

ON CHARACTER TABLES FOR FUSION SYSTEMS

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ABSTRACT. A character table X for a saturated fusion system $\mathcal{F}$ on a finite p -group S is the square matrix of values associated to a basis of virtual $\mathcal{F}$ -stable ordinary characters of S . We investigate a conjecture of the second author which equates the p -part of $|\det(X)|^2$ with the product of the orders of S -centralisers of fully $\mathcal{F}$ -centralised $\mathcal{F}$ -class representatives. This statement is exactly column orthogonality for the character table of S when $\mathcal{F} = \mathcal{F}_S(S)$. We prove the conjecture when $\mathcal{F} = \mathcal{F}_S(G)$ is realised by some finite group G with Sylow p -subgroup S , and for all simple fusion systems when $|S| \leq p^4$.

1. INTRODUCTION

Let $\mathcal{F}$ be a fusion system on a finite p -group S and let $\mathcal{F}^z \subseteq S$ be a set of $\mathcal{F}$ -conjugacy class representatives of fully $\mathcal{F}$ -centralised elements. Let B be any basis for the ring of virtual $\mathcal{F}$ -stable ordinary characters of S . Since $|B| = |\mathcal{F}^z|$ (see [3, Lem. 2.1]) we may write

$$X = X_B(\mathcal{F}) \in M_n(\mathbb{C}), \quad X_{\chi, s} = \chi(s)$$

for the corresponding *character table* of $\mathcal{F}$ with respect to B .

Motivated by supporting computational evidence, the second author was led to conjecture the following:

Conjecture 1.1 (Semeraro). *Let $\mathcal{F}$ be a fusion system on a finite p -group S and B be a basis for the ring of virtual $\mathcal{F}$ -stable ordinary characters of S . If $\mathcal{F}$ is saturated then*

$$|X_B(\mathcal{F}) \overline{X_B(\mathcal{F})}^T|_p = \prod_{s \in \mathcal{F}^z} |C_S(s)|.$$

Here it is implicit that $|X_B(\mathcal{F}) \overline{X_B(\mathcal{F})}^T| \in \mathbb{Z}$, and this follows quickly from the well-known fact that $\mathcal{F}$ is always realised by a finite group in which S is not necessarily Sylow (see Remark 3.1). It is then clear that the left hand side of the displayed equality is independent of the choice of B . Note also that the conjecture is equivalent to the orthogonality of columns in the character table of S when $\mathcal{F} = \mathcal{F}_S(S)$. More generally, the following is true:

Theorem 1.2 (Lawrence–Olsson). *Conjecture 1.1 holds when $\mathcal{F} = \mathcal{F}_S(G)$ is the fusion system of a finite group G in which S is a Sylow p -subgroup.*

In Section 3 we present a proof of Theorem 1.2, using elementary applications of Brauer’s characterisation of characters and column orthogonality for G . A version of this proof first appeared in [12], and Benjamin Sambale subsequently pointed out that a similar observation had already been made by Olsson in [14] using π -blocks. It is worth remarking that all proofs use that S is a Sylow p -subgroup of G in an essential way; indeed the saturation hypothesis in Conjecture 1.1 is necessary (see Example 2.6).

Thus to prove Conjecture 1.1 we may assume that $\mathcal{F}$ is exotic. Recall that $\mathcal{F}$ is *transitive* on S if $\mathcal{F}^z = \{1, z\}$ for some $1 \neq z \in Z(S)$. For such a fusion system we may take $B = \{1_S, \text{reg}_S - 1_S\}$ where reg_S denotes the regular character of S (which is always $\mathcal{F}$ -stable) so that $X_B(\mathcal{F}) = \begin{pmatrix} 1 & 1 \\ |S|-1 & -1 \end{pmatrix}$ and Conjecture 1.1 holds trivially. This shows, in particular, that Conjecture 1.1 holds for the three Ruiz–Viruel exotic fusion systems on 7-groups of order 7^3 appearing in [17]. Thus by [17, Thm 1.1], Conjecture 1.1 holds if $|S| = p^3$, so we may also assume that $|S| \geq p^4$. Our second main theorem is the following:

Theorem 1.3. *Conjecture 1.1 holds if $\mathcal{F}$ is simple and $|S| \leq p^4$.*

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Suppose $\mathcal{F}$ is a simple saturated fusion system on a p -group S of order p^4 . If $p = 2$ then $\mathcal{F}$ is realisable by [1, Thm. 4.1] so we may assume that $p \geq 3$. For such p , the list of possibilities for $\mathcal{F}$ was compiled by Moragues Moncho in [13] and this is recalled in Section 4. There are three such systems, all on a Sylow p -subgroup of $\mathrm{PSp}_4(p)$, two of which are exotic and referred to here as $O^{p'}(\mathcal{F}_{(1)})$ and $\mathcal{G}$ (see Section 4 for details). For $p \geq 7$ these are the only exotic examples which occur, whereas for $p = 3, 5$ some further exceptional cases arise. To verify Conjecture 1.1 we apply an inductive approach which we expect to work more broadly for fusion systems on p -groups which contain an abelian subgroup of index p . Briefly, we choose a suitable realisable subsystem $\mathcal{N} \subseteq \mathcal{F}$ and construct a basis $B_{\mathcal{F}}$ of $\mathcal{F}$ -stable characters by taking linear combinations of an $\mathcal{N}$ -stable basis $B_{\mathcal{N}}$ of $\mathcal{N}$ -stable characters. Row operations on $X_{B_{\mathcal{N}}}(\mathcal{N})$ then allow us to express $|X_{B_{\mathcal{N}}}(\mathcal{N})|$ in terms of $|X_{B_{\mathcal{F}}}(\mathcal{F})|$ in such a way that Conjecture 1.1 for $\mathcal{F}$ follows from Theorem 1.2 for $\mathcal{N}$ (see Proposition 2.5).

Structure of the paper: We start by recalling some background results on $\mathcal{F}$ -stable characters in Section 2. Theorem 1.2 is proved in Section 3, following the approach taken in [12]. In Section 4, we introduce notation to describe simple exotic fusion systems on p -groups of order p^4 , and recall their classification (Theorem 4.1). Theorem 1.3 is then established via a case-by-case check in Section 5.

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2. THE RING OF $\mathcal{F}$ -STABLE VIRTUAL CHARACTERS

For a fusion system $\mathcal{F}$ on S , let $\mathrm{Ch}(S)^{\mathcal{F}}$ denote the subring of $\mathbb{Z}[\mathrm{Irr}(S)]$ consisting of $\mathcal{F}$ -stable virtual characters. We say that a character $\chi \in \mathrm{Ch}(S)^{\mathcal{F}}$ is *indecomposable* if whenever $\chi = \chi_1 + \chi_2$ for a pair of characters $\chi_1, \chi_2 \in \mathrm{Ch}(S)^{\mathcal{F}}$ we have $\chi = \chi_1$ or χ_2 . Following [18], write $\mathrm{Ind}(\mathcal{F})$ for the set of indecomposable characters of $\mathrm{Ch}(S)^{\mathcal{F}}$. The following is an elementary consequence of this definition.

Lemma 2.1. *We have, $\langle \mathrm{Ind}(\mathcal{F}) \rangle_{\mathbb{Z}} = \mathrm{Ch}(S)^{\mathcal{F}}$.*

Proof. Any character $\chi \in \mathrm{Ch}(S)^{\mathcal{F}}$ which does not lie in the $\mathbb{Z}$ -span of $\mathrm{Ind}(\mathcal{F})$ must contain an $\mathcal{F}$ -indecomposable constituent with the same property, a contradiction. $\square$

Let $\mathbf{k}(\mathcal{F})$ denote the number of $\mathcal{F}$ -conjugacy classes of elements of S . By [3, Lem. 2.1], $\mathrm{rk}_{\mathbb{Z}}(\mathrm{Ch}(S)^{\mathcal{F}}) = \mathbf{k}(\mathcal{F})$, and in [18, Thm. 1] it is shown that $\mathbf{k}(\mathcal{F}) \leq |\mathrm{Ind}(\mathcal{F})|$. Using Brauer's characterisation of characters, one can easily establish the following:

Proposition 2.2. *Suppose $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with Sylow p -subgroup S . Then $\mathrm{Ch}(S)^{\mathcal{F}} = \mathbb{Z}[\chi|_S \mid \chi \in \mathrm{Irr}(G)]$.*

Proof. See, for example, [18, Thm. 4]. $\square$

In the special case when $S \trianglelefteq G$, we have the following stronger result, which is an elementary consequence of Clifford theory:

Lemma 2.3. *Suppose G is finite group with a normal Sylow p -subgroup S and let $\mathcal{F} = \mathcal{F}_S(G)$. Then $\mathrm{Ind}(\mathcal{F}) = \{\chi|_S \mid \chi \in \mathrm{Irr}(G)\}$. In particular, $|\mathrm{Ind}(\mathcal{F})| = \mathbf{k}(\mathcal{F})$.*

Proof. See [18, Thm. 2]. $\square$

If L is a $\mathbb{Z}$ -lattice, its *volume* $\mathrm{Vol}(L)$ is the modulus of the determinant of a basis. We require the following basic fact concerning this invariant:

Lemma 2.4. *If $M \subseteq L$ are $\mathbb{Z}$ -lattices of equal rank then as abelian groups $|L : M| = \mathrm{Vol}(L)/\mathrm{Vol}(M)$.*

Proof. See [8, Lem. 1]. $\square$

The following application of Lemma 2.4 is the main technical result that we will use to prove Theorem 1.3.

