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The two-mass contributions to the three-loop massive operator matrix elements $\tilde{A}_{Qg}^{(3)}$ and $\Delta\tilde{A}_{Qg}^{(3)}$

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Abstract

We calculate the two-mass three-loop contributions to the unpolarized and polarized massive operator matrix elements $\tilde{A}_{Qg}^{(3)}$ and $\Delta\tilde{A}_{Qg}^{(3)}$ in x-space for a general mass ratio by using a semi-analytic approach. We also compute Mellin moments up to $N = 2000(3000)$ by an independent method, to which we compare the results in x-space. In the polarized case, we work in the Larin scheme. We present numerical results. The two-mass contributions amount to about 50% of the full $O(T_F^2)$ and $O(T_F^3)$ terms contributing to the operator matrix elements. The present result completes the calculation of all unpolarized and polarized massive three-loop operator matrix elements.

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1 Introduction

The massive operator matrix elements (OMEs) of local composite operators in Quantum Chromodynamics (QCD) are of central importance to describe the heavy-flavor contributions to the massive Wilson coefficients in deep-inelastic scattering in the single-mass case in the asymptotic region $Q^2 \gg m_Q^2$ [1–12], where m_Q denotes a heavy quark mass. From two-loop order onward, two-mass effects contribute [4, 13]. The three-loop corrections have been calculated analytically in Refs. [14–18], except for the OMEs $\tilde{A}_{Qg}^{(3)}$ and $\Delta\tilde{A}_{Qg}^{(3)}$. The further transition functions in the single-mass variable flavor number scheme (VFNS) were calculated in Refs. [19–24]. In the polarized case, we work in the Larin scheme [25].

The two-mass effects require to extend the single heavy mass VFNS [19, 26] to the two-mass one [14]. This is phenomenologically motivated since the mass ratio of the charm and the bottom quarks

$$\eta = \frac{m_c^2}{m_b^2} \sim \frac{1}{9} \tag{1.1}$$

is not a very small quantity.

The moments $N = 2, 4, 6$ for the constant part of the unrenormalized OMEs $\tilde{A}_{Qg}^{(3)}$, $\tilde{a}_{Qg}^{(3)}$, have been calculated in [27, 28] expanding in the parameter η with the code `Q2E/Exp` [29, 30]. In the expanded form, much higher Mellin moments have been calculated in Ref. [31]. An analytic conversion of these results to x-space cannot be performed straightforwardly since with higher powers in η the number of N-dependent terms in the expansion grows. Therefore, we will compute the two-mass corrections in x-space directly. Both, in N- and x-space, elliptic contributions are present, as it is already the case for the single-mass amplitudes, cf. Refs. [11, 12,32,33]. Because of this we will calculate the quantities $\tilde{A}_{Qg}^{(3)}$ and $\Delta\tilde{A}_{Qg}^{(3)}$ by using a semi-analytic method, see also Refs. [11, 12, 34–36].

The paper is organized as follows. In Section 2, we present a series of specific aspects for the renormalization of the OME $(\Delta)\tilde{A}_{Qg}^{(3)}$ in the two-mass case. We describe the main steps of the calculation in Section 3. The calculation of the two-mass OME $(\Delta)\tilde{A}_{Qg}^{(3)}$ in Mellin-N space is described in Section 4. We also discuss the limitations of this approach for the present OME.[2] In Section 5, we compute the constant part of the unrenormalized OMEs $\tilde{A}_{Qg}^{(3)}$ and $\Delta\tilde{A}_{Qg}^{(3)}$ in x-space, based on the coupled system of first order differential equations obtained for the master integrals. In the generating function representation [37, 38] they depend on the variable t emerging in the linear propagators introduced to perform the integration-by-parts (IBP) reduction [39, 40]. Section 6 contains the conclusions. In the Appendix, we list a series of moments for $\tilde{a}_{Qg}^{(3)}$ and $\Delta\tilde{a}_{Qg}^{(3)}$ both expanded and unexpanded in the mass ratio η.

2 Renormalization

2.1 General Formalism

The renormalization of the operator matrix elements for deep-inelastic scattering up to $O(a_s^3)$ has been carried out in the single-mass case in Ref. [41] and in the two-mass case in Ref. [14].

[2] Also for the pure-singlet two-mass OMEs $\tilde{A}_{Qq}^{(3),\mathrm{PS}}$ and $\Delta\tilde{A}_{Qq}^{(3),\mathrm{PS}}$ [15,17] it turned out that the N-space solution was not possible, since the corresponding difference equations did not factorize at first order, unlike the differential equations in x-space.

Here $a_s = \alpha_s/(4\pi) = g_s^2/(16\pi^2)$ denotes the strong coupling constant. In the following, we collect the essential steps for the renormalization of the OME A_{Qg} to three–loop order. In general, the renormalization in the two-mass case has to include that of the single-mass cases. Later on, the pure two-mass part of the renormalized OME is separated. Some steps of the renormalization in [14] were given for the power-expanded form in the parameter η since the full analytic results had not been available yet.

The Feynman integrals contributing to the various operator matrix elements contain mass, coupling, ultraviolet operator singularities, as well as collinear divergences, due to massless sub-graphs. They are regularized in $D = 4 + \varepsilon$ dimensions. The singularities appear as poles in the Laurent series in ε, with the highest pole corresponding to the loop order. At one and two-loop order, there are no irreducible contributions to the OMEs $(\Delta)\tilde{A}_{ij}$, cf. Refs. [1,3,8,19,20,42–44]. The first single-particle irreducible diagrams with two masses emerge at $O(a_s^3)$. In the following, we consider the renormalization of the two-mass contributions in individual terms together with the genuine two-mass contributions. The latter terms will then be obtained subtracting the former ones, cf. Ref. [41]. The unrenormalized OMEs are given by

$$\hat{\hat{A}}_{ij}^{(l)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) = \hat{\hat{A}}_{ij}^{(l)}\left(\frac{m_1^2}{\mu^2}\right) + \hat{\hat{A}}_{ij}^{(l)}\left(\frac{m_2^2}{\mu^2}\right) + \hat{\hat{\tilde{A}}}_{ij}^{(l)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right), \tag{2.1}$$

where $\hat{\hat{A}}_{ij}^{(l)}\left(\frac{m_i^2}{\mu^2}\right)$ are the single-mass OMEs [41] and $\hat{\hat{\tilde{A}}}_{ij}^{(l)}$ are the two-mass contributions. A change in the renormalization scheme like in Eqs. (2.40) and (2.41) generally introduces a mixing between the different components of Eq. (2.1). Here we introduced the two masses m_1 and m_2. We set $m_2 < m_1$ so that $\eta < 1$. In the following we will also refer to the logarithms

$$L_1 = \ln\left(\frac{m_1^2}{\mu^2}\right), \qquad L_2 = \ln\left(\frac{m_2^2}{\mu^2}\right), \qquad L_\eta = \ln(\eta) = L_2 - L_1, \tag{2.2}$$

where μ is the renormalization scale.

The renormalization procedure follows the one outlined in Ref. [41], incorporating the necessary modifications for the two-mass case. Here the case of N_F massless and two massive quark flavors is considered as this covers the physical case of contributions due to the charm and bottom quarks. The large mass gap to the top quark in general allows to decouple it after the charm and the bottom quark and thus does not have to be included into a scheme with N_F massless and three massive quarks. Since in this chapter again two massive quarks are considered, we will use the notation

$$\tilde{f}(\xi) = \frac{f(\xi)}{\xi}, \tag{2.3}$$

$$\hat{f}(\xi) = f(\xi + 2) - f(\xi) \tag{2.4}$$

to abbreviate certain expressions. Note that these notations are not meant to apply to OMEs, but only to the anomalous dimensions γ_{ij} in this section. The differences with respect to [14] mainly lie in the use of Eq. (2.4) instead of the convention with one heavy quark. Furthermore, the notation $\hat{\hat{f}}(x)$ was used to represent the coefficient of the N_F^2 dependent term of certain anomalous dimensions. However, this is not in accordance with the conventions for anomalous dimensions which start with N_F^0. These terms will be therefore denoted by $f^{N_F^2}$ in the following. Also the inclusion of reducible contributions introduces some subtleties for OMEs with external gluons. The renormalization concerns the heavy quark masses, the strong coupling [45–61], the local operators [5–7,21,22,62–79], and the subtraction of the collinear singularities due to massless sub-graphs [41], which occurs at one-loop order in the present case, unlike the single-mass OMEs.

The schemes most frequently used for the mass renormalization are the $\overline{\rm MS}$- and the on-mass shell scheme (OMS). In the following, the mass is renormalized in the OMS and the finite renormalization to switch to the $\overline{\rm MS}$-mass is provided at a later stage. The mass renormalization is applied first, i.e., the respective expressions are still containing the bare coupling $\hat{a}_s = \hat{g}_s^2/(4\pi)^2$.[3]

The bare masses $\hat{m}_i,\ i \in \{1,2\}$ are expressed by the renormalized on-shell masses m_i via

$$\hat{m}_i = Z_{m,i}(m_1,m_2)\ m_i = m_i\Big[1 + \hat{a}_s\Big(\frac{m_i^2}{\mu^2}\Big)^{\varepsilon/2}\delta m_1 + \hat{a}_s^2\Big(\frac{m_i^2}{\mu^2}\Big)^{\varepsilon}\delta m_{2,i}\,(m_1,m_2)\Big] + O(\hat{a}_s^3)\ , \quad (2.5)$$

and

$$\delta m_{2,i}\,(m_1,m_2) = \delta m_2^0 + \tilde{\delta} {m_2}^i(m_1,m_2)\ . \quad (2.6)$$

Here δm_2^0 is the single mass-contribution, whereas $\tilde{\delta}{m_2}^i$ denotes the additional contribution emerging in the case of two massive flavors. Note that from order $O(\hat{a}_s^2)$ onward the Z-factor renormalizing $\hat{m}_1$ depends on m_2 and vice versa. For the massive operator matrix elements this can be observed at 3-loop order for the first time. The coefficients δm_1 and δm_2 have been derived in Refs. [80, 81] up to $O(\varepsilon^0)$ and $O(\varepsilon^{-1})$, respectively. The constant part of δm_2 was given in Refs. [46, 82, 83] and the $O(\varepsilon)$-term of δm_1 in Ref. [41]. One obtains

$$\begin{aligned}
\delta m_1 &= C_F\left[\frac{6}{\varepsilon} - 4 + \left(4 + \frac{3}{4}\zeta_2\right)\varepsilon\right] && (2.7)\\
&\equiv \frac{\delta m_1^{(-1)}}{\varepsilon} + \delta m_1^{(0)} + \delta m_1^{(1)}\varepsilon\ , && (2.8)\\
\delta m_2^0 &= C_F\Bigg[\frac{1}{\varepsilon^2}\left(18C_F - 22C_A + 8T_F(N_F+1)\right) + \frac{1}{\varepsilon}\Bigg(-\frac{45}{2}C_F + \frac{91}{2}C_A \\
&\quad -14T_F(N_F+1)\Bigg) + C_F\left(\frac{199}{8} - \frac{51}{2}\zeta_2 + 48\ln(2)\zeta_2 - 12\zeta_3\right) + C_A\Bigg(-\frac{605}{8} \\
&\quad +\frac{5}{2}\zeta_2 - 24\ln(2)\zeta_2 + 6\zeta_3\Bigg) + T_F\left[N_F\left(\frac{45}{2} + 10\zeta_2\right) + \frac{69}{2} - 14\zeta_2\right]\Bigg] && (2.9)\\
&\equiv \frac{\delta m_2^{0,(-2)}}{\varepsilon^2} + \frac{\delta m_2^{0,(-1)}}{\varepsilon} + \delta m_2^{0,(0)}\ , && (2.10)\\
\tilde{\delta}{m_2}^i(m_1,m_2) &= C_F T_F\Bigg\{\frac{8}{\varepsilon^2} - \frac{14}{\varepsilon} + 8r_i^4{\rm H}_0^2(r_i) - 8(r_i+1)^2\left(r_i^2 - r_i + 1\right){\rm H}_{-1,0}(r_i) \\
&\quad +8(r_i-1)^2\left(r_i^2 + r_i + 1\right){\rm H}_{1,0}(r_i) + 8r_i^2{\rm H}_0(r_i) + \frac{3}{2}\left(8r_i^2 + 15\right) \\
&\quad +2\Big[4r_i^4 - 12r_i^3 - 12r_i + 5\Big]\zeta_2\Bigg\} && (2.11)\\
&\equiv \frac{\tilde{\delta}{m_2}^{(-2)}}{\varepsilon^2} + \frac{\tilde{\delta}{m_2}^{(-1)}}{\varepsilon} + \tilde{\delta}{m_2}^{i,(0)}\ , && (2.12)
\end{aligned}$$

cf. Ref. [83], $i \in \{1,2\}$ and

$$r_1 = \sqrt{\eta} \quad \text{and} \quad r_2 = \frac{1}{\sqrt{\eta}}. \quad (2.13)$$

[3]Note that this notation therefore agrees with Ref. [83], but, e.g., differs form the notation in [45,84,85], where also the charge renormalization has been carried out.

The superscript i for the coefficients $\tilde{\delta} m_2^{(-2)}$ and $\tilde{\delta} m_2^{(-1)}$ has been dropped as they are independent of the renormalized mass m_i. The color factors of QCD are given by $C_A = N_c = 3$, $C_F = (N_c^2-1)/(2N_c) = 4/3$, $T_F = 1/2$. The constants $\zeta_n,\ \ n \geq 2$, denote the Riemann ζ function [86,87] evaluated at integer argument n,

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \quad n \geq 2, \quad n \in \mathbb{N}. \tag{2.14}$$

The harmonic polylogarithms [88] used to express the result in Eq. (2.11) are iterated integrals of the kind

$$\mathrm{H}_{b,\vec{a}}(x) = \int_0^x dy f_b(y) \mathrm{H}_{\vec{a}}(y), \quad f_b(x) \in \left\{\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}\right\}, a_i, b \in \{-1, 0, 1\} \tag{2.15}$$

and if all k entries of the indices in H are zero, one defines the harmonic polylogarithm by $\ln^k(x)/k!$.

Applying Eq. (2.5) we obtain the mass renormalized operator matrix elements by

$$\begin{aligned}
\hat{\hat{A}}_{ij}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, \varepsilon, N\right) \quad = \quad & \delta_{ij} + \hat{a}_s \ \hat{\hat{A}}_{ij}^{(1)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, \varepsilon, N\right) + \hat{a}_s^2 \Biggl\{ \hat{\hat{A}}_{ij}^{(2)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, \varepsilon, N\right) \\
& + \delta m_1 \left[\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} m_1 \frac{d}{dm_1} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2} m_2 \frac{d}{dm_2}\right] \hat{\hat{A}}_{ij}^{(1)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, \varepsilon, N\right) \Biggr\} \\
& + \hat{a}_s^3 \Biggl\{ \hat{\hat{A}}_{ij}^{(3)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, \varepsilon, N\right) \\
& + \delta m_1 \left[\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} m_1 \frac{d}{dm_1} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2} m_2 \frac{d}{dm_2}\right] \hat{\hat{A}}_{ij}^{(2)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, \varepsilon, N\right) \\
& + \delta m_{2,1}(m_1, m_2) \left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon} m_1 \frac{d}{dm_1} \hat{\hat{A}}_{ij}^{(1)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, \varepsilon, N\right) \\
& + \delta m_{2,2}(m_1, m_2) \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon} m_2 \frac{d}{dm_2} \hat{\hat{A}}_{ij}^{(1)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, \varepsilon, N\right) \\
& + \frac{(\delta m_1)^2}{2} \left[\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon} m_1^2 \frac{d^2}{{dm_1}^2} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon} m_2^2 \frac{d^2}{{dm_2}^2}\right] \hat{\hat{A}}_{ij}^{(1)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, \varepsilon, N\right) \\
& + (\delta m_1)^2 \left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2} m_1 \frac{d}{dm_1} m_2 \frac{d}{dm_2} \hat{\hat{A}}_{ij}^{(1)}\left(\frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}, \varepsilon, N\right) \Biggr\} ,
\end{aligned} \tag{2.16}$$

which generalizes Eq. (3.10) of Ref. [41]. The OMEs are symmetric under the interchange of the masses m_1 and m_2.

We now turn to the renormalization of the strong coupling constant. It is important to note that the factorization relation for the OMEs strictly requires the external massless partonic legs of the operator matrix elements to be on-shell, i.e.

$$p^2 = 0 \ , \tag{2.17}$$

with p the external momentum. This condition would be violated by naively applying massive loop corrections to the gluon propagator. Following [41] it is possible to absorb these corrections uniquely into the coupling constant by using the background field method [89–91] to maintain the Slavnov-Taylor identities of QCD. In this way, one first obtains the coupling constant in a MOM-scheme. A finite renormalization to transform to the $\overline{\rm MS}$-scheme is applied subsequently.

The light flavor contributions to the unrenormalized coupling constant in terms of the renormalized coupling constant in the $\overline{\rm MS}$-scheme read

$$
\begin{aligned}
\hat{a}_s &= Z_g^{\overline{\rm MS}^2}(\varepsilon, N_F) a_s^{\overline{\rm MS}}(\mu^2) \\
&= a_s^{\overline{\rm MS}}(\mu^2)\left[1 + \delta a_{s,1}^{\overline{\rm MS}}(N_F) a_s^{\overline{\rm MS}}(\mu^2) + \delta a_{s,2}^{\overline{\rm MS}}(N_F) a_s^{\overline{\rm MS}^2}(\mu)\right] + O(a_s^{\overline{\rm MS}^3}) \;. \qquad (2.18)
\end{aligned}
$$

Here the coefficients $\delta a_{s,i}^{\overline{\rm MS}}(N_F)$ are given by

$$
\begin{aligned}
\delta a_{s,1}^{\overline{\rm MS}}(N_F) &= \frac{2}{\varepsilon}\beta_0(N_F) \;, \qquad (2.19)\\
\delta a_{s,2}^{\overline{\rm MS}}(N_F) &= \frac{4}{\varepsilon^2}\beta_0^2(N_F) + \frac{1}{\varepsilon}\beta_1(N_F), \qquad (2.20)
\end{aligned}
$$

with $\beta_k(N_F)$ the expansion coefficients of the QCD β-function [92–96]

$$
\begin{aligned}
\beta_0(N_F) &= \frac{11}{3}C_A - \frac{4}{3}T_F N_F \;, \qquad (2.21)\\
\beta_1(N_F) &= \frac{34}{3}C_A^2 - 4\left(\frac{5}{3}C_A + C_F\right) T_F N_F \;. \qquad (2.22)
\end{aligned}
$$

The renormalized gluon self-energy Π can be split into the purely light and the heavy-flavor contributions, Π_L and Π_H,

$$
\Pi\left(p^2, m_1^2, m_2^2\right) = \Pi_L\left(p^2\right) + \Pi_H\left(p^2, m_1^2, m_2^2\right) \;. \qquad (2.23)
$$

The heavy quarks are required to decouple from the running coupling constant and the renormalized OMEs for $\mu^2 < m_1^2, m_2^2$, which implies [1]

$$
\Pi_H(0, m_1^2, m_2^2) = 0 \;. \qquad (2.24)
$$

Applying the background field method has the advantage of producing gauge-invariant results also for off-shell Green's functions, [89, 97]. We obtain for the heavy-flavor contributions to the unrenormalized gluon polarization function

$$
\begin{aligned}
\hat{\Pi}^{\mu\nu}_{H,ab,\rm BF}(p^2, m_1^2, m_2^2, \mu^2, \varepsilon, \hat{a}_s) &= i(-p^2 g^{\mu\nu} + p^\mu p^\nu)\delta_{ab}\hat{\Pi}_{H,\rm BF}(p^2, m_1^2, m_2^2, \mu^2, \varepsilon, \hat{a}_s) \;, \qquad (2.25)\\
\hat{\Pi}_{H,\rm BF}(0, m_1^2, m_2^2, \mu^2, \varepsilon, \hat{a}_s) &= \hat{a}_s \frac{2\beta_{0,Q}}{\varepsilon}\left[\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2}\right] \exp\Bigl(\sum_{i=2}^{\infty}\frac{\zeta_i}{i}\left(\frac{\varepsilon}{2}\right)^i\Bigr) \\
&\quad + \hat{a}_s^2\left[\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon}\right]\Biggl[\frac{1}{\varepsilon}\Bigl(-\frac{20}{3}T_F C_A - 4T_F C_F\Bigr) \\
&\quad - \frac{32}{9}T_F C_A + 15 T_F C_F \\
&\quad + \varepsilon\Bigl(-\frac{86}{27}T_F C_A - \frac{31}{4}T_F C_F - \frac{5}{3}\zeta_2 T_F C_A - \zeta_2 T_F C_F\Bigr) \\
&\quad + 2\left(\frac{2\beta_{0,Q}}{\varepsilon}\right)^2 \left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2}\left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2} \exp\Bigl(2\sum_{i=2}^{\infty}\frac{\zeta_i}{i}\left(\frac{\varepsilon}{2}\right)^i\Bigr)\Biggr]
\end{aligned}
$$

$$+O(\hat{a}_s^3) \ , \tag{2.26}$$

where the masses m_1 and m_2 have been renormalized in the on-shell scheme, cf. Eq. (2.5). In order to write the relation in Eq. (2.26) in a more compact form, the following notation

$$f(\varepsilon) \equiv \left[\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2}\right] \exp\left[\sum_{i=2}^{\infty} \frac{\zeta_i}{i}\left(\frac{\varepsilon}{2}\right)^i\right] \ , \tag{2.27}$$

is used. The expression $f(\varepsilon)$ is kept unexpanded in the dimensional regularization parameter ε for the moment. Furthermore, the contributions to the QCD β-function coefficients are denoted by $\beta_{i,Q}^{(j)}$ [1, 41, 92–96]

$$\beta_{0,Q} = -\frac{4}{3} T_F \ , \tag{2.28}$$

$$\beta_{1,Q} = -4\left(\frac{5}{3} C_A + C_F\right) T_F \ , \tag{2.29}$$

$$\beta_{1,Q}^{(1)} = -\frac{32}{9} T_F C_A + 15 T_F C_F \ , \tag{2.30}$$

$$\beta_{1,Q}^{(2)} = -\frac{86}{27} T_F C_A - \frac{31}{4} T_F C_F - \zeta_2 \left(\frac{5}{3} T_F C_A + T_F C_F\right) \ . \tag{2.31}$$

Eq. (2.26) differs from the sum of the two individual single-mass contributions [41] by the last term only, which is due to additional reducible Feynman diagrams in the cases of two heavy quark flavors of different mass.

