}|0\rangle_s^0 T_{cs}, \quad (125)$$

where

$$T_{cs} = P_f \left[4 \frac{\sum_m \cos q_m^{f0} - \sum_m \cos q_m^{00} + \cos(P_s^0 - P_s^f)}{\tan\left(\frac{P_s^f - P_s^0}{2}\right)} + 2i \frac{\sum_m e^{iq_m^{00}} - \sum_m e^{-iq_m^{f0}}}{N} + 2ie^{i(P_s^f - P_s^0)} \right]. \quad (126)$$

The purely imaginary contributions were omitted since they do not contribute to the linear order of the modulus squared of this matrix element $|\langle f|c_{1\uparrow}|0\rangle|^2$ needed for observables in this work, and the terms were rearranged for compactness. The contributions with $x_{m'}$ are added by shifting the spin quasimomenta in the spin contribution to the linear order in Eq. (109) as

$$\mathbf{q}_{0/f}^1 \rightarrow \mathbf{q}_{0/f}^1 + 4P_f \frac{\cos \mathbf{q}_{0/f}^0 - 1}{N}. \quad (127)$$

The result in Eq. (126) is presented in Eq. (9) of the main text.

The matrix element of the creation operators $c_{k\uparrow}^\dagger$, $\langle f|c_{1\uparrow}^\dagger|0\rangle$, can be expanded in the same way in the Taylor series in t/U . In the zeroth order, the expressions are the same as in Eqs. (73–79) and in the linear order the expressions are the same as in Eqs. (89, 109, 126), in which, in both orders, the quasimomenta are swapped as $\mathbf{k}^f, \mathbf{q}^f \leftrightarrow \mathbf{k}^0, \mathbf{q}^0$ and the numbers of particles and spins are increased by one, $N \rightarrow N + 1$ and $M \rightarrow M + 1$.

IV. t/U CORRECTION TO NORMALISATION FACTOR

We have selected the normalisation factor for the Lieb-Wu wave function in Eq. (1) to be unity in the $U = \infty$ limit, $\langle \Psi^0 | \Psi^0 \rangle = 1$, and have used it to evaluate the linear order of the t/U expansion of the matrix element in the previous section. However, in the t/U expansion calculated in this work, deviations of the normalisation factor from unity also have to be accounted for upto the same linear order,

$$\langle \Psi | \Psi \rangle = 1 + \delta Z \frac{t}{U}. \quad (128)$$

Under substitution of the t/U expansion of the wave function up to the linear order in Eq. (38) in the l.h.s. of the above equation, we obtain the linear coefficient δZ in its r.h.s. as

$$\delta Z = \delta Z_c + \delta Z_{cs} + \delta Z_s, \quad (129)$$

where the three contributions correspond to the three contributions in the linear term of the wave function in Eq. (39).

The first term in Eq. (129), the charge part, has two contributions originating from the linear terms of the bra and ket wave functions of the charge type in Eq. (40), similarly to the charge part of the matrix element in Eq. (81),

$$\delta Z_c = \frac{1}{L^N N!} \sum_{Q, Q', \mathbf{j}} (-1)^{Q+Q'} e^{i(Q\mathbf{k}^0 - Q'\mathbf{k}^0) \cdot \mathbf{j}} (Q\mathbf{k}^1 - Q'\mathbf{k}^1) \cdot \mathbf{j}. \quad (130)$$

