ON THE STABILITY OF THE BISHOP'S PROPERTY(β) UNDER COMPACT PERTURBATIONS

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ABSTRACT. Let $B(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space X. Are sums and products of commuting decomposable operators on Banach spaces decomposable? This is one of the most important open problems in the local spectral theory of operators on Banach spaces. Similarly, it is not known if local spectral properties such as the single valued extension property, Dunfords property (C), Bishops property (β), or the decomposition property (δ) are preserved under sums and products of commuting operators. But it is shown by Bourhim and Muller that the single-valued extension property is not preserved under the sums and products of commuting operators. On the positive side, Sun proved that the sum and the product of two commuting operators with Dunfords property (C) have the single-valued extension property. Very recently, Aiena and Muller showed that the (localized) single-valued extension property is stable under commuting Riesz perturbations. In this paper, we show that Bishops property (β), the decomposition property (δ), or decomposable operators $T \in B(X)$ are stable under quasinilpotent, compact, and algebraic commuting perturbations.

1. INTRODUCTION

Let $B(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space X. An operator $T \in B(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc D of λ_0, the only analytic function $f : D \Rightarrow X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in D$ is the function $f \equiv 0$. An operator $T \in B(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. The study of operators satisfying Bishops property (β) is of significant interest and is currently being done by a number of mathematicians around the world [17, 18, ?, 20, 21]. Let $D(\lambda; v)$ be the open disc centred at $\lambda \in \mathbb{C}$ with radius $v > 0$, and let $O(U; X)$ denote the Fréchet algebra of all X-valued functions on the open subset of $U \in \mathbb{C}$ endowed with the uniform convergence on compact subsets of U. An operator $T \in B(X)$ has Bishop's property (β) at λ_0 if there

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exists $v > 0$ such that for every open subset $U \subset D(\lambda_0; v)$ and for any sequence $\{f_n\}_{n=1}^{\infty} \subset O(U; X)$, $\lim_{n\to\infty}(S - \lambda I)f_n(\lambda) = 0$ in $O(U; X)$ implies $\lim_{n\to\infty} f_n(\lambda) = 0$ in $O(U; X)$. An operator $T \in B(X)$ is said to have the decomposition property (δ) if T^* satisfies Bishops property (β), where T^* is the dual operator of T. In [6], Foias showed that every decomposable operator (and therefore spectral operators in the sense of Dunford, all generalized scalar operators in the sense of Colojoara and Foias [6], compact operators, and unitary, normal, and self-adjoint operators on a Hilbert space) has decomposition property (δ). It is well known that T is decomposable if and only if T satisfies both Bishop's property (β) and decomposition property $\delta)$. The left shift operator L on $l_2(N)$ has the decomposition property (δ). Indeed, since the right shift operator R on $l_2(N)$ is subnormal as the restriction of the bilateral right R shift on $l_2(Z)$. Clearly, R has the Bishop's property (β). Since L is the adjoint of R, it follows that L has the decomposition property $(delta)$. The left shift operator L on $l_2(N)$ is an example of a bounded linear operator that has decomposition property $\delta)$, but whose adjoint does not. This shows that the decomposition property $\delta)$ is not preserved under the adjoint operation. The natural related operator in the context of the spectral theory is the restriction operator. We give an example of an operator T and a T-invariant subspace M such that T has decomposition property (δ); but $T|_M$ does not. Let T be the right bilateral shift operator on $l_2(\mathbb{Z})$; and let $Y := span\{e_i : i = 1; 2\}^-$, where $\{e_i : i \in \mathbb{Z}\}$ is the usual orthonormal basis for $l_2(\mathbb{Z})$. Now T is unitary and so certainly has decomposition property (δ), but $T|_M$ is isomorphic to the right shift on $l_2(N)$, and hence does not have decomposition property (δ). In particular, the single valued extension property of operators was first introduced by N. Dunford to investigate the class of spectral operators which is another important generalization of normal operators (see [8]). In the local spectral theory, for given an operator T on a complex Banach space X and a vector $x \in X$, one is often interested in the existence and the uniqueness of analytic solution $f(.) : U \to X$ of the local resolvent equation