The background field is renormalized using the Z-factor Z_A which is split into light and heavy quark contributions, $Z_{A,L}$ and $Z_{A,H}$. It is related to the Z-factor renormalizing the coupling constant g via

$$Z_g = Z_A^{-\frac{1}{2}} = \frac{1}{(Z_{A,L} + Z_{A,H})^{1/2}} \ . \tag{2.32}$$

Concerning the light flavors, we require the renormalization to correspond to the $\overline{\text{MS}}$-scheme with N_F light flavors

$$Z_{A,l}(N_F) = Z_g^{\overline{\text{MS}}^{1/2}} \ . \tag{2.33}$$

The heavy-flavor contributions are fixed by condition (2.24), which implies

$$\Pi_{H,\text{BF}}(0, \mu^2, a_s, m_1^2, m_2^2) + Z_{A,H} \equiv 0 \ . \tag{2.34}$$

The Z-factor in the MOM-scheme is read off by combining Eqs. (2.32),(2.24),(2.26) and (2.34)

$$Z_g^{\text{MOM}}(\varepsilon, N_F + 2, \mu, m_1^2, m_2^2) \equiv \frac{1}{(Z_{A,l} + Z_{A,H})^{1/2}} \ . \tag{2.35}$$

Up to $O(a_s^{\text{MOM}^3})$, one obtains the renormalization constant

$$\begin{aligned} Z_g^{\text{MOM}^2}(\varepsilon, N_F + 2, \mu, m_1^2, m_2^2) &= 1 + a_s^{\text{MOM}}(\mu^2)\Big[\frac{2}{\varepsilon}(\beta_0(N_F) + \beta_{0,Q} f(\varepsilon))\Big] \\ &\quad + a_s^{\text{MOM}^2}(\mu^2)\Big[\frac{\beta_1(N_F)}{\varepsilon} + \frac{4}{\varepsilon^2}(\beta_0(N_F) + \beta_{0,Q} f(\varepsilon))^2 \end{aligned}$$

$$
+\frac{1}{\varepsilon}\left(\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon}+\left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon}\right)\left(\beta_{1,Q}+\varepsilon\beta_{1,Q}^{(1)}+\varepsilon^2\beta_{1,Q}^{(2)}\right)\Big]
$$
$$
+O(a_s^{\mathrm{MOM}^3}) \; . \tag{2.36}
$$

.

The coefficients of the MOM-scheme Z-factor, $\delta a_{s,1}^{\mathrm{MOM}}$ and $\delta a_{s,2}^{\mathrm{MOM}}$, are defined analogously to those of the $\overline{\mathrm{MS}}$-coefficients in Eq. (2.18)

$$
\delta a_{s,1}^{\mathrm{MOM}} = \frac{2\beta_0(N_F)}{\varepsilon}+\frac{2\beta_{0,Q}}{\varepsilon}f(\varepsilon) \; , \tag{2.37}
$$
$$
\begin{aligned}
\delta a_{s,2}^{\mathrm{MOM}} &= \frac{\beta_1(N_F)}{\varepsilon}+\left(\frac{2\beta_0(N_F)}{\varepsilon}+\frac{2\beta_{0,Q}}{\varepsilon}f(\varepsilon)\right)^2 \\
&\quad +\frac{1}{\varepsilon}\left(\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon}+\left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon}\right)\left(\beta_{1,Q}+\varepsilon\beta_{1,Q}^{(1)}+\varepsilon^2\beta_{1,Q}^{(2)}\right)+O(\varepsilon^2) \; .
\end{aligned} \tag{2.38}
$$

Finally, we express our results in the $\overline{\mathrm{MS}}$-scheme. For this transition the decoupling of the heavy quark flavors is assumed. The transformation to the $\overline{\mathrm{MS}}$ scheme is then implied by

$$
Z_g^{\overline{\mathrm{MS}}^2}(\varepsilon,N_F+2)a_s^{\overline{\mathrm{MS}}}(\mu^2)=Z_g^{\mathrm{MOM}^2}(\varepsilon,N_F+2,\mu,m_1^2,m_2^2)a_s^{\mathrm{MOM}}(\mu^2) \; . \tag{2.39}
$$

Solving Eq. (2.39) perturbatively one obtains

$$
\begin{aligned}
a_s^{\mathrm{MOM}} &= a_s^{\overline{\mathrm{MS}}}-\beta_{0,Q}\left(\ln\left(\frac{m_1^2}{\mu^2}\right)+\ln\left(\frac{m_2^2}{\mu^2}\right)\right)a_s^{\overline{\mathrm{MS}}^2}+\Bigg[\beta_{0,Q}^2\left(\ln\left(\frac{m_1^2}{\mu^2}\right)+\ln\left(\frac{m_2^2}{\mu^2}\right)\right)^2 \\
&\quad -\beta_{1,Q}\left(\ln\left(\frac{m_1^2}{\mu^2}\right)+\ln\left(\frac{m_2^2}{\mu^2}\right)\right)-2\beta_{1,Q}^{(1)}\Bigg]a_s^{\overline{\mathrm{MS}}^3}+O\left(a_s^{\overline{\mathrm{MS}}^4}\right) \; ,
\end{aligned} \tag{2.40}
$$

or,

$$
\begin{aligned}
a_s^{\overline{\mathrm{MS}}} &= a_s^{\mathrm{MOM}}+a_s^{\mathrm{MOM}^2}\left(\delta a_{s,1}^{\mathrm{MOM}}-\delta a_{s,1}^{\overline{\mathrm{MS}}}(N_F+2)\right)+a_s^{\mathrm{MOM}^3}\Bigg(\delta a_{s,2}^{\mathrm{MOM}}-\delta a_{s,2}^{\overline{\mathrm{MS}}}(N_F+2) \\
&\quad -2\delta a_{s,1}^{\overline{\mathrm{MS}}}(N_F+2)\left[\delta a_{s,1}^{\mathrm{MOM}}-\delta a_{s,1}^{\overline{\mathrm{MS}}}(N_F+2)\right]\Bigg)+O(a_s^{\mathrm{MOM}^4}) \; .
\end{aligned} \tag{2.41}
$$

Note that, unlike in Eq. (2.18), in Eqs. (2.40) and (2.41) $a_s^{\overline{\mathrm{MS}}} \equiv a_s^{\overline{\mathrm{MS}}}(N_F+2)$. Applying the coupling renormalization, cf. Eq. (2.36), to Eq. (2.16), the OME after mass and coupling renormalization is obtained by

$$
\begin{aligned}
\hat{A}_{ij} &= \delta_{ij}+a_s^{\mathrm{MOM}}\,\hat{\hat{A}}_{ij}^{(1)}+a_s^{\mathrm{MOM}^2}\Bigg[\hat{\hat{A}}_{ij}^{(2)}+\delta a_{s,1}^{\mathrm{MOM}}\hat{\hat{A}}_{ij}^{(1)} \\
&\quad +\delta m_1\left(\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2}m_1\frac{d}{dm_1}+\left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2}m_2\frac{d}{dm_2}\right)\hat{\hat{A}}_{ij}^{(1)}\Bigg] \\
&\quad +a_s^{\mathrm{MOM}^3}\Bigg[\hat{\hat{A}}_{ij}^{(3)}+\delta a_{s,2}^{\mathrm{MOM}}\hat{\hat{A}}_{ij}^{(1)}+2\delta a_{s,1}^{\mathrm{MOM}}\Bigg[\hat{\hat{A}}_{ij}^{(2)} \\
&\quad +\delta m_1\left(\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2}m_1\frac{d}{dm_1}+\left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2}m_2\frac{d}{dm_2}\right)\hat{\hat{A}}_{ij}^{(1)}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\delta m_1 \left(\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} m_1 \frac{d}{dm_1} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2} m_2 \frac{d}{dm_2} \right) \hat{\hat{A}}_{ij}^{(2)} \\
&+ \left(\delta m_{2,1}(m_1,m_2) \left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon} m_1 \frac{d}{dm_1} + \delta m_{2,2}(m_1,m_2) \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon} m_2 \frac{d}{dm_2} \right) \hat{\hat{A}}_{ij}^{(1)} \\
&+\frac{(\delta m_1)^2}{2} \left(\left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon} m_1^2 \frac{d^2}{dm_1{}^2} + \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon} m_2^2 \frac{d^2}{dm_2{}^2} \right) \hat{\hat{A}}_{ij}^{(1)} \\
&+(\delta m_1)^2 \left(\frac{m_1^2}{\mu^2}\right)^{\varepsilon/2} \left(\frac{m_2^2}{\mu^2}\right)^{\varepsilon/2} m_1 \frac{d}{dm_1} m_2 \frac{d}{dm_2} \hat{\hat{A}}_{ij}^{(1)} \Bigg] , \qquad (2.42)
\end{aligned}
$$

where the dependence on the masses, ε and N in the arguments of the OMEs has been suppressed for brevity.

The renormalization steps for the local operators and removing the collinear singularities proceed very closely to Ref. [41].

Finally, one has to account for the one-particle reducible contributions. In the present case, these are gluon self-energy contributions to the external legs of lower order one-particle irreducible diagrams. From 3-loop order onward, the reducible contributions to A_{Qg} may contain three different heavy flavors, while this is not the case for the irreducible contributions. Note that the inclusion of the top quark in a loop of the irreducible terms for $A_{ij}^{(3)}$ would demand to consider the energy range $Q^2 \gg m_t^2$. At a scale $\mu^2 \simeq m_t^2$, both charm and bottom can be dealt with as effectively massless. The emergence of massive top loops in the reducible contributions is accounted for by renormalization. In the following, we will strictly consider the case of two heavy flavors only.

The scalar self-energies are obtained by projecting out the Lorentz-structure

$$
\begin{aligned}
\hat{\Pi}_{\mu\nu}^{ab}(p^2, \hat{m}_1^2, \hat{m}_2^2, \mu^2, \hat{a}_s) &= i\delta^{ab} \left[-g_{\mu\nu}p^2 + p_\mu p_\nu \right] \hat{\Pi}(p^2, \hat{m}_1^2, \hat{m}_2^2, \mu^2, \hat{a}_s) , \qquad (2.43) \\
\hat{\Pi}(p^2, \hat{m}_1^2, \hat{m}_2^2, \mu^2, \hat{a}_s) &= \sum_{k=1}^{\infty} \hat{a}_s^k \hat{\Pi}^{(k)}(p^2, \hat{m}_1^2, \hat{m}_2^2, \mu^2) . \qquad (2.44)
\end{aligned}
$$

In the same way as the OMEs themselves, the irreducible two-mass self-energies can be divided into contributions which depend on one mass only and an additional part stemming from diagrams containing both heavy quark flavors

$$
\hat{\Pi}^{(k)}\left(p^2, \hat{m}_1^2, \hat{m}_2^2, \mu^2\right) = \hat{\Pi}^{(k)}\left(p^2, \frac{\hat{m}_1^2}{\mu^2}\right) + \hat{\Pi}^{(k)}\left(p^2, \frac{\hat{m}_2^2}{\mu^2}\right) + \hat{\tilde{\Pi}}^{(k)}\left(p^2, \hat{m}_1^2, \hat{m}_2^2, \mu^2\right) . \qquad (2.45)
$$

Up to two-loop order no diagrams with two heavy flavors contribute

$$
\hat{\tilde{\Pi}}^{(k)}(p^2, \hat{m}_1^2, \hat{m}_2^2, \mu^2) = 0 \quad \text{for } k \in \{1, 2\} . \qquad (2.46)
$$

The single-mass contributions for the gluon are known from [41, 47, 48, 98]

$$
\begin{aligned}
\hat{\Pi}^{(1)}\left(0, \frac{\hat{m}^2}{\mu^2}\right) &= T_F \left(\frac{\hat{m}^2}{\mu^2}\right)^{\varepsilon/2} \left[\frac{8}{3\varepsilon} \exp\left(\sum_{i=2}^{\infty} \frac{\zeta_i}{i} \left(\frac{\varepsilon}{2}\right)^i \right) \right] , \qquad (2.47) \\
\hat{\Pi}^{(2)}\left(0, \frac{\hat{m}^2}{\mu^2}\right) &= T_F \left(\frac{\hat{m}^2}{\mu^2}\right)^{\varepsilon} \Bigg\{ -\frac{4}{\varepsilon^2} C_A + \frac{1}{\varepsilon} (5C_A - 12C_F) + C_A \left(\frac{13}{12} - \zeta_2 \right) - \frac{13}{3} C_F \\
&\quad + \varepsilon \left[C_A \left(\frac{169}{144} + \frac{5}{4}\zeta_2 - \frac{\zeta_3}{3} \right) - C_F \left(\frac{35}{12} + 3\zeta_2 \right) \right] \Bigg\} + O(\varepsilon^2) , \qquad (2.48)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Pi}^{(3)}\Big(0,\frac{\hat{m}^2}{\mu^2}\Big) \;=\;& T_F\Big(\frac{\hat{m}^2}{\mu^2}\Big)^{3\varepsilon/2}\Bigg\{\frac{1}{\varepsilon^3}\left[-\frac{32}{9}T_FC_A\left(2N_F+1\right)+\frac{164}{9}C_A^2\right] \\
&+\frac{1}{\varepsilon^2}\left[\frac{80}{27}(C_A-6C_F)N_FT_F+\frac{8}{27}(35C_A-48C_F)T_F-\frac{781}{27}C_A^2\right. \\
&\left.+\frac{712}{9}C_AC_F\right]+\frac{1}{\varepsilon}\left[\frac{4}{27}(C_A(-101-18\zeta_2)-62C_F)N_FT_F\right. \\
&-\frac{2}{27}(C_A(37+18\zeta_2)+80C_F)T_F+C_A^2\Big(-12\zeta_3+\frac{41}{6}\zeta_2+\frac{3181}{108}\Big) \\
&\left.+C_AC_F\Big(16\zeta_3-\frac{1570}{27}\Big)+\frac{272}{3}C_F^2\right] \\
&+N_FT_F\left[C_A\Big(\frac{56}{9}\zeta_3+\frac{10}{9}\zeta_2-\frac{3203}{243}\Big)-C_F\Big(\frac{20}{3}\zeta_2+\frac{1942}{81}\Big)\right] \\
&+T_F\left[C_A\Big(-\frac{295}{18}\zeta_3+\frac{35}{9}\zeta_2+\frac{6361}{486}\Big)-C_F\Big(7\zeta_3+\frac{16}{3}\zeta_2+\frac{218}{81}\Big)\right] \\
&+C_A^2\Big(4\mathrm{B}_4-27\zeta_4+\frac{1969}{72}\zeta_3-\frac{781}{72}\zeta_2+\frac{42799}{3888}\Big) \\
&+C_AC_F\Big(-8\mathrm{B}_4+36\zeta_4-\frac{1957}{12}\zeta_3+\frac{89}{3}\zeta_2+\frac{10633}{81}\Big) \\
&+C_F^2\Big(\frac{95}{3}\zeta_3+\frac{274}{9}\Big)\Bigg\}+O(\varepsilon)\;. \qquad (2.49)
\end{aligned}
$$

Similarly to other massive processes [41,99–103], the constant

$$
\mathrm{B}_4 \;=\; -4\zeta_2\ln^2(2)+\frac{2}{3}\ln^4(2)-\frac{13}{2}\zeta_4+16\mathrm{Li}_4\Big(\frac{1}{2}\Big)\;\approx\;-1.762800093... \qquad (2.50)
$$

emerges in Eq. (2.49). At $O(a_s^3)$, irreducible diagrams with two different masses contribute for the first time. In [14] the gluonic case was calculated to $O(\eta^3)$ using the codes `Q2E/Exp` [29,30]. However, the full η dependence is needed in the following. All the diagrams can be expressed through a one-dimensional Mellin-Barnes integral, and the residue sums are easily evaluated using the `Mathematica` package `EvaluateMultiSums` [104] which is built on `Sigma` [105,106] and `HarmonicSums` [88,107–124]. The result is given by [125]

$$
\begin{aligned}
\hat{\hat{\Pi}}^{(3)}(0,m_1^2,m_2^2,\mu^2) \;=\;& -C_FT_F^2\Bigg\{\frac{256}{9\varepsilon^2}+\frac{64}{3\varepsilon}\left[L_1+L_2+\frac{5}{9}\right]-5\eta-\frac{5}{\eta} \\
&+\Big(-\frac{5\eta}{8}-\frac{5}{8\eta}+\frac{51}{4}\Big)\ln^2(\eta)+\Big(\frac{5}{2\eta}-\frac{5\eta}{2}\Big)\ln(\eta)+\frac{32\zeta_2}{3} \\
&+32L_1L_2+\frac{80}{9}L_1+\frac{80}{9}L_2+\frac{1246}{81} \\
&+\Big(\frac{5\eta^{3/2}}{2}+\frac{5}{2\eta^{3/2}}+\frac{3\sqrt{\eta}}{2}+\frac{3}{2\sqrt{\eta}}\Big)\left[\frac{1}{8}\ln\Big(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\Big)\ln^2(\eta)\right. \\
&\left.-\mathrm{Li}_3\left(-\sqrt{\eta}\right)+\mathrm{Li}_3\left(\sqrt{\eta}\right)-\frac{1}{2}\ln(\eta)\left(\mathrm{Li}_2\left(\sqrt{\eta}\right)-\mathrm{Li}_2\left(-\sqrt{\eta}\right)\right)\right]\Bigg\} \\
&-C_AT_F^2\Bigg\{\frac{64}{9\varepsilon^3}+\frac{16}{3\varepsilon^2}\left[(L_1+L_2)-\frac{35}{9}\right]+\frac{4}{\varepsilon}\left[L_1^2+L_2^2-\frac{35}{9}L_1\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{35}{9}L_2+\frac{2}{3}\zeta_2+\frac{37}{27}\Biggr]+2\left(L_1^3+L_2^3\right)-\frac{70}{3}L_1L_2-\frac{4}{9}\ln^3(\eta)\\
&+\left(2\zeta_2+\frac{37}{9}\right)(L_1+L_2)+\left[\frac{8}{3}\ln(1-\eta)-\frac{2}{3}\left(\eta+\frac{1}{\eta}\right)-\frac{179}{18}\right]\\
&\times\ln^2(\eta)-\frac{16}{3}\left(\eta+\frac{1}{\eta}\right)-\frac{70}{9}\zeta_2-\frac{56}{9}\zeta_3-\frac{3769}{243}\\
&+\frac{8}{3}\left(\frac{1}{\eta}-\eta\right)\ln(\eta)+\frac{16}{3}\left(\mathrm{Li}_2(\eta)\ln(\eta)-\mathrm{Li}_3(\eta)\right)\\
&+\left[8\frac{1+\eta^3}{3\eta^{3/2}}+10\frac{1+\eta}{\sqrt{\eta}}\right]\left[\frac{1}{8}\ln\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\right)\ln^2(\eta)-\mathrm{Li}_3\left(-\sqrt{\eta}\right)\right.\\
&\left.+\mathrm{Li}_3\left(\sqrt{\eta}\right)-\frac{1}{2}\ln(\eta)\left(\mathrm{Li}_2\left(\sqrt{\eta}\right)-\mathrm{Li}_2\left(-\sqrt{\eta}\right)\right)\right]+O(\varepsilon)\Biggr\}.
\end{aligned}
\tag{2.51}
$$

Here the classical polylogarithm [127–129] is defined by

$$
\mathrm{Li}_n(x)=\sum_{k=1}^{\infty}\frac{x^k}{k^n},\quad n\geq 0,\quad x\in[-1,1].
\tag{2.52}
$$

After adjusting notations complete agreement is found with the earlier calculation in Ref. [126]. For the reducible parts in the case of external massless fermion lines, see e.g. [31] and references therein.