The scalar product in the last factor in the summand in the above expression is a sum over N terms with the coordinate of only one particle in each term. The sum over the remaining $N - 1$ coordinates in each term produces a product of $N - 1$ delta-functions in the charge quasimomenta as $\delta(Qk_j - Q'k_j)$ so that the N^{th} charge quasimomenta in both permutation $Q\mathbf{k}$ and $Q'\mathbf{k}$ have to coincide producing, altogether, the δ -function in the whole permutations, $\delta(Q - Q')$, under the sum over the charge coordinates $\mathbf{j}$. Then, the sum over Q' resolves the $\delta(Q - Q')$ making the last factor in the summand and the whole sum over the charge coordinates $\mathbf{j}$ and the permutations Q and Q' in Eq. (130) zero,

$$\delta Z_c = 0. \quad (131)$$

The third term in Eq. (129), the spin part, has two complex-conjugated contributions also originating from the linear terms in the bra and ket wave functions of the spin type in Eq. (42),

$$\delta Z_s = 2\text{Re}B_s, \quad (132)$$

where one of the contributions is

$$B_s = \frac{1}{Z^2} \sum_{R, R', \mathbf{x}} e^{i \sum_{l < m} \varphi_{R_l R_m} - i \sum_{l < m} \varphi_{R'_l R'_m} + i(R\mathbf{q}^0 - R'\mathbf{q}^0) \cdot \mathbf{x}} \left[i2 \sum_{l < m} \frac{Rq_l^1 (1 - \cos Rq_m^0) - Rq_m^1 (1 - \cos Rq_l^0)}{(e^{iRq_l^0} + e^{-iRq_m^0} - 2)(e^{-iRq_l^0} + e^{iRq_m^0} - 2)} + iR\mathbf{q}^1 \cdot \mathbf{x} \right]. \quad (133)$$

The sum over the spin coordinates in the last expression can be expressed through the scalar product of two Bethe states using the trick with the shift by $g\mathbf{q}^1$ and the limit of a derivative in Eqs. (90), similarly to the spin contribution to the linear term of the matrix element in Eq (103),

$$B_s = \lim_{g \rightarrow 0} d_g \langle \mathbf{q}^0 | \mathbf{q}^0 + g\mathbf{q}^1 \rangle. \quad (134)$$

The scalar product of two Bethe states in the last expression can be evaluated using the algebraic Bethe-ansatz method, giving the Slavnov formula in Eq. (66). Taking the $\Delta \rightarrow 1$ limit of this result, as in Eq. (71), and dividing it by the square root of the prefactor in front of the determinant in Eq. (72) for the bra and ket states to account for the conversion of the spin normalisation factor from the algebraic to the coordinate representation, we obtain

$$\langle \mathbf{q}^0 | \mathbf{q}^0 + g\mathbf{q}^1 \rangle = \frac{1}{Z^2} \frac{\prod_{l \neq m}^M \left(\cot \frac{q_l^0 + gq_l^1}{2} - \cot \frac{q_m^0}{2} + 2i \right) \det \hat{Q}}{\prod_{l \neq m}^M \sqrt{\cot \frac{q_l^0 + gq_l^1}{2} - \cot \frac{q_m^0 + gq_m^1}{2} + 2i} \prod_{l \neq m}^M \sqrt{\cot \frac{q_l^0}{2} - \cot \frac{q_m^0}{2} + 2i}}, \quad (135)$$

where the matrix elements of $\hat{Q}$ are

$$Q_{ab} = \frac{i \left(\cot \frac{q_a^0 + gq_a^1}{2} - \cot \frac{q_a^0}{2} + 2i \right) \left(e^{iN(q_b^0 + gq_b^1)} \prod_{l=1 \neq a}^M \frac{\cot \frac{q_b^0 + gq_b^1}{2} - \cot \frac{q_l^0}{2} - 2i}{\cot \frac{q_b^0 + gq_b^1}{2} - \cot \frac{q_l^0}{2} + 2i} - 1 \right)}{2 \sin \frac{q_a^0 + gq_a^1}{2} \sin \frac{q_a^0}{2} \left(\cot \frac{q_b^0 + gq_b^1}{2} - \cot \frac{q_a^0}{2} \right) \left(\cot \frac{q_b^0 + gq_b^1}{2} - \cot \frac{q_a^0}{2} + 2i \right)}. \quad (136)$$

