

A CLASSIFICATION ALGORITHM FOR REFLEXIVE SIMPLICES

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ABSTRACT. We present a general classification algorithm for reflexive simplices, which allows us to determine all reflexive simplices in dimensions five and six. In terms of algebraic geometry this means that we classify the Gorenstein fake weighted projective spaces in dimensions five and six. As a byproduct of our methods, we obtain explicit formulae for the Picard group and the Gorenstein index of any fake weighted projective space.

1. INTRODUCTION

A *reflexive polytope* is a lattice polytope having the origin in its interior and whose dual is also a lattice polytope. This concept showed up in [1], where among other things, Batyrev determines all reflexive polygons up to unimodular equivalence. Kreuzer and Skarke extended this classification in dimensions three and four, see [2, 3]. A complete classification of all reflexive polygons in dimension 5 or higher seems to be out of reach. However, Schöller and Skarke could at least compute the *weight systems* of the reflexive polytopes in dimension five, see [4].

In the present note, we consider the particular case of *reflexive simplices*. The aforementioned classifications comprise in particular the 5 reflexive triangles, the 48 reflexive 3-simplices and the 1561 reflexive 4-simplices. The main result of this article is the general classification algorithm 4.17 for reflexive simplices. It allows us in particular to obtain the following.

Classification 1.1. *Up to unimodular equivalence, there are 220 794 reflexive simplices in dimension five and 309 019 970 reflexive simplices in dimension six. The complete data are available at [11].*

On a midrange computer, with 16 threads, the algorithm terminates in less than one minute for dimension five, and in approximately 20 days for dimension six.

We succeed by working mostly in terms of *degree matrices*, a Gale dual encoding of lattice simplices by means of a matrix with columns in an abelian group $\mathbb{Z} \times \Gamma$ with Γ finite; see Section 3 for the precise formulation. This framework has two main advantages. First, unimodular equivalences of the lattice simplices correspond to automorphisms of $\mathbb{Z} \times \Gamma$. Theorem 2.1 provides explicit generators of the automorphism group of $\mathbb{Z} \times \Gamma$, which allows early pruning of the possible degree matrices. Second, Proposition 4.1 turns the reflexivity condition for the simplex into explicit conditions on the degree matrix: one condition involves only the free row, and the others one torsion row at a time. Hence, we find all reflexive degree matrices by computing the admissible free rows, enumerating the torsion rows compatible with each free row, and suitably combining them.

Our results directly apply to toric geometry. Recall that the reflexive simplices are exactly the Fano polytopes of the Gorenstein *fake weighted projective spaces*

(fwps), that means, the $\mathbb{Q}$ -factorial toric varieties of Picard number one whose anticanonical divisor is ample and Cartier. So, in the setting of toric varieties, Classification 1.1 tells us the following.

Corollary 1.2. *Up to isomorphism, there are 220 794 Gorenstein fwps of dimension five and 309 019 970 Gorenstein fwps of dimension six.*

Moreover, the simplices of the classification lists directly yield the defining fan of the corresponding fwps: the maximal cones are the cones over the facets of the simplex. Beyond that, we present explicit formulae for the Picard group and the Gorenstein index of a fwps in terms of its degree matrix, see Theorem 3.3.

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2. AUTOMORPHISMS OF FINITELY GENERATED ABELIAN GROUPS

The main result of this section explicitly provides generators of the automorphism group of a finitely generated abelian group G . A first description of $\text{Aut}(G)$ was given in [6] by Ranum. In [7], Hillar and Rhea characterized $\text{Aut}(G)$ when G is a finite abelian group, and provided a formula for its cardinality. Both articles used the primary decomposition of the group G . We will instead present any finitely generated abelian group G in *invariant factor form*, that means that

$$G = \mathbb{Z}^k \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}, \quad \mu_r \mid \mu_{r-1}, \dots, \mu_2 \mid \mu_1.$$

This presentation involves in general fewer cyclic factors. This in turn has significant impact on the performance of the algorithms presented later.