The two-mass OMEs at one-loop order and the irreducible OMEs at $O(a_s^2)$ are defined by

$$
\hat{\hat{A}}_{ij}^{(1)}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)=\hat{\hat{A}}_{ij}^{(1)}\left(\frac{\hat{m}_1^2}{\mu^2}\right)+\hat{\hat{A}}_{ij}^{(1)}\left(\frac{\hat{m}_2^2}{\mu^2}\right)\ ,
\tag{2.53}
$$

$$
\hat{\hat{A}}_{ij}^{(\prime),(2),\mathrm{irr}}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)=\hat{\hat{A}}_{ij}^{(\prime),(2),\mathrm{irr}}\left(\frac{\hat{m}_1^2}{\mu^2}\right)+\hat{\hat{A}}_{ij}^{(\prime),(2),\mathrm{irr}}\left(\frac{\hat{m}_2^2}{\mu^2}\right)\ ,
\tag{2.54}
$$

where the A_{ij}'s with one argument denote the usual single-mass OMEs. The irreducible contributions from OMEs with external gluons need a further discussion. These have to be included using a consistent projection, either physical or unphysical, since the ghost contributions restore gauge invariance globally. Therefore the superscript $'$ is included for irreducible OMEs with external gluons. Using previous definitions the reducible massive operator matrix elements at $O(a_s^2)$ are composed by

$$
\hat{\hat{A}}_{qq}^{(2),\mathrm{NS}}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)=\hat{\hat{A}}_{qq}^{(2),\mathrm{NS,irr}}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)-\hat{\Sigma}^{(2)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right)\ ,
\tag{2.55}
$$

$$
\hat{\hat{A}}_{Qg}^{(2)}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)=\hat{\hat{A}}_{Qg}^{\prime,(2),\mathrm{irr}}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)-\hat{\hat{A}}_{Qg}^{(1)}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)\hat{\Pi}^{(1)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right)\ ,
\tag{2.56}
$$

$$
\begin{aligned}
\hat{\hat{A}}_{gg}^{(2)}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)&=\hat{\hat{A}}_{gg}^{\prime,(2),\mathrm{irr}}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)-\hat{\Pi}^{(2)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right)\\
&\quad-\hat{\hat{A}}_{gg}^{(1)}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)\hat{\Pi}^{(1)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right)\ ,
\end{aligned}
\tag{2.57}
$$

and at $O(a_s^3)$ by

$$
\hat{\hat{A}}_{qq}^{(3),\mathrm{NS}}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)=\hat{\hat{A}}_{qq}^{(3),\mathrm{NS,irr}}\left(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\right)-\hat{\Sigma}^{(3)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right)
\tag{2.58}
$$

$$
\begin{aligned}
\hat{\hat{A}}_{Qg}^{(3)}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr) &= \hat{\hat{A}}_{Qg}^{\prime,(3),\rm irr}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr)-\hat{\hat{A}}_{Qg}^{(2)}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr)\hat{\Pi}^{(1)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right) \\
&\quad -\hat{\hat{A}}_{Qg}^{(1)}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr)\hat{\Pi}^{(2)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right) \qquad (2.59)\\
\hat{\hat{A}}_{gg}^{(3)}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr) &= \hat{\hat{A}}_{gg}^{\prime,(3),\rm irr}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr)-\hat{\hat{A}}_{gg}^{(2)}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr)\hat{\Pi}^{(1)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right) \\
&\quad -\hat{\hat{A}}_{gg}^{(1)}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr)\hat{\Pi}^{(2)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right)-\hat{\Pi}^{(3)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right) . \qquad (2.60)
\end{aligned}
$$

One can subtract the single-mass contributions to these equations using Eq. (2.1), keeping only the genuine two-mass contributions. At three loops one obtains

$$
\begin{aligned}
\hat{\tilde{\hat{A}}}_{qq}^{(3),\rm NS}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr) &= \hat{\tilde{\hat{A}}}_{qq}^{(3),\rm NS,irr}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr)-\hat{\tilde{\Sigma}}^{(3)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right) \qquad (2.61)\\
\hat{\tilde{\hat{A}}}_{Qg}^{(3)}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr) &= \hat{\tilde{\hat{A}}}_{Qg}^{\prime,(3),\rm irr}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr) \\
&\quad +\hat{\hat{A}}_{Qg}^{(1)}\left(\frac{\hat{m}_1^2}{\mu^2}\right)\left[2\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_1^2}{\mu^2}\Bigr)+\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_2^2}{\mu^2}\Bigr)\right]\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_2^2}{\mu^2}\Bigr) \\
&\quad +\hat{\hat{A}}_{Qg}^{(1)}\left(\frac{\hat{m}_2^2}{\mu^2}\right)\left[2\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_2^2}{\mu^2}\Bigr)+\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_1^2}{\mu^2}\Bigr)\right]\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_1^2}{\mu^2}\Bigr) \\
&\quad -\hat{\hat{A}}_{Qg}^{(2)}\left(\frac{\hat{m}_1^2}{\mu^2}\right)\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_2^2}{\mu^2}\Bigr)-\hat{\hat{A}}_{Qg}^{(2)}\left(\frac{\hat{m}_2^2}{\mu^2}\right)\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_1^2}{\mu^2}\Bigr) \\
&\quad -\hat{\hat{A}}_{Qg}^{(1)}\left(\frac{\hat{m}_1^2}{\mu^2}\right)\hat{\Pi}^{(2)}\Bigl(0,\frac{\hat{m}_2^2}{\mu^2}\Bigr)-\hat{\hat{A}}_{Qg}^{(1)}\left(\frac{\hat{m}_2^2}{\mu^2}\right)\hat{\Pi}^{(2)}\Bigl(0,\frac{\hat{m}_1^2}{\mu^2}\Bigr) \qquad (2.62)\\
\hat{\tilde{\hat{A}}}_{gg}^{(3)}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr) &= \hat{\tilde{\hat{A}}}_{gg}^{\prime,(3),\rm irr}\Bigl(\frac{\hat{m}_1^2}{\mu^2},\frac{\hat{m}_2^2}{\mu^2}\Bigr)-\hat{\tilde{\Pi}}^{(3)}\left(0,\hat{m}_1^2,\hat{m}_2^2,\mu^2\right) \\
&\quad -\hat{\hat{A}}_{gg}^{\prime,(2),\rm irr}\left(\frac{\hat{m}_1^2}{\mu^2}\right)\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_2^2}{\mu^2}\Bigr)-\hat{\hat{A}}_{gg}^{\prime,(2),\rm irr}\left(\frac{\hat{m}_2^2}{\mu^2}\right)\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_1^2}{\mu^2}\Bigr) \\
&\quad -2\hat{\hat{A}}_{gg}^{(1)}\left(\frac{\hat{m}_1^2}{\mu^2}\right)\hat{\Pi}^{(2)}\Bigl(0,\frac{\hat{m}_2^2}{\mu^2}\Bigr)-2\hat{\hat{A}}_{gg}^{(1)}\left(\frac{\hat{m}_2^2}{\mu^2}\right)\hat{\Pi}^{(2)}\Bigl(0,\frac{\hat{m}_1^2}{\mu^2}\Bigr) \\
&\quad +\hat{\hat{A}}_{gg}^{(1)}\left(\frac{\hat{m}_1^2}{\mu^2}\right)\left[2\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_1^2}{\mu^2}\Bigr)+\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_2^2}{\mu^2}\Bigr)\right]\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_2^2}{\mu^2}\Bigr) \\
&\quad +\hat{\hat{A}}_{gg}^{(1)}\left(\frac{\hat{m}_2^2}{\mu^2}\right)\left[2\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_2^2}{\mu^2}\Bigr)+\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_1^2}{\mu^2}\Bigr)\right]\hat{\Pi}^{(1)}\Bigl(0,\frac{\hat{m}_1^2}{\mu^2}\Bigr) . \qquad (2.63)
\end{aligned}
$$

2.2 The structure of $(\Delta)A_{Qg}$

The single-mass and two-mass contributions to the OME A_{Qg} are given by

$$
\begin{aligned}
A_{Qg} &= a_s A_{Qg}^{(1)} + a_s^2 A_{Qg}^{(2)} + a_s^3 A_{Qg}^{(3)} + O(a_s^4), \qquad (2.64)\\
A_{qg,Q} &= a_s^3 A_{qg,Q}^{(3)} + O(a_s^4), \qquad (2.65)
\end{aligned}
$$

depending on whether the operator couples to a light or heavy quark. Of these OMEs, only A_{Qg} contains two-flavor contributions starting from $O(a_s^2)$

$$
\tilde{A}_{Qg} = a_s^2 \tilde{A}_{Qg}^{(2)} + a_s^3 \tilde{A}_{Qg}^{(3)} + O(a_s^4) . \qquad (2.66)
$$

In Eq. (2.66) the $O(a_s^2)$ contribution consists of one-particle reducible diagrams only, see Eq. (2.56). As a consequence, the flavor dependence factorizes in the $O(a_s^2)$ terms.

In the renormalized MOM-scheme, cf. Ref. [41], the two-loop contribution is obtained by

$$\begin{aligned} A_{Qg}^{(2),\mathrm{MOM}} &= \hat{A}_{Qg}^{(2),\mathrm{MOM}} + Z_{qg}^{-1,(2)}(N_F+2) - Z_{qg}^{-1,(2)}(N_F) + Z_{qg}^{-1,(1)}(N_F+2)\hat{A}_{gg,Q}^{(1),\mathrm{MOM}} \\ &\quad + Z_{qq}^{-1,(1)}(N_F+2)\hat{A}_{Qg}^{(1),\mathrm{MOM}} + \Big[\hat{A}_{Qg}^{(1),\mathrm{MOM}} + Z_{qg}^{-1,(1)}(N_F+2) \\ &\quad - Z_{qg}^{-1,(1)}(N_F)\Big]\Gamma_{gg}^{-1,(1)}(N_F)~. \end{aligned} \tag{2.67}$$

The unrenormalized terms are given by

$$\begin{aligned} \hat{\hat{A}}_{Qg}^{(2)} &= -\frac{2}{\varepsilon^2}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)} - \frac{1}{\varepsilon}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\left(L_1+L_2\right) + \tilde{a}_{Qg}^{(2)} \\ &\quad + \varepsilon\overline{\tilde{a}}_{Qg}^{(2)}~. \end{aligned} \tag{2.68}$$

The coefficients $\tilde{a}_{Qg}^{(2)}$ and $\overline{\tilde{a}}_{Qg}^{(2)}$ are read off from Eq. (2.56)

$$\tilde{a}_{Qg}^{(2)} = -\frac{1}{2}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\left\{\frac{1}{2}\left(L_1+L_2\right)^2 + \zeta_2\right\}~, \tag{2.69}$$

$$\overline{\tilde{a}}_{Qg}^{(2)} = \frac{1}{2}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\left\{-\frac{1}{12}\left(L_1+L_2\right)^3 - \frac{1}{2}\zeta_2\left(L_1+L_2\right) - \frac{1}{3}\zeta_3\right\}~. \tag{2.70}$$

The renormalized expression at 2 loops then reads

$$\begin{aligned} \tilde{A}_{Qg}^{(2),\overline{\mathrm{MS}}} &= \frac{1}{4}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\left(L_1^2+L_2^2\right) + \frac{1}{2}\zeta_2\beta_{0,Q}\hat{\gamma}_{qg}^{(0)} + \tilde{a}_{Qg}^{(2)} \\ &= -\frac{1}{2}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}L_1\,L_2~. \end{aligned} \tag{2.71}$$

The renormalized 3-loop OMEs in the MOM-scheme are obtained from the charge- and mass-renormalized OMEs by

$$\begin{aligned} & A_{Qg}^{(3),\mathrm{MOM}} + A_{gg,Q}^{(3),\mathrm{MOM}} = \hat{A}_{Qg}^{(3),\mathrm{MOM}} + \hat{A}_{gg,Q}^{(3),\mathrm{MOM}} + Z_{qg}^{-1,(3)}(N_F+2) - Z_{qg}^{-1,(3)}(N_F) \\ &\quad + Z_{qg}^{-1,(2)}(N_F+2)\hat{A}_{gg,Q}^{(1),\mathrm{MOM}} + Z_{qg}^{-1,(1)}(N_F+2)\hat{A}_{gg,Q}^{(2),\mathrm{MOM}} + Z_{qq}^{-1,(2)}(N_F+2)\hat{A}_{Qg}^{(1),\mathrm{MOM}} \\ &\quad + Z_{qq}^{-1,(1)}(N_F+2)\hat{A}_{Qg}^{(2),\mathrm{MOM}} + \Big[\hat{A}_{Qg}^{(1),\mathrm{MOM}} + Z_{qg}^{-1,(1)}(N_F+2) \\ &\quad - Z_{qg}^{-1,(1)}(N_F)\Big]\Gamma_{gg}^{-1,(2)}(N_F) + \Big[\hat{A}_{Qg}^{(2),\mathrm{MOM}} + Z_{qg}^{-1,(2)}(N_F+2) - Z_{qg}^{-1,(2)}(N_F) \\ &\quad + Z_{qq}^{-1,(1)}(N_F+2)A_{Qg}^{(1),\mathrm{MOM}} + Z_{qg}^{-1,(1)}(N_F+2)A_{gg,Q}^{(1),\mathrm{MOM}}\Big]\Gamma_{gg}^{-1,(1)}(N_F) \\ &\quad + \Big[\hat{A}_{Qq}^{(2),\mathrm{PS,MOM}} + Z_{qq}^{-1,(2),\mathrm{PS}}(N_F+2) - Z_{qq}^{-1,(2),\mathrm{PS}}(N_F)\Big]\Gamma_{qg}^{-1,(1)}(N_F) \\ &\quad + \Big[\hat{A}_{qq,Q}^{(2),\mathrm{NS,MOM}} + Z_{qq}^{-1,(2),\mathrm{NS}}(N_F+2) - Z_{qq}^{-1,(2),\mathrm{NS}}(N_F)\Big]\Gamma_{qg}^{-1,(1)}(N_F)~. \end{aligned} \tag{2.72}$$

It is explicitly given by[4]

$$\hat{\hat{A}}_{Qg}^{(3)} = \frac{1}{\varepsilon^3}\left[\frac{14}{3}\beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)} - \frac{4}{3}\hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q} + \frac{7}{3}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)} + 12\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)} + \frac{1}{12}\gamma_{gq}^{(0)}\left(\hat{\gamma}_{qg}^{(0)}\right)^2\right]$$

[4]The different symbols used here are defined in Ref. [41] and above.

$$
\begin{aligned}
&+\frac{1}{\varepsilon^2}\Biggl[\Biggl\{\frac{1}{16}\gamma_{gq}^{(0)}\left(\hat{\gamma}_{qg}^{(0)}\right)^2+9\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)}+\frac{7}{2}\beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}-\hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q}+\frac{7}{4}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}\Biggr\}\\
&\times\left(L_1+L_2\right)+\frac{1}{12}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{PS},(1)}+\frac{1}{12}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{NS},(1)}-\frac{5}{3}\beta_{0,Q}\hat{\gamma}_{qg}^{(1)}+\frac{1}{6}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gg}^{(1)}-\frac{1}{3}\hat{\gamma}_{qg}^{(0)}\beta_{1,Q}\\
&+5\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\delta m_1^{(-1)}\Biggr]+\frac{1}{\varepsilon}\Biggl[\Biggl\{\frac{1}{16}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{NS},(1)}+\frac{15}{4}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\delta m_1^{(-1)}+\frac{1}{16}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{PS},(1)}-\frac{5}{4}\beta_{0,Q}\hat{\gamma}_{qg}^{(1)}\\
&+\frac{1}{8}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gg}^{(1)}-\frac{1}{4}\hat{\gamma}_{qg}^{(0)}\beta_{1,Q}\Biggr\}\left(L_1+L_2\right)+\Biggl\{\frac{13}{8}\beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}+\frac{13}{16}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}+\frac{15}{4}\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)}\\
&+\frac{3}{64}\gamma_{gq}^{(0)}\left(\hat{\gamma}_{qg}^{(0)}\right)^2-\frac{1}{2}\hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q}\Biggr\}\left(L_1^2+L_2^2\right)+\Biggl\{-\frac{1}{2}\hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q}+2\beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}+6\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)}\\
&+\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}\Biggr\}L_1L_2+\frac{2}{3}\gamma_{qg}^{(2),N_F^2}-8\beta_{0,Q}a_{Qg}^{(2)}-\frac{1}{32}\left(\hat{\gamma}_{qg}^{(0)}\right)^2\zeta_2\gamma_{gq}^{(0)}+\hat{\gamma}_{qg}^{(0)}a_{gg,Q}^{(2)}-\hat{\gamma}_{qg}^{(0)}\tilde{\delta}m_2^{(-1)}\\
&+\frac{9}{2}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}^2+\frac{1}{8}\beta_{0,Q}\zeta_2\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}+\frac{1}{4}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}\beta_0+4\delta m_1^{(0)}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\Biggr]\\
&+\tilde{a}_{Qg}^{(3)}\left(m_1^2,m_2^2,\mu^2\right)\ . \qquad (2.73)
\end{aligned}
$$

For the renormalized operator matrix element in the $\overline{\mathrm{MS}}$ scheme one finally obtains,

$$
\begin{aligned}
\tilde{A}_{Qg}^{(3),\overline{\mathrm{MS}}} &= \Biggl\{-\frac{9}{8}\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)}-\frac{7}{384}\gamma_{gq}^{(0)}\left(\hat{\gamma}_{qg}^{(0)}\right)^2+\frac{1}{6}\hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q}-\frac{25}{96}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}-\frac{25}{48}\beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\Biggr\}\\
&\times\left(L_1^3+L_2^3\right)+\Biggl\{\frac{1}{8}\hat{\gamma}_{qg}^{(0)}\gamma_{qq}^{(0)}\beta_{0,Q}-\frac{1}{2}\beta_0\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}-\frac{1}{4}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}-\frac{3}{2}\beta_{0,Q}^2\hat{\gamma}_{qg}^{(0)}\Biggr\}\\
&\times\left(L_1^2L_2+L_2^2L_1\right)+\Biggl\{-\frac{1}{64}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{PS},(1)}-\frac{1}{64}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{qq}^{\mathrm{NS},(1)}+\frac{9}{16}\beta_{0,Q}\hat{\gamma}_{qg}^{(1)}-\frac{1}{16}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gg}^{(1)}\\
&-\frac{29}{16}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\delta m_1^{(-1)}+\frac{1}{16}\hat{\gamma}_{qg}^{(0)}\beta_{1,Q}\Biggr\}\left(L_1^2+L_2^2\right)-2L_1L_2\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\delta m_1^{(-1)}+\Biggl\{\frac{3}{4}\hat{\gamma}_{qg}^{(0)}\tilde{\delta}m_2^{(-1)}\\
&+\frac{1}{128}\left(\hat{\gamma}_{qg}^{(0)}\right)^2\zeta_2\gamma_{gq}^{(0)}+\frac{1}{8}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}\gamma_{qq}^{(0)}-3\delta m_1^{(0)}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}-\frac{1}{2}\hat{\gamma}_{qg}^{(0)}a_{gg,Q}^{(2)}-\frac{9}{32}\beta_{0,Q}\zeta_2\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}\\
&-\frac{27}{8}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}^2-\frac{9}{16}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}\beta_0+4\beta_{0,Q}a_{Qg}^{(2)}\Biggr\}\left(L_1+L_2\right)+8\overline{a}_{Qg}^{(2)}\beta_{0,Q}-\frac{1}{32}\hat{\gamma}_{qg}^{(0)}\zeta_2\hat{\gamma}_{qq}^{\mathrm{PS},(1)}\\
&-\frac{1}{32}\hat{\gamma}_{qg}^{(0)}\zeta_2\hat{\gamma}_{qq}^{\mathrm{NS},(1)}+\frac{1}{96}\left(\hat{\gamma}_{qg}^{(0)}\right)^2\zeta_3\gamma_{gq}^{(0)}-\frac{3}{2}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}^2\zeta_3+\frac{1}{8}\hat{\gamma}_{qg}^{(1)}\beta_{0,Q}\zeta_2-4\delta m_1^{(1)}\beta_{0,Q}\hat{\gamma}_{qg}^{(0)}\\
&+\frac{1}{8}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{1,Q}-\hat{\gamma}_{qg}^{(0)}\overline{a}_{gg,Q}^{(2)}-\frac{1}{12}\hat{\gamma}_{qg}^{(0)}\beta_0\beta_{0,Q}\zeta_3-\frac{1}{24}\beta_{0,Q}\zeta_3\hat{\gamma}_{qg}^{(0)}\gamma_{gg}^{(0)}+\frac{1}{2}\hat{\gamma}_{qg}^{(0)}\Bigl(\tilde{\delta}m_2{}^{1,(0)}\\
&+\tilde{\delta}m_2{}^{2,(0)}\Bigr)-\frac{9}{8}\hat{\gamma}_{qg}^{(0)}\zeta_2\beta_{0,Q}\delta m_1^{(-1)}+\tilde{a}_{Qg}^{(3)}\left(m_1^2,m_2^2,\mu^2\right)\ . \qquad (2.74)
\end{aligned}
$$

3 The main steps of the calculation

The main technical steps of the calculation of the OMEs in x-space are very similar to those described in Ref. [11]. Typical Feynman diagrams contributing to the two-mass corrections of

$(\Delta)\tilde{A}_{Qg}^{(3)}$ are shown in Figure 1.

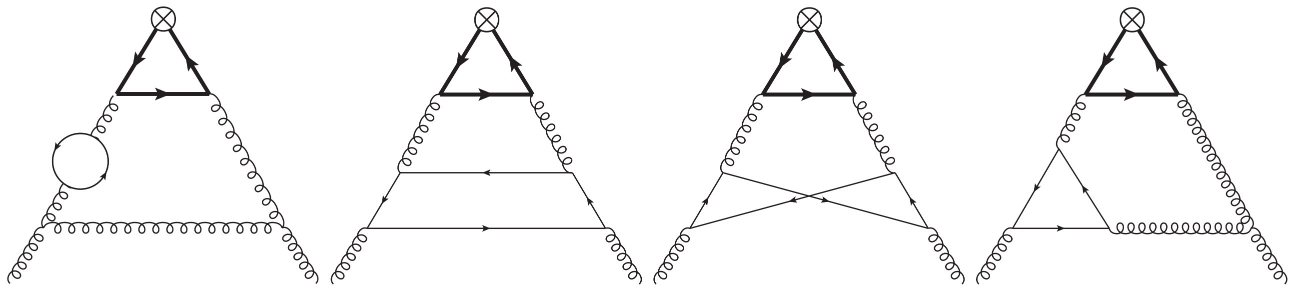

Figure 1: Sample diagrams contributing to the two-mass contributions to the operator matrix element $\tilde{A}_{Qg}^{(3)}$. The thin fermionic lines correspond to heavy quarks of a mass m_1 and the thick ones to quarks of a mass m_2. The amplitudes are symmetric under the interchange $m_1 \leftrightarrow m_2$.

We use the workflow described in Refs. [41,130–152] and the packages `QGRAF` [153], `Form` [154, 155], and `color` [156] to construct the amplitude. In order to work with (effective) propagators instead of local operator insertions, we compute the generating function $F(t,\eta)$ [37,38]

$$F(t,\eta) = \sum_{N=1}^{\infty} t^N F(N,\eta), \text{ with } t \in \mathbb{R}. \tag{3.1}$$

This turns, e.g., the local operator

$$\mathrm{OP}_1^{(n)}(\tilde{p}_1) \;\; = \;\; (\Delta.\tilde{p}_1)^n, \tag{3.2}$$

into the following effective propagator

$$\sum_{N=0}^{\infty} t^N \mathrm{OP}_1^{(N)}(\tilde{p}_1) = \sum_{N=0}^{\infty} t^N (\Delta.\tilde{p}_1)^N = \frac{1}{1 - t\Delta.\tilde{p}_1}. \tag{3.3}$$

The Cauchy product

$$\sum_{i=0}^{\infty} a_i \sum_{j=0}^{\infty} b_j = \sum_{i=0}^{\infty} \sum_{j=0}^{i} a_j b_{i-j} \tag{3.4}$$

allows to resum more involved structures. To find the Mellin space result, one has to expand the generating functions around $t = 0$ and extract the Nth coefficient of the series. Alternatively, one can use the method described in Ref. [33] to directly relate the generating function in t to the solution in x-space.