Repeating the steps as for the spin contribution to the linear term of the matrix element, we expand the matrix $\hat{Q}$ in the last expression in a Taylor series in g up to the linear order

$$\hat{Q}_0 + g\hat{Q}_1, \quad (137)$$

as in Eq. (104). Here, however, we need to evaluate the expansion coefficients with a bit more care, as limits of $g \rightarrow 0$, since both numerator and denominator in the diagonal matrix elements Q_{aa} in Eq. (136) become zero for $g = 0$. We evaluate the zeroth-order term in Eq. (137) as

$$\hat{Q}_0 = \lim_{g \rightarrow 0} \hat{Q} \quad (138)$$

and obtain the same matrix as in the Gaudin normalisation factor [6] in Eq. (20). Calculating the coefficient in the linear order as

$$\hat{Q}_1 = \lim_{g \rightarrow 0} d_g \hat{Q}, \quad (139)$$

we obtain

$$Q_{aa}^1 = \frac{q_a^1}{2} \left[\sum_{l \neq a} \frac{\sin^2 \frac{q_l^0}{2} (2 \sin q_a^0 - \sin(q_a^0 + q_l^0))}{\left(\frac{3}{2} + \frac{\cos(q_a^0 + q_l^0)}{2} - \cos q_a^0 - \cos q_l^0 \right)^2} + i \left(N - \sum_{l \neq a} \frac{1 - \cos q_l^0}{\frac{3}{2} + \frac{\cos(q_a^0 + q_l^0)}{2} - \cos q_a^0 - \cos q_l^0} \right)^2 \right], \quad (140)$$

$$Q_{ab}^1 = i \frac{1 - \cos q_b^0}{3 + \cos(q_a^0 + q_b^0) - 2 \cos q_a^0 - 2 \cos q_b^0} \left[q_b^1 \left(N - \frac{1}{2} - \sum_{l \neq a, b} \frac{1 - \cos q_l^0}{\frac{3}{2} + \frac{\cos(q_b^0 + q_l^0)}{2} - \cos q_b^0 - \cos q_l^0} \right) \right. \\ \left. - 2 \frac{e^{i \frac{q_a^0 - 3q_b^0}{2}} \sin \frac{q_a^0}{2} \left(1 + e^{iq_a^0} + e^{iq_b^0} (e^{iq_a^0} - 3) \right)}{\sin \frac{q_b^0}{2} (e^{iq_a^0} + e^{-iq_b^0} - 2)^2} \right) \frac{e^{iq_a^0 + iq_b^0} + 1 - 2e^{iq_b^0}}{e^{iq_a^0} - e^{iq_b^0}} + q_a^1 \frac{1 + i \sin q_a^0}{1 - \cos q_a^0} \right] \quad (141)$$

for the diagonal and off-diagonal $a \neq b$ matrix elements of $\hat{Q}_1$ respectively. In the substitution of the expansion in Eq. (137) into Eq. (134) we use the Jacobi formula for the derivative of a determinant as in Eq. (86). Then we substitute the result for B_s into Eq. (132) and obtain

$$\delta Z_s = 2\text{ReTr} \left(\hat{Q}_0^{-1} \hat{Q}_1 \right), \quad (142)$$

where the linear term in g coming from the Taylor expansion of the prefactor in front of the determinant in Eq. (135) does not contribute since it has zero real part.

The second term in Eq. (129), in which the sums over the spin and charge coordinates are mixed with each other, has two complex-conjugated contributions also originating from the linear terms in the bra and ket wave functions of the spin-charge mixing type in Eq. (41),

$$\delta Z_{sc} = 2\text{Re} B_{sc}. \quad (143)$$

Since $|\Psi_{cs}^1\rangle$ in Eq. (41) is linear in the charge coordinates and in the charge quasimomenta, the sum over the charge coordinates in B_{sc} can be evaluated independently of the spin coordinates, giving the “average” value of the charge quasimomenta over the charge state that depend on the coordinate on the spin chain,

$$\bar{k}_l^0 = \frac{1}{L^N N!} \sum_{j,Q,Q'} (-1)^{Q+Q'} e^{i(Q\mathbf{k}^0 - Q'\mathbf{k}^0) \cdot \mathbf{O}_j} Q k_l^0, \quad (144)$$

which in the thermodynamic limit gives

$$\bar{k}_l^0 = \frac{P}{N} + i \frac{N}{2L} (\delta_{l,1} - \delta_{l,N}). \quad (145)$$