Proposition 2.5. *Suppose that $\mathcal{F}$ is a saturated fusion system on S , and let $\mathcal{N} \subsetneq \mathcal{F}$ be a saturated subsystem on S for which Conjecture 1.1 holds. Let $B_{\mathcal{N}}$ be an $\mathcal{N}$ -stable basis of characters for $\mathrm{Ch}(S)^{\mathcal{N}}$ and $B_{\mathcal{F}}$ be a $\mathbb{Z}$ -linearly independent set of $\mathcal{F}$ -stable characters of size $\mathbf{k}(\mathcal{F})$ given by linear combinations of $B_{\mathcal{N}}$. Suppose, moreover, that:*

- (a) $\mathbf{k}(\mathcal{F}) + 1 = \mathbf{k}(\mathcal{N})$ and there exists $\mathbf{u} \in S$ and $\mathbf{z} \in Z(S)$ such that $\mathbf{u}^{\mathcal{N}} \neq \mathbf{z}^{\mathcal{N}}$, $\mathbf{u}^{\mathcal{F}} = \mathbf{z}^{\mathcal{F}}$ and $|C_S(\mathbf{u})| = p^2$.
- (b) There exists a subset $X_{\mathcal{N}} \subsetneq B_{\mathcal{N}}$ and a bijection $\Phi : X_{\mathcal{N}} \rightarrow B_{\mathcal{F}}$ such that χ occurs with multiplicity 1 in $\Phi(\chi)$ for each $\chi \in X_{\mathcal{N}}$, the unique element $\eta \in B_{\mathcal{N}} \setminus X_{\mathcal{N}}$ satisfies $\eta(\mathbf{u}) - \eta(\mathbf{z}) = \pm p$ and $\text{Ch}(S)^{\mathcal{F}} \oplus \langle \eta \rangle \subsetneq \text{Ch}(S)^{\mathcal{N}}$.

Then $B_{\mathcal{F}}$ is a basis for $\text{Ch}(S)^{\mathcal{F}}$ and Conjecture 1.1 holds for $\mathcal{F}$.

Proof. Since $|C_S(\mathbf{u})| = p^2$ and $\mathbf{z}$ and $\mathbf{u}$ are the only $\mathcal{N}$ -classes fused in $\mathcal{F}$ (by the assumption in part (a)),

$$p^2 \cdot \prod_{s \in S^{\mathcal{F}}} |C_S(s)| = \prod_{s \in S^{\mathcal{N}}} |C_S(s)|.$$

Thus, to prove Conjecture 1.1 holds for $\mathcal{F}$, it suffices to show that $B_{\mathcal{F}}$ is an $\mathcal{F}$ -stable basis for which

$$(2.1) \quad |X_{B_{\mathcal{N}}}(\mathcal{N})| = \pm p \cdot |X_{B_{\mathcal{F}}}(\mathcal{F})|.$$

The condition in part (b) implies that there exists a sequence of elementary row operations which transforms the matrix $X_{B_{\mathcal{N}}}(\mathcal{N})$ into a matrix X' which contains $X_{B_{\mathcal{F}}}(\mathcal{F})$ as minor with respect to the position $(\eta, \mathbf{u})$. Since all elements of $B_{\mathcal{F}}$ are $\mathcal{F}$ -stable, subtracting the column corresponding to $\mathbf{z}$ from that corresponding to $\mathbf{u}$ in X' produces a column of zeros, except for the $(\eta, \mathbf{u})$ position which has value $\eta(\mathbf{u}) - \eta(\mathbf{z}) = \pm p$. Taking the determinant of the resulting matrix now delivers (2.1).

It remains to prove that $B_{\mathcal{F}}$ is a basis of $\text{Ch}(S)^{\mathcal{F}}$. Since $B_{\mathcal{N}}$ is a basis of $\text{Ch}(S)^{\mathcal{N}}$, (2.1) also shows that

$$\text{Vol}(\text{Ch}(S)^{\mathcal{N}}) = p \text{Vol}(\langle B_{\mathcal{F}} \rangle).$$

As abelian groups, we have $\langle B_{\mathcal{F}} \rangle \oplus \langle \eta \rangle \subseteq \text{Ch}(S)^{\mathcal{F}} \oplus \langle \eta \rangle \subsetneq \text{Ch}(S)^{\mathcal{N}}$. Thus taking volumes and using that $\text{Vol}(\langle B_{\mathcal{F}} \rangle)$ divides $\text{Vol}(\langle B_{\mathcal{F}} \rangle \oplus \langle \eta \rangle)$ we obtain:

$$\text{Vol}(\langle B_{\mathcal{F}} \rangle \oplus \langle \eta \rangle) = \text{Vol}(\text{Ch}(S)^{\mathcal{F}} \oplus \langle \eta \rangle)$$

by Lemma 2.4. Thus $\text{Vol}(\langle B_{\mathcal{F}} \rangle) = \text{Vol}(\text{Ch}(S)^{\mathcal{F}})$, so $\langle B_{\mathcal{F}} \rangle = \text{Ch}(S)^{\mathcal{F}}$ and $B_{\mathcal{F}}$ is a basis for $\text{Ch}(S)^{\mathcal{F}}$, as needed. $\square$

We close this section by showing that the conclusion of Conjecture 1.1 is false for non-saturated fusion systems.

Example 2.6. Suppose $S = \langle \mathbf{a} \rangle \cong C_8$, set $P := \langle \mathbf{a}^2 \rangle$ and $\mathcal{F} := \langle \mathcal{F}_S(S); \text{Aut}(P) \rangle_S$. Then $\mathcal{F}$ is not saturated since the non-trivial element in $\text{Aut}(P)$ which fuses $\mathbf{a}^2$ to $\mathbf{a}^6$ does not extend to S . Let $\omega := \exp(2\pi i/8)$ and let $\chi_j \in \text{Irr}(S)$ be given by $\chi_j(\mathbf{a}^k) := \omega^{jk}$ for $0 \leq j \leq 7$. One easily calculates that:

$$\text{Ind}(\mathcal{F}) = \{\chi_0, \chi_2, \chi_4, \chi_6, \chi_1 + \chi_3, \chi_3 + \chi_5, \chi_5 + \chi_7, \chi_7 + \chi_1\}$$

and so $\text{Ch}(S)^{\mathcal{F}} = \langle B \rangle$ where B consists of the first 7 of these characters by Lemma 2.1. Now,

$$|X_B(\mathcal{F}) \overline{X_B(\mathcal{F})^T}| = 2^{22} \neq 2^{21} = \prod_{s \in \mathcal{F}^z} |C_S(s)|.$$

3. PROOF OF THEOREM 1.2

We follow the approach taken in [12], pointing out some overlap with that taken by Olsson in [14]. Let G be a finite group, fix a (not necessarily Sylow) p -subgroup S of G and let $\mathcal{F} = \mathcal{F}_S(G)$ be the associated fusion system. Let $\text{Ch}(S) = \mathbb{Z}[\text{Irr}(S)]$ be the ring of virtual S -characters and $\text{Ch}(S)^G := \text{Ch}(S)^{\mathcal{F}_S(G)}$ be the subring consisting of those which are G -stable. Let $n = |S^G|$ be the number of G -classes of elements in S and set $m = |\text{Irr}(G)|$. Note that $\text{rk}_{\mathbb{Z}}(\text{Ch}(S)^G) = n$ by the discussion in Section 2.