We generate the amplitude with quarks of masses m_a and m_b, where the operator always sits on a heavy quark line of mass m_a. This means we have to calculate the amplitude twice. Once for the mass assignment $m_a = m_1$, $m_b = m_2$ and once for the assignment $m_a = m_2$, $m_b = m_1$. The two-mass assignments are related by the symmetry $\eta \leftrightarrow 1/\eta$.

We need three integral families, B_1, B_2 and B_3, to cover all diagrams contributing to $(\Delta)\tilde{A}_{Qg}^{(3)}$. After resummation their respective inverse propagators are given by

$$P_{B_1,1} = k_1^2 - m_a^2, \qquad P_{B_2,1} = k_1^2 - m_a^2, \qquad P_{B_3,1} = k_1^2 - m_a^2,$$

$$
\begin{aligned}
P_{B_1,2} &= (k_1-p)^2 - m_a^2, & P_{B_2,2} &= (k_1-p)^2 - m_a^2, & P_{B_3,2} &= (k_1-p)^2 - m_a^2,\\
P_{B_1,3} &= k_2^2 - m_b^2, & P_{B_2,3} &= k_2^2 - m_b^2, & P_{B_3,3} &= k_2^2 - m_a^2,\\
P_{B_1,4} &= (k_2-p)^2 - m_b^2, & P_{B_2,4} &= (k_2-p)^2 - m_b^2, & P_{B_3,4} &= (k_2-p)^2 - m_a^2,\\
P_{B_1,5} &= k_3^2, & P_{B_2,5} &= k_3^2, & P_{B_3,5} &= k_3^2,\\
P_{B_1,6} &= (k_1-k_3)^2 - m_a^2, & P_{B_2,6} &= (k_1-k_3)^2 - m_a^2, & P_{B_3,6} &= (k_1-k_3)^2 - m_b^2,\\
P_{B_1,7} &= (k_2-k_3)^2 - m_b^2, & P_{B_2,7} &= (k_2-k_3)^2 - m_b^2, & P_{B_3,7} &= (k_2-k_3)^2 - m_b^2,\\
P_{B_1,8} &= (k_1-k_2)^2, & P_{B_2,8} &= (k_1-k_2)^2, & P_{B_3,8} &= (k_1-k_2)^2,\\
P_{B_1,9} &= (k_3-p)^2, & P_{B_2,9} &= (k_3-p)^2, & P_{B_3,9} &= (k_3-p)^2,\\
P_{B_1,10} &= 1 - t\,\Delta.k_1, & P_{B_2,10} &= 1 - t\,\Delta.k_1, & P_{B_3,10} &= 1 - t\,\Delta.k_1,\\
P_{B_1,11} &= 1 - t\,\Delta.k_3, & P_{B_2,11} &= 1 - t\,\Delta.\left(k_1-k_3\right), & P_{B_3,11} &= 1 - t\,\Delta.k_3,\\
P_{B_1,12} &= 1 - t\,\Delta.k_2, & P_{B_2,12} &= 1 - t\,\Delta.k_2, & P_{B_3,12} &= 1 - t\,\Delta.k_2.
\end{aligned}
\tag{3.5}
$$

Furthermore, the crossed families, where p goes to $-p$, are needed for the reduction. At the moment level, these integrals generally have an additional factor of $(-1)^N$ compared to the non-crossed ones. At the level of generating functions one has to make the replacement $t \to -t$. The diagrams with operator insertion on an external gluon need further considerations. Here terms like

$$
\frac{1}{(1-t\,\Delta.k_1)\,(1-t\,(\Delta.k_1-\Delta.p))}
\tag{3.6}
$$

contribute. Since the propagators $P_{i,10}$ to $P_{i,12}$ do not involve the external momentum p, these terms cannot be attributed to any of the integral families. However, we can use partial fractioning to obtain relations like

$$
\frac{1}{(1-t\,\Delta.k_1)\,(1-t\,(\Delta.k_1-\Delta.p))} = \frac{1}{t\,\Delta.p}\left(\frac{1}{1-t\,\Delta.k_1} - \frac{1}{1-t\,\Delta.(k_1-p)}\right).
\tag{3.7}
$$

The two terms can now be handled separately and mapped to one or even several different integral families. Operator insertions on three and four gluons can be treated similarly and lead to three and four different terms respectively. We use the code `Reduze 2` [39, 40] for the reduction to master integrals and find a set of 150 master integrals, for which we generate a system of differential equations in the variables t and m_b.

One may compute the Mellin moments for $(\Delta)\tilde{a}_{Qg}^{(3)}$ by either setting N to the desired integer in the operator insertion or by expanding the resummed amplitude in t to the respective order. The results in N- and in x-space are related by a Mellin transform [86, 157–160]

$$
\mathbf{M}[f(x)](N) = \int_0^1 dx x^{N-1} f(x).
\tag{3.8}
$$

When expanding the amplitude in the small mass ratio η, low Mellin moments can be obtained by using the code `Q2E/Exp`. The method of arbitrarily high moments [161] allows to obtain a large number of moments expanded in η to a certain degree. We give details on this approach in Section 4. One can also compute the moments not expanding in η. These moments can be used to compare to the to Mellin moments computed numerically from the semi-analytic solutions of $(\Delta)\tilde{a}_{Qg}^{(3)}(x)$ we present in Section 5. The lowest moments expanded and unexpanded in η are given in Appendix A.

4 The Operator Matrix Elements $\tilde{A}_{Qg}^{(3)}$ in Mellin space

The direct calculation of the two-mass contributions in Mellin space similar to the single-mass case is not possible at present. Here recurrences related to the purely rational and ζ_3 terms of the moments could not be solved in a closed form yet since analytic solution methods for non-first-order-factorizable difference equations are not known at present [11]. Still, we may analyze the OME $\tilde{A}_{Qg}^{(3)}$ in Mellin space.[5] We use an adapted version of the algorithm of arbitrary high Mellin moments described in [161] and calculate a large number of moments expanding in η. Then we use guessing techniques [162–164] to find the recurrences for the first few orders of the η-expansion. It turns out that closed solutions in terms of nested sums and products can only be found in the region $N > N_0(\eta)$, where $N_0(\eta)$ grows with the depth of the η-expansion. I.e. we do *not* obtain a closed form solution for the whole range of N, but still a partial result. Its analytic continuation into x-space seems not to be possible, but all even or odd integer moments can be described.

Let us describe our method of calculating the moments. To calculate the master integrals, one first considers their system of coupled differential equations in the variable t which can be written as

$$\frac{d}{dt}\begin{pmatrix} I_1(t,\eta) \\ I_2(t,\eta) \\ \vdots \\ I_m(t,\eta) \end{pmatrix} = M \begin{pmatrix} I_1(t,\eta) \\ I_2(t,\eta) \\ \vdots \\ I_m(t,\eta) \end{pmatrix} + \begin{pmatrix} r_1(t,\eta) \\ r_2(t,\eta) \\ \vdots \\ r_m(t,\eta) \end{pmatrix}. \tag{4.1}$$

Here M is an $m \times m$ matrix with entries consisting of rational functions in ε, η and t. This system of differential equations can be decoupled to a single higher order differential equation using e.g. Zürchers's or other algorithms [165, 166] implemented in the package `Oresys` [167]. The master integrals $I_j(t,\eta)$ and the inhomogeneities $r_j(t,\eta)$ can be expanded in a Laurent expansion in ε, a Taylor series in the resummation parameter t and a logarithmically generalized Laurent series in η, yielding the general expressions

$$I_j(t,\eta) = \sum_{n=0}^{\infty}\left[\sum_{k=o}^{\infty}\left(\sum_{m=0}^{3}\left\{\sum_{l=o'}^{\infty} I_j^{(k,m,l)} \ln^m(\eta)\right\}\eta^l\right)\varepsilon^k\right]t^n, \tag{4.2}$$

$$r_j(t,\eta) = \sum_{n=0}^{\infty}\left[\sum_{k=o}^{\infty}\left(\sum_{m=0}^{3}\left\{\sum_{l=o'}^{\infty} r_j^{(k,m,l)} \ln^m(\eta)\right\}\eta^l\right)\varepsilon^k\right]t^n. \tag{4.3}$$

By inserting this ansatz into the decoupled differential equation, it is possible to find recurrences for the different expansion coefficients. Provided enough initial values are known, these recurrences can be used to iteratively calculate higher and higher moments of the master integrals. Although the ansatz in Eq. (4.3) generalizes the algorithm designed for single scale quantities, the generalization is rather immediate.

These sets of Mellin moments provide the basis to guess [162–164] the recurrences for the respective functions of the Mellin variable N. If these recurrences turn out to be first-order factorizable, we can find the closed form solutions with `Sigma` [105, 106] using the algorithms described in [130–143] under certain conditions.

Let us now turn to the calculation of the necessary initial values to solve the above system. In Ref. [168] a method to calculate initial values based on dimensional shifts was introduced. Here

[5] We will only consider the unpolarized case here. The results in the polarized case are rather similar.

the master integrals for fixed values of N can be reduced to a small set of scalar integrals without operator insertion in shifted dimensions. This small set of scalar integrals can then be calculated using direct integration techniques like hypergeometric methods [169–187] and Mellin-Barnes integrals [158, 188–190]. The drawback of this procedure is that for every higher moment the scalar integrals need to be calculated in a higher dimension, i.e. the shift $N \to N+1$ leads to the dimensional shift $d \to d+2$.

Another method can be established using the Mellin-Barnes representations of the master integrals. In the current case, it was possible to find a one-dimensional contour integral representation for all master integrals, which can be calculated analytically in terms of a linear combination of single infinite sums of the kind

$$\sum_{k=0}^{\infty} \eta^{k+j\pm a\varepsilon} f(k,\varepsilon), \tag{4.4}$$

with $j \in \mathbb{Z}$ and $a \in \left[\frac{1}{2}, 1, \frac{3}{2}\right]$. In the case of more involved topologies, where the operator polynomial has to be split up, further finite sums over Eq. (4.4) have to be applied. Then the function $f(k,\varepsilon)$ will also depend on the new summation quantifiers. Fixing the value of N to an integer will lead to a collapse of the finite sums into many terms. Since we are only interested in the η-expansion of the initial values, we can cut off the infinite sum in k to the desired order in η. Now only the expansion in ε has to be calculated in order to arrive at the initial values of the master integrals. The truncation of the infinite sums and high values of N will lead to a proliferation of terms. However, the last step is a simple ε-expansion of ratios of Γ-functions, which can be implemented very efficiently and massively parallelized, making this method of calculating initial values for the master integrals quite efficient.

Let us consider a series of master integrals. For example, one finds

$$\begin{aligned}
B_2^{1,1,1,0,0,1,1,0,0,1,1,0}(N) = {}& \frac{iS_\varepsilon^3}{(4\pi)^6} e^{\frac{3(4-D)}{2}\gamma_E} \Gamma(5-\tfrac{3D}{2}) \sum_{l=0}^{N}\sum_{i=0}^{l} \binom{l}{i} \int_0^1 dz_1 \int_0^1 dz_2 \int_0^1 dz_3 \int_0^1 dz_4 \Biggl\{ \\
& z_1^N \left[z_2(1-z_2)\right]^{\frac{D}{2}-2} z_3^{\frac{D}{2}-3+l-i} (1-z_3)^{N-l+i+\frac{D}{2}-2} \\
& z_4^{2+l-i-\frac{D}{2}} (1-z_4)^{N-l+1-\frac{D}{2}} \left[\frac{z_4 m_a^2}{z_3(1-z_3)} + \frac{(1-z_4)m_b^2}{z_2(1-z_2)}\right]^{\frac{3D}{2}-5} \Biggr\}.
\end{aligned} \tag{4.5}$$

This allows to calculate the following first initial values up to $O(\eta^5)$. For each master integral, we have to consider mass assignment A ($m_a = m_1$, $m_b = m_2$) and mass assignment B ($m_b = m_1$, $m_a = m_2$), which are represented in the superscript of the integral.

$$\begin{aligned}
B_2^{(1,1,1,0,0,1,1,0,0,1,1,0),A}(0) = {}& \left(\frac{m_1^2}{\mu^2}\right)^{3\varepsilon/2} m_1^2 S_\varepsilon^3 \Biggl\{ -\frac{1}{\varepsilon^3}\frac{8(2+\eta)}{3\eta} + \frac{1}{\varepsilon^2}\left[\frac{8}{9\eta} + \frac{4(5+3\ln(\eta))}{3\eta}\right] \\
& + \frac{1}{\varepsilon}\left[-\frac{2}{3} - \zeta_2 - \frac{1}{\eta}\left(\frac{16}{3} + 2\zeta_2 + 4\ln(\eta) + \ln^2(\eta)\right)\right] - \frac{16}{3} + 2\ln(\eta) \\
& - \ln^2(\eta) + \frac{1}{3}\ln^3(\eta) + \zeta_2 + \frac{7}{3}\zeta_3 + \frac{1}{\eta}\left(\frac{5}{3} - \frac{10}{3}\zeta_3 + \ln(\eta) - \frac{1}{6}\ln^3(\eta) \right. \\
& \left. + \frac{1}{2}\zeta_2(5+3\ln(\eta))\right) + \eta\left(\frac{7}{4} - \frac{3}{2}\ln(\eta) + \frac{1}{2}\ln^2(\eta)\right) + \eta^2\left(\frac{19}{108} \right. \\
& \left. - \frac{5}{18}\ln(\eta) + \frac{1}{6}\ln^2(\eta)\right) + \eta^3\left(\frac{37}{864} - \frac{7}{72}\ln(\eta) + \frac{1}{12}\ln^2(\eta)\right)
\end{aligned}$$

$$+\eta^4\left(\frac{61}{4000}-\frac{9}{200}\ln(\eta)+\frac{1}{20}\ln^2(\eta)\right)$$

$$+\eta^5\left(\frac{91}{13500}-\frac{11}{450}\ln(\eta)+\frac{1}{30}\ln^2(\eta)\right)\Bigg\}+O(\eta^6\ln^2(\eta)), \tag{4.6}$$

$$B_2^{(1,1,1,0,0,1,1,0,0,1,1,0),B}(0)=\left(\frac{m_2^2}{\mu^2}\right)^{3\varepsilon/2}m_2^2S_\varepsilon^3\Bigg\{-\frac{1}{\varepsilon^3}\frac{8(1+2\eta)}{3}+\frac{1}{\varepsilon^2}\left[\frac{8}{3}+\frac{4}{3}\eta\big(5-3\ln(\eta)\big)\right]$$

$$+\frac{1}{\varepsilon}\left[-\frac{2}{3}-\zeta_2-\eta\left(\frac{16}{3}+2\zeta_2-4\ln(\eta)+\ln(\eta)^2\right)\right]-\frac{10}{3}+\zeta_2+\frac{7}{3}\zeta_3$$

$$+\eta\left(\frac{11}{3}+\frac{1}{2}\zeta_2\big(5-3\ln(\eta)\big)-\frac{10}{3}\zeta_3-3\ln(\eta)+\ln^2(\eta)-\frac{1}{6}\ln^3(\eta)\right)$$

$$+\eta^2\left(-\frac{7}{4}+\frac{3}{2}\ln(\eta)-\frac{1}{2}\ln^2(\eta)\right)+\eta^3\left(-\frac{19}{108}+\frac{5}{18}\ln(\eta)-\frac{1}{6}\ln^2(\eta)\right)$$

$$+\eta^4\left(-\frac{37}{864}+\frac{7}{72}\ln(\eta)-\frac{1}{12}\ln^2(\eta)\right)+\eta^5\Bigg(-\frac{61}{4000}$$

$$+\frac{9}{200}\ln(\eta)-\frac{1}{20}\ln^2(\eta)\Bigg)\Bigg\}+O(\eta^6\ln^2(\eta)), \tag{4.7}$$

$$B_2^{(1,1,1,0,0,1,1,0,0,1,1,0),A}(1)=\left(\frac{m_1^2}{\mu^2}\right)^{3\varepsilon/2}m_1^2S_\varepsilon^3\Bigg\{-\frac{1}{\varepsilon^3}\frac{4(3+\eta)}{3\eta}+\frac{1}{\varepsilon^2}\left[1+\frac{13+9\ln(\eta)}{3\eta}\right]$$

$$+\frac{1}{\varepsilon}\left[\frac{3}{4}-\frac{1}{2}\zeta_2-\frac{1}{\eta}\left(\frac{11}{4}+\frac{9}{4}\ln(\eta)+\frac{3}{4}\ln^2(\eta)+\frac{3}{2}\zeta_2\right)\right]-\frac{73}{48}-\frac{1}{2}\ln(\eta)$$

$$-\frac{1}{4}\ln^2(\eta)+\frac{1}{6}\ln^3(\eta)+\frac{3}{8}\zeta_2+\frac{7}{6}\zeta_3-\frac{1}{\eta}\Bigg(\frac{23}{48}+\frac{5}{2}\zeta_3+\frac{11}{16}\ln(\eta)$$

$$+\frac{7}{16}\ln^2(\eta)+\frac{1}{8}\ln^3(\eta)-\frac{1}{8}\zeta_2(13+9\ln(\eta))\Bigg)+\eta\Bigg(\frac{415}{432}-\frac{61}{72}\ln(\eta)$$

$$+\frac{7}{24}\ln^2(\eta)\Bigg)+\eta^2\left(\frac{913}{9000}-\frac{49}{300}\ln(\eta)+\frac{1}{10}\ln^2(\eta)\right)+\eta^3\Bigg(\frac{29945}{1185408}$$

$$-\frac{821}{14112}\ln(\eta)+\frac{17}{336}\ln^2(\eta)\Bigg)+\eta^4\left(\frac{53141}{5832000}-\frac{881}{32400}\ln(\eta)+\frac{11}{360}\ln^2(\eta)\right)$$

$$+\eta^5\left(\frac{97289}{23958000}-\frac{1079}{72600}\ln(\eta)+\frac{9}{440}\ln^2(\eta)\right)\Bigg\}+O(\eta^6\ln^2(\eta)), \tag{4.8}$$

$$B_2^{(1,1,1,0,0,1,1,0,0,1,1,0),B}(1)=\left(\frac{m_2^2}{\mu^2}\right)^{3\varepsilon/2}m_2^2S_\varepsilon^3\Bigg\{-\frac{1}{\varepsilon^3}\frac{4(1+3\eta)}{3}+\frac{1}{\varepsilon^2}\left[1+\eta\big(\frac{13}{3}-3\ln(\eta)\big)\right]$$

$$+\frac{1}{\varepsilon}\left[\frac{3}{4}-\frac{1}{2}\zeta_2-\eta\big(\frac{11}{4}-\frac{9}{4}\ln(\eta)+\frac{3}{4}\ln^2(\eta)+\frac{3}{2}\zeta_2\big)\right]-\frac{193}{48}+\frac{3}{8}\zeta_2$$

$$+\frac{7}{6}\zeta_3+\eta\Bigg(\frac{217}{48}-\frac{37}{16}\ln(\eta)+\frac{9}{16}\ln^2(\eta)-\frac{1}{8}\ln^3(\eta)+\frac{1}{8}\zeta_2\big(13-9\ln(\eta)\big)$$

$$-\frac{5}{2}\zeta_3\Bigg)+\eta^2\left(-\frac{265}{216}+\frac{37}{36}\ln(\eta)-\frac{1}{3}\ln^2(\eta)\right)+\eta^3\Bigg(-\frac{6397}{54000}$$

$$+\frac{331}{1800}\ln(\eta)-\frac{13}{120}\ln^2(\eta)\Bigg)+\eta^4\left(-\frac{5585}{197568}+\frac{149}{2352}\ln(\eta)-\frac{3}{56}\ln^2(\eta)\right)$$

$$+\eta^5\left(-\frac{116063}{11664000}+\frac{1883}{64800}\ln(\eta)-\frac{23}{720}\ln^2(\eta)\right)\Bigg\}+O(\eta^6\ln^2(\eta)). \quad (4.9)$$

We can also choose not to expand in η in order to obtain moments exact in the mass ratio. To obtain initial values in this case, we do not truncate the infinite sum as before, but evaluate it with `HarmonicSums` and `Sigma`. The resulting infinite sums can be rewritten in terms of harmonic polylogarithms using the capabilities of `HarmonicSums`.

However, a more effective approach is to perform the expansion in $t \to 0$ and $p^2 = 0$, which leads to two-mass tadpoles, and employ an IBP reduction. We reveal four master integrals in the limit $t \to 0$ which are easily computed from their Mellin-Barnes representation.

When we do not expand in the mass ratio η, higher moments in N develop larger and larger rational function coefficients in η, leading to very large final expressions for the moments. We have calculated the moments up to $N = 60$ to cross-check the x-space results presented in the next section. We note that the expansion in η for a fixed moment converges very quickly.

The initial values allow to compute a large number of moments. We generated 2000 Mellin moments by expanding the master integrals to $O(\eta^5)$. This number is sufficient to determine the corresponding recurrences for the color-ζ-η-log expressions up to $O(\eta^3)$, because of additional poles in η in the amplitudes. For the cases $C_A T_F^2$ and $C_A T_F^2 \eta$, we needed 3000 moments.