Write (w, η) for the elements of G , where $w \in \mathbb{Z}^k$ and $\eta \in \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$. Observe that we obtain well-defined automorphisms of G , each modifying the i -th coordinate of either w or η , by

$$\begin{aligned} \psi_i(w, \eta) &:= (w_1, \dots, -w_i, \dots, w_k, \eta), & 1 \leq i \leq k, \\ \psi_{i,u}(w, \eta) &:= (w, \eta_1, \dots, u\eta_i, \dots, \eta_r), & 1 \leq i \leq r, u \in (\mathbb{Z}/\mu_i\mathbb{Z})^*, \\ \alpha_{i,j}(w, \eta) &:= (w_1, \dots, w_j + w_i, \dots, w_k, \eta), & 1 \leq i, j \leq k, i \neq j, \\ \beta_{i,j}(w, \eta) &:= (w, \eta_1, \dots, w_j + \eta_i, \dots, \eta_r), & 1 \leq j \leq k, 1 \leq i \leq r, \\ \gamma_{i,j}(w, \eta) &:= (w, \eta_1, \dots, \eta_j + \eta_i, \dots, \eta_r), & 1 \leq j < i \leq r, \\ \delta_{i,j}(w, \eta) &:= (w, \eta_1, \dots, \frac{\mu_i}{\mu_j}\eta_j + \eta_i, \dots, \eta_r), & 1 \leq i < j \leq r. \end{aligned}$$

Theorem 2.1. *Let $G = \mathbb{Z}^k \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ be in invariant factor form. Then $\text{Aut}(G)$ is generated by the automorphisms $\psi_i, \psi_{i,u}, \alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}$ and $\delta_{i,j}$.*

The proof of Theorem 2.1 is given at the end of this section. As a first preparatory step, we develop a matrix calculus for the $\mathbb{Z}$ -algebra of endomorphisms of an abelian group in invariant factor form.

Definition 2.2. Let $G = \mathbb{Z}^k \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ be in invariant factor form. By a G -matrix we mean a square matrix of the shape

$$A = (a_{ij}) = \begin{bmatrix} A_1 & 0 \\ * & A_2 \end{bmatrix},$$

where the columns of A are elements of G , the block A_1 is a $k \times k$ square matrix, the block A_2 is an $r \times r$ matrix and the entries of A_2 satisfy

$$\mu_i a_{k+i, k+j} = 0 \in \mathbb{Z}/\mu_j\mathbb{Z}, \quad 1 \leq i < j \leq r.$$

Remark 2.3. Let G be an abelian group in invariant factor form, ω an element of G and A, A' two G -matrices. The conditions on G -matrices ensure that the formal matrix-vector product $A \cdot \omega$ is a well defined element of G and the formal matrix-matrix product $A \cdot A'$ is a G -matrix.

Construction 2.4. Consider $G = \mathbb{Z}^k \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ in invariant factor form and denote by $e_i \in G$ the element with i -th component 1 and all others 0. With any endomorphism $\varphi: G \rightarrow G$ we associate the $(k+r) \times (k+r)$ square matrix

$$A(\varphi) := (\varphi_{ij}) := [\varphi(e_1), \dots, \varphi(e_{k+r})].$$

Lemma 2.5. Let $G = \mathbb{Z}^k \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ be in invariant factor form, and let $\varphi \in \text{End}(G)$. Then the matrix $A(\varphi)$ from Construction 2.4 is a G -matrix.

Proof. The assertion follows directly from $0 = \varphi(0) = \varphi(\mu_i e_{k+i}) = \mu_i \varphi(e_{k+i})$. $\square$

Remark 2.6. For any abelian group G in invariant factor form, the matrices from Construction 2.4 represent the endomorphisms: for all $\varphi, \varphi_1, \varphi_2 \in \text{End}(G)$ and $\omega \in G$, we have

$$\varphi(\omega) = A(\varphi) \cdot \omega, \quad A(\varphi_2 \circ \varphi_1) = A(\varphi_2) \cdot A(\varphi_1).$$

Conversely, every G -matrix defines an endomorphism via matrix-vector multiplication and these assignments are inverse to each other.

Example 2.7. The matrices associated with $\psi_i, \psi_{i,u}, \alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}$ and $\delta_{i,j}$ from Theorem 2.1 are G -matrices. Multiplication of these G -matrices from the left (right) to a given G -matrix yields the elementary row (column) operations preserving the structure of G -matrix.