Fix a basis B of $\text{Ch}(S)^G$ and let $\mathcal{F}^z$ be a set of fully $\mathcal{F}$ -centralised $\mathcal{F}$ -conjugacy class representatives of S . Thus if ρ is an $\mathcal{F}$ -stable character of S , we may write

$$\rho = \sum_{\psi \in B} \lambda_{\rho, \psi} \psi$$

for unique $\lambda_{\rho, \psi} \in \mathbb{Z}$ depending on B . Let $D = D_B(G) \in M_{m \times n}(\mathbb{Z})$ and $X = X_B(G) \in M_n(\mathbb{C})$ be the matrices given for $\chi \in \text{Irr}(G)$, $\psi \in B$ and $s \in \mathcal{F}^z$ by:

$$(D_B(G))_{\chi, \psi} := \lambda_{\chi|_S, \psi} \text{ and } (X_B(G))_{\psi, s} := \psi(s).$$

Since $(DX)_{\chi,s} = \sum_{\psi \in B} \lambda_{\chi|s,\psi} \psi(s) = \chi(s)$,

$$(3.1) \quad ((\overline{DX})^T DX)_{s,t} = \sum_{\chi \in \text{Irr}(G)} \overline{\chi(s)} \chi(t) = \delta_{st} |C_G(s)|$$

by column orthogonality. Thus writing $C := \overline{D}^T D$, we have $|(\overline{DX})^T DX| = |C| \cdot |X \overline{X}^T|$, and so

$$(3.2) \quad |X \overline{X}^T| = \frac{1}{|C|} \prod_{s \in \mathcal{F}^z} |C_G(s)|.$$

Remark 3.1. As was first noted in [12, Cor. 3.18], since $|C| \in \mathbb{Z}$, (3.2) implies that $|X \overline{X}^T|$ is a rational algebraic integer, and hence an integer.

We assume for the remainder of this section that S is a Sylow p -subgroup of G . The next result is essentially [14, Folg. 6.9]; here we provide a self-contained proof.

Proposition 3.2. *If S is a Sylow p -subgroup of G then $(|C|, p) = 1$. Consequently, Theorem 1.2 holds.*

Proof. Writing Δ for the diagonal matrix $\Delta_{s,t} = \delta_{s,t} |C_G(s)|$ for $s, t \in \mathcal{F}^z$, (3.3) implies

$$\overline{X}^T C X = \overline{X}^T D^T D X = (\overline{DX})^T DX = \Delta$$

so that $C^{-1} = X \Delta^{-1} \overline{X}^T$ and explicitly for $\psi, \phi \in B$ we have,

$$(C^{-1})_{\psi,\phi} = \sum_{s \in \mathcal{F}^z} \frac{\psi(s) \overline{\phi(s)}}{|C_G(s)|}.$$

For $\psi \in B$ write ψ_0 for the G -class function given by

$$\psi_0(g) := \begin{cases} |G|_{p'} \psi(s) & \text{if } g \text{ is } G\text{-conjugate to } s \in S \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi_0 \in \mathbb{Z}[\text{Irr}(G)]$ by Brauer's characterisation of characters. Indeed, if E is p -elementary then $E = O_p(E) \times O_{p'}(E)$ and conjugating ψ_0 if necessary (using that ψ is G -stable) we may assume that $O_p(E) \leq S$ since S is a Sylow p -subgroup of G . Then for any $\rho \in \text{Irr}(E)$,

$$\langle \psi_0|_E, \rho \rangle = \frac{1}{|E|} \sum_{g \in E} \psi_0(g) \overline{\rho(g)} = \frac{1}{|E|} \sum_{g \in O_p(E)} |G|_{p'} \psi(g) \overline{\rho(g)} = \frac{|G|_{p'}}{|O_{p'}(E)|} \langle \psi|_{O_p(E)}, \rho|_{O_p(E)} \rangle \in \mathbb{Z}.$$

Calculating, we obtain:

$$|G|_{p'}^2 (C^{-1})_{\psi,\phi} = |G|_{p'}^2 \sum_{s \in \mathcal{F}^z} \frac{\psi(s) \overline{\phi(s)}}{|C_G(s)|} = \sum_{s \in \mathcal{F}^z} \frac{\psi_0(s) \overline{\phi_0(s)}}{|C_G(s)|} = \langle \psi_0, \phi_0 \rangle \in \mathbb{Z}.$$

Hence $|G_{p'}|^2 C^{-1} \in M_n(\mathbb{Z})$ from which it follows that $(|C|, p) = 1$. Finally, for $\mathcal{F} = \mathcal{F}_S(G)$, $C_S(s) \in \text{Syl}_p(C_G(s))$ if and only if s is fully $\mathcal{F}$ -centralised. Thus by (3.2) we have

$$|X \overline{X}^T|_p = \prod_{s \in \mathcal{F}^z} |C_G(s)|_p = \prod_{s \in \mathcal{F}^z} |C_S(s)|,$$

as required. □

3.1. Characters in the principal block. There is an analogue D_0 of D for the principal p -block B_0 of G given by restricting its rows to $\text{Irr}(B_0)$. Let $X_B(G)$ be as above and let $D_0 = D_B(B_0)$ be given for $\chi \in \text{Irr}(B_0)$, and $\psi \in B$ by $(D_B(B_0))_{\chi,\psi} := \lambda_{\chi|s,\psi}$ as before. Set $C_0 = \overline{D_0}^T D_0$. Using orthogonality of characters in B_0 , we show the following:

Proposition 3.3. *For $D_0 = D_B(B_0)$ as above, we have:*

$$(3.3) \quad ((\overline{D_0 X})^T D_0 X)_{s,t} = \delta_{st} \dim(B_0(s))$$

where $B_0(s)$ is the principal p -block of $C_G(s)$. Consequently $(|C_0|, p) = 1$.

Proof. Since $(D_0X)_{\chi,s} = \sum_{\psi \in B} \lambda_{\chi|s,\psi} \psi(s) = \chi(s)$,

$$((D_0X)^T D_0X)_{s,t} = \sum_{\chi \in \text{Irr}(B_0)} \overline{\chi(s)} \chi(t) = \delta_{st} \dim(B_0(s))$$

by a refinement of block orthogonality (see, e.g., [11, Thm. 2.1]). Now,

$$\frac{1}{|C|} \prod_{s \in \mathcal{F}^z} |C_G(s)| = |X \overline{X^T}| = \frac{1}{|C_0|} \prod_{s \in \mathcal{F}^z} \dim(B_0(s)).$$

Since $(|C|, p) = 1$ by Proposition 3.2 and since $v_\ell(\dim(B_0(s))) = v_\ell(|C_G(s)|)$, for each $s \in \mathcal{F}^z$ we must also have $(|C_0|, p) = 1$. $\square$

4. SIMPLE EXOTIC FUSION SYSTEMS ON p -GROUPS OF ORDER p^4

For an odd prime p , in this section we introduce notation to describe fusion systems on a Sylow p -subgroup of $\text{PSp}_4(p)$. Let

$$V := \{f(x, y) \in \mathbb{F}_p[x, y] \mid \deg(f) = 2\} = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle, \text{ for } \mathbf{v}_1 = x^2, \mathbf{v}_2 = xy \text{ and } \mathbf{v}_3 = y^2,$$

and regard V as an $(\mathbb{F}_p^\times \times \text{GL}_2(p))$ -module via the action

$$f(x, y) \cdot \left(\lambda, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \lambda \cdot f(ax + by, cx + dy).$$

Then setting $U := \langle \mathbf{u} \rangle$ for $\mathbf{u} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, define $S := V \rtimes U$ so that S is isomorphic to a Sylow p -subgroup of $\text{PSp}_4(p)$. Let $\mathbf{z} := \mathbf{v}_1$ (so that $\langle \mathbf{z} \rangle = Z(S)$) and $E := \langle \mathbf{z}, \mathbf{u} \rangle \cong C_p \times C_p$. Let $\chi : \mathbb{F}_p^\times \rightarrow \{\pm 1\}$ be the unique non-trivial homomorphism, define subgroups of $\mathbb{F}_p^\times \times \text{GL}_2(p)$

$$H := \{(\det(A)^{-1}, A) \mid A \in \text{GL}_2(p)\} \text{ and } H^* := \{(\chi(\det(A)) \cdot \det(A)^{-1}, A) \mid A \in \text{GL}_2(p)\}$$

and observe that neither of these acts faithfully on V (scalar matrices act trivially) and so their images $\overline{H}$ and $\overline{H}^*$ in $\text{GL}(V)$ are isomorphic to $\text{PGL}_2(p)$. Choosing a primitive element $\lambda \in \mathbb{F}_p^\times$ for $e \mid p-1$, define subgroups of $\text{GL}(V)$:

$$\Gamma_{(e)} := \overline{H(\langle \lambda^e \rangle \times \langle I_2 \rangle)} \text{ and } \Gamma_{(e)}^* := \overline{H^*(\langle \lambda^e \rangle \times \langle I_2 \rangle)}.$$