All recurrences can be solved by `Sigma` in product-sum expressions for $N > N_0$, $\ N_0 \in \mathbb{N}$ in terms of generalized harmonic sums [107, 191]. The boundary value N_0 grows with the power of the expansion parameter η. I.e. no closed analytic representation can be given for all even moments in N in the unpolarized case. Therefore, the present representation in Mellin N space does not allow the inversion to x-space. Table 1 summarizes the characteristics of the different contributions. The corresponding recurrences are valid for $N \geq N_0$. In most of the cases, non-removable poles limit the validity of the recurrence below N_0, but not in all cases.

5 The semi-analytic approach in x–space

In the previous section, we have seen that the expansion around $\eta = 0$ introduces severe problems for the transition of the results from N- to x-space. One way to find a viable x-space solution is to fix η to a physical value and use the semi-analytic approach already used in the single mass case to calculate a numerical approximation. While this is viable, phenomenological applications need to scan over different values of the masses or want to use different renormalization schemes so that the differential equation needs to be solved for several different values of the quark masses in order to obtain a grid. Another technical issue are poles in the differential equation which move very close to $t = 1$ for physical quark masses. This makes the precise numerical solution of the system a non-trivial task.

In order to find a more flexible representation of the OMEs we will not expand them around $\eta \to 0$, but around the unphysical limit $\eta \to 1$ for which we introduce the expansion variable

$$\delta = 1 - \frac{m_c}{m_b}. \quad (5.1)$$

This approach was already successfully applied to the calculation of inclusive semi-leptonic b quark decays in Ref. [192]. We can also have a look at the analytically known moments to examine the convergence in this limit. Using the results for fixed moments (see Appendix A) and setting

$$m_c = 1.59 \text{ GeV}, \qquad m_b = 4.78 \text{ GeV}, \quad (5.2)$$

cf. [193, 194], and, $\mu = 10$ GeV

Unpolarized case Coefficient	# moments	rec. order	rec. degree	N_0	pole position
$C_F T_F^2$	1550	17	320	2	
$C_F T_F^2 \ln(\eta)$	168	4	43	2	
$C_F T_F^2 \ln^2(\eta)$	56	2	19	2	
$C_F T_F^2 \ln^3(\eta)$	36	2	9	2	
$C_F T_F^2 \zeta_2$	294	7	69	2	
$C_F T_F^2 \zeta_3$	126	4	31	2	
$C_F T_F^2 \zeta_2 \ln(\eta)$	36	2	9	2	
$C_F T_F^2 \eta$	384	8	95	4	2
$C_F T_F^2 \eta \ln(\eta)$	72	2	23	4	
$C_F T_F^2 \eta \ln^2(\eta)$	32	2	8	2	
$C_F T_F^2 \eta^2$	600	13	192	4	2
$C_F T_F^2 \eta^2 \ln(\eta)$	336	8	84	4	2
$C_F T_F^2 \eta^2 \ln^2(\eta)$	42	2	13	4	
$C_F T_F^2 \eta^3$	561	9	151	6	2,4
$C_F T_F^2 \eta^3 \ln(\eta)$	221	5	71	6	2,4
$C_F T_F^2 \eta^3 \ln^2(\eta)$	54	2	17	6	
$C_A T_F^2$	2622	26	513	2	
$C_A T_F^2 \ln(\eta)$	748	12	161	2	
$C_A T_F^2 \ln^2(\eta)$	308	8	70	2	
$C_A T_F^2 \ln^3(\eta)$	108	4	27	2	
$C_A T_F^2 \zeta_2$	360	10	76	2	
$C_A T_F^2 \zeta_3$	108	4	26	2	
$C_A T_F^2 \zeta_2 \ln(\eta)$	108	4	27	2	
$C_A T_F^2 \eta$	2080	23	400	4	2
$C_A T_F^2 \eta \ln(\eta)$	840	14	160	4	
$C_A T_F^2 \eta \ln^2(\eta)$	280	8	55	2	
$C_A T_F^2 \eta^2$	1548	20	311	4	2
$C_A T_F^2 \eta^2 \ln(\eta)$	500	10	113	4	2
$C_A T_F^2 \eta^2 \ln^2(\eta)$	117	4	31	4	
$C_A T_F^2 \eta^3$	1457	17	324	6	2,4
$C_A T_F^2 \eta^3 \ln(\eta)$	525	9	128	6	2,4
$C_A T_F^2 \eta^3 \ln^2(\eta)$	117	4	33	6	

Table 1: Characteristics of the recurrences and the parameter $N_0 \leq N$, N even, for which the respective recurrence is valid in the unpolarized case The poles of the solution of the recurrence are also given.

we obtain for the 10th moment

$$\tilde{A}_{Qg}^{(3)}(N = 10) = -74.215863973188462672...$$

$$
\begin{aligned}
&\approx 4.69095... - 52.87648...\chi - 13.51460...\chi^2 - 5.77578...\chi^3 - 2.91093...\chi^4 \\
&- 1.58651...\chi^5 - 0.90329...\chi^6 - 0.52844...\chi^7 - 0.31488...\chi^8 - 0.19014...\chi^9 \\
&- 0.11599...\chi^{10} - 0.07134...\chi^{11} - 0.04417...\chi^{12} - 0.02750...\chi^{13} \\
&- 0.01720...\chi^{14} - 0.01081...\chi^{15} - 0.00681...\chi^{16} - 0.00431...\chi^{17} \\
&- 0.00273...\chi^{18} - 0.00174...\chi^{19} + \mathcal{O}(\delta^{20}) \\
&= -74.212774664755273018..., \qquad (5.3)
\end{aligned}
$$

where we set

$$\delta = \left(1 - \frac{m_c}{m_b}\right)\chi \qquad (5.4)$$

in order to see the individual terms in the expansion.

We see that keeping the first 20 terms in the expansion allows for an accuracy of 0.004% for the individual moment. Using Aitken extrapolation [195] we can push the accuracy even further. Using the partial sums

$$\tilde{A}_{Qg}^{(3)}(N = 10) = \sum_{j=0}^{\infty} \tilde{A}_{Qg}^{(3),\delta,k}(N = 10)\delta^k \ , \qquad (5.5)$$

$$\tilde{A}_{Qg}^{(3),k}(N = 10) = \sum_{j=0}^{k} \tilde{A}_{Qg}^{(3),\delta,k}(N = 10)\delta^k, \qquad (5.6)$$

we use one step in the series acceleration to obtain

$$\tilde{A}_{Qg}^{(3),\mathrm{acc}}(N = 10) = \frac{\left[\tilde{A}_{Qg}^{(3),k}(N = 10)\right]^2 - \left[\tilde{A}_{Qg}^{(3),k-1}(N = 10)\right]^2}{\tilde{A}_{Qg}^{(3),k}(N = 10) - 2\tilde{A}_{Qg}^{(3),k-1}(N = 10) + \tilde{A}_{Qg}^{(3),k-2}(N = 10)} \ ,$$

which converges to the same limit as $k \to \infty$, but usually shows a better convergence. In our case we find

$$\tilde{A}_{Qg}^{(3),\mathrm{acc}}(N = 10) = -74.215816598482... \ , \qquad (5.7)$$

which shows a relative deviation of only 0.00006% from the exact result. In principle, this procedure can be repeated, however, one step of the sum acceleration already leads to sufficiently precise predictions.

To obtain the OMEs in an expansion around the equal mass limit, we proceed as follows.

- We use the differential equation in m_b for the two-mass master integrals to obtain an expansion to 21 terms in the equal mass limit. To do this, we plug in an ansatz of the form

$$\vec{I}(t, m_a, m_b, \varepsilon) = \sum_{i=0}^{\infty} \delta^i \vec{I}_i(t, m_a, m_a, \varepsilon) \ , \qquad (5.8)$$

  where $\vec{I}$ is the vector of master integrals, into the differential equation, and solve for the coefficients $\vec{I}_0(t, m_a, m_a, \varepsilon)$ as boundary conditions.

- The boundary conditions $\vec{I}_0(t, m_a, m_a, \varepsilon)$ are the integrals we already encountered in the single mass case (cf. Ref. [11, 12]), where they have been calculated using semi-analytic methods.

- We plug all expansions back into our amplitude expression and expand consistently in δ. This allows us to compute $\tilde{A}_{Qg}^{(3)}$ to order δ^{19} and, due to higher poles in the amplitude, $\Delta\tilde{A}_{Qg}^{(3)}$ to order δ^{18}. If more precise results are needed in the future, we can extend the expansion in δ easily.

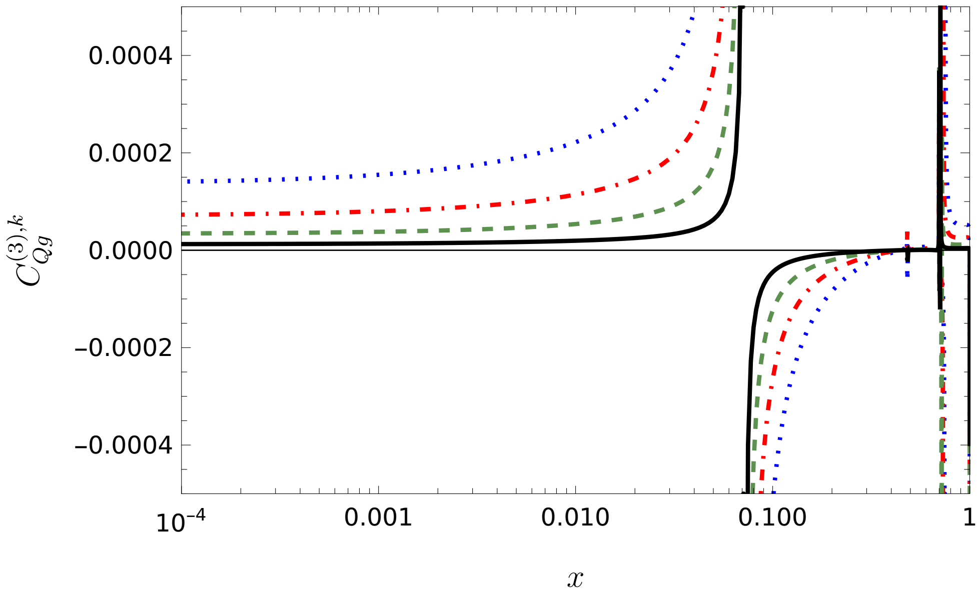

Figure 2: The relative difference between different expansions depths of $\tilde{A}_{Qg}^{(3)}$ in δ to our best approximation which uses 20 expansion terms for $Q^2 = 50\ \mathrm{GeV}^2$. The curves show the relative difference to the expansion up to δ^{15}, $k = 15$ (blue dotted line), δ^{16}, $k = 16$ (red dash-dotted line), δ^{17}, $k = 17$ (green dashed line), δ^{18}, $k = 18$ (black full line).

N	$\tilde{A}_{Qg}^{(3),\text{exact}}(N)$	$r_{Qg}^{(3)}(N)[\%]$	N	$\Delta\tilde{A}_{Qg}^{(3),\text{exact}}(N)$	$\Delta r_{Qg}^{(3)}(N)[\%]$
2	–4236.5	0.00004	1	0	(0.00004)
6	–3091.6	0.00005	5	–2988.0	0.00008
12	–1970.7	0.00005	11	–2026.4	0.00008

Table 2: Relative deviation of exact moments and the ones obtained by numerically integrating over the x-space approximation as defined in Eq. (5.10). Since $\Delta\tilde{A}_{Qg}^{(3),\text{exact}}(N = 1)$ vanishes, we give the absolute difference for this value.

The approximation shows good perturbative convergence, which is illustrated in Figure 2, where we plot the quantity

$$C_{Qg}^{(3),k}(x, Q^2) = 1 - \frac{\tilde{A}_{Qg}^{(3),k}(x, Q^2)}{\tilde{A}_{Qg}^{(3),19}(x, Q^2)}. \tag{5.9}$$

Here, we borrow the notation of Eq. (5.6) also for the partial sums in x-space. Since the denominator vanishes at $x \sim 0.07$ the relative difference diverges there. However, the value of all approximations and the absolute differences are small in this region. Overall, one sees that higher expansion orders converge homogeneously. At $x \sim 0.75$, we see a slower convergence of the series. However, the relative difference between the two last expansion terms is smaller than 0.001, sufficient for any practical purpose. The convergence in the polarized case is similar.

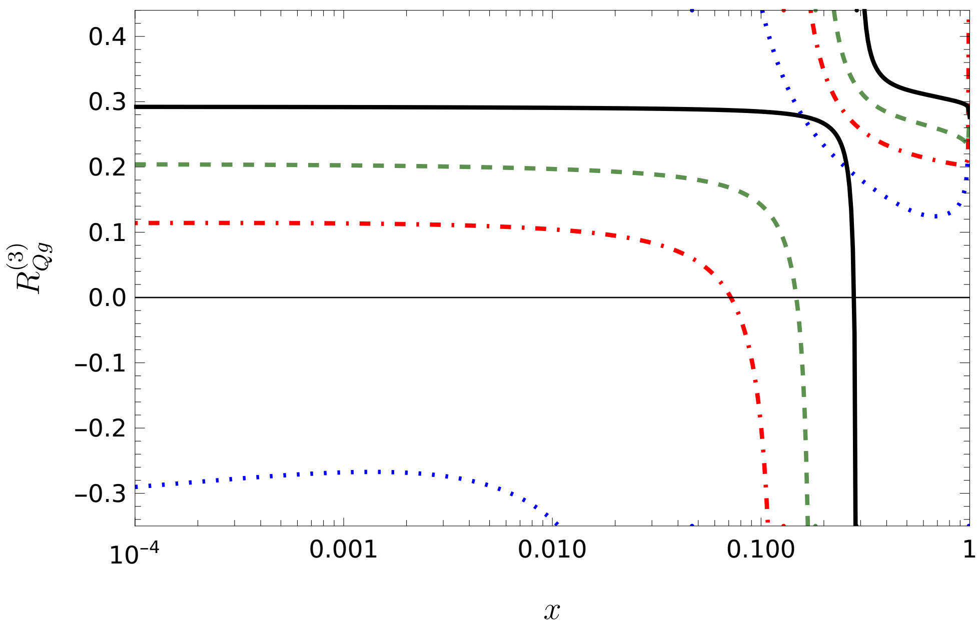

Figure 3: The ratio $R_{Qg}^{(3)}(x, Q^2)$ of the 2-mass contributions to $\tilde{A}_{Qg}^{(3)}$ to the complete contribution of $O(T_F^2)$ and $O(T_F^3)$ to $A_{Qg}^{(3)}$ as defined in Eq. (5.11) in the unpolarized case. $Q^2 = 30 \text{ GeV}^2$ dotted line (blue), $Q^2 = 50 \text{ GeV}^2$ dash-dotted line (red), $Q^2 = 100 \text{ GeV}^2$ dashed line (green), $Q^2 = 1000 \text{ GeV}^2$ full line (black) .

Note that for the evaluation of the OMEs, we use one step of the sum acceleration at each point of x separately. From the difference between the approximations using our highest available expansion depth and one term less we can even estimate the residual error of our approximation. However, due to the high expansion orders already available, the residual error is rather small and will not be considered further. As a check, we can compare the relative difference between fixed moments which we have computed analytically and the moments we obtain by numerically integrating over our approximation

$$(\Delta) r_{Qg}^{(3)}(N, Q^2) = \frac{(\Delta)\tilde{A}_{Qg}^{(3),\text{exact}}(N, Q^2) - (\Delta)\tilde{A}_{Qg}^{(3),\text{approx}}(N, Q^2)}{(\Delta)\tilde{A}_{Qg}^{(3),\text{exact}}(N, Q^2)} \,. \tag{5.10}$$

The results for some moments are shown in Table 2. There is excellent agreement between the exact moments of the OME and the ones obtained by numerically integrating over the x-space approximation obtained after expanding in δ. Note that the achieved relative error is of similar size as the one obtained from the expansion of the moment directly. This shows that the errors due to the approximate solution of the x-space through generalized series expansions are negligible.

In order to illustrate the phenomenological results, we show the ratio of the two-mass contributions over the contributions of the color factors T_F^2 and T_F^3 to the full OME

$$(\Delta)R_{Qg}^{(3)}(x,Q^2) = \frac{(\Delta)\tilde{A}_{Qg}^{(3)}(x,Q^2)}{(\Delta)A_{Qg}^{(3),T_F^2+T_F^3}(x,Q^2)}\ , \tag{5.11}$$

in the unpolarized case in Figure 3 and in the polarized case in Figure 4, where we work in the Larin scheme. We see that the two-mass contributions make up -30% to 30% in the small x region moving from Q^2 from 30 GeV^2 to 1000 GeV^2. At large x, the ratio varies between 20% and 50%. In the polarized case the picture is similar and one observes variations between -20% to 40%.

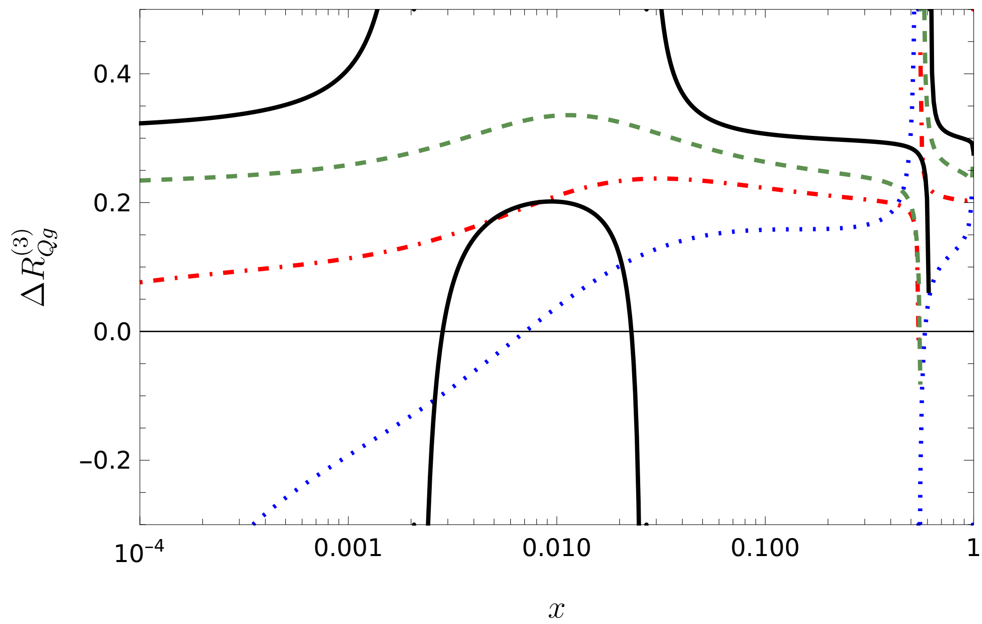

Figure 4: The ratio $\Delta R_{Qg}^{(3)}(x,Q^2)$ of the 2-mass contributions to $\Delta\tilde{A}_{Qg}^{(3)}$ to the complete contribution of $O(T_F^2)$ and $O(T_F^3)$ to $\Delta A_{Qg}^{(3)}$ as defined in Eq. (5.11) in the polarized case in the Larin scheme. $Q^2 = 30\ \mathrm{GeV}^2$ dotted line (blue), $Q^2 = 50\ \mathrm{GeV}^2$ dash-dotted line (red), $Q^2 = 100\ \mathrm{GeV}^2$ dashed line (green), $Q^2 = 1000\ \mathrm{GeV}^2$ full line (black) .

The polarized OME $\Delta\tilde{A}_{Qg}$ contributes to the 2-mass variable flavor number scheme [14] and the polarized heavy-flavor Wilson coefficients for $Q^2 \gg m_Q^2$. By referring to the polarized massless Wilson coefficients [79], the anomalous dimensions [24,65], and the parton distributions [196], one can represent the structure function $g_1(x,Q^2)$ as an observable.

6 Conclusions

We have calculated the two-mass corrections to the constant part of the three-loop massive OMEs $\tilde{A}_{Qg}^{(3)}$ and $\Delta\tilde{A}_{Qg}^{(3)}$ using a semi-analytic method in x-space. Initially, we have considered the structure of this contribution in Mellin N-space by expanding in the variable η to the first few powers. It turned out that a closed N-space solution covering all physical values of $N = 2k, k \in \mathbb{N}$ in the unpolarized case and of $N = 2k+1$ in the polarized case cannot be found at present. Like known in the x-space solution, one may conjecture that the N-space solution contains letters which are higher transcendental functions in the whole range of N. Sum-product solutions are only possible for $N > N_0$, where the value of N_0 is not always caused by poles in the corresponding expressions. In general, it is expected that in the case of the present amplitude, non-first-order-factorizing contributions are present, which will not thoroughly lead to sum-product solutions, since these obey first-order-factorizing recurrences. Nevertheless, by employing first-order differential equations of the master integrals in the generating-function-variable t, and by fixing the value of η, we can solve the two-parameter problem. Like in the single-mass case Ref. [12] for $x \in]0,1]$, highly precise local series expansions can be derived. The ratios $(\Delta)\tilde{A}_{Qg}^{(3)}/(\Delta)A_{Qg}^{(3),\mathrm{T}_\mathrm{F}^2}$ illustrate the importance of this contribution. They amount to about -30% to 50% of all $T_F^2 C_i$ contributions, with $C_i = C_F, C_A$ or T_F. A numerical program to evaluate the two-mass three-loop contributions to the unpolarized and polarized massive OMEs will be published in a later paper.