The following statement gives a characterization of generator systems of the group G . It will be essentially used in the proof of Theorem 2.1 and in the later algorithmic Section 4.

Proposition 2.8. Let $G = \mathbb{Z}^k \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ be in invariant factor form, and let $\omega_i = (w_i, \eta_i) \in G$ for $i = 1, \dots, n$. For $j = 0, \dots, r$ set

$$Q_j := \begin{bmatrix} w_1 & \dots & w_n \\ \eta_{11} & \dots & \eta_{n1} \\ \vdots & & \vdots \\ \eta_{1j} & \dots & \eta_{nj} \end{bmatrix}.$$

Then $G = \langle \omega_1, \dots, \omega_n \rangle$ if and only if the maximal minors of Q_0 generate $\mathbb{Z}$ and the maximal minors of Q_j generate $\mathbb{Z}/\mu_j\mathbb{Z}$ for any $j = 1, \dots, r$.

Proof. Let $G = \langle \omega_1, \dots, \omega_n \rangle$. By classical determinantal divisors theory, the maximal minors of Q_0 generate $\mathbb{Z}$. Fix $j = 1, \dots, r$, and consider the prime factorization $\mu_j = p_1^{m_1} \cdots p_t^{m_t}$. Now, fix $i = 1, \dots, t$, and consider the projection onto the first $k + j$ coordinates followed by the coordinate-wise projection onto $\mathbb{Z}/p_i\mathbb{Z}$:

$$\pi: G \rightarrow (\mathbb{Z}/p_i\mathbb{Z})^{k+i}.$$

Then $\pi(\omega_1), \dots, \pi(\omega_n)$ forms a basis of the vector space $(\mathbb{Z}/p_i\mathbb{Z})^{k+j}$, encoded by the matrix Q_j modulo p_i . In particular, there exists a maximal minor of Q_j generating $\mathbb{Z}/p_i\mathbb{Z}$, and hence $\mathbb{Z}/p_1^{m_1}\mathbb{Z}$. The assertion follows since, by the Chinese remainder theorem, we have an isomorphism

$$\mathbb{Z}/\mu_j\mathbb{Z} \rightarrow \mathbb{Z}/p_1^{m_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_t^{m_t}\mathbb{Z}, \quad \eta \mapsto (\eta \bmod p_1^{m_1}, \dots, \eta \bmod p_t^{m_t}).$$

Now assume that the maximal minors of Q_0 generate $\mathbb{Z}$ and the maximal minors of Q_j generate $\mathbb{Z}/\mu_j\mathbb{Z}$ for any $j = 1, \dots, r$. Let $\eta'_{ij} \in \mathbb{Z}_{\geq 0}$ be the representing integer of η_{ij} between 0 and $\mu_j - 1$, and consider the integer matrix

$$Q'_j := \begin{bmatrix} w_1 & \cdots & w_n \\ \eta'_{11} & \cdots & \eta'_{n1} \\ \vdots & & \vdots \\ \eta'_{1j} & \cdots & \eta'_{nj} \end{bmatrix}.$$

Denote by q_j the greatest common divisor of all maximal minors of Q'_j . By assumption, we have $q_0 = 1$ and $\gcd(q_j, \mu_j) = 1$. With the vectors $b_i := \mu_i e_{k+i}$ we define the $(k+r) \times (n+r)$ -matrix

$$M := [Q'_r, B], \quad B := [b_1, \dots, b_r].$$

Denote by Δ the greatest common divisor of all maximal minors of M . Notice that Δ divides $\mu_r \cdots \mu_{i+1} q_i$ for all $i = 0, \dots, r$. In particular, Δ divides

$$\begin{aligned} & \gcd(q_r, \mu_r q_{r-1}, \mu_r \mu_{r-1} q_{r-2}, \dots, \mu_r \cdots \mu_1 q_0) \\ &= \gcd(q_r, \mu_r \gcd(q_{r-1}, \mu_{r-1} q_{r-2}, \dots, \mu_{r-1} \cdots \mu_1 q_0)). \end{aligned}$$