In particular $\Gamma := \Gamma_{(1)} = \Gamma_{(1)}^* \cong (C_{p-1} \times \text{PGL}_2(p))$ and we may form $N := V \rtimes \Gamma$. By [9, Thm. 4.1], there is a fusion system

$$\mathcal{F}_{(1)} := \langle \mathcal{N}, \mathcal{H} \rangle_S, \text{ where } \mathcal{N} = \mathcal{F}_S(N) \text{ and } \mathcal{H} = \mathcal{F}_{N_S(E)}(Y)$$

with $Y \cong (C_p \times C_p) \rtimes \text{GL}_2(p)$ (in the notation of [9], $\mathbf{E}_0 = \mathcal{H}_0$ is equal to $E^\mathcal{F}$). Moreover, from the analysis in the proof of [13, Thm. 7.1], there are simple subsystems $\mathcal{F}_{(2)}, \mathcal{F}_{(4)}^*$ of $\mathcal{F}_{(1)}$ such that:

$$O^{p'}(\mathcal{F}_{(1)}) = \begin{cases} \mathcal{F}_{(2)} & \text{if } p \equiv 3 \pmod{4} \\ \mathcal{F}_{(4)}^* & \text{if } p \equiv 1 \pmod{4} \end{cases}$$

where $\mathcal{F}_{(2)}$ and $\mathcal{F}_{(4)}^*$ are defined by the properties that:

$$N_{\mathcal{F}_{(2)}}(V) = \mathcal{F}_S(V \rtimes \Gamma_{(2)}) \text{ and } N_{\mathcal{F}_{(4)}^*}(V) = \mathcal{F}_S(V \rtimes \Gamma_{(4)}^*).$$

It is furthermore clear from the above description that the only $N_{O^{p'}(\mathcal{F}_{(1)})}(V)$ -classes fused in $O^{p'}(\mathcal{F}_{(1)})$ are those represented by $\mathbf{z}$ and $\mathbf{u}$.

There is an additional simple subsystem $\mathcal{G}$ of $\mathcal{F}_{(1)}$ obtained by *pruning* V (see [15]). Explicitly, set $\mathbf{b} := (b, \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix})$ for some (any) primitive $b \in \mathbb{F}_p^\times$ and note that $\mathbf{b}$ acts on $V = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ via the matrix $\begin{pmatrix} b & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & b^3 \end{pmatrix}$. Let $N := S : \langle \mathbf{b} \rangle$. By [15, Lem. 6.4], there is a fusion system

$$\mathcal{G} := \langle \mathcal{N}, \mathcal{H} \rangle_S, \text{ where } \mathcal{N} = \mathcal{F}_S(N) \text{ and } \mathcal{H} = \mathcal{F}_{N_S(E)}(Y)$$

with $Y \cong E \rtimes \text{SL}_2(p)$. Again, the only pair of $\mathcal{N}$ -classes fused in $\mathcal{G}$ are those represented by $\mathbf{z}$ and $\mathbf{u}$. With these descriptions, the following is proven in [13].

Theorem 4.1. *Suppose $p \geq 7$, and $\mathcal{F}$ is a simple exotic saturated fusion system a p -group of order p^4 . Then either $\mathcal{F} \cong O^{p'}(\mathcal{F}_{(1)})$ or $\mathcal{F} \cong \mathcal{G}$.*

Proof. See [13, Thm. 7.1 and Tables 7.1, 7.2]. $\square$

TABLE 1. S -restricted irreducible characters of N when $\mathcal{F} = \mathcal{G}$

χ	conditions	$\#\{\chi\}$	$\#\{\chi _S\}$	$\chi(1)$	$\chi(\mathbf{u})$	$\chi(\mathbf{z})$
χ_i	$1 \leq i \leq p-1$	$p-1$	1	1	1	1
$\chi_{i,j}$	$0 \leq i, j \leq p-1, i \neq 0$	p	p	$p-1$	-1	$p-1$
$\chi_{0,s}^t$	$p \equiv 1 \pmod{3}, 0 \leq s, t \leq 2$	9	3	$(p-1)/3$	$(p-1)/3$	$(p-1)/3$
$\chi_{0,1}$	$p \equiv 2 \pmod{3}$	1	1	$p-1$	$p-1$	$p-1$
$\chi_{i,j,k}$	$(i,j) \neq (0,0)$	p	p	$p(p-1)$	0	$-p$
$\chi_{0,s,0}^t$	$0 \leq s, t \leq 1$	4	2	$p(p-1)/2$	0	$p(p-1)/2$

When $p \leq 5$ and $\mathcal{F}$ is a fusion system as in Theorem 4.1, further examples arise in addition to $O^{p'}(\mathcal{F}_{(1)})$ and $\mathcal{G}$. These have all been calculated using the **MAGMA** [4] package **FusionSystems** developed jointly by the second author and Parker (see [15]) and are listed explicitly in [15, Tables 1-4].

5. PROOF OF THEOREM 1.3

Here we prove Theorem 1.3 via a case-by-case check. After dealing with $\mathcal{F} = \mathcal{G}$ and $\mathcal{F} = \mathcal{F}_{(1)}$ in Sections 5.1 and 5.2 respectively, we deduce Conjecture 1.1 in Section 5.3 for $\mathcal{F} = O^{p'}(\mathcal{F}_{(1)})$ as a consequence of the latter. Finally, in Section 5.4 we deal with all remaining cases when $p = 3, 5$ with computer assistance.

5.1. The case $\mathcal{F} = \mathcal{G}$. Suppose first that $p \geq 5$ and $\mathcal{F} = \mathcal{G}$ is as described in Section 4. In particular, $N_{\mathcal{F}}(V) = N_{\mathcal{F}}(S) = \mathcal{F}_S(N)$ where $N = V : \langle \mathbf{u}, \mathbf{b} \rangle = S : \langle \mathbf{b} \rangle$.

Notation 5.1. We introduce the following notation to describe the characters of N .

- (a) Write $\chi_1, \dots, \chi_{p-1}$ for the linear characters obtained by inflation from $N/S \cong \langle \mathbf{b} \rangle$.
- (b) Write $\chi_{i,j} := \psi_{i,j}^N$ where $\{\psi_{i,j}\}$ is a complete set of $\langle \mathbf{b} \rangle$ -orbit representatives of $\text{Irr}(S/[S, S]) \cong \langle \bar{\mathbf{u}}, \bar{\mathbf{v}}_3 \rangle$ for $i, j \in \mathbb{F}_p$. If $i \neq 0$ then $\chi_{i,j}$ is irreducible, leading to p distinct characters of degree $p-1$. If $p \equiv 2 \pmod{3}$ then $\chi_{0,1}$ is also irreducible of degree $p-1$. If $p \equiv 1 \pmod{3}$, $\psi_{0,1}, \psi_{0,b}$ and ψ_{0,b^2} are three separate $\langle \mathbf{b} \rangle$ -orbit representatives, each with a stabiliser of order 3 which yields 9 further irreducible characters $\chi_{0,s}^t := (\widehat{\psi_{0,b^s}} \otimes \rho_t)^N$ each of degree $(p-1)/3$ for $0 \leq s, t \leq 2$ where $\rho_t \in \text{Irr}(I_{\langle \mathbf{b} \rangle}(\psi_{0,b^s}))$ and $\widehat{\psi}$ denotes the extension of ψ to $V \rtimes I_{\langle \mathbf{b} \rangle}(\psi)$.
- (c) For $i, j, k \in \mathbb{F}_p$, write $\chi_{i,j,k} := \psi_{i,j,k}^N$ where $\{\psi_{i,j,k}\}$ is a complete set of $\langle \mathbf{b} \rangle$ -orbit representatives of degree p characters of $\text{Irr}(S)$ and $\psi_{i,j,k} = \rho_{i,j,k}^S$ for some $\rho_{i,j,k} \in \text{Irr}(V)$. Since $\psi_{0,1,0}$ and $\psi_{0,b,0}$ both have $\langle \mathbf{b} \rangle$ -stabiliser of order 2, we obtain 4 irreducible characters $\chi_{0,s,0}^t := (\widehat{\psi_{0,b^s,0}} \otimes \rho_t)^N$ each of degree $p(p-1)/2$ for $0 \leq s, t \leq 1$ and $\rho_t \in \text{Irr}(I_{\langle \mathbf{b} \rangle}(\psi_{0,b^s,0}))$.