A Mellin moments for $\tilde{a}_{Qg}^{(3)}$ and $\Delta\tilde{a}_{Qg}^{(3)}$ at fixed values of N

The Mellin moments given in Refs. [27, 28] concerned only the contributions due to irreducible Feynman diagrams in the unpolarized case up to terms of $O(\eta^3)$. Here we present the complete results including terms of $O(\eta^5)$. We use the abbreviations

$$L_1 = \ln\left(\frac{m_b^2}{\mu^2}\right), \quad L_2 = \ln\left(\frac{m_c^2}{\mu^2}\right), \quad L_\eta = \ln\left(\frac{m_c^2}{m_b^2}\right), \quad \xi = \frac{m_c}{m_b}. \tag{A.1}$$

One obtains

$$\begin{aligned}
\tilde{a}_{Qg}^{(3)}(N=2) \quad \propto \quad & T_F^2 \Biggl\{ C_A \Biggl[\frac{59314}{2187} + \frac{1964L_1}{243} - \frac{4009L_1^2}{216} + \frac{1276L_1^3}{81} + \frac{1964L_2}{243} - \frac{1351L_1L_2}{108} \\
& + \frac{440L_1^2L_2}{27} - \frac{4009L_2^2}{216} + \frac{616L_1L_2^2}{27} + \frac{1100L_2^3}{81} - \frac{4880L_\eta}{243} + \frac{3307L_\eta^2}{648} \\
& - \frac{8\eta^3}{843908625}\bigl(- 7010842 - 6475770L_\eta + 31652775L_\eta^2\bigr) \\
& - \frac{4\eta^2}{496125}\bigl(- 391259 + 157710L_\eta + 22050L_\eta^2\bigr) + \eta\Biggl(\frac{256304}{10125} + \frac{161L_1}{108} - \frac{161L_1^2}{432} \\
& - \frac{161L_2}{108} + \frac{161L_1L_2}{216} - \frac{161L_2^2}{432} - \frac{8237L_\eta}{900} + \frac{421L_\eta^2}{2160}\Biggr) \\
& + \Biggl(- \frac{1340}{81} + \frac{308L_1}{9} + \frac{308L_2}{9}\Biggr)\zeta_2 - \frac{176}{81}\zeta_3\Biggr] \\
& + C_F \Biggl[\frac{25556}{729} + \frac{13636L_1}{243} + \frac{16957L_1^2}{324} - \frac{992L_1^3}{81} + \frac{13636L_2}{243} + \frac{10691L_1L_2}{162} \\
& - \frac{64L_1^2L_2}{27} + \frac{16957L_2^2}{324} - \frac{320L_1L_2^2}{27} - \frac{736L_2^3}{81} + \frac{3184L_\eta}{243} - \frac{925L_\eta^2}{324} \\
& + \frac{32\eta^3}{843908625}\bigl(25837687 - 58011030L_\eta + 74021850L_\eta^2\bigr) \\
& + \frac{32\eta^2}{10418625}\bigl(5309152 - 3373755L_\eta + 1642725L_\eta^2\bigr) + \eta\Biggl(- \frac{758944}{30375} + \frac{409L_1}{54}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{409L_1^2}{216}-\frac{409L_2}{54}+\frac{409L_1L_2}{108}-\frac{409L_2^2}{216}-\frac{15277L_\eta}{4050}+\frac{5629L_\eta^2}{1080}\Bigg) \\
& +\left(\frac{416}{9}-\frac{160L_1}{9}-\frac{160L_2}{9}\right)\zeta_2+\frac{1408}{81}\zeta_3\Bigg]\Bigg\} \\
& +T_F^3\left[-\frac{32L_1^3}{3}-\frac{64L_1^2L_2}{3}-\frac{64L_1L_2^2}{3}-\frac{32L_2^3}{3}-32(L_1+L_2)\zeta_2-\frac{128}{9}\zeta_3\right], \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{a}_{Qg}^{(3)}(N=4) \quad \propto \quad & T_F^2\Bigg\{C_A\Bigg[\frac{4366284317}{36450000}+\frac{64863121L_1}{972000}+\frac{965364173L_1^2}{82944000}+\frac{37642L_1^3}{2025}+\frac{64817671L_2}{972000} \\
& +\frac{1259765299L_1L_2}{41472000}+\frac{2596L_1^2L_2}{135}+\frac{965364173L_2^2}{82944000}+\frac{18172L_1L_2^2}{675}+\frac{1298L_2^3}{81} \\
& -\frac{120332519L_\eta}{4860000}+\frac{518749747L_\eta^2}{82944000}+\eta^3\Bigg(\frac{250077164867}{11232423798750}+\frac{101L_1}{2160}-\frac{101L_1^2}{1440} \\
& -\frac{101L_2}{2160}+\frac{101L_1L_2}{720}-\frac{101L_2^2}{1440}+\frac{2461291549L_\eta}{25933446000}-\frac{4904369L_\eta^2}{14968800}\Bigg) \\
& +\eta^2\Bigg(\frac{1255194149}{468838125}+\frac{80203L_1}{2764800}-\frac{468043L_1^2}{11059200}-\frac{80203L_2}{2764800}+\frac{468043L_1L_2}{5529600} \\
& -\frac{468043L_2^2}{11059200}-\frac{6519186689L_\eta}{6096384000}-\frac{17662451L_\eta^2}{77414400}\Bigg) \\
& +\eta\Bigg(\frac{496855133}{14883750}+\frac{559627L_1}{460800}-\frac{144839L_1^2}{460800}-\frac{559627L_2}{460800} \\
& +\frac{144839L_1L_2}{230400}-\frac{144839L_2^2}{460800}-\frac{1746174071L_\eta}{145152000}+\frac{796501L_\eta^2}{1382400}\Bigg)+\Bigg(\frac{120721}{6750} \\
& +\frac{9086L_1}{225}+\frac{9086L_2}{225}\Bigg)\zeta_2-\frac{5192\zeta_3}{2025}\Bigg] \\
& +C_F\Bigg[-\frac{12930316237}{2187000000}+\frac{417741347L_1}{24300000}+\frac{352861499L_1^2}{17280000}-\frac{360041L_1^3}{40500} \\
& +\frac{424203347L_2}{24300000}+\frac{14360107L_1L_2}{576000}-\frac{33209L_1^2L_2}{6750}+\frac{352861499L_2^2}{17280000} \\
& -\frac{70411L_1L_2^2}{6750}-\frac{285637L_2^3}{40500}+\frac{25621379L_\eta}{3037500}-\frac{98948977L_\eta^2}{51840000} \\
& +\eta^3\Bigg(\frac{23024568781}{44929695195}-\frac{359L_1}{1350}+\frac{359L_1^2}{900}+\frac{359L_2}{1350} \\
& -\frac{359L_1L_2}{450}+\frac{359L_2^2}{900}-\frac{742502683L_\eta}{648336150}+\frac{2772871L_\eta^2}{1871100}\Bigg)+\eta^2\Bigg(\frac{582667691}{75014100} \\
& -\frac{441313L_1}{1728000}+\frac{1819873L_1^2}{6912000}+\frac{441313L_2}{1728000}-\frac{1819873L_1L_2}{3456000}+\frac{1819873L_2^2}{6912000}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{3876440473L_\eta}{762048000}+\frac{14002001L_\eta^2}{5376000}\Bigg)+\eta\Bigg(-\frac{59657237}{4134375}+\frac{2875369L_1}{864000}-\frac{55159L_1^2}{72000} \\
& -\frac{2875369L_2}{864000}+\frac{55159L_1L_2}{36000}-\frac{55159L_2^2}{72000}-\frac{40838437L_\eta}{30240000}+\frac{65917L_\eta^2}{24000}\Bigg) \\
& +\left(\frac{6503111}{405000}-\frac{70411L_1}{4500}-\frac{70411L_2}{4500}\right)\zeta_2+\frac{78397\zeta_3}{10125}\Bigg]\Bigg\} \\
& +T_F^3\Bigg[-\frac{88L_1^3}{15}-\frac{176L_1^2L_2}{15}-\frac{176L_1L_2^2}{15}-\frac{88L_2^3}{15}+\left(-\frac{88L_1}{5}-\frac{88L_2}{5}\right)\zeta_2 \\
& -\frac{352}{45}\zeta_3\Bigg], \qquad\qquad (A.3)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{a}_{Qg}^{(3)}(N=6) \quad \propto \quad & T_F^2\Bigg\{C_A\Bigg[\frac{8699108665601}{73513818000}+\frac{499483057039L_1}{7468070400}+\frac{2934742201991L_1^2}{182078668800}+\frac{128557L_1^3}{7938} \\
& +\frac{499300102897L_2}{7468070400}+\frac{3078895880569L_1L_2}{91039334400}+\frac{22165L_1^2L_2}{1323}+\frac{2934742201991L_2^2}{182078668800} \\
& +\frac{4433L_1L_2^2}{189}+\frac{110825L_2^3}{7938}-\frac{154918449457L_\eta}{7468070400}+\frac{949106952569L_\eta^2}{182078668800} \\
& +\eta^3\Bigg(-\frac{84840004938801319}{1381947564807810000}+\frac{217742347L_1}{4335206400}-\frac{282650969L_1^2}{5780275200}-\frac{217742347L_2}{4335206400} \\
& +\frac{282650969L_1L_2}{2890137600}-\frac{282650969L_2^2}{5780275200}+\frac{50096069455446961L_\eta}{251324552134656000} \\
& -\frac{9178336044211L_\eta^2}{22317642547200}\Bigg)+\eta^2\Bigg(\frac{755537213056}{624023544375}+\frac{62549797L_1}{4335206400}-\frac{398602763L_1^2}{17340825600} \\
& -\frac{62549797L_2}{4335206400}+\frac{398602763L_1L_2}{8670412800}-\frac{398602763L_2^2}{17340825600}-\frac{6551048225287L_\eta}{23605198848000} \\
& -\frac{86215219591L_\eta^2}{190749081600}\Bigg)+\eta\Bigg(\frac{832369820129}{29172150000}+\frac{448073153L_1}{867041280}-\frac{2325070613L_1^2}{17340825600} \\
& -\frac{448073153L_2}{867041280}+\frac{2325070613L_1L_2}{8670412800}-\frac{2325070613L_2^2}{17340825600}-\frac{21862063585279L_\eta}{2275983360000} \\
& +\frac{19009228649L_\eta^2}{86704128000}\Bigg)+\left(\frac{6117389}{277830}+\frac{4433L_1}{126}+\frac{4433L_2}{126}\right)\zeta_2 \\
& -\frac{8866\zeta_3}{3969}\Bigg] \\
& +C_F\Bigg[-\frac{9883289655671}{428830605000}+\frac{275930321L_1}{131712000}+\frac{25268241945419L_1^2}{2549101363200}-\frac{1106501L_1^3}{138915} \\
& +\frac{903763853L_2}{395136000}+\frac{1959440025187L_1L_2}{182078668800}-\frac{43142L_1^2L_2}{9261}+\frac{25268241945419L_2^2}{2549101363200} \\
& -\frac{439406L_1L_2^2}{46305}-\frac{25223L_2^3}{3969}+\frac{4266955293521L_\eta}{522764928000}-\frac{4852991775563L_\eta^2}{2549101363200}
\end{aligned}
$$

$$
\begin{aligned}
&+\eta^3\Biggl(\frac{990283034941336}{2467763508585375}-\frac{12005277251L_1}{40461926400}+\frac{18378891457L_1^2}{53949235200}\\
&+\frac{12005277251L_2}{40461926400}-\frac{18378891457L_1L_2}{26974617600}+\frac{18378891457L_2^2}{53949235200}\\
&-\frac{174872727202100857L_\eta}{201059641707724800}+\frac{145121184651853L_\eta^2}{124978798264320}\Biggr)+\eta^2\Biggl(\frac{524351089261}{97070329125}\\
&-\frac{99760601L_1}{421478400}+\frac{50312021201L_1^2}{242771558400}+\frac{99760601L_2}{421478400}-\frac{50312021201L_1L_2}{121385779200}\\
&+\frac{50312021201L_2^2}{242771558400}-\frac{5027545889867L_\eta}{1376969932800}+\frac{212963348429L_\eta^2}{106819485696}\Biggr)\\
&+\eta\Biggl(-\frac{32427817736}{2552563125}+\frac{273911981947L_1}{121385779200}-\frac{251102611049L_1^2}{485543116800}\\
&-\frac{273911981947L_2}{121385779200}+\frac{251102611049L_1L_2}{242771558400}-\frac{251102611049L_2^2}{485543116800}\\
&-\frac{4936024379021L_\eta}{12745506816000}+\frac{1040084747881L_\eta^2}{485543116800}\Biggr)+\Biggl(\frac{29196929}{4862025}-\frac{219703L_1}{15435}\\
&-\frac{219703L_2}{15435}\Biggr)\zeta_2+\frac{130108\zeta_3}{19845}\Biggr]\Biggr\}+T_F^3\Biggl[-\frac{88L_1^3}{21}-\frac{176L_1^2L_2}{21}-\frac{176L_1L_2^2}{21}\\
&-\frac{88L_2^3}{21}+\Biggl(-\frac{88L_1}{7}-\frac{88L_2}{7}\Biggr)\zeta_2-\frac{352}{63}\zeta_3\Biggr]. \qquad \text{(A.4)}
\end{aligned}
$$

The corresponding unexpanded quantities are given by

$$
\begin{aligned}
&\tilde{a}_{Qg}^{(3)}(N=2)=\\
&T_F^3P_{29}+T_F^2\Biggl\{C_A\Biggl[-\frac{28043}{2187}+\frac{1276L_1^3}{81}+\frac{440L_1^2L_2}{27}+\frac{616L_1L_2^2}{27}+\frac{1100L_2^3}{81}-\frac{161}{54\xi^2}\\
&-\frac{161\xi^2}{54}+\frac{L_1^2\bigl(-161-8018\xi^2-161\xi^4\bigr)}{432\xi^2}+\frac{L_2^2\bigl(-161-8018\xi^2-161\xi^4\bigr)}{432\xi^2}\\
&+\frac{7L_1L_2\bigl(23-386\xi^2+23\xi^4\bigr)}{216\xi^2}+\frac{H_{1,0,0}(\xi)P_1}{108\xi^3}+\frac{H_{-1,0,0}(\xi)P_2}{108\xi^3}\\
&+L_2\Biggl(\frac{1449+7856\xi^2-1449\xi^4}{972\xi^2}+\frac{308\zeta_2}{9}\Biggr)+L_1\Biggl(\frac{-1449+7856\xi^2+1449\xi^4}{972\xi^2}+\frac{308\zeta_2}{9}\Biggr)\\
&-\frac{1340}{81}\zeta_2+\frac{248}{27}\zeta_3-\frac{920\zeta_3}{81}\Biggr]+C_F\Biggl[\frac{45478}{729}-\frac{992L_1^3}{81}-\frac{64L_1^2L_2}{27}-\frac{320L_1L_2^2}{27}\\
&-\frac{736L_2^3}{81}-\frac{409}{27\xi^2}-\frac{409\xi^2}{27}+\frac{L_1^2\bigl(-1227+33914\xi^2-1227\xi^4\bigr)}{648\xi^2}\\
&+\frac{L_2^2\bigl(-1227+33914\xi^2-1227\xi^4\bigr)}{648\xi^2}+\frac{L_1L_2\bigl(1227+21382\xi^2+1227\xi^4\bigr)}{324\xi^2}\\
&+\frac{H_{-1,0,0}(\xi)P_3}{54\xi^3}+\frac{H_{1,0,0}(\xi)P_4}{54\xi^3}+L_2\Biggl(\frac{3681+27272\xi^2-3681\xi^4}{486\xi^2}-\frac{160\zeta_2}{9}\Biggr)
\end{aligned}
$$

$$+L_1\left(\frac{-3681+27272\xi^2+3681\xi^4}{486\xi^2}-\frac{160\zeta_2}{9}\right)+\frac{416}{9}\zeta_2+\frac{1408}{81}\zeta_3\Bigg]\Bigg\}, \tag{A.5}$$

$$\tilde{a}_{Qg}^{(3)}(N=4)=$$

$$\frac{11T_F^3P_{29}}{20}+T_F^2\Bigg\{C_A\Bigg[\frac{32864913953}{466560000}+\frac{37642L_1^3}{2025}+\frac{2596L_1^2L_2}{135}+\frac{18172L_1L_2^2}{675}+\frac{1298L_2^3}{81}$$
$$+\frac{5453}{153600\xi^4}-\frac{303011}{129600\xi^2}-\frac{303011\xi^2}{129600}+\frac{5453\xi^4}{153600}+\frac{H_{1,0,0}(\xi)P_7}{2764800\xi^5}+\frac{H_{-1,0,0}(\xi)P_8}{2764800\xi^5}$$
$$+\frac{L_1L_2P_{13}}{82944000(1-\xi^2)^2\xi^4}+\frac{L_1^2P_{16}}{165888000(1-\xi^2)^2\xi^4}+\frac{L_2^2P_{16}}{165888000(1-\xi^2)^2\xi^4}$$
$$+L_1\left(\frac{P_6}{124416000(\xi-1)\xi^4(1+\xi)}+\frac{9086\zeta_2}{225}\right)+L_2\left(\frac{P_{11}}{124416000(\xi-1)\xi^4(1+\xi)}+\frac{9086\zeta_2}{225}\right)$$
$$+\frac{120721\zeta_2}{6750}+\frac{6182}{675}\zeta_3-\frac{23738\zeta_3}{2025}\Bigg]+C_F\Bigg[\frac{10051995341}{874800000}-\frac{360041L_1^3}{40500}-\frac{33209L_1^2L_2}{6750}$$
$$-\frac{70411L_1L_2^2}{6750}-\frac{285637L_2^3}{40500}-\frac{2023}{96000\xi^4}-\frac{2328341}{324000\xi^2}-\frac{2328341\xi^2}{324000}-\frac{2023\xi^4}{96000}+\frac{H_{-1,0,0}(\xi)P_9}{1728000\xi^5}$$
$$+\frac{H_{1,0,0}(\xi)P_{10}}{1728000\xi^5}+\frac{L_1^2P_{14}}{34560000(1-\xi^2)^2\xi^4}+\frac{L_2^2P_{14}}{34560000(1-\xi^2)^2\xi^4}+\frac{L_1L_2P_{15}}{3456000(1-\xi^2)^2\xi^4}$$
$$+L_2\left(\frac{P_5}{388800000(\xi^2-1)\xi^4}-\frac{70411\zeta_2}{4500}\right)+L_1\left(\frac{P_{12}}{388800000(\xi^2-1)\xi^4}-\frac{70411\zeta_2}{4500}\right)$$
$$+\frac{6503111\zeta_2}{405000}+\frac{78397}{10125}\zeta_3\Bigg]\Bigg\}, \tag{A.6}$$

$$\tilde{a}_{Qg}^{(3)}(N=6)=$$

$$\frac{11T_F^3P_{29}}{28}+T_F^2\Bigg\{C_A\Bigg[\frac{128557L_1^3}{7938}+\frac{22165L_1^2L_2}{1323}+\frac{4433L_1L_2^2}{189}+\frac{110825L_2^3}{7938}$$
$$+\frac{237644145113224327}{301112598528000(1-\xi^2)^2}-\frac{18297}{3211264(1-\xi^2)^2\xi^6}+\frac{503795}{9633792(1-\xi^2)^2\xi^4}$$
$$-\frac{17392516271}{16257024000(1-\xi^2)^2\xi^2}-\frac{4691336790174373\xi^2}{30111259852800(1-\xi^2)^2}+\frac{237644145113224327\xi^4}{301112598528000(1-\xi^2)^2}$$
$$-\frac{17392516271\xi^6}{16257024000(1-\xi^2)^2}+\frac{503795\xi^8}{9633792(1-\xi^2)^2}-\frac{18297\xi^{10}}{3211264(1-\xi^2)^2}+\frac{H_{1,0,0}(\xi)P_{19}}{4335206400\xi^7}$$
$$+\frac{H_{-1,0,0}(\xi)P_{20}}{4335206400\xi^7}+\frac{L_1^2P_{26}}{364157337600(1-\xi^2)^4\xi^6}+\frac{L_2^2P_{26}}{364157337600(1-\xi^2)^4\xi^6}$$
$$+\frac{L_1L_2P_{28}}{182078668800(1-\xi^2)^4\xi^6}+L_2\left(\frac{P_{21}}{1911826022400(\xi^2-1)^3\xi^6}+\frac{4433\zeta_2}{126}\right)$$
$$+L_1\left(\frac{P_{24}}{1911826022400(\xi^2-1)^3\xi^6}+\frac{4433\zeta_2}{126}\right)+\frac{6117389\zeta_2}{277830}+\frac{10252\zeta_3}{1323}-\frac{39622\zeta_3}{3969(1-\xi^2)^2}$$
$$+\frac{79244\xi^2\zeta_3}{3969(1-\xi^2)^2}-\frac{39622\xi^4\zeta_3}{3969(1-\xi^2)^2}\Bigg]$$

$$
\begin{aligned}
&+C_F\Bigg[-\frac{1106501L_1^3}{138915}-\frac{43142L_1^2L_2}{9261}-\frac{439406L_1L_2^2}{46305}-\frac{25223L_2^3}{3969}\\
&+\frac{6110987626095533}{1756490158080000(1-\xi^2)^2}-\frac{33885}{89915392(1-\xi^2)^2\xi^6}-\frac{189583}{56197120(1-\xi^2)^2\xi^4}\\
&-\frac{74541151381}{15173222400(1-\xi^2)^2\xi^2}+\frac{20279035268810711\xi^2}{7025960632320000(1-\xi^2)^2}+\frac{6110987626095533\xi^4}{1756490158080000(1-\xi^2)^2}\\
&-\frac{74541151381\xi^6}{15173222400(1-\xi^2)^2}-\frac{189583\xi^8}{56197120(1-\xi^2)^2}-\frac{33885\xi^{10}}{89915392(1-\xi^2)^2}+\frac{H_{-1,0,0}(\xi)P_{17}}{24277155840\xi^7}\\
&+\frac{H_{1,0,0}(\xi)P_{18}}{24277155840\xi^7}+\frac{L_1^2P_{25}}{10196405452800(1-\xi^2)^4\xi^6}+\frac{L_2^2P_{25}}{10196405452800(1-\xi^2)^4\xi^6}\\
&+\frac{L_1L_2P_{27}}{728314675200(1-\xi^2)^4\xi^6}+L_2\left(\frac{P_{22}}{606928896000(\xi^2-1)^3\xi^6}-\frac{219703\zeta_2}{15435}\right)\\
&+L_1\left(\frac{P_{23}}{606928896000(\xi^2-1)^3\xi^6}-\frac{219703\zeta_2}{15435}\right)+\frac{29196929\zeta_2}{4862025}-\frac{31196\zeta_3}{6615}+\frac{223696\zeta_3}{19845(1-\xi^2)^2}\\
&-\frac{447392\xi^2\zeta_3}{19845(1-\xi^2)^2}+\frac{223696\xi^4\zeta_3}{19845(1-\xi^2)^2}\Bigg]\Bigg\}. \qquad \text{(A.7)}
\end{aligned}
$$