$$(T - \lambda)f(\lambda) = x$$

on a suitable open subset U of $\mathbb{C}$. Obviously, if T has SVEP, then the existence of analytic solution to any local resolvent equation (related to T) implies the uniqueness of its analytic solution. The SVEP is possessed by many important classes of operators such as hyponormal operators and decomposable operators [7, 14]. To emphasize the significance of Bishop's property (β); we mention the important connections to sheaf theory and the spectral theory of several commuting operators from the monograph by Eschmeier and Putinar [9]. There are also interesting applications to invariant subspaces [9], harmonic analysis [10], and the theory of automatic continuity [15]. Unfortunately, but perhaps not surprisingly, the direct verification of property (β) in con- crete cases tends to be a difficult task. It is therefore desirable to have sufficient conditions for property (β) which are easier to handle. Are sums and products of commuting decomposable operators on Banach spaces decomposable? This is one of the most important open problems in the local spectral theory of operators on Banach spaces. Similarly, it is not known if local spectral properties such as Dunfords property (C), Bishops property (β), or the decomposition property (δ) are preserved under sums and products of commuting operators. But it is shown in [3] that the single-valued extension property is not preserved under the sums and products of commuting operators; see also [1]. On the positive side, Sun [24] proved that the sum and the product of two commuting operators with Dunfords property (C) have the single-valued extension property. Very recently, Aiena and Muller [2] showed that the (localized) single-valued extension property is stable under commuting Riesz perturbations. In this paper, we show that Bishops property (β), the decomposition property (δ), or decomposable operators are stable under quasinilpotent, compact, and algebraic commuting perturbations.

In [12], J.W. Helton initiated the study of operators $T \in B(H)$ which satisfy an identity of the form

$$T^{*m} - \binom{m}{1} T^{*m-1}T + ... + (-1)^m T^m = 0. \tag{1}$$

Further study of this class of operators is needed. Let $R, S \in B(X)$ and let $C(R,S)^k(I) : X \to X$ be defined by $C(R,S)(A) = RA - AS$. Then

$$C(R,S)^k(I) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} R^j S^{k-j}. \tag{2}$$

Thus we have the following definition

Definition 1.1. *Let $R \in B(X)$. If there is an integer $k \geq 1$ such that an operator S satisfies $C(R,S)^k(I) = 0$, we say that S belongs to the Helton class of R. We denote this by $S \in Helton_k(R)$.*

We remark that $C(R,S)^k(I) = 0$ does not imply $C(S,R)^k(I) = 0$ in general.

Example 1.1.

$$S = \begin{pmatrix} 0 & A & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; R = \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix},$$

where A, B, C and D are bounded linear operators defined on H. Then it is easy to calculate that $C(R,S)^2(I) = 0$, but $C(S,R)2(I) \neq 0$.

2. Main Results

We will consider the Helton class of an operator which has Bishop's property (β).

Theorem 2.1. *Let $R \in B(X)$ has Bishop's property (β) at λ_0. If $S \in Helton_k(R)$, then S has Bishop's property (β) at λ_0.*

Proof. Suppose that R has Bishop's property (β)). Assume that there exists $v > 0$ such that for every open subset $U \subset D(\lambda_0; v)$ and for any sequence $\{f_n\}_{n=1}^{\infty} \subset O(U; X)$, $\lim_{n\to\infty}(\lambda I - S)f_n(\lambda) = 0$ in $O(U; X)$. Since the terms of the below equation equal to zero when $j + s \neq r$, it suffices to consider only the case of $j + s = r$ Then we have Let $f_n : U \to X$ be any sequence of analytic function on U (any open set) such that $(\lambda I - S)f_n\lambda \to 0$ as $n \to 0$ uniformly on all compact subsets of U. Since the terms of the bellow equation equal to zero when $j + s \neq r$, it suffices to consider only the case of $j + s = r$. Then we have