Since $\gcd(q_r, \mu_r) = 1$, we have

$$\begin{aligned} & \gcd(q_r, \mu_r \gcd(q_{r-1}, \mu_{r-1} q_{r-2}, \dots, \mu_{r-1} \cdots \mu_1 q_0)) \\ &= \gcd(q_r, q_{r-1}, \mu_{r-1} q_{r-2}, \dots, \mu_{r-1} \cdots \mu_1 q_0). \end{aligned}$$

Iterating the argument for $\mu_{r-1}, \dots, \mu_1$, we see that Δ divides $\gcd(q_r, \dots, q_0)$. Since $q_0 = 1$, it follows that $\Delta = 1$. By classical determinantal divisors theory, it follows that $\mathbb{Z}^{k+r} = \langle M_{*,1}, \dots, M_{*,n+r} \rangle$. Thus, the equality $G = \langle \omega_1, \dots, \omega_n \rangle$ follows from

$$\frac{G}{\langle \omega_1, \dots, \omega_r \rangle} \cong \frac{\mathbb{Z}^{k+r}}{\langle M_{*,1}, \dots, M_{*,n+r} \rangle}.$$

□

Lemma 2.9. *Let $a, b, c \in \mathbb{Z}$ such that $\gcd(a, b, c) = 1$. Then there exists $k \in \mathbb{Z}$ such that $\gcd(a + kb, c) = 1$.*

Proof. Let $p_1, \dots, p_t$ be the prime factors of c . For every $i = 1, \dots, t$, we find an integer $k_i \in \mathbb{Z}$ such that p_i does not divide $a + k_i b$. If p_i does not divide a , we set $k_i := 0$. If p_i divides a , then it does not divide $a + b$, and we set $k_i := 1$. By the Chinese remainder theorem, the system

$$\{k \equiv k_i \bmod p_i, \quad i = 1, \dots, t\}$$

admits a solution $k \in \mathbb{Z}$. We conclude that $\gcd(a + kb, c) = 1$ since, for any prime factor p_i of c , we have

$$a + kb \equiv a + k_i b \not\equiv 0 \pmod{p_i}.$$

□

The idea for proving Theorem 2.1 is simply to turn the matrix $A(\varphi)$ associated with $\varphi \in \text{Aut}(G)$ into the identity matrix by suitably applying the elementary matrices from Example 2.7.

Proof of Theorem 2.1. Let $\varphi \in \text{Aut}(G)$, and let $A(\varphi)$ be the matrix from Construction 2.4. By Lemma 2.5, we have

$$A(\varphi) = \begin{bmatrix} A_1 & 0 \\ * & A_2 \end{bmatrix},$$

where A_1 and A_2 are from Definition 2.2. Since φ is an automorphism, $A(\varphi)$ is invertible. In particular, the integer matrix A_1 is also invertible. Hence, suitably applying α_{ij} and ψ_i to φ turns A_1 into the identity matrix. Then, suitably applying β_{ij} turns $A(\varphi)$ into the matrix

$$\begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix}.$$

By Lemma 2.5, there exist $x_i \in \mathbb{Z}/\mu_i\mathbb{Z}$ for $i = 1, \dots, r$ such that the last column of A_2 is given as

$$\left(\frac{\mu_1}{\mu_r} x_1, \dots, \frac{\mu_{r-1}}{\mu_r} x_{r-1}, x_r \right).$$

Let $x'_i \in \mathbb{Z}_{\geq 0}$ be the representing integer of x_i between 0 and $\mu_i - 1$. By Proposition 2.8, the determinant of A_2 generates $\mathbb{Z}/\mu_r\mathbb{Z}$. In particular, we have

$$\gcd\left(\frac{\mu_1}{\mu_r} x'_1, \dots, \frac{\mu_{r-1}}{\mu_r} x'_{r-1}, x'_r, \mu_r\right) = 1.$$

By Lemma 2.9, there exists $k \in \mathbb{Z}$ such that

$$x'_r + k \gcd\left(\frac{\mu_1}{\mu_r} x'_1, \dots, \frac{\mu_{r-1}}{\mu_r} x'_{r-1}\right) = u \in (\mathbb{Z}/\mu_r\mathbb{Z})^\times.$$