The polynomials P_i are

$$P_1 = 161\xi^6+2151\xi^4-5632\xi^3+2151\xi^2+161, \qquad \text{(A.8)}$$

$$P_2 = 161\xi^6+2151\xi^4+5632\xi^3+2151\xi^2+161, \qquad \text{(A.9)}$$

$$P_3 = 409\xi^6-753\xi^4-4096\xi^3-753\xi^2+409, \qquad \text{(A.10)}$$

$$P_4 = 409\xi^6-753\xi^4+4096\xi^3-753\xi^2+409, \qquad \text{(A.11)}$$

$$P_5 = -4096575\xi^{10}-1393211475\xi^8+8081169602\xi^6-5389945502\xi^4-1393211475\xi^2 -4096575, \qquad \text{(A.12)}$$

$$P_6 = -2208465\xi^{10}+147490155\xi^8+8151380198\xi^6-8447761178\xi^4+147490155\xi^2 -2208465, \qquad \text{(A.13)}$$

$$P_7 = -49077\xi^{10}+3233935\xi^8+68098470\xi^6-170131456\xi^5+68098470\xi^4+3233935\xi^2 -49077, \qquad \text{(A.14)}$$

$$P_8 = -49077\xi^{10}+3233935\xi^8+68098470\xi^6+170131456\xi^5+68098470\xi^4+3233935\xi^2 -49077, \qquad \text{(A.15)}$$

$$P_9 = 18207\xi^{10}+6208235\xi^8-15266250\xi^6-76189696\xi^5-15266250\xi^4+6208235\xi^2 +18207, \qquad \text{(A.16)}$$

$$P_{10} = 18207\xi^{10}+6208235\xi^8-15266250\xi^6+76189696\xi^5-15266250\xi^4+6208235\xi^2 +18207, \qquad \text{(A.17)}$$

$$P_{11} = 2208465\xi^{10}-147490155\xi^8+8447761178\xi^6-8151380198\xi^4-147490155\xi^2 +2208465, \qquad \text{(A.18)}$$

$$P_{12} = 4096575\xi^{10}+1393211475\xi^8+5389945502\xi^6-8081169602\xi^4+1393211475\xi^2 +4096575, \qquad \text{(A.19)}$$

$$P_{13} = -736155\xi^{12}+49735950\xi^{10}+2422267163\xi^8-4938655516\xi^6+2422267163\xi^4 +49735950\xi^2-736155, \qquad \text{(A.20)}$$

$$
\begin{aligned}
P_{14} &= -91035\xi^{12} - 30889450\xi^{10} + 767775003\xi^8 - 1468993836\xi^6 + 767775003\xi^4 \\
&\quad -30889450\xi^2 - 91035, & \text{(A.21)} \\
P_{15} &= 18207\xi^{12} + 6177890\xi^{10} + 73750241\xi^8 - 160811716\xi^6 + 73750241\xi^4 + 6177890\xi^2 \\
&\quad +18207, & \text{(A.22)} \\
P_{16} &= 736155\xi^{12} - 49735950\xi^{10} + 2027991781\xi^8 - 3961862372\xi^6 + 2027991781\xi^4 \\
&\quad -49735950\xi^2 + 736155, & \text{(A.23)} \\
P_{17} &= 4574475\xi^{14} + 49929453\xi^{12} + 59726658551\xi^{10} - 204795402735\xi^8 - 938249027584\xi^7 \\
&\quad -204795402735\xi^6 + 59726658551\xi^4 + 49929453\xi^2 + 4574475, & \text{(A.24)} \\
P_{18} &= 4574475\xi^{14} + 49929453\xi^{12} + 59726658551\xi^{10} - 204795402735\xi^8 + 938249027584\xi^7 \\
&\quad -204795402735\xi^6 + 59726658551\xi^4 + 49929453\xi^2 + 4574475, & \text{(A.25)} \\
P_{19} &= 12350475\xi^{14} - 89110350\xi^{12} + 2132547424\xi^{10} + 89696346915\xi^8 - 232416870400\xi^7 \\
&\quad +89696346915\xi^6 + 2132547424\xi^4 - 89110350\xi^2 + 12350475, & \text{(A.26)} \\
P_{20} &= 12350475\xi^{14} - 89110350\xi^{12} + 2132547424\xi^{10} + 89696346915\xi^8 + 232416870400\xi^7 \\
&\quad +89696346915\xi^6 + 2132547424\xi^4 - 89110350\xi^2 + 12350475, & \text{(A.27)} \\
P_{21} &= -5446559475\xi^{18} + 55032169500\xi^{16} - 1068722020863\xi^{14} + 130798101242323\xi^{12} \\
&\quad -385462593602301\xi^{10} + 381602873228547\xi^8 - 124890387701293\xi^6 \\
&\quad -1068722020863\xi^4 + 55032169500\xi^2 - 5446559475, & \text{(A.28)} \\
P_{22} &= -114361875\xi^{18} - 917857575\xi^{16} - 1489869987200\xi^{14} + 5747747644968\xi^{12} \\
&\quad -6923569098094\xi^{10} + 1055435494034\xi^8 + 3088079447592\xi^6 - 1489869987200\xi^4 \\
&\quad -917857575\xi^2 - 114361875, & \text{(A.29)} \\
P_{23} &= 114361875\xi^{18} + 917857575\xi^{16} + 1489869987200\xi^{14} - 3088079447592\xi^{12} \\
&\quad -1055435494034\xi^{10} + 6923569098094\xi^8 - 5747747644968\xi^6 + 1489869987200\xi^4 \\
&\quad +917857575\xi^2 + 114361875, & \text{(A.30)} \\
P_{24} &= 5446559475\xi^{18} - 55032169500\xi^{16} + 1068722020863\xi^{14} + 124890387701293\xi^{12} \\
&\quad -381602873228547\xi^{10} + 385462593602301\xi^8 - 130798101242323\xi^6 + 1068722020863\xi^4 \\
&\quad -55032169500\xi^2 + 5446559475, & \text{(A.31)} \\
P_{25} &= -480319875\xi^{20} - 3481419690\xi^{18} - 6254413865175\xi^{16} + 126135043836776\xi^{14} \\
&\quad -447182751937958\xi^{12} + 654730699180164\xi^{10} - 447182751937958\xi^8 \\
&\quad +126135043836776\xi^6 - 6254413865175\xi^4 - 3481419690\xi^2 - 480319875, & \text{(A.32)} \\
P_{26} &= -259359975\xi^{20} + 2822303925\xi^{18} - 52907211399\xi^{16} + 6058077409828\xi^{14} \\
&\quad -23799432487570\xi^{12} + 35582472943182\xi^{10} - 23799432487570\xi^8 + 6058077409828\xi^6 \\
&\quad -52907211399\xi^4 + 2822303925\xi^2 - 259359975, & \text{(A.33)} \\
P_{27} &= 68617125\xi^{20} + 497345670\xi^{18} + 893487695025\xi^{16} + 4257463521448\xi^{14} \\
&\quad -25223771715670\xi^{12} + 40127575963044\xi^{10} - 25223771715670\xi^8 + 4257463521448\xi^6 \\
&\quad +893487695025\xi^4 + 497345670\xi^2 + 68617125, & \text{(A.34)} \\
P_{28} &= 259359975\xi^{20} - 2822303925\xi^{18} + 52907211399\xi^{16} + 5969198755292\xi^{14} \\
&\quad -24309672172910\xi^{12} + 36581184047538\xi^{10} - 24309672172910\xi^8 + 5969198755292\xi^6 \\
&\quad +52907211399\xi^4 - 2822303925\xi^2 + 259359975, & \text{(A.35)} \\
P_{29} &= -\frac{32L_1^3}{3} - \frac{64L_1^2L_2}{3} - \frac{64L_1L_2^2}{3} - 32L_1\zeta_2 - \frac{32L_2^3}{3} - 32L_2\zeta_2 - \frac{128\zeta_3}{9}. & \text{(A.36)}
\end{aligned}
$$

The first expanded moments in the polarized case are

$$\Delta\tilde{a}_{Qg}^{(3)}(N=3) =$$

$$T_F^2\Biggl\{C_A\Biggl[\frac{1377865}{17496}+\frac{311183L_1}{7776}+\frac{73615L_1^2}{13824}+\frac{1160L_1^3}{81}+\frac{310913L_2}{7776}+\frac{127345L_1L_2}{6912}$$
$$+\frac{400L_1^2L_2}{27}+\frac{73615L_2^2}{13824}+\frac{560L_1L_2^2}{27}+\frac{1000L_2^3}{81}-\frac{149369L_\eta}{7776}+\frac{202835L_\eta^2}{41472}$$
$$+\eta^3\biggl(\frac{83626153}{843908625}+\frac{5L_1}{144}-\frac{5L_1^2}{96}-\frac{5L_2}{144}+\frac{5L_1L_2}{48}-\frac{5L_2^2}{96}-\frac{1276873L_\eta}{42865200}-\frac{45377L_\eta^2}{272160}\biggr)$$
$$+\eta^2\biggl(\frac{35770738}{10418625}+\frac{5L_1}{144}-\frac{5L_1^2}{144}-\frac{5L_2}{144}+\frac{5L_1L_2}{72}-\frac{5L_2^2}{144}-\frac{2446699L_\eta}{1587600}+\frac{461L_\eta^2}{15120}\biggr)$$
$$+\eta\biggl(\frac{742837}{30375}+\frac{11269L_1}{6912}-\frac{11509L_1^2}{27648}-\frac{11269L_2}{6912}+\frac{11509L_1L_2}{13824}$$
$$-\frac{11509L_2^2}{27648}-\frac{4851337L_\eta}{518400}+\frac{68809L_\eta^2}{138240}\biggr)+\biggl(\frac{785}{81}+\frac{280L_1}{9}+\frac{280L_2}{9}\biggr)\zeta_2-\frac{160}{81}\zeta_3\Biggr]$$
$$+C_F\Biggl[\frac{463987}{23328}+\frac{124775L_1}{3888}+\frac{216931L_1^2}{10368}-\frac{1055L_1^3}{162}+\frac{125315L_2}{3888}+\frac{142349L_1L_2}{5184}$$
$$-\frac{95L_1^2L_2}{27}+\frac{216931L_2^2}{10368}-\frac{205L_1L_2^2}{27}-\frac{835L_2^3}{162}+\frac{8233L_\eta}{1944}-\frac{9523L_\eta^2}{10368}$$
$$+\eta^3\biggl(\frac{335165512}{843908625}-\frac{5L_1}{36}+\frac{5L_1^2}{24}+\frac{5L_2}{36}-\frac{5L_1L_2}{12}+\frac{5L_2^2}{24}-\frac{10119223L_\eta}{10716300}+\frac{85153L_\eta^2}{68040}\biggr)$$
$$+\eta^2\biggl(\frac{13141981}{2315250}-\frac{5L_1}{36}+\frac{5L_1^2}{36}+\frac{5L_2}{36}-\frac{5L_1L_2}{18}+\frac{5L_2^2}{36}-\frac{57787L_\eta}{14700}+\frac{2717L_\eta^2}{1260}\biggr)$$
$$+\eta\biggl(-\frac{21016}{3375}+\frac{4123L_1}{1728}-\frac{3883L_1^2}{6912}-\frac{4123L_2}{1728}+\frac{3883L_1L_2}{3456}-\frac{3883L_2^2}{6912}-\frac{139613L_\eta}{43200}$$
$$+\frac{93143L_\eta^2}{34560}\biggr)+\biggl(\frac{1919}{108}-\frac{205L_1}{18}-\frac{205L_2}{18}\biggr)\zeta_2+\frac{470}{81}\zeta_3\Biggr]\Biggr\}$$
$$+T_F^3\Biggl[-\frac{16L_1^3}{3}-\frac{32L_1^2L_2}{3}-\frac{32L_1L_2^2}{3}-\frac{16L_2^3}{3}+(-16L_1-16L_2)\zeta_2-\frac{64}{9}\zeta_3\Biggr], \tag{A.37}$$

$$\Delta\tilde{a}_{Qg}^{(3)}(N=5) =$$

$$T_F^2\Biggl\{C_A\Biggl[\frac{2875905577}{27337500}+\frac{450503027L_1}{7776000}+\frac{9047504867L_1^2}{663552000}+\frac{30856L_1^3}{2025}+\frac{450408077L_2}{7776000}$$
$$+\frac{9835907101L_1L_2}{331776000}+\frac{2128L_1^2L_2}{135}+\frac{9047504867L_2^2}{663552000}+\frac{14896L_1L_2^2}{675}+\frac{1064L_2^3}{81}-\frac{757990081L_\eta}{38880000}$$
$$+\frac{3256879133L_\eta^2}{663552000}+\eta^2\biggl(\frac{676581544}{468838125}+\frac{153173L_1}{22118400}-\frac{66139L_1^2}{5898240}-\frac{153173L_2}{22118400}+\frac{66139L_1L_2}{2949120}$$
$$-\frac{66139L_2^2}{5898240}-\frac{20502832607L_\eta}{48771072000}-\frac{237897901L_\eta^2}{619315200}\biggr)+\eta^3\biggl(-\frac{91188920849}{5616211899375}+\frac{1811L_1}{86400}$$

$$
-\frac{1307L_1^2}{57600}-\frac{1811L_2}{86400}+\frac{1307L_1L_2}{28800}-\frac{1307L_2^2}{57600}+\frac{33142148327L_\eta}{207467568000}-\frac{44268859L_\eta^2}{119750400}\Biggr)
$$

$$
+\eta\Biggl(\frac{983886373}{37209375}+\frac{7044007L_1}{11059200}-\frac{1788229L_1^2}{11059200}-\frac{7044007L_2}{11059200}+\frac{1788229L_1L_2}{5529600}
$$

$$
-\frac{1788229L_2^2}{11059200}-\frac{52251929221L_\eta}{5806080000}+\frac{11128409L_\eta^2}{55296000}\Biggr)+\Biggl(\frac{192092}{10125}+\frac{7448L_1}{225}+\frac{7448L_2}{225}\Biggr)\zeta_2
$$

$$
-\frac{4256\zeta_3}{2025}\Biggr]
$$

$$
+C_F\Biggl[-\frac{221066081}{22781250}+\frac{87833107L_1}{8100000}+\frac{1227439897L_1^2}{103680000}-\frac{74032L_1^3}{10125}+\frac{88285357L_2}{8100000}
$$

$$
+\frac{146865403L_1L_2}{10368000}-\frac{14336L_1^2L_2}{3375}+\frac{1227439897L_2^2}{103680000}-\frac{29344L_1L_2^2}{3375}-\frac{59024L_2^3}{10125}
$$

$$
+\frac{159295001L_\eta}{24300000}-\frac{157462297L_\eta^2}{103680000}+\eta^3\Biggl(\frac{402156040976}{1123242379875}-\frac{109L_1}{1200}+\frac{81L_1^2}{800}+\frac{109L_2}{1200}
$$

$$
-\frac{81L_1L_2}{400}+\frac{81L_2^2}{800}-\frac{4378699309L_\eta}{5186689200}+\frac{17063353L_\eta^2}{14968800}\Biggr)+\eta^2\Biggl(\frac{151435948}{31255875}-\frac{82681L_1}{1152000}
$$

$$
+\frac{844843L_1^2}{13824000}+\frac{82681L_2}{1152000}-\frac{844843L_1L_2}{6912000}+\frac{844843L_2^2}{13824000}-\frac{1728724721L_\eta}{508032000}+\frac{187253459L_\eta^2}{96768000}\Biggr)
$$

$$
+\eta\Biggl(-\frac{672368}{70875}+\frac{1816457L_1}{864000}-\frac{1179509L_1^2}{2304000}-\frac{1816457L_2}{864000}+\frac{1179509L_1L_2}{1152000}-\frac{1179509L_2^2}{2304000}
$$

$$
-\frac{1235063L_\eta}{864000}+\frac{1758503L_\eta^2}{768000}\Biggr)+\Biggl(\frac{422596}{50625}-\frac{14672L_1}{1125}-\frac{14672L_2}{1125}\Biggr)\zeta_2+\frac{61376\zeta_3}{10125}\Biggr]\Biggr\}
$$

$$
+T_F^3\Biggl[-\frac{64L_1^3}{15}-\frac{128L_1^2L_2}{15}-\frac{128L_1L_2^2}{15}-\frac{64L_2^3}{15}+\Biggl(-\frac{64L_1}{5}-\frac{64L_2}{5}\Biggr)\zeta_2-\frac{256}{45}\zeta_3\Biggr], \tag{A.38}
$$

$$
\Delta\tilde{a}_{Qg}^{(3)}(N=7)=
$$

$$
T_F^2\Biggl\{C_A\Biggl[\frac{23334644017021}{217818720000}+\frac{64027714809077L_1}{1062125568000}+\frac{125636708578021L_1^2}{7768689868800}+\frac{10382L_1^3}{735}
$$

$$
+\frac{64003435143947L_2}{1062125568000}+\frac{124025409719579L_1L_2}{3884344934400}+\frac{716L_1^2L_2}{49}+\frac{125636708578021L_2^2}{7768689868800}
$$

$$
+\frac{716L_1L_2^2}{35}+\frac{1790L_2^3}{147}-\frac{56215293776161L_\eta}{3186376704000}+\frac{34360837098779L_\eta^2}{7768689868800}
$$

$$
+\eta^2\Biggl(\frac{14352439212829}{23296878990000}+\frac{1592793023L_1}{123312537600}-\frac{613388477L_1^2}{29595009024}-\frac{1592793023L_2}{123312537600}
$$

$$
+\frac{613388477L_1L_2}{14797504512}-\frac{613388477L_2^2}{29595009024}+\frac{334835304263447L_\eta}{14100172111872000}-\frac{4130848867577L_\eta^2}{8138627481600}\Biggr)
$$

$$
+\eta^3\Biggl(-\frac{14624127854214079}{230324594134635000}+\frac{7986605831L_1}{246625075200}-\frac{2535517577L_1^2}{65766686720}-\frac{7986605831L_2}{246625075200}
$$

$$
+\frac{2535517577L_1L_2}{32883343360}-\frac{2535517577L_2^2}{65766686720}+\frac{942075107613860599 9L_\eta}{428927235643146240 00}
$$
$$
-\frac{1585041603186541L_\eta^2}{3808877661388800}\Bigg)+\eta\Bigg(\frac{166013524319}{6806835000}+\frac{188096578493L_1}{739875225600}
$$
$$
-\frac{202758388859L_1^2}{2959500902400}-\frac{188096578493L_2}{739875225600}+\frac{202758388859L_1L_2}{1479750451200}-\frac{202758388859L_2^2}{2959500902400}
$$
$$
-\frac{4292717431702621L_\eta}{543808290816000}+\frac{800363669341L_\eta^2}{20716506316800}\Bigg)+\Bigg(\frac{529103}{24696}+\frac{1074L_1}{35}+\frac{1074L_2}{35}\Bigg)\zeta_2-\frac{1432}{735}\zeta_3\Bigg]
$$
$$
+C_F\Bigg[-\frac{292368887666357}{12197848320000}-\frac{1944703337L_1}{2458624000}+\frac{167874076860551L_1^2}{27190414540800}-\frac{115915L_1^3}{16464}
$$
$$
-\frac{15058164463L_2}{22127616000}+\frac{11776150078591L_1L_2}{1942172467200}-\frac{57107L_1^2L_2}{13720}+\frac{167874076860551L_2^2}{27190414540800}
$$
$$
-\frac{115429L_1L_2^2}{13720}-\frac{66133L_2^3}{11760}+\frac{2005296301549L_\eta}{278807961600}-\frac{9248094576667L_\eta^2}{5438082908160}
$$
$$
+\eta^3\Bigg(\frac{787942400862074}{2617324933348125}-\frac{18467618651L_1}{129478164480}+\frac{156484147949L_1^2}{863187763200}+\frac{18467618651L_2}{129478164480}
$$
$$
-\frac{156484147949L_1L_2}{431593881600}+\frac{156484147949L_2^2}{863187763200}-\frac{348761937329434169L_\eta}{487417313230848000}
$$
$$
+\frac{296565060769037L_\eta^2}{302978904883200}\Bigg)+\eta^2\Bigg(\frac{2256072641521}{575231580000}-\frac{16002091283L_1}{129478164480}+\frac{297330438653L_1^2}{2589563289600}
$$
$$
+\frac{16002091283L_2}{129478164480}-\frac{297330438653L_1L_2}{1294781644800}+\frac{297330438653L_2^2}{2589563289600}-\frac{156623887360933L_\eta}{55953063936000}
$$
$$
+\frac{46763955788321L_\eta^2}{28485196185600}\Bigg)+\eta\Bigg(-\frac{119203663}{11576250}+\frac{1062066777019L_1}{647390822400}-\frac{991362614047L_1^2}{2589563289600}
$$
$$
-\frac{1062066777019L_2}{647390822400}+\frac{991362614047L_1L_2}{1294781644800}-\frac{991362614047L_2^2}{2589563289600}-\frac{392516586611L_\eta}{1078984704000}
$$
$$
+\frac{4858443793183L_\eta^2}{2589563289600}\Bigg)+\Bigg(\frac{31215727}{11524800}-\frac{346287L_1}{27440}-\frac{346287L_2}{27440}\Bigg)\zeta_2+\frac{16837\zeta_3}{2940}\Bigg]\Bigg\}
$$
$$
+T_F^3\Bigg[-\frac{24L_1^3}{7}-\frac{48L_1^2L_2}{7}-\frac{48L_1L_2^2}{7}-\frac{24L_2^3}{7}+\Bigg(-\frac{72L_1}{7}-\frac{72L_2}{7}\Bigg)\zeta_2-\frac{32}{7}\zeta_3\Bigg]. \tag{A.39}
$$