Substitution of the latter into the remaining sum over the spin coordinates gives

$$B_{sc} = -\frac{4i}{Z^2} \sum_{R,R',\mathbf{x}} e^{i \sum_{l < m} \varphi_{R_l R_m} - i \sum_{l < m} \varphi_{R'_l R'_m} + i(R\mathbf{q}_0 - R'\mathbf{q}_0) \cdot \mathbf{x}} i \sum_{m'} \frac{N}{2L} (1 - \cos R q_{m'}^0) \quad (146)$$

where δ -functions originating from Eq. (145) are resolved by the sums in Eq. (41) and the thermodynamic limit is also taken, similarly to the spin-charge mixing part in linear order of the matrix element in Eq. (113). The sum over m' in the last expression can be taken outside the sum over the permutations R , since under the sum over m' it does not depend on R . Then, the remaining sum over the spin coordinates is the normalisation factor Z^2 that cancels the Z^2 in the denominator, and we obtain

$$B_{sc} = -\frac{2N}{L} \sum_m (\cos q_m^0 - 1). \quad (147)$$

Substitution of this expression back into Eq. (143) gives the spin-charge mixing contribution as

$$\delta Z_{sc} = -\frac{4N}{L} \sum_m (\cos q_m^0 - 1). \quad (148)$$

Substitution of the three results in Eqs. (131, 142, 148) into Eq. (129) gives the linear order of the t/U expansion of the normalisation factor as

$$\delta Z = 2\text{ReTr}(\hat{Q}_0^{-1} \hat{Q}_1) - \frac{4N}{L} \sum_m (\cos q_m^0 - 1). \quad (149)$$

The result in Eq. (149) is presented in Eqs. (10) of the main paper.

V. TRANSPORT THEORY FOR THE FREE SYSTEM

In order to assess expectations for the conductance measured in the transport-spectroscopy experiment in the nonlinear regime, we present here the corresponding transport theory for the noninteracting 1D system [18]. Then

we compare its prediction for the nonlinear charge mode in the particle sector with the data measured for our wires with Coulomb interactions.

The current between the two wells in the weak-tunneling regime for the device in the inset in Fig. 3A of the main paper is given by the convolution of the two spectral functions as [19]

$$I(B, V_{\text{dc}}) = \int d^2k dE (f_T^{\text{UW}}(E - eV_{\text{dc}}) - f_T^{\text{LW}}(E)) A_{\text{UW}}(\mathbf{k}, E) A_{\text{LW}}\left(\mathbf{k} + \frac{ed}{\hbar}(\mathbf{n} \times \mathbf{B}), E - eV_{\text{dc}}\right), \quad (150)$$

where $A_{\text{UW/LW}}(\mathbf{k}, E)$ and $f_T^{\text{UW/LW}}(E)$ are the spectral and Fermi functions for the upper/lower wells (UW/LW), $-e$ is the electron charge, d is the distance between the centers of the wells, and $\mathbf{n} = \hat{\mathbf{z}}$ is the normal to the 2D plane of the wells. When a DC bias V_{dc} is applied between the wells, the energy offset acquired between the two electronic systems is eV_{dc} and an in-plane magnetic field $\mathbf{B} = -B\hat{\mathbf{y}}$ shifts the momentum k in the tunneling process due to the Lorentz force by edB in the x -direction.