Hence, we can use suitable $\gamma_{i,r}$ and then normalize the last coordinate with $\psi_{r,u-1}$ to turn the last column of A_2 into

$$\left(\frac{\mu_1}{\mu_r} x_1, \dots, \frac{\mu_{r-1}}{\mu_r} x_{r-1}, 1 \right).$$

Then, suitably applying $\delta_{i,r}$ turns the last column of A_2 into $(0, \dots, 0, 1)$. Iterating the procedure for the other columns turns A_2 into a lower diagonal matrix with diagonal entries equal to one. Finally, we use suitable $\gamma_{i,j}$ to turn A_2 into the identity matrix. □

3. PICARD GROUP AND GORENSTEIN INDEX OF FWPS

In this section we provide explicit formulae for the Picard group and the Gorenstein index of a *fake weighted projective space (fwps)*, that means a $\mathbb{Q}$ -factorial toric Fano variety of Picard number one. Every fwps of dimension n is encoded by an $n \times (n+1)$ *generator matrix*, that means a matrix

$$P = \begin{bmatrix} v_0 & \dots & v_n \end{bmatrix},$$

the columns of which are pairwise distinct primitive vectors generating $\mathbb{Q}^n$ as a convex cone. The fwps associated with P is the toric Fano variety $Z(P)$ whose Fano polytope has vertices $v_0, \dots, v_n$. The class group of Z is isomorphic to $\mathbb{Z}^{n+1}/\text{im}(P^*)$, where P^* denotes the transpose of P . We present $\text{Cl}(Z)$ in invariant factor form

$$\mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}.$$

Consider the projection $Q: \mathbb{Z}^{n+1} \rightarrow \text{Cl}(Z)$, and let $\omega_i = (w_i, \eta_i) := Q(e_i)$, where $1 \leq w_0 \leq \dots \leq w_n$ and $\eta_i \in \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$. We view Q as a *degree matrix* in $\text{Cl}(Z)$, that means

$$Q = \begin{bmatrix} \omega_0 & \dots & \omega_n \end{bmatrix}$$

where any n of the ω_i generate $\text{Cl}(Z)$ as a group. One can directly gain Z as a quotient of $\mathbb{K}^{n+1}$ by the diagonal action of $H = \mathbb{K}^* \times F$ with weights ω_i , where F is finite and H is the quasitorus with character group $\text{Cl}(Z)$, see [8, Sec. 2].

Example 3.1. We consider the fwps $Z = Z(P)$ with generator matrix P and degree matrix Q in $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$:

$$P = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -2 & -2 & 0 & 1 \\ -3 & 2 & 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 & 1 & 4 \\ \bar{0} & \bar{1} & \bar{2} & \bar{2} \end{bmatrix}.$$

The quasitorus associated to the grading is $H = \mathbb{K}^* \times \{\pm 1, \pm\sqrt{-1}\}$, and Z is isomorphic to the quotient of $\mathbb{K}^4$ by the action of H :

$$(t, \sqrt{-1}) \cdot z := [tz_1, \sqrt{-1}tz_2, -tz_3, -t^4z_4].$$

Notation 3.2. Let $G = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ be in invariant factor form, and let $\omega_i = (w_i, \eta_i) \in G$ for $i = 0, \dots, n$. We denote by $\eta'_{ij} \in \mathbb{Z}$ the representing integer of η_{ij} between 0 and $\mu_j - 1$, and we set

$$\begin{aligned} L &:= \text{lcm}(w_0, \dots, w_n), \\ L_{ij} &:= \frac{L}{w_i} \eta'_{ij}, \quad 0 \leq i \leq n, \quad 1 \leq j \leq r, \\ M_i &:= \frac{\mu_i}{\gcd(\mu_i, L_{0i}, \dots, L_{ni})}, \quad 1 \leq i \leq r, \\ M &:= \text{lcm}(M_1, \dots, M_r). \end{aligned}$$

Theorem 3.3. *Let Z be a fwps with $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ and degree matrix $Q = \begin{bmatrix} \omega_0 & \dots & \omega_n \end{bmatrix}$. With Notation 3.2, we have*

$$\begin{aligned} \text{Pic}(Z) &= \langle (LM, 0) \rangle, \\ \iota(Z) &= \text{lcm} \left(\frac{LM}{\gcd(LM, \sum_i w_i)}, \frac{\mu_j}{\gcd(\mu_j, \sum_i \eta'_{ij})}; \quad j = 1, \dots, r \right). \end{aligned}$$

We obtain Theorem 3.3 from the following statement about finitely generated abelian groups of rank one.