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(R-\lambda)^j(\lambda-S)^{k-j}$$

$$= \sum_{j=0}^{k}\sum_{r=0}^{j}\sum_{s=0}^{k-j}(-1)^{k-(s+r)}\binom{k}{j}\binom{j}{r}\binom{k-j}{s}R^jS^{k-j}$$

$$= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}R^jS^{k-j}.$$

Thus, we have

$$\sum_{j=0}^{k-j}\binom{k}{j}R^jS^{k-j} - (R-\lambda)^k f_n(\lambda)$$

$$= \sum_{j=0}^{k}\binom{k}{j}(R-\lambda)^j(\lambda-S)^{k-j} - (R-\lambda)^k f_n(\lambda)$$

$$= \sum_{j=0}^{k-1}\binom{k}{j}(R-\lambda)^j(\lambda-S)^{k-j}f_n(\lambda)$$

$$= \sum_{j=0}^{k-1}\binom{k}{j}(R-\lambda)^j(\lambda-S)^{k-j-1}(\lambda-S)f_n(\lambda) \to 0$$

as $n \to \infty$ uniformly on all compact subsets of U. Since

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}R^jS^{k-j} = 0,$$

we have $(R-\lambda)^k f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Since R has the Bishop's property (β), $(R-\lambda)^{k-1}f_n(\lambda) \to 0$ as

$n \to \infty$ uniformly on all compact subsets of U. By induction we get that $f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Hence, S has the Bishop's property β. $\square$

If $S; T \in B(X)$ have Bishop's property (β) at λ_0, does $S + T$ have Bishop's property (β) at λ_0? So far we do not know the answer about this question. In the following theorem we study a special case of this question.

Theorem 2.2. *If R has Bishop's property (β) at λ_0, $S \in Helton_k(R)$ and $RS = SR$, then $T = R + S$ has Bishop's property (β) at λ_0.*

Proof. It is easily seen that $C(2R; S)^k(I) = C(R; S)^k(I) = 0$. Hence $T = R + S \in Helton_k(2R)$. Since $2R$ has Bishop's property (β), it follows from Theorem 2.1 that T has Bishop's property (β). $\square$

Note that if there exists an integer $k \in \mathbb{N}$ for which $C(R; S)^k(I) = C(S; R)^k(I) = 0$, then the operators S and R are said to be nilpotent equivalent. For R; $S \in B(X)$ with $RS = SR$, it is easily seen that $C(R; S^k(I) = (R - S)$ for all $k \in \mathbb{N}$. Thus R and S are nilpotent equivalent precisely when $R - S$ is nilpotent.

Theorem 2.3. *If $S; T \in B(X)$ are nilpotent equivalent, then T has Bishop's property (β) at λ_0 if and only if S has Bishop's property (β) at λ_0.*

Proof. It suffices to show by symmetry that Bishop's property (β) is transferred from S to T. Assume that S has Bishop's property (β) at λ_0, that is, there exists $v > 0$ such that for every open subset $U \subset D(\lambda_0; v)$ and for any sequence $\{f_n\}_{n=1}^{\infty} \subset O(U; X)$, $\lim_{n\to\infty}(S - \lambda I)f_n(\lambda) = 0$ in $O(U; X)$ implies $\lim_{n\to\infty} f_n(\lambda) = 0$ in $O(U; X)$. Then, we have

$$C(S,T)^k(I)f_n(\lambda) - (s-\lambda I)^k f_n(\lambda) = C(S-\lambda I, T-\lambda I)^i f_n(\lambda) - (S-\lambda I)^k f_n(\lambda)$$

$$= \sum_{i=1}^{k} (-1)^i (S - \lambda I)^{k-i} (T - \lambda I)^i$$

$$= [\sum_{i=1}^{k} (-1)^i (S - \lambda I)^{k-i} (T - \lambda I)^{i-1}](T - \lambda I)f_n(\lambda) \to 0$$