Lemma 5.2. *When $\mathcal{F} = \mathcal{G}$, the characters listed in Table 1 form a basis B_N for $\text{Ch}(S)^N$, and the values on $\mathbf{u}$ and $\mathbf{z}$ are as given.*

Proof. Note first that since $S \trianglelefteq N$, $\text{Ind}(\mathcal{N}) = \{\chi|_S \mid \chi \in \text{Irr}(N)\}$ is a basis for $\text{Ch}(S)^N$ by Lemma 2.3. The characters described in Notation 5.1 are distinct and the squares of their degrees sum to $|N|$. It thus suffices to determine the values of these characters on $\mathbf{u}$ and $\mathbf{z}$. The values of χ_i are clear. Plainly $\chi_{i,j}(\mathbf{z}) = \psi_{i,j}^N(\mathbf{z}) = p-1$ and

$$\chi_{i,j}(\mathbf{u}) = \psi_{i,j}^N(\mathbf{u}) = \sum_{b \in \langle \mathbf{b} \rangle} \psi_{i,j}(\mathbf{u}^b) = \begin{cases} \sum_{i=1}^{p-1} \zeta^i = -1 & \text{if } i \neq 0 \\ \sum_{i=1}^{p-1} 1 = p-1 & \text{if } i = 0 \end{cases},$$

where ζ is a p 'th root of unity. Similar arguments give the stated values for $\chi_{0,s}^t$ for $0 \leq s, t \leq 2$ when $p \equiv 1 \pmod{3}$. If $(i,j) \neq (0,0)$ then $\chi_{i,j,k}(\mathbf{u}) = \psi_{i,j,k}^N(\mathbf{u}) = 0$ since $\psi_{i,j,k}(\mathbf{u}) = 0$ and

$$\chi_{i,j,k}(\mathbf{z}) = \sum_{b \in \langle \mathbf{b} \rangle} \psi_{i,j,k}(\mathbf{z}^b) = \sum_{i=1}^{p-1} \zeta^i p = -p.$$

TABLE 2. A basis $B_{\mathcal{F}}$ for $\text{Ch}(S)^{\mathcal{F}}$ when $\mathcal{F} = \mathcal{G}$

χ	conditions	$\#\{\chi _S\}$	$\chi(1)$	$\chi(\mathbf{z}) = \chi(\mathbf{u})$
χ_0	—	1	1	1
$\chi_{i,j} + \chi_1$	$\chi_{i,j} \neq \chi_2$	$p - 1$	$p^2 - 1$	-1
$\chi_{i,j,k} + \chi_2$	$\chi_{i,j,k} \neq \chi_1$	$p - 1$	$p^2 - 1$	-1
$\chi_1 + \chi_2$	—	1	$p^2 - 1$	-1
$\chi_{0,s}^t$	$p \equiv 1 \pmod{3}$	3	$(p-1)/3$	$(p-1)/3$
$\chi_{0,1}$	$p \equiv 2 \pmod{3}$	1	$p - 1$	$p - 1$
$\chi_{0,s,0}^t + \frac{p-1}{2} \cdot \chi_1$	—	2	$p^2(p-1)/2$	0

Similarly, $\chi_{0,s,0}^t(\mathbf{u}) = 0$ for $0 \leq s, t \leq 1$, and we have

$$\chi_{0,s,0}^t(\mathbf{z}) = (\widehat{\psi_{0,b^s,0}} \otimes \rho_t)^N(\mathbf{z}) = |N/\langle \mathbf{b}^{(p-1)/2} \rangle| \psi_{0,s,0}(\mathbf{z}) = p(p-1)/2,$$

since $\psi_{0,s,0}(\mathbf{z}) = p$. □

Proposition 5.3. *For all $p \geq 5$, Conjecture 1.1 holds when $\mathcal{F} = \mathcal{G}$.*

Proof. Let χ_1 be any of the characters $\chi_{i,j,k}$ with $(i, j) \neq (0, 0)$ and χ_2 be any of the characters $\chi_{i,j}$ with $i \neq 0$. Using Table 1, it is elementary to observe that Table 2 lists a set $B_{\mathcal{F}}$ of $|S^{\mathcal{F}}| = |S^{\mathcal{N}}| - 1$ linearly independent $\mathcal{F}$ -stable characters, each of which is an $\mathbb{N}$ -linear combination of $B_{\mathcal{N}}$. Part (a) of Proposition 2.5 holds by the description of $\mathcal{G}$ given in Section 4 and part (b) also holds with $\eta = \chi_1$. The result follows. □

5.2. The case $\mathcal{F} = \mathcal{F}_{(1)}$. Now suppose that $p \geq 3$ and $\mathcal{F} = \mathcal{F}_{(1)}$ is as in Section 4. In particular $N = V \rtimes \Gamma$ with $\Gamma := \Gamma_{(1)} \cong C_{p-1} \times \text{PGL}_2(p)$. We first determine the orbits of Γ on V . Fix $\epsilon \in \mathbb{F}_p^\times$ such that $\sqrt{-\epsilon} \notin \mathbb{F}_p^\times$.

Lemma 5.4. *There are exactly three non-trivial orbits of Γ on V represented by x^2 , xy , and $x^2 + \epsilon y^2$ with Γ -stabilisers isomorphic with $C_p \rtimes C_{p-1}$, $D_{2(p-1)}$ and $D_{2(p+1)}$ respectively. Consequently, $S^{\mathcal{F}} = \{1, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1 + \epsilon \mathbf{v}_3, \mathbf{u}, \mathbf{u} \mathbf{v}_1\}$.*

Proof. Any non-zero 2-homogeneous polynomial $f \in V$ is either irreducible, separable or the square of a linear polynomial. An elementary counting argument shows that the set of all polynomials of such a type is respectively a Γ -orbit of V of size $p(p-1)^2/2$, $p(p^2-1)/2$ or p^2-1 . To compute the stabilisers, let $\lambda \in \mathbb{F}_p^\times$ be primitive and $\mu = a + bu$ be a primitive element of $\mathbb{F}_p[u]/(u^2 + \epsilon) \cong \mathbb{F}_{p^2}$. Thus $N(\mu) = \mu^{p+1} = a^2 + b^2\epsilon \in \mathbb{F}_p^\times$. Observe that:

$$\begin{aligned} \text{stab}_\Gamma(x^2 + \epsilon y^2) &\geq \overline{\left\langle \left(-1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \left(N(\mu)^{-1}, \begin{pmatrix} a & b\epsilon \\ -b & a \end{pmatrix}\right) \right\rangle} \cong D_{2(p+1)} \\ \text{stab}_\Gamma(xy) &\geq \overline{\left\langle \left(-1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \left(\lambda^{-1}, \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\right) \right\rangle} \cong D_{2(p-1)} \\ \text{stab}_\Gamma(x^2) &\geq \overline{\left\langle \left(d^{-1}, \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right) \mid c \in \mathbb{F}_p, d \in \mathbb{F}_p^\times \right\rangle} \cong C_p \rtimes C_{p-1}. \end{aligned}$$

The first statement of the lemma now follows. The second statement also follows, together with the fact that there are exactly p N -classes of elements in $S \setminus V$ represented by $\mathbf{u}(\mathbf{v}_2)^i$ for $0 \leq i \leq p-1$ which fall into two $\mathcal{F}$ -classes represented by $\mathbf{u}$ and $\mathbf{u} \mathbf{v}_2$ in $V \rtimes \Gamma$ (see [9, Not. 2.4]). □

Notation 5.5. We now adopt the following notation to describe the characters of N .

- (a) For $k = p-1, p$ or $p+1$ let θ_k denote the restriction to S of any irreducible representation of $\text{PGL}_2(p)$ of degree k .
- (b) Write $\text{Irr}(V) = \{\psi_{i,j,k} \mid 0 \leq i, j, k \leq p-1\}$.
- (c) For $\psi \in \text{Irr}(V)$ and $\rho \in \text{Irr}(I_\Gamma(V))$ write $\widehat{\psi}$ for its extension to $V \cdot I_\Gamma(V)$ and set $\chi(\psi, \rho) := (\widehat{\psi} \otimes \rho)^\Gamma|_S$ and $\chi(\psi) := \chi(\psi, 1)$.