The 2D electrons in the lower well are described by the spectral function of the Fermi gas as

$$A_{\text{LW}}(\mathbf{k}, E) = \frac{1}{\pi} \frac{\Gamma}{\Gamma^2 + \left(E - \frac{\hbar^2(k - k_{\text{F}}^{2\text{D}})^2}{2m_{2\text{D}}^*}\right)^2}, \quad (151)$$

where $k_{\text{F}}^{2\text{D}}$ is its Fermi momentum, $m_{2\text{D}}^* = 0.062m_e$ [20] is the Fermi-liquid's effective electron mass, m_e is the free electron mass, and Γ is the width of the inhomogeneous broadening which we assume to be larger than the interaction broadening. The 1D electrons in the upper well are described by the spectral function of a Fermi system without interactions,

$$A_{\text{UW}}(\mathbf{k}, E) = \frac{1}{\pi} \frac{\Gamma}{\Gamma^2 + \left(E - \frac{\hbar^2(k_x - k_{\text{F}}^{1\text{D}})^2}{2m_0}\right)^2}, \quad (152)$$

with the same inhomogeneous broadening Γ and a free mass m_0 .

The numerical evaluation of the integrals in Eq. (150) with the spectral functions in Eqs. (151, 152) for the parameters comparable with our semiconductor experiment ($m_0 = 0.93m_e$, $\Gamma = 0.3\text{ meV}$, $k_{\text{F}}^{1\text{D}} = 73.6\text{ }\mu\text{m}^{-1}$, and $k_{\text{F}}^{2\text{D}} = 99.7\text{ }\mu\text{m}^{-1}$) is presented in Figs. 1D-F as the differential conductance $G = dI/dV_{\text{dc}}$ and its derivatives dG/dB and dG/dV_{dc} . We observe that a symmetrical line width in the 1D spectral function without interaction in Eq. (152) does not lead to any perceptible asymmetry of the 1D line in G in the charge sector enclosed by the olive-yellow dashed line. On the other hand, the measured conductance G for electrons with the Coulomb interaction, see Fig. 1A-C, shows a large asymmetry of this line, which is a manifestation of the large continuum of nonlinear many-body excitations predicted in Fig. 2A of the main paper around this line in the particle sector.