in $O(U, X)$ as $n \to \infty$. Since $C(S, T)^k(I) = 0$, we have

$$\lim_{n\to\infty} (S - \lambda I)^k f_n(\lambda) = 0$$

in $O(U, X)$. Since S has the Bishop's property (β) at λ_0, we have

$$\lim_{n\to\infty} (S - \lambda I)^{k-1} f_n(\lambda) = 0$$

in $O(U, X)$. By induction, we get $\lim_{n\to\infty} f_n\lambda = 0$in $O(U, X)$. Hence T has Bishop's property (β) at λ_0. $\square$

Corollary 2.1. *If T has Bishop's property (β) at λ_0 and N is nilpotent such that $TN = NT$, then $T + N$ has Bishop's property (β) at λ_0.*

Unfortunately, but perhaps not surprisingly, the direct verification of property (β) in concrete cases tends to be a difficult task. It is therefore desirable to have sufficient conditions for Bishop's property (β) which are easier to handle. in the following theorem we give a necessary and sufficient condition for 2×2 operator matrix to have Bishop's property (β).

Theorem 2.4. *Let $T_1 \in B(X_1)$ and let $T_2 \in B(X_2)$, where X_1 and X_2 are two Banach spaces. If T_1 and T_2 have Bishop's property (β) at λ_0, then $T_1 \oplus T_2$ has Bishop's property (β) at λ_0.*

Proof. Assume that T_1 and T_2 have Bishop's property (β) at λ_0. Then there exists $v_i > 0; i = 1; 2$, such that for every open subset $U_i \subset D(\lambda_0; v_i)$ and for any sequence $\{f_n^i\}_{n=1}^{\infty} \subset O(U_i; X_i)$, $\lim_{n\to\infty}(T_i - \lambda I)f_n^i(\lambda) = 0$ in $O(U_i; X_i)$ implies $\lim_{n\to\infty} f_n^i(\lambda) = 0$ in $O(U_i; X_i)$. Let $f_n = f_n^1 \oplus f_n^2 \subset O(U; X_1 \oplus X_2)$ be an analytic sequence, where $U \subset D(\lambda_0; min\{v_1; v_2\})$; $i = 1; 2$ is an arbitrary open subset and $f_n^i \subset O(U; X_i)$; $i = 1; 2$. Since $\lim_{n\to\infty}(\lambda I - T_1 \oplus T_2)f_n(\lambda) = 0$ in $O(U; X_1 \oplus X_2)$, then $lim_{n\to\infty}(\lambda I - T_i)f^i(\lambda) = 0$ in $O(U; X_i)$; $i = 1; 2$. Since $T_i; i = 1; 2$ have Bishop's property (β), then $\lim_{n\to\infty} f_n^i(\lambda) = 0$ in $O(U; X_i); i = 1; 2$. Hence $lim_{n\to\infty} f_n(\lambda) = 0$ in $O(U; X_1 \oplus X_2)$. $\square$

Recall that an operator $K \in B(X)$ is said to be algebraic if there exists a non trivial polynomial p such that $p(K) = 0$. Trivially, every nilpotent operator is algebraic and it is known that every finite- dimensional operator is algebraic. It is also known that every algebraic operator has a finite spectrum. In the following theorem we study the Bishop's property (β) under commuting algebraic perturbations.