TABLE 3. Pairs $f_{v,\psi}(g)$, $n_{v,\psi}$ for $g = \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \in \mathrm{PGL}_2(p)$

v/ψ	$\psi_{1,0,0}$	$\psi_{0,1,0}$	$\psi_{1,0,\epsilon}$
$\mathbf{v}_1$	$a^2, p(p-1)$	$ac, 2p(p-1)$	$a^2 + \epsilon c^2, 0$
$\mathbf{v}_2$	$2ab, 2p(p-1)$	$ad + bc, (p-1)^2$	$2(ab + \epsilon cd), p^2 - 1$
$\mathbf{v}_1 + \epsilon \mathbf{v}_3$	$a^2 + \epsilon b^2, 0$	$ac + \epsilon bd, p^2 - 1$	$a^2 + \epsilon b^2 + \epsilon c^2 + \epsilon^2 d^2, (p+1)^2$

(d) Write $\rho_{p-1} \in \mathrm{Irr}(I_\Gamma(\psi_{1,0,0}))$ for the unique character of degree $p-1$ and ρ_2 for any character of degree 2 in $\mathrm{Irr}(I_\Gamma(\psi_{0,1,0}))$ or $\mathrm{Irr}(I_\Gamma(\psi_{1,0,\epsilon}))$.

The following function will be of assistance when computing characters of N .

Lemma 5.6. *Suppose $\psi \in \mathrm{Irr}(V)/\Gamma$, $\rho \in \mathrm{Irr}(I_\Gamma(\psi))$ and $v \in V/\Gamma$. Then*

$$(\widehat{\psi} \otimes \rho)^\Gamma(v) = \frac{p \cdot \rho(1)}{|I_\Gamma(\psi)|} (n_{v,\psi} - p^2 + 1)$$

where $n_{v,\psi}$ is exactly as given in Table 3.

Proof. We have,

$$(\widehat{\psi} \otimes \rho)^\Gamma(v) = \frac{\rho(1)}{|I_\Gamma(\psi)|} \sum_{g \in \Gamma} \psi(g^{-1}vg) = \frac{\rho(1)}{|I_\Gamma(\psi)|} \sum_{g \in \mathrm{PGL}_2(p)} \sum_{\lambda \in \mathbb{F}_p^\times} \omega^{\lambda f_{v,\psi}(g)},$$

where ω is a primitive p 'th root of unity and $f_{v,\psi} : \mathrm{PGL}_2(p) \rightarrow \mathbb{F}_p$ is one of the functions listed in Table 3. Writing $n_{v,\psi} = |\{g \in \mathrm{PGL}_2(p) \mid f_{v,\psi}(g) = 0\}|$ and separating the sum, this becomes

$$\frac{\rho(1)}{|I_\Gamma(\psi)|} ((p-1)n_{v,\psi} - (|\mathrm{PGL}_2(p)| - n_{v,\psi}))$$

exactly as in the statement of the lemma. The values of $n_{v,\psi}$ in Table 3 are determined by elementary counting in almost all cases. One exception is the case $v = \mathbf{v}_1 + \epsilon \mathbf{v}_3$ and $\psi = \psi_{1,0,\epsilon}$ where $f_{v,\psi}$ is a non-degenerate smooth projective quadric on $\mathbb{P}^3 \mathbb{F}_p$. Any point on this quadric must arise from an element of $\mathrm{PGL}_2(p)$. Indeed, if $ad = bc$ and $a \neq 0$ then

$$0 = (a^2 + \epsilon b^2)(a^2 + \epsilon c^2),$$

a contradiction (the case $a = 0$ is dealt with similarly). Thus the assertion that $n_{v,\psi} = (p+1)^2$ follows from the main results of [16]. $\square$

Lemma 5.7. *When $\mathcal{F} = \mathcal{F}_{(1)}$, the first 6 characters listed in Table 4 form a basis $B_{\mathcal{N}}$ for $\mathrm{Ch}(S)^{\mathcal{N}}$, and the values are as presented there.*

Proof. By Gallagher's theorem, the characters described in the first column of Table 4 coincide with the set $\{\chi|_S \mid \chi \in \mathrm{Irr}(N)\}$. The values of characters of form $\chi(\psi, \rho)$ in columns two through four of Table 4 are quickly computed using Lemma 5.6 and Table 3. All values of characters not of this form are deduced from [10, Table 11.2]. The values of characters of form $\chi(\psi, \rho)$ in the final two columns are plainly 0 when $\psi \in \{\psi_{1,0,\epsilon}, \psi_{0,1,0}\}$ (since $p \nmid I_\Gamma(\psi)$ in these cases). The values when $\psi = \psi_{1,0,0}$ may be obtained directly via a straightforward calculation and we omit the details. Finally, since the last 4 rows of Table 4 are linear combinations of the first 6, the lemma follows from Proposition 2.2. $\square$

Proposition 5.8. *For all $p \geq 3$ Conjecture 1.1 holds when $\mathcal{F} = \mathcal{F}_{(1)}$.*

Proof. Table 5 lists a linearly independent subset $B_{\mathcal{F}} \subseteq \mathrm{Ch}(S)^{\mathcal{F}}$ of size $5 = |S^{\mathcal{F}}| = |S^{\mathcal{N}}| - 1$ and we conclude using Proposition 2.5 with $\eta = \theta_{p-1}$. $\square$

Remark 5.9. Following [5], we say that a fusion system $\mathcal{F}$ is *factorial* if every character in $\mathrm{Ch}(S)^{\mathcal{F}}$ decomposes uniquely as a sum of indecomposable characters, and this holds if and only if $|\mathrm{Ind}(\mathcal{F})| = \mathbf{k}(\mathcal{F})$ by [18, Lem. 5].

TABLE 4. S -restricted irreducible characters of N when $\mathcal{F} = \mathcal{F}_{(1)}$

χ	$\chi(1)$	$\chi(\mathbf{v}_1)$	$\chi(\mathbf{v}_2)$	$\chi(\mathbf{v}_1 + \epsilon \mathbf{v}_3)$	$\chi(\mathbf{u})$	$\chi(\mathbf{u} \cdot \mathbf{v}_2)$
1_S	1	1	1	1	1	1
θ_{p-1}	$p-1$	$p-1$	$p-1$	$p-1$	-1	-1
$\chi(\psi_{1,0,0})$	p^2-1	-1	$p-1$	$-(p+1)$	$p-1$	-1
$\chi(\psi_{1,0,0}, \rho_{p-1})$	$(p^2-1)(p-1)$	$-(p-1)$	$(p-1)^2$	$-(p^2-1)$	$-(p-1)$	1
$\chi(\psi_{1,0,\epsilon})$	$p(p-1)^2/2$	$-p(p-1)/2$	0	p	0	0
$\chi(\psi_{0,1,0})$	$p(p^2-1)/2$	$p(p-1)/2$	$-p$	0	0	0
$\theta_p = \theta_{p-1} + 1_S$	p	p	p	p	0	0
$\theta_{p+1} = \theta_{p-1} + 2 \cdot 1_S$	$p+1$	$p+1$	$p+1$	$p+1$	1	1
$\chi(\psi_{1,0,\epsilon}, \rho_2) = 2\chi(\psi_{1,0,\epsilon})$	$p(p-1)^2$	$-p(p-1)$	0	$2p$	0	0
$\chi(\psi_{0,1,0}, \rho_2) = 2\chi(\psi_{0,1,0})$	$p(p^2-1)$	$p(p-1)$	$-2p$	0	0	0

TABLE 5. A basis $B_{\mathcal{F}}$ for $\text{Ch}(S)^{\mathcal{F}}$ when $\mathcal{F} = \mathcal{F}_{(1)}$

χ	$\chi(1)$	$\chi(\mathbf{v}_1) = \chi(\mathbf{u})$
1_S	1	1
$\chi(\psi_{1,0,0}, \rho_{p-1})$	$(p^2-1)(p-1)$	$-(p-1)$
$\chi(\psi_{0,1,0}) + \chi(\psi_{1,0,\epsilon})$	$p^2(p-1)$	0
$\theta_{p-1} + \chi(\psi_{1,0,0})$	$(p-1)(p+2)$	$p-2$
$\frac{p-1}{2}(\chi(\psi_{1,0,0})) + \chi(\psi_{0,1,0})$	$(p^2-1)(2p-1)/2$	$(p-1)^2/2$

For $\mathcal{F} = \mathcal{F}_{(1)}$ and $\mathcal{N} = \mathcal{N}_{(1)}$ as above, observe that

$$\text{reg}_S = 1_S + \theta_{p-1} + \chi(\psi_{1,0,0}) + \chi(\psi_{1,0,0}, \rho_{p-1}) + p\chi(\psi_{1,0,\epsilon}) + p\chi(\psi_{0,1,0}),$$

where reg_S denotes the regular representation of S . This, together with [5, Prop. 2.11], implies that $\mathcal{N}$ is a factorial fusion system with $\text{Ind}(\mathcal{N}) = B_{\mathcal{N}}$. However, while the 5 characters listed in Table 5 all lie in $\text{Ind}(\mathcal{F})$, it is not true that $\mathcal{F}$ is factorial since the character $\frac{p-1}{2} \cdot \theta_{p-1} + \chi(\psi_{1,0,\epsilon})$ is also $\mathcal{F}$ -indecomposable (and we conjecture that there are no further elements of $\text{Ind}(\mathcal{F})$). We thus obtain an infinite family of non-factorial exotic fusion systems, supplementing the realisable examples of such systems constructed in [6].