Proposition 3.4. *Let $G = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ be in invariant factor form, and let $\omega_i = (w_i, \eta_i) \in G$ for $i = 0, \dots, n$. Suppose $\omega_0, \dots, \omega_n$ generate G . Then, with Notation 3.2, we have*

$$\bigcap_{i=0}^n \langle \omega_i \rangle = \langle (LM, 0) \rangle.$$

Proof. The inclusion " $\subseteq$ " follows from $LM\omega_i = w_i(LM, 0)$ for $i = 1, \dots, n$.

We show " $\supseteq$ ". Let $\omega = (w, \zeta) \in G$ such that $\omega = k_i\omega_i$ for integers $k_i \in \mathbb{Z}$, $i = 0, \dots, n$. We first prove that $\zeta_1 = 0 \in \mathbb{Z}/\mu_1\mathbb{Z}$. Let μ_1 factor as $p_1^{m_1} \cdots p_s^{m_s}$. Consider the projection onto the first two coordinates followed by the coordinate-wise projection onto $\mathbb{Z}/p_i^{m_i}\mathbb{Z}$:

$$\pi: G \rightarrow (\mathbb{Z}/p_i^{m_i}\mathbb{Z})^2$$

By Nakayama's lemma, $\pi(\omega_j), \pi(\omega_\ell)$ generate $(\mathbb{Z}/p_i^{m_i}\mathbb{Z})^2$ for some $1 \leq j, \ell \leq n$. It follows that

$$\det \begin{bmatrix} \pi(\omega_j) & \pi(\omega_\ell) \end{bmatrix} \in (\mathbb{Z}/p_i^{m_i}\mathbb{Z})^\times.$$

From $k_j\omega_j = k_\ell\omega_\ell$ we have

$$0 = \det \begin{bmatrix} \pi(k_j\omega_j - k_\ell\omega_\ell) & \pi(\omega_\ell) \end{bmatrix} = k_j \det \begin{bmatrix} \pi(\omega_j) & \pi(\omega_\ell) \end{bmatrix}.$$

Hence, $p_i^{m_i}$ divides k_j . In particular $\zeta_1 = k_j\eta_{j1} \equiv 0 \pmod{p_i^{m_i}}$. By the Chinese remainder theorem, we have $\zeta_1 = 0 \in \mathbb{Z}/\mu_1\mathbb{Z}$. Iterating the argument for $\mu_2, \dots, \mu_r$, we prove that $\zeta = 0$.

Now, since w_i divides w for all i , there exists $k \in \mathbb{Z}$ such that $w = Lk$. In particular, we have

$$kL_{ij} = k_i\eta_{ij} = 0 \in \mathbb{Z}/\mu_j\mathbb{Z}.$$

It follows that μ_j divides $k \gcd(L_{0j}, \dots, L_{nj})$. Hence, M_j divides k for all j . In turn, this implies that LM divides w . $\square$

Proof of Theorem 3.3. Since $\omega_0, \dots, \omega_n$ generate $\text{Cl}(Z)$, the first assertion follows from Proposition 3.4. Now, let $k \in \mathbb{Z}_{\geq 0}$ and recall that $-\mathcal{K}_Z = \sum_i \omega_i$. Then $k(-\mathcal{K})$ is Cartier if and only if

$$\begin{cases} LM & | & k \sum_i w_i, \\ \mu_j & | & k \sum_i \eta'_{ij}, \quad j = 1, \dots, r. \end{cases}$$

$\square$

Example 3.5. Let Z be the fwps from Example 3.1. Then $\text{Pic}(Z) = \langle (8, \bar{0}) \rangle$ and $\iota(Z) = 8$.