Theorem 2.5. *Let $T \in B(X)$ and let $K \in B(X)$ be an algebraic operator such that $TK = KT$. If T has Bishop's property (β) at each of the zero of p, where p is a non zero polynomial for which $p(K) = 0$, then $T + K$ has Bishop's property (β) at 0. In particular, if T has Bishop's property (β), then $T + K$ has Bishop's property (β).*

Proof. Since K is algebraic, $\sigma(K)$ is finite. Let $\sigma(K) = \{\lambda_1, ..., \lambda_n\}$ and let P_i the spectral projection associated with K and the spectral set $\{\lambda_i\}$. Set $Y_i = R(P_i)$. Thus, $Y_1, ..., Y_n$ are closed linear subspaces of X each of which is invariant under both K and T, and $X = Y_1 \oplus, ..., \oplus Y_n$ by the classical spectral decomposition. Also, for arbitrary $i = 1, ..., n$, $K_i = K|_{Y_i}$ and $T_i = T|_{Y_i}$ commute and we have $\sigma(K_i) = \{\lambda_i\}$. We claim that $N_i = \lambda_i I - K_i$ is nilpotent for every $i = 1, ..., n$. Since $p(\{\lambda_i\}) = p(\sigma(K_i)) = \{0\}$, it follows that $p(\lambda_i) = 0$. Let $p(\mu) = (\lambda_i - \mu)^r q(\mu)$ with $q(\lambda_i \neq 0)$. Then $(\lambda_i - K_i)^r q(K_i) = p(K_i) = 0$ with $q(K_i)$ invertible. Hence $(\lambda_i - K_i)^r = 0$ and $N_i = \lambda_i - K_i$ is nilpotent for every $i = 1, 2..., n$.

We have $T_i - K_i = (T_i - \lambda I) - (K_i - \lambda_i I) = T_i - \lambda_i I - N_i$. It is easily seen that if T has Bishop's property (β) at λ_0, then $T - \lambda_0 I$ has Bishop's property (β) at 0. Since Bishop's property (β) is inherited by restrictions to closed invariant subspaces, we have $T_i - \lambda_i$ has Bishop's property (β) at 0. Therefore, $T_i - K_i = T_i - \lambda_i - N_i$ has Bishop's property (β) at 0 for all $i = 1; ...; n$ by Corollary 2.1. Then it follows from Theorem 2.4 that $T - K = (T_1 - K_1) \oplus, ..., \oplus (T_n - K_n)$ has Bishop's property (β) at 0. The final claim can be established by the main result □

. Now we will apply Theorem 2.5 to show that a decomposable operator is stable under algebraic commuting perturbations.

Corollary 2.2. *Let $T \in B(X)$. If T has the decomposition property (β) and $K \in B(X)$ is an algebraic operator which commutes with T, then $T + K$ has decomposition property (δ).*

Proof. Assume that T has the decomposition property (δ), Then T^* has Bishop's property (β). It is obvious that K^* is algebraic and commutes with T^*. Hence $(T + K)^* = T^* + K^*$ has Bishop's property (β). Therefore, $T + K$ has decomposition property (δ). □

Corollary 2.3. *Let $T \in B(X)$. If T is decomposable and $K \in B(X)$ is an algebraic operator which commutes with T, then $T + K$ is decomposable.*

Corollary 2.4. *If T is decomposable and N is nilpotent such that $TN = NT$, then $T + N$ is decomposable.*

Recall [5] that if $T \in B(X)$ is decomposable, and f is an analytic scalar-valued function on some neighborhood of $\sigma(T)$, then $f(T)$ is decomposable. Thus, we have the following corollaries.

Corollary 2.5. *If T is decomposable and K is algebraic with $TK = KT$, then $f(T + K)$ is decomposable, where f is an analytic scalar-valued function on some neighborhood of $\sigma(T)$.*

Corollary 2.6. *If T has Bishop's property (β) and K is algebraic with $TK = KT$, then $f(T + K)$ has Bishop's property (β), where f is an analytic scalar-valued function on some neighborhood of $\sigma(T)$.*

Under stronger hypotheses we can conclude that the decomposition property (δ) is preserved under certain types of perturbations. In [16], it is shown that if T is decomposable and S is an operator that commutes with T such that S has totally disconnected spectrum, then $T+S$ is decomposable. By using this, we have the following immediate corollary.