5.3. The case $\mathcal{F} = O^{p'}(\mathcal{F}_{(1)})$. Recall that $O^{p'}(\mathcal{F}_{(1)})$ is respectively $\mathcal{F}_{(2)}$ or $\mathcal{F}_{(4)}^*$ according to whether $p \equiv 3 \pmod{4}$ or $p \equiv 1 \pmod{4}$. We handle these two cases separately, starting with the former.

Lemma 5.10. *Suppose $p \equiv 3 \pmod{4}$. There are exactly three non-trivial orbits of $\Gamma_{(2)}$ on V represented by x^2 , xy , and $x^2 + \epsilon y^2$ with stabilisers isomorphic with $C_p \rtimes C_{\frac{p-1}{2}}$, C_{p-1} and C_{p+1} respectively.*

Proof. Since $p \equiv 3 \pmod{4}$, $-1 \notin \langle \lambda^2 \rangle$ and the structure of $\text{stab}_{\Gamma_{(2)}}(\mathbf{v}) = \Gamma_{(2)} \cap \text{stab}_{\Gamma}(\mathbf{v})$ follows from the explicit description of $\text{stab}_{\Gamma}(\mathbf{v})$ given in the proof of Lemma 5.4 for $\mathbf{v} \in \{x^2, xy, x^2 + \epsilon y^2\}$. There can be no further orbits by counting, and the lemma follows. $\square$

Proposition 5.11. *For all $p \geq 3$ with $p \equiv 3 \pmod{4}$, Conjecture 1.1 holds when $\mathcal{F} = O^{p'}(\mathcal{F}_{(1)})$.*

Proof. Suppose that $\mathcal{F} = \mathcal{F}_{(2)}$, $\mathcal{N} = \mathcal{N}_{(2)}$ and $N = N_{(2)}$. The S -restricted irreducible characters of N are determined by those of $N_{(1)}$ via Clifford theory. Specifically, we see that all characters listed in Table 4 remain irreducible, except for those in the last two rows, and $\chi(\psi_{1,0,0}, \rho_{p-1})$ which splits as $\chi(\psi_{1,0,0}, \rho_a) + \chi(\psi_{1,0,0}, \rho_b)$, where $\rho_a, \rho_b \in \text{Irr}(I_{\Gamma_{(2)}}(\psi_{1,0,0}))$ both have degree $(p-1)/2$. Arguing as in the proof of Lemma 5.7, we see that the seven characters

$$1_S, \theta_{p-1}, \chi(\psi_{1,0,0}), \chi(\psi_{1,0,0}, \rho_a), \chi(\psi_{1,0,0}, \rho_b), \chi(\psi_{1,0,\epsilon}), \chi(\psi_{0,1,0})$$

form a basis for $B_{\mathcal{N}}$. Now, the set $B_{\mathcal{F}}$ given by

$$\{1_S, \chi(\psi_{1,0,0}, \rho_a), \chi(\psi_{1,0,0}, \rho_b), \chi(\psi_{1,0,\epsilon}) + \chi(\psi_{0,1,0}), \theta_{p-1} + \chi(\psi_{1,0,0}), \frac{p-1}{2}\chi(\psi_{1,0,0}) + \chi(\psi_{0,1,0})\}$$

is $\mathcal{F}_{(2)}$ -stable and satisfies the conditions of Proposition 2.5 with $\eta = \theta_{p-1}$, so we conclude. $\square$

Lemma 5.12. *Suppose $p \equiv 1 \pmod{4}$ and let λ be a primitive element of $\mathbb{F}_p^\times$. There are exactly five non-trivial orbits of $\Gamma_{(4)}$ on V represented by x^2 , xy , λxy , $x^2 + \epsilon y^2$ and $\lambda(x^2 + \epsilon y^2)$ with stabilisers isomorphic with $C_p \rtimes C_{\frac{p-1}{4}}$, D_{p-1} , D_{p-1} , D_{p+1} and D_{p+1} respectively.*

Proof. Adopting the same notation as in Lemma 5.4, $N(\mu)$ and λ are plainly non-squares so the structure of $\text{stab}_{\Gamma_{(4)}^*}(\mathbf{v}) = \Gamma_{(4)}^* \cap \text{stab}_\Gamma(\mathbf{v})$ follows from Lemma 5.4 for $\mathbf{v} \in \{x^2, xy, x^2 + \epsilon y^2\}$. The structure of $\text{stab}_{\Gamma_{(4)}^*}(\mathbf{v})$ for $\mathbf{v} \in \{\lambda xy, \lambda(x^2 + \epsilon y^2)\}$ is determined similarly and there can be no further orbits by counting. The result follows. $\square$

Proposition 5.13. *For all $p \geq 5$ with $p \equiv 1 \pmod{4}$, Conjecture 1.1 holds when $\mathcal{F} = O^{p'}(\mathcal{F}_{(1)})$.*

Proof. Suppose that $\mathcal{F} = \mathcal{F}_{(4)}^*$ and $\mathcal{N} = \mathcal{N}_{(4)}^*$. By Clifford theory, and Lemma 5.12, the character $\chi(\psi_{1,0,0}, \rho_{p-1})$ in Table 4 splits into four irreducible characters on restriction to $N_{(4)}^*$ while $\chi(\psi_{1,0,\epsilon})$, $\chi(\psi_{0,1,0})$, $\chi(\psi_{1,0,\epsilon}, \rho_2)$ and $\chi(\psi_{0,1,0}, \rho_2)$ each split into two. Thus writing

$$\begin{aligned} \chi(\psi_{1,0,0}, \rho_{p-1}) &= \chi_a(\psi_{1,0,0}, \rho_{p-1}) + \chi_b(\psi_{1,0,0}, \rho_{p-1}) + \chi_c(\psi_{1,0,0}, \rho_{p-1}) + \chi_d(\psi_{1,0,0}, \rho_{p-1}) \\ \chi(\psi_{1,0,\epsilon}) &= \chi_a(\psi_{1,0,\epsilon}) + \chi_b(\psi_{1,0,\epsilon}) \\ \chi(\psi_{0,1,0}) &= \chi_a(\psi_{0,1,0}) + \chi_b(\psi_{0,1,0}) \end{aligned}$$

we see that the eleven characters

$$1_S, \theta_{p-1}, \chi(\psi_{1,0,0}), \chi_*(\psi_{1,0,0}, \rho_{p-1}), \chi_*(\psi_{1,0,\epsilon}), \chi_*(\psi_{0,1,0})$$

form a basis $B_{\mathcal{N}}$ for $\text{Ch}(S)^{\mathcal{N}}$ and the set $B_{\mathcal{F}}$ of ten characters given by

$$\{1_S, \chi_*(\psi_{1,0,0}, \rho_{p-1}), \chi_*(\psi_{0,1,0}) + \chi_*(\psi_{1,0,\epsilon}), \theta_{p-1} + \chi(\psi_{1,0,0}), \frac{p-1}{4}\chi(\psi_{1,0,0}) + \chi_*(\psi_{0,1,0})\}$$

is $\mathcal{F}$ -stable and satisfies the conditions of Proposition 2.5 with $\eta = \theta_{p-1}$ as required. $\square$

5.4. The case $p \leq 5$. To complete the proof of Theorem 1.3, it remains to verify Conjecture 1.1 for the simple fusion systems listed in [15, Tables 1-4] which are not covered by Propositions 5.3, 5.11 or 5.13. As in [15], we specify the j 'th fusion system on $\text{SmallGroup}(p^n, i)$ using the notation $\mathcal{F}(p^n, i, j)$.