4. CLASSIFICATION OF REFLEXIVE SIMPLICES

As a direct consequence of Theorem 3.3, we have the following characterization of Gorenstein fwps in terms of their degree matrix.

Proposition 4.1. *Let Z be a fwps with $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ and degree matrix $Q = [\omega_0 \ \cdots \ \omega_n]$. Then Z is Gorenstein if and only if, with Notation 3.2, we have*

$$\begin{cases} L & | \sum_i w_i, \\ M_j & | \frac{\sum_i w_i}{L}, \quad j = 1, \dots, r, \\ \eta_{nj} & = -(\eta_{0j} + \cdots + \eta_{n-1j}) \in \mathbb{Z}/\mu_j\mathbb{Z}, \quad j = 1, \dots, r. \end{cases}$$

Notice that the first condition of Proposition 4.1 involves only the torsion-free row of Q . In turn, the other two conditions are the same for all torsion rows, and only involve one torsion row at a time. This justifies the following definitions.

Definition 4.2. We call $w = [w_0, \dots, w_n] \in \mathbb{Z}_{\geq 1}^{n+1}$ a (Gorenstein) *weight vector* if it is the degree matrix of a (Gorenstein) fwps $Z = Z(w)$ with $\text{Cl}(Z) = \mathbb{Z}$. In this case, we write $\mathbb{P}(w_0, \dots, w_n) := Z(w)$ and call it a *weighted projective space (wps)*.

Definition 4.3. Let $w = [w_0, \dots, w_n]$ be a weight vector and $\mu \in \mathbb{Z}_{\geq 2}$. We call $\eta = [\eta_0, \dots, \eta_n] \in (\mathbb{Z}/\mu\mathbb{Z})^{n+1}$ a (Gorenstein) *torsion vector of order μ for w* if the matrix

$$Q = \begin{bmatrix} w_0 & \cdots & w_n \\ \eta_0 & \cdots & \eta_n \end{bmatrix}$$

is the degree matrix of a (Gorenstein) fwps $Z = Z(Q)$ with $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$.

Definition 4.4. Let $w = [w_0, \dots, w_n]$ be a weight vector and $\mu \in \mathbb{Z}_{\geq 2}$. We say that two torsion vectors η and ζ of order μ for w are *equivalent* if there exists an automorphism of $\mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$ sending (w_i, η_i) to (w_i, ζ_i) for $i = 0, \dots, n$. We write $\eta \leq \zeta$ if $[\eta'_0, \dots, \eta'_n] \leq_{\text{lex}} [\zeta'_0, \dots, \zeta'_n]$, where η'_i and ζ'_i are the representative integers for η_i and ζ_i between 0 and $\mu - 1$. We say that η is *minimal* if $\eta \leq \zeta$ for all ζ equivalent to η .

Proposition 4.5. *Let Z be a Gorenstein fwps with $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$. Then $Z = Z(Q)$ for a degree matrix*

$$Q = \begin{bmatrix} w_0 & \cdots & w_n \\ \eta_{01} & \cdots & \eta_{n1} \\ \vdots & & \vdots \\ \eta_{0r} & \cdots & \eta_{nr} \end{bmatrix},$$

where $w = [w_0, \dots, w_n]$ is a Gorenstein weight vector, $\eta_{*i} := [\eta_{0i}, \dots, \eta_{ni}]$ is a minimal Gorenstein torsion vector of order μ_i for w , and $\eta_{*i} < \eta_{*j}$ for all $i < j$ such that $\mu_i = \mu_j$.

Proof. Let Q be any degree matrix for Z . By Theorem 3.3, the first row w of Q is a Gorenstein weight vector, and the $(i+1)$ -th row of Q is a Gorenstein torsion vector of order μ_i for w , for $i = 1, \dots, r$. By Theorem 2.1, there exists an automorphism of $\text{Cl}(Z)$ that turns all torsion vectors minimal and orders increasingly those with the same torsion order. $\square$

We turn to our classification algorithm 4.17. First we present the necessary ingredients, which then fit together in the final algorithm.

The following, presented for instance in [1, Sec. 5.4], gives a bijection between Gorenstein weight vectors and the unit fraction decompositions of one.