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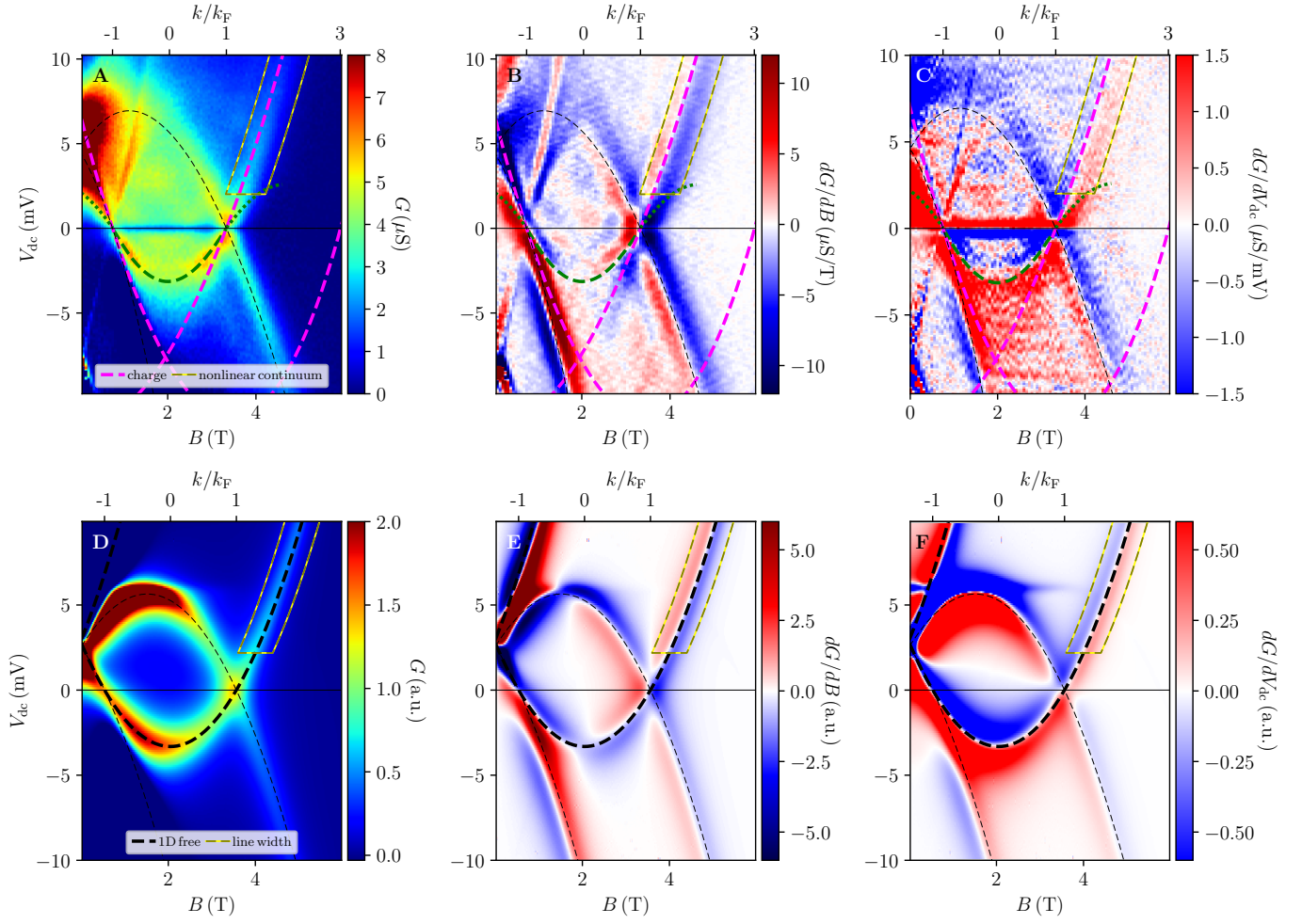


Figure 1. **A** The conductance $G(B, V_{dc})$ measured for $V_{FG} = -664$ mV. The lines are the same as in Fig. 3A of the main paper. **B** and **C** are the dG/dB and dG/dV_{dc} derivatives of the conductance in **A**. **D** Numerical evaluation of the conductance for the transport theory for the non-interacting 1D system in Eqs. (150, 151, 152) using $m_0 = 0.93m_e$, $\Gamma = 0.3$ meV, $k_F^{1D} = 73.6 \mu\text{m}^{-1}$, and $k_F^{2D} = 99.7 \mu\text{m}^{-1}$. The additional thick black dashed line is the 1D dispersion in Eq. (152). **E** and **F** are the dG/dB and dG/dV_{dc} derivatives of the conductance in **D**.