The first unexpanded moments read

$$
\Delta\tilde{a}_{Qg}^{(3)}(N=3)=
$$
$$
T_F^3Q_{37}+T_F^2\Bigg\{C_A\Bigg[\frac{5678723}{139968}+\frac{1160L_1^3}{81}+\frac{400L_1^2L_2}{27}+\frac{560L_1L_2^2}{27}+\frac{1000L_2^3}{81}-\frac{11029}{3456\xi^2}
$$
$$
-\frac{11029\xi^2}{3456}+\frac{L_2Q_1}{62208(\xi^2-1)\xi^2}+\frac{H_{1,0,0}(\xi)Q_5}{6912\xi^3}+\frac{H_{-1,0,0}(\xi)Q_6}{6912\xi^3}+\frac{L_1Q_8}{62208(\xi^2-1)\xi^2}
$$
$$
+\frac{L_1^2Q_{10}}{27648(\xi^2-1)^2\xi^2}+\frac{L_2^2Q_{10}}{27648(\xi^2-1)^2\xi^2}+\frac{L_1L_2Q_{11}}{13824(\xi^2-1)^2\xi^2}+\Bigg(\frac{785}{81}+\frac{280L_1}{9}
$$

$$+\frac{280L_2}{9}\Big)\zeta_2-\frac{160}{81}\zeta_3\Big]+C_F\Big[\frac{223435}{7776}-\frac{1055L_1^3}{162}-\frac{95L_1^2L_2}{27}-\frac{205L_1L_2^2}{27}-\frac{835L_2^3}{162}$$
$$-\frac{4363}{864\xi^2}-\frac{4363\xi^2}{864}+\frac{L_2Q_2}{15552(\xi^2-1)\xi^2}+\frac{H_{-1,0,0}(\xi)Q_3}{1728\xi^3}+\frac{H_{1,0,0}(\xi)Q_4}{1728\xi^3}+\frac{L_1Q_7}{15552(\xi^2-1)\xi^2}$$
$$+\frac{L_1^2Q_9}{20736(\xi^2-1)^2\xi^2}+\frac{L_2^2Q_9}{20736(\xi^2-1)^2\xi^2}+\frac{L_1L_2Q_{12}}{10368(\xi^2-1)^2\xi^2}+\Big(\frac{1919}{108}-\frac{205L_1}{18}$$
$$-\frac{205L_2}{18}\Big)\zeta_2+\frac{470}{81}\zeta_3\Big]\Big\} \tag{A.40}$$
$$\Delta\tilde{a}_{Qg}^{(3)}(N=5)=$$
$$\frac{4T_F^3Q_{37}}{5}+T_F^2\Big\{C_A\Big[\frac{30856L_1^3}{2025}+\frac{2128L_1^2L_2}{135}+\frac{14896L_1L_2^2}{675}+\frac{1064L_2^3}{81}$$
$$+\frac{1541144309981}{22394880000(\xi^2-1)^2}+\frac{22547}{1228800(\xi^2-1)^2\xi^4}-\frac{21325961}{16588800(\xi^2-1)^2\xi^2}$$
$$-\frac{756388034053\xi^2}{5598720000(\xi^2-1)^2}+\frac{1541144309981\xi^4}{22394880000(\xi^2-1)^2}-\frac{21325961\xi^6}{16588800(\xi^2-1)^2}$$
$$+\frac{22547\xi^8}{1228800(\xi^2-1)^2}+\frac{H_{1,0,0}(\xi)Q_{13}}{22118400\xi^5}+\frac{H_{-1,0,0}(\xi)Q_{14}}{22118400\xi^5}+\frac{L_1Q_{17}}{995328000(\xi^2-1)^3\xi^4}$$
$$+\frac{L_2Q_{22}}{995328000(\xi^2-1)^3\xi^4}+\frac{L_1L_2Q_{25}}{663552000(\xi^2-1)^4\xi^4}+\frac{L_1^2Q_{28}}{1327104000(\xi^2-1)^4\xi^4}$$
$$+\frac{L_2^2Q_{28}}{1327104000(\xi^2-1)^4\xi^4}+\Big(\frac{192092}{10125}+\frac{7448L_1}{225}+\frac{7448L_2}{225}\Big)\zeta_2+\Big(\frac{4976}{675}$$
$$-\frac{19184}{2025(\xi^2-1)^2}+\frac{38368\xi^2}{2025(\xi^2-1)^2}-\frac{19184\xi^4}{2025(\xi^2-1)^2}\Big)\zeta_3\Big]$$
$$+C_F\Big[-\frac{74032L_1^3}{10125}-\frac{14336L_1^2L_2}{3375}-\frac{29344L_1L_2^2}{3375}-\frac{59024L_2^3}{10125}+\frac{4811953349}{388800000(\xi^2-1)^2}$$
$$+\frac{49}{64000(\xi^2-1)^2\xi^4}-\frac{1868743}{432000(\xi^2-1)^2\xi^2}-\frac{782406581\xi^2}{48600000(\xi^2-1)^2}$$
$$+\frac{4811953349\xi^4}{388800000(\xi^2-1)^2}-\frac{1868743\xi^6}{432000(\xi^2-1)^2}+\frac{49\xi^8}{64000(\xi^2-1)^2}+\frac{H_{-1,0,0}(\xi)Q_{15}}{3456000\xi^5}$$
$$+\frac{H_{1,0,0}(\xi)Q_{16}}{3456000\xi^5}+\frac{L_1Q_{20}}{259200000(\xi^2-1)^3\xi^4}+\frac{L_2Q_{21}}{259200000(\xi^2-1)^3\xi^4}$$
$$+\frac{L_1L_2Q_{26}}{20736000(\xi^2-1)^4\xi^4}+\frac{L_1^2Q_{27}}{207360000(\xi^2-1)^4\xi^4}+\frac{L_2^2Q_{27}}{207360000(\xi^2-1)^4\xi^4}$$
$$+\Big(\frac{422596}{50625}-\frac{14672L_1}{1125}-\frac{14672L_2}{1125}\Big)\zeta_2+\Big(-\frac{2912}{675}+\frac{105056}{10125(\xi^2-1)^2}$$
$$-\frac{210112\xi^2}{10125(\xi^2-1)^2}+\frac{105056\xi^4}{10125(\xi^2-1)^2}\Big)\zeta_3\Big]\Big\} \tag{A.41}$$
$$\Delta\tilde{a}_{Qg}^{(3)}(N=7)=$$

$$
\begin{aligned}
&\frac{9T_F^3Q_{37}}{14}+T_F^2\Bigg\{C_A\Bigg[\frac{10382L_1^3}{735}+\frac{716L_1^2L_2}{49}+\frac{716L_1L_2^2}{35}+\frac{1790L_2^3}{147}\\
&+\frac{3956117166430889 09}{5353112862720000(\xi^2-1)^4}-\frac{4958901}{1644167168(\xi^2-1)^4\xi^6}+\frac{40076219}{1027604480(\xi^2-1)^4\xi^4}\\
&-\frac{3267395061527}{5549064192000(\xi^2-1)^4\xi^2}-\frac{1245615918676222 2973\xi^2}{42824902901760000(\xi^2-1)^4}\\
&+\frac{776244888381666553\xi^4}{1784370954240000(\xi^2-1)^4}-\frac{12456159186762222973\xi^6}{42824902901760000(\xi^2-1)^4}\\
&+\frac{395611716643088909\xi^8}{5353112862720000(\xi^2-1)^4}-\frac{3267395061527\xi^{10}}{5549064192000(\xi^2-1)^4}+\frac{40076219\xi^{12}}{1027604480(\xi^2-1)^4}\\
&-\frac{4958901\xi^{14}}{1644167168(\xi^2-1)^4}+\frac{H_{1,0,0}(\xi)Q_{23}}{147975045120\xi^7}+\frac{H_{-1,0,0}(\xi)Q_{24}}{147975045120\xi^7}\\
&+\frac{L_2Q_{29}}{543808290816000(\xi^2-1)^5\xi^6}+\frac{L_1Q_{32}}{543808290816000(\xi^2-1)^5\xi^6}\\
&+\frac{L_1^2Q_{33}}{62149518950400(\xi^2-1)^6\xi^6}+\frac{L_2^2Q_{33}}{62149518950400(\xi^2-1)^6\xi^6}\\
&+\frac{L_1L_2Q_{36}}{31074759475200(\xi^2-1)^6\xi^6}+\left(\frac{529103}{24696}+\frac{1074L_1}{35}+\frac{1074L_2}{35}\right)\zeta_2\\
&+\Bigg(\frac{1642}{245}-\frac{6358}{735(\xi^2-1)^4}+\frac{25432\xi^2}{735(\xi^2-1)^4}-\frac{12716\xi^4}{245(\xi^2-1)^4}+\frac{25432\xi^6}{735(\xi^2-1)^4}\\
&-\frac{6358\xi^8}{735(\xi^2-1)^4}\Bigg)\zeta_3\Bigg]+C_F\Bigg[-\frac{115915L_1^3}{16464}-\frac{57107L_1^2L_2}{13720}-\frac{115429L_1L_2^2}{13720}\\
&-\frac{66133L_2^3}{11760}+\frac{88119076890915193}{18735895019520000(\xi^2-1)^4}+\frac{30375}{1438646272(\xi^2-1)^4\xi^6}\\
&-\frac{1750311}{1438646272(\xi^2-1)^4\xi^4}-\frac{226840183453}{64739082240(\xi^2-1)^4\xi^2}+\frac{477564512027081881\xi^2}{37471790039040000(\xi^2-1)^4}\\
&-\frac{348302141626761803\xi^4}{12490596679680000(\xi^2-1)^4}+\frac{477564512027081881\xi^6}{37471790039040000(\xi^2-1)^4}\\
&+\frac{88119076890915193\xi^8}{18735895019520000(\xi^2-1)^4}-\frac{226840183453\xi^{10}}{64739082240(\xi^2-1)^4}-\frac{1750311\xi^{12}}{1438646272(\xi^2-1)^4}\\
&+\frac{30375\xi^{14}}{1438646272(\xi^2-1)^4}+\frac{H_{-1,0,0}(\xi)Q_{18}}{129478164480\xi^7}+\frac{H_{1,0,0}(\xi)Q_{19}}{129478164480\xi^7}\\
&+\frac{L_1Q_{30}}{22658678784000(\xi^2-1)^5\xi^6}+\frac{L_2Q_{31}}{22658678784000(\xi^2-1)^5\xi^6}\\
&+\frac{L_1L_2Q_{34}}{3884344934400(\xi^2-1)^6\xi^6}+\frac{L_1^2Q_{35}}{54380829081600(\xi^2-1)^6\xi^6}\\
&+\frac{L_2^2Q_{35}}{54380829081600(\xi^2-1)^6\xi^6}+\left(\frac{31215727}{11524800}-\frac{346287L_1}{27440}-\frac{346287L_2}{27440}\right)\zeta_2\\
&+\Bigg(-\frac{1027}{245}+\frac{29161}{2940(\xi^2-1)^4}-\frac{29161\xi^2}{735(\xi^2-1)^4}+\frac{29161\xi^4}{490(\xi^2-1)^4}-\frac{29161\xi^6}{735(\xi^2-1)^4}
\end{aligned}
$$

$$+\frac{29161\xi^8}{2940(\xi^2-1)^4}\Bigg)\zeta_3\Bigg]\Bigg\}, \tag{A.42}$$

with the polynomials Q_i

$$Q_1 = -99261\xi^6 + 2588725\xi^4 - 2388043\xi^2 - 99261, \tag{A.43}$$

$$Q_2 = -39267\xi^6 + 538367\xi^4 - 461993\xi^2 - 39267, \tag{A.44}$$

$$Q_3 = 4363\xi^6 - 7803\xi^4 - 56320\xi^3 - 7803\xi^2 + 4363, \tag{A.45}$$

$$Q_4 = 4363\xi^6 - 7803\xi^4 + 56320\xi^3 - 7803\xi^2 + 4363, \tag{A.46}$$

$$Q_5 = 11029\xi^6 + 131547\xi^4 - 327680\xi^3 + 131547\xi^2 + 11029, \tag{A.47}$$

$$Q_6 = 11029\xi^6 + 131547\xi^4 + 327680\xi^3 + 131547\xi^2 + 11029, \tag{A.48}$$

$$Q_7 = 39267\xi^6 + 461993\xi^4 - 538367\xi^2 + 39267, \tag{A.49}$$

$$Q_8 = 99261\xi^6 + 2388043\xi^4 - 2588725\xi^2 + 99261, \tag{A.50}$$

$$Q_9 = -13089\xi^8 + 460040\xi^6 - 892462\xi^4 + 460040\xi^2 - 13089, \tag{A.51}$$

$$Q_{10} = -11029\xi^8 + 169288\xi^6 - 316998\xi^4 + 169288\xi^2 - 11029, \tag{A.52}$$

$$Q_{11} = 11029\xi^8 + 232632\xi^6 - 486842\xi^4 + 232632\xi^2 + 11029, \tag{A.53}$$

$$Q_{12} = 13089\xi^8 + 258520\xi^6 - 544658\xi^4 + 258520\xi^2 + 13089, \tag{A.54}$$

$$\begin{aligned} Q_{13} = {} & -202923\xi^{10} + 13818977\xi^8 + 429684810\xi^6 - 1115684864\xi^5 + 429684810\xi^4 \\ & +13818977\xi^2 - 202923, \end{aligned} \tag{A.55}$$

$$\begin{aligned} Q_{14} = {} & -202923\xi^{10} + 13818977\xi^8 + 429684810\xi^6 + 1115684864\xi^5 + 429684810\xi^4 \\ & +13818977\xi^2 - 202923, \end{aligned} \tag{A.56}$$

$$\begin{aligned} Q_{15} = {} & -1323\xi^{10} + 7472375\xi^8 - 23485500\xi^6 - 122945536\xi^5 - 23485500\xi^4 \\ & +7472375\xi^2 - 1323, \end{aligned} \tag{A.57}$$

$$\begin{aligned} Q_{16} = {} & -1323\xi^{10} + 7472375\xi^8 - 23485500\xi^6 + 122945536\xi^5 - 23485500\xi^4 \\ & +7472375\xi^2 - 1323, \end{aligned} \tag{A.58}$$

$$\begin{aligned} Q_{17} = {} & -9131535\xi^{14} + 648233955\xi^{12} + 55762321201\xi^{10} - 171719012613\xi^8 \\ & +174230851323\xi^6 - 59554300111\xi^4 + 648233955\xi^2 - 9131535, \end{aligned} \tag{A.59}$$

$$\begin{aligned} Q_{18} = {} & -1366875\xi^{14} + 73347120\xi^{12} + 227138865058\xi^{10} - 956498341239\xi^8 \\ & -4403163561984\xi^7 - 956498341239\xi^6 + 227138865058\xi^4 + 73347120\xi^2 \\ & -1366875, \end{aligned} \tag{A.60}$$

$$\begin{aligned} Q_{19} = {} & -1366875\xi^{14} + 73347120\xi^{12} + 227138865058\xi^{10} - 956498341239\xi^8 + 4403163561984\xi^7 \\ & -956498341239\xi^6 + 227138865058\xi^4 + 73347120\xi^2 - 1366875, \end{aligned} \tag{A.61}$$

$$\begin{aligned} Q_{20} = {} & -99225\xi^{14} + 560714775\xi^{12} + 1143582449\xi^{10} - 7338980847\xi^8 + 9568391697\xi^6 \\ & -4492208399\xi^4 + 560714775\xi^2 - 99225, \end{aligned} \tag{A.62}$$

$$\begin{aligned} Q_{21} = {} & 99225\xi^{14} - 560714775\xi^{12} + 4492208399\xi^{10} - 9568391697\xi^8 + 7338980847\xi^6 \\ & -1143582449\xi^4 - 560714775\xi^2 + 99225, \end{aligned} \tag{A.63}$$

$$\begin{aligned} Q_{22} = {} & 9131535\xi^{14} - 648233955\xi^{12} + 59554300111\xi^{10} - 174230851323\xi^8 + 171719012613\xi^6 \\ & -55762321201\xi^4 - 648233955\xi^2 + 9131535, \end{aligned} \tag{A.64}$$

$$\begin{aligned} Q_{23} = {} & 223150545\xi^{14} - 2001150423\xi^{12} + 34327153343\xi^{10} + 2606894507847\xi^8 \\ & -6919192313856\xi^7 + 2606894507847\xi^6 + 34327153343\xi^4 - 2001150423\xi^2 \\ & +223150545, \end{aligned} \tag{A.65}$$

$$Q_{24} = 223150545\xi^{14} - 2001150423\xi^{12} + 34327153343\xi^{10} + 2606894507847\xi^8$$

$$
\begin{aligned}
& +6919192313856\xi^7+2606894507847\xi^6+34327153343\xi^4-2001150423\xi^2 \\
& +223150545, \qquad (A.66) \\
Q_{25} &= -3043845\xi^{16}+218445420\xi^{14}+18828470972\xi^{12}-77222873708\xi^{10} \\
& +116359292562\xi^8-77222873708\xi^6+18828470972\xi^4+218445420\xi^2-3043845, \qquad (A.67) \\
Q_{26} &= -3969\xi^{16}+22431678\xi^{14}+204043784\xi^{12}-1019181374\xi^{10}+1585258482\xi^8 \\
& -1019181374\xi^6+204043784\xi^4+22431678\xi^2-3969, \qquad (A.68) \\
Q_{27} &= 19845\xi^{16}-112158390\xi^{14}+2903314904\xi^{12}-10598228426\xi^{10}+15614910534\xi^8 \\
& -10598228426\xi^6+2903314904\xi^4-112158390\xi^2+19845, \qquad (A.69) \\
Q_{28} &= 3043845\xi^{16}-218445420\xi^{14}+18938352964\xi^{12}-73844422036\xi^{10} \\
& +110241651054\xi^8-73844422036\xi^6+18938352964\xi^4-218445420\xi^2+3043845, \qquad (A.70) \\
Q_{29} &= -820078252875\xi^{22}+11363499263025\xi^{20}-169884277764915\xi^{18} \\
& +3348986162097724 9\xi^{16}-165103109949724790\xi^{14}+328428484284419170\xi^{12} \\
& -327091003475063710\xi^{10}+162656633930016650\xi^8-32062087154971039\xi^6 \\
& -169884277764915\xi^4+11363499263025\xi^2-820078252875, \qquad (A.71) \\
Q_{30} &= -239203125\xi^{22}+14005183500\xi^{20}+39684280140400\xi^{18}-214042666755237\xi^{16} \\
& +437303357270770\xi^{14}-362976447216020\xi^{12}-29556983576980\xi^{10} \\
& +270593625451250\xi^8-180700720391333\xi^6+39684280140400\xi^4+14005183500\xi^2 \\
& -239203125, \qquad (A.72) \\
Q_{31} &= 239203125\xi^{22}-14005183500\xi^{20}-39684280140400\xi^{18}+180700720391333\xi^{16} \\
& -270593625451250\xi^{14}+29556983576980\xi^{12}+362976447216020\xi^{10} \\
& -437303357270770\xi^8+214042666755237\xi^6-39684280140400\xi^4-14005183500\xi^2 \\
& +239203125, \qquad (A.73) \\
Q_{32} &= 820078252875\xi^{22}-11363499263025\xi^{20}+169884277764915\xi^{18} \\
& +32062087154971039\xi^{16}-162656633930016650\xi^{14}+327091003475063710\xi^{12} \\
& -328428484284419170\xi^{10}+165103109949724790\xi^8-33489861620977249\xi^6 \\
& +169884277764915\xi^4-11363499263025\xi^2+820078252875, \qquad (A.74) \\
Q_{33} &= -23430807225\xi^{24}+342895368690\xi^{22}-5104322258070\xi^{20}+1029830924634698\xi^{18} \\
& -6092302565491047\xi^{16}+15174619653233604\xi^{14}-20214731883198100\xi^{12} \\
& +15174619653233604\xi^{10}-6092302565491047\xi^8+1029830924634698\xi^6 \\
& -5104322258070\xi^4+342895368690\xi^2-23430807225, \qquad (A.75) \\
Q_{34} &= -20503125\xi^{24}+1216391175\xi^{22}+3400577829420\xi^{20}+3124437140987\xi^{18} \\
& -87250203933651\xi^{16}+266415595377150\xi^{14}-371385991522312\xi^{12} \\
& +266415595377150\xi^{10}-87250203933651\xi^8+3124437140987\xi^6+3400577829420\xi^4 \\
& +1216391175\xi^2-20503125, \qquad (A.76) \\
Q_{35} &= 143521875\xi^{24}-8514738225\xi^{22}-23804044805940\xi^{20}+478743194834467\xi^{18} \\
& -2392934101392699\xi^{16}+5644304654680590\xi^{14}-7412583155771336\xi^{12} \\
& +5644304654680590\xi^{10}-2392934101392699\xi^8+478743194834467\xi^6 \\
& -23804044805940\xi^4-8514738225\xi^2+143521875, \qquad (A.77) \\
Q_{36} &= 23430807225\xi^{24}-342895368690\xi^{22}+5104322258070\xi^{20}+967466021746102\xi^{18} \\
& -5891479112793753\xi^{16}+14784834542478396\xi^{14}-19731207044417900\xi^{12}
\end{aligned}
$$

$$+14784834542478396\xi^{10} - 5891479112793753\xi^8 + 967466021746102\xi^6$$
$$+5104322258070\xi^4 - 342895368690\xi^2 + 23430807225, \tag{A.78}$$
$$Q_{37} = -\frac{16L_1^3}{3} - \frac{32L_1^2L_2}{3} - \frac{32L_1L_2^2}{3} - 16(L_1+L_2)\zeta_2 - \frac{16L_2^3}{3} - \frac{64}{9}\zeta_3. \tag{A.79}$$

The first moment $\Delta\tilde{a}_{Qg}^{(3)}(N=1)$ vanishes.

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