Corollary 2.7. *Let T be a decomposable operator on $B(X)$, and let $S \in B(X)$ that commutes with T. If*

1. S is a quasinilpotent operator, or

2. S is a compact operator, or

3. S has discrete spectrum, then $T+S$ is decomposable. In particular $T+S$ has decomposition property (δ).

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References

[1] P. Aiena and M.M. Neumann, On the stability of the localized single-valued extension property under commuting perturbations, Proc. Amer. Math. Soc, 141(2013), 2039-2050.

[2] P. AIENA, V. MULLER. The localized single-valued extension property and Riesz operators. To appear in Proc. Amer. Math. Soc. [3] A.

[3] BOURHIM, V. G. MILLER. The single-valued extension property is not preserved under sums and products of commuting operators. Glasg. Math. J., 2007, 49: 99-104.

[4] E. Bishop. A duality theorem for an arbitrary operator. Pacific J. Math, 9: 379-424, 1959.

[5] J. Eschmeier, K. B. Laursen and M. M. Neumann. Multipliers with natural local spectra on commutative Banach algebras. J. Funct. Anal, 138(1996) 273-294.

[6] I. Colojoara and C. Foias, Theory of Generallized spectral Operators, Mathematics and its Applications, Vol. 9, Gordon and Breach, Science Publishers, New York-London-Paris, 1968.

[7] H.R.Dowsow. Spectral theory of linear operators. Academic Press, London 1973.

[8] N.Dunford. A survey of the theory of spectral operators. Bull. Amer. Math. Soc, 64: 217-274, 1958.

[9] J. Eschmeier and M. Putinar. Spectral Decompositions and Analytic Sheaves. London Mathematical Society Monographs, No. 10, Clarendon Press, Oxford, 1996.

[10] J. Eschmeier, K. B. Laursen and M. M. Neumann. Multipliers with natural local spectra on commutative Banach algebras. J. Funct. Anal, 138: 273-294, 1996.

[11] D.A.Herrero. Approximation of Hilbert space operators. Pitnam Res. Notes Math. Ser., vol 224, Longman Scientific and Technical, Harlow, 1989.

[12] J. W. Helton. Operators with a representation as multiplication by x on a Sobolev space. Colloquia Math. Soc. Janos Bolyai 5, Hilbert space operators, Thany, Hungary 279-287, 1970.

[13] E. Ko. On Hypo-Jordan operators. Glasg. Math. J, 43: 411-418, 2001.

[14] K.B. Laursen and M.M. Neumann. An introduction to Local Spectral Theory. London Mathematical society Monographs, Oxford 2000.

[15] K. B. Laursen and M. M. Neumann. Automatic continuity of intertwining linear operators on Banach spaces. Rend. Circ. Mat. Palermo II, 40: 325-341, 1991.

[16] Duality and asymptotic spectral decompositions, Pacific J. Math. 121 (1986), 93-108.

[17] S. Mecheri, Weyl type theorems for posinormal operators, Math. Proc. Royal Irish. Acad 107B (2007), 11-17.

[18] S. Mecheri, On k-quasi-M-hyponormal operators, Math. Ineq. Appl., 16(2013) 895-902.

[19] S. Mecheri, A. Uchiyama, An extension of the Fuglede-Putnam's theorem to class A operators,13(2010, 57-61.

[20] S. Mecheri, Bishops property (β), hypercyclicity and hyperinvariant subspaces, Operators and Matrices 8(2014), 555-562.

[21] S.Mecheri, Bishop's property, SVEP and Dunford property (C), Elect. Jour. Lin.Alg , 23 (2012), 523-529.

[22] S. Mecheri, A. Mansour, On the operator equation $AXB - XD = E$, Lobachevskii Journal of Mathematics, 30(2009), 224-228

[23] M. Putinar, Hyponormal operators are subscalar, J. Operator Theory 12 (1984), no. 2, 385-395.

[24] Shanli SUN. The sum and product of decomposable operators (in chinese). Northeast. Math. J., 5(1989),105-117

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