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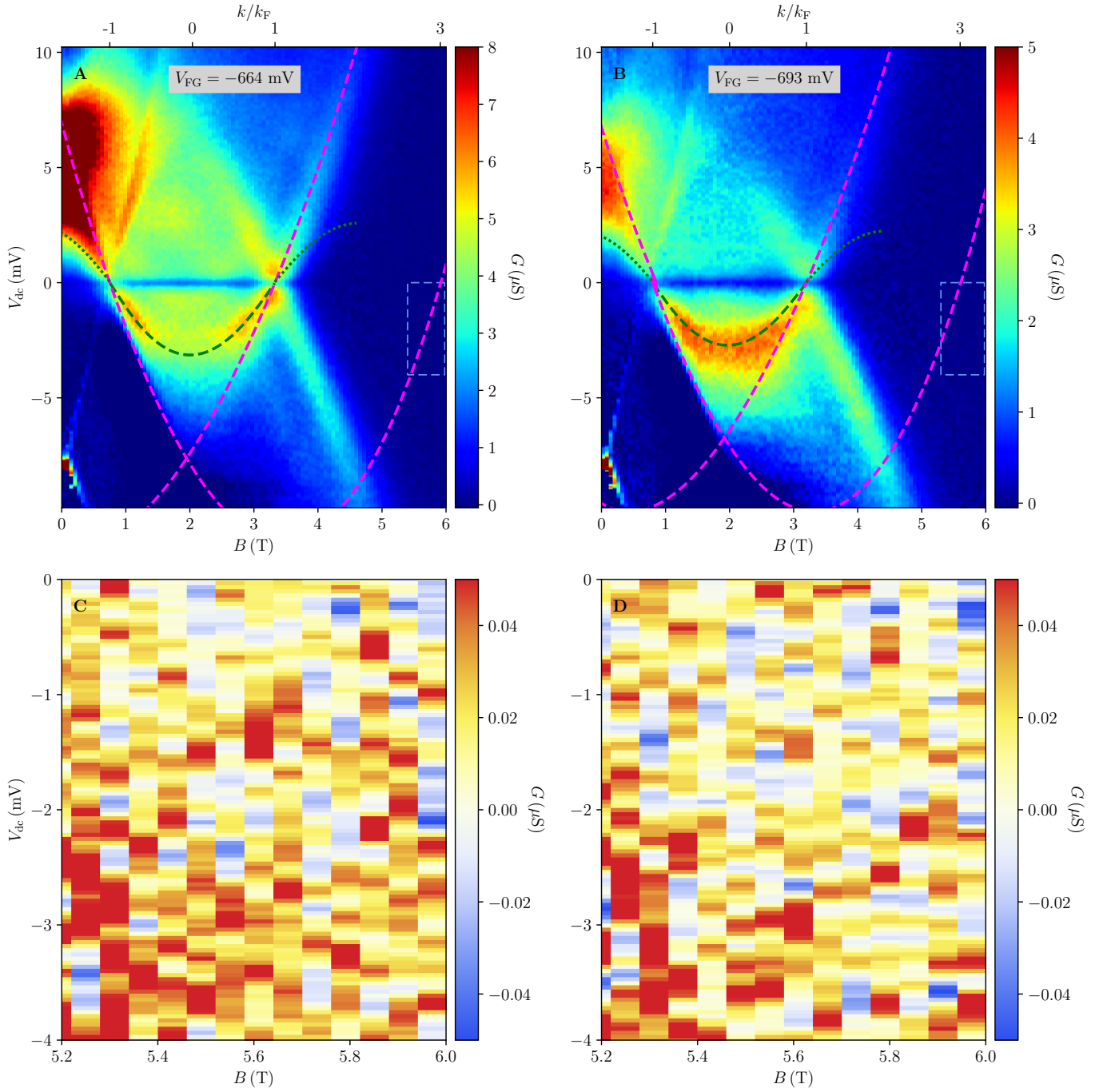


Figure 2. **A** and **B** are the conductance $G(B, V_{dc})$ measured for $V_{FG} = -664$ mV and $V_{FG} = -693$ mV. The green and magenta dashed lines are the dispersions of the spin and charge modes obtained from the full Lieb-Wu equations for $\gamma = 1.25$ in **A** and for $\gamma = 1.30$ in **B**, and were corrected for capacitance using $c_{UL} = 5.6 \text{ mFm}^{-2}$ and $c_{UW} = 4.7 \text{ mFm}^{-2}$, see details in [21], for both applied V_{FG} . The upper horizontal axis is the linear transformation of B using the two crossing points with the $V_{dc} = 0$ line as $B_{lo} = 0.75$ T is $-k_F$ and $B_{hi} = 3.33$ T is k_F in **A** and $B_{lo} = 0.80$ T is $-k_F$ and $B_{hi} = 3.20$ T is k_F in **B**. **C** and **D** are the zoomed-in conductances in the light-blue dashed rectangle in **A** and **B** respectively. The noise floor in both measurements is $|G| \sim 0.05 \mu S$.