Proposition 5.14. *Suppose that one of the following holds:*

- (a) $p = 3$ and $\mathcal{F}$ is $\mathcal{F}(3^4, 7, 2)$, $\mathcal{F}(3^4, 8, 1)$, or $\mathcal{F}(3^4, 9, 3)$;
- (b) $p = 3$ and $\mathcal{F}$ is $\mathcal{F}(3^4, 9, 2)$;
- (c) $p = 5$ and $\mathcal{F}$ is $\mathcal{F}(5^4, 7, i)$ for some $1 \leq i \leq 10$.

Then Conjecture 1.1 holds for $\mathcal{F}$.

Proof. In cases (a) and (c) we apply the same strategy as carried out previously. That is, for a well-chosen subgroup $V \leq S$ of index p , we obtain a basis for $B_{\mathcal{N}}$ with $\mathcal{N} = N_{\mathcal{F}}(V)$ and construct a linearly independent subset $B_{\mathcal{F}}$ of characters given by linear combinations of $B_{\mathcal{N}}$ which satisfies the hypotheses of Proposition 2.5. In case (a) these calculations are completed entirely on MAGMA [4] and we omit the details. In case (c), note first that there are containments:

$$\mathcal{F}(\text{PSU}_5(4)) = \mathcal{F}(5^4, 7, 1) \supset \mathcal{F}(5^4, 7, 2) \supset \mathcal{F}(5^4, 7, 4) \supset \mathcal{F}(5^4, 7, 6) \supset \mathcal{F}(5^4, 7, 9) = \mathcal{F}_{(4)}^*.$$

Setting $\zeta = \frac{-1+\sqrt{5}}{2}$ and $\bar{\zeta} = \frac{-1-\sqrt{5}}{2}$ we label the characters of $\mathcal{N}_{(4)}^*$ when $p = 5$ as in Table 6, and construct stable sets of characters for these fusion systems as follows:

$$\begin{aligned} \mathcal{F}(5^4, 7, 9) &: \{1_S, \sigma_* + \sigma'_*, \chi_1, \chi_2, \chi_3, \chi_4, \chi_0 + \theta_4, \chi_0 + \sigma_*\} \\ \mathcal{F}(5^4, 7, 6) &: \{1_S, \sigma_* + \sigma'_*, \chi_2, \chi_3, \chi_4, \chi_0 + \chi_1 + \theta_4, \chi_0 + \chi_1 + \sigma_*\} \\ \mathcal{F}(5^4, 7, 4) &: \{1_S, \sigma_* + \sigma'_*, \chi_3, \chi_4, \chi_0 + \chi_1 + \chi_2 + \theta_4, \chi_0 + \chi_1 + \chi_2 + \sigma_*\} \\ \mathcal{F}(5^4, 7, 2) &: \{1_S, \sigma_* + \sigma'_*, \chi_4, \chi_0 + \chi_1 + \chi_2 + \chi_3 + \theta_4, \chi_0 + \chi_1 + \chi_2 + \chi_3 + \sigma_*\} \end{aligned}$$

of sizes 10, 9, 8, 7. We now iteratively apply Proposition 2.5 with

$$\eta = \chi_0, \chi_0 + \chi_1, \chi_0 + \chi_1 + \chi_2, \chi_0 + \chi_1 + \chi_2 + \chi_3$$

to see that Conjecture 1.1 holds for these examples. We similarly have containments:

$$\mathcal{F}(5^4, 7, 10) \supset \mathcal{F}(5^4, 7, 8) \supset \mathcal{F}(5^4, 7, 7) \supset \mathcal{F}(5^4, 7, 5) \supset \mathcal{F}(5^4, 7, 3) = \mathcal{G}.$$

Conjecture 1.1 is established via an identical approach in these cases and so we omit the details.

TABLE 6. S -restricted irreducible characters of $N_{(4)}^*$ when $p = 5$

χ	1	$\mathbf{v}_1$	$\mathbf{v}_2$	$\lambda\mathbf{v}_2$	$\mathbf{v}_1 + \epsilon\mathbf{v}_3$	$\lambda(\mathbf{v}_1 + \epsilon\mathbf{v}_3)$	$\mathbf{u}$	$\mathbf{u}\mathbf{v}_2$	$\mathbf{u}(\mathbf{v}_2)^2$	$\mathbf{u}(\mathbf{v}_2)^3$	$\mathbf{u}(\mathbf{v}_2)^4$
1_S	1	1	1	1	1	1	1	1	1	1	1
θ_4	4	4	4	4	4	4	-1	-1	-1	-1	-1
$\chi_0 := \chi(\psi_{1,0,0})$	24	-1	4	4	-6	-6	4	-1	-1	-1	-1
$\chi_1 := \chi_a(\psi_{1,0,0}, \rho_4)$	24	-1	4	4	-6	-6	-1	4	-1	-1	-1
$\chi_2 := \chi_b(\psi_{1,0,0}, \rho_4)$	24	-1	4	4	-6	-6	-1	-1	4	-1	-1
$\chi_3 := \chi_c(\psi_{1,0,0}, \rho_4)$	24	-1	4	4	-6	-6	-1	-1	-1	4	-1
$\chi_4 := \chi_d(\psi_{1,0,0}, \rho_4)$	24	-1	4	4	-6	-6	-1	-1	-1	-1	4
$\sigma'_1 := \chi_a(\psi_{1,0,\epsilon})$	20	-5	0	0	-5ζ	$-5\bar{\zeta}$	0	0	0	0	0
$\sigma'_2 := \chi_b(\psi_{1,0,\epsilon})$	20	-5	0	0	$-5\bar{\zeta}$	-5ζ	0	0	0	0	0
$\sigma_1 := \chi_a(\psi_{0,1,0})$	30	5	5ζ	$5\bar{\zeta}$	0	0	0	0	0	0	0
$\sigma_2 := \chi_b(\psi_{0,1,0})$	30	5	$5\bar{\zeta}$	5ζ	0	0	0	0	0	0	0

TABLE 7. S -restricted irreducible characters of N when $\mathcal{F} = \mathcal{F}(3^4, 9, 2)$

χ	g_1	$g_2 = \mathbf{z}$	g_3	g_4	g_5	$g_6 = \mathbf{u}$	$g_7 = \mathbf{u}'$	g_8	g_9	g_{10}
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	2	2	2	2	2	-1	-1	-1	-1	-1
χ_3	2	2	2	2	-1	2	-1	-1	-1	-1
χ_4	2	2	2	2	-1	-1	2	-1	-1	-1
χ_5	2	2	2	2	-1	-1	-1	2	2	2
χ_6	3	3	3ω	$3\bar{\omega}$	0	0	0	0	0	0
χ_7	3	3	$3\bar{\omega}$	3ω	0	0	0	0	0	0
χ_8	6	-3	0	0	0	0	0	α	β	γ
χ_9	6	-3	0	0	0	0	0	β	γ	α
χ_{10}	6	-3	0	0	0	0	0	γ	α	β

We are unable to apply Proposition 2.5 in case (b) because three $N_{\mathcal{F}}(S)$ -classes are fused in $\mathcal{F}$; instead, we give a direct argument. $N_{\mathcal{F}}(S)$ is realised by a group N of order 162, and Table 7 lists the S -restricted irreducible characters N where ω is a primitive cube root of unity and

$$\alpha = 2\sqrt{3}\cos(\pi/18), \quad \beta = 2\sqrt{3}\cos(13\pi/18), \quad \gamma = 2\sqrt{3}\cos(25\pi/18)$$

are the three distinct roots of $x^3 - 9x - 9$. Since $\mathbf{z}$, $\mathbf{u}$, $\mathbf{u}'$ are fused in $\mathcal{F}$, it is easy to see that any indecomposable $\mathcal{F}$ -stable character must be of the form $\rho + \psi$ for $\rho \in \{\chi_1, \chi_2, \chi_3 + \chi_4, \chi_5, \chi_6, \chi_7\}$ and $\psi \in \{\chi_8, \chi_9, \chi_{10}\}$. Thus, for example, the set

$$B_{\mathcal{F}} = \{\chi_1, \chi_5 + \chi_8, \chi_2 + \chi_8, \chi_6 + \chi_8, \chi_7 + \chi_8, \chi_7 + \chi_9, \chi_7 + \chi_{10}, \chi_3 + \chi_4 + \chi_8\}$$

is a basis for $\text{Ch}(S)^{\mathcal{F}}$ by Lemma 2.1. The result now follows by direct calculation. $\square$

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