Remark 4.6. Let W be the set of all Gorenstein weight vectors $(w_0, \dots, w_n)$, and

$$U := \left\{ (u_0, \dots, u_n) \in \mathbb{Z}_{\geq 1}^{n+1}; u_0 \geq \dots \geq u_n, \frac{1}{u_0} + \dots + \frac{1}{u_n} = 1 \right\}.$$

Then, with $u := \text{lcm}(u_0, \dots, u_n)$ and $S := w_0 + \dots + w_n$, we have mutually inverse bijections

$$\begin{aligned} U &\longleftrightarrow W \\ (u_0, \dots, u_n) &\mapsto \left(\frac{u}{u_0}, \dots, \frac{u}{u_n} \right), \\ \left(\frac{S}{w_0}, \dots, \frac{S}{w_n} \right) &\leftarrow (w_0, \dots, w_n). \end{aligned}$$

There exist several algorithms for computing the set U , see for instance [1, Algorithm 5.6]. Moreover, the lists for $n = 1, \dots, 8$ are available at [10].

We now determine for which $\mu \in \mathbb{Z}_{\geq 2}$ there exists at least one Gorenstein torsion vector of order μ for a weight vector w .

Remark 4.7. Let Z be a fwps with $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ and degree matrix $Q = [\omega_0 \ \dots \ \omega_n]$. By Proposition 2.8, $\eta_{0j}, \dots, \eta_{nj}$ generate $\mathbb{Z}/\mu_j\mathbb{Z}$ for $j = 1, \dots, r$. Hence, with Notation 3.2, we have

- (1) $\frac{\mu_j}{M_j} = \gcd(\mu_j, L_{0j}, \dots, L_{nj}) \mid L$;
- (2) if Z is Gorenstein, then $\mu \mid S$.

By Remark 4.7, all the orders admitting at least one torsion vector for a Gorenstein weight vector w are of the form $\mu = ab$, where a divides L and b divides $\frac{S}{L}$. The following lemma provides a criterion to quickly exclude some pairs (a, b) , improving significantly the performance of our algorithm.

Lemma 4.8. *Let Z be a fwps with $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ and degree matrix $Q = [\omega_0 \ \dots \ \omega_n]$. With Notation 3.2, set $\alpha_k := \frac{LM_k}{\mu_k}$ for $k = 1, \dots, r$. Then*

$$\begin{aligned} \frac{w_i}{\gcd(\alpha_k, w_i)} &\mid \eta'_{ik}, \quad i = 0, \dots, n, \\ \gcd\left(\mu_k, \frac{w_i w_j}{\gcd(\alpha_k, w_i w_j)}; i, j = 0, \dots, n\right) &= 1. \end{aligned}$$

Proof. For the first assertion, observe

$$\frac{w_i}{\gcd(\alpha_k, w_i)} = \frac{\frac{L}{\alpha_k}}{\frac{L}{\text{lcm}(\alpha_k, w_i)}} = \frac{\gcd(\mu_k, L_{1k}, \dots, L_{nk})}{\gcd\left(\mu_k, L_{1k}, \dots, L_{nk}, \frac{L}{w_i}\right)} \mid \eta'_{ik}.$$

In particular, for any $i, j = 0, \dots, n$, $k = 1, \dots, r$ we have

$$\frac{w_i w_j}{\gcd(\alpha_k, w_i w_j)} \mid w_i \eta'_{jk} - w_j \eta'_{ik}.$$

The second assertion follows since, by Proposition 2.8, we have

$$\gcd(\mu_k, w_i \eta'_{jk} - w_j \eta'_{ik}; i, j = 0, \dots, n) = 1.$$

□

Lemma 4.9. *Let $w = [w_0, \dots, w_n]$ be a weight vector, $\mu \in \mathbb{Z}_{\geq 2}$ and $\eta = [\eta_0, \dots, \eta_n]$ a torsion vector of order μ for w . Write η'_i for the representative integer of η_i between 0 and $\mu - 1$, and set*

$$d_i := \frac{\mu \gcd(\mu, w_0, \dots, w_i)}{\gcd(\mu, w_0, \dots, w_{i-1})}.$$

Then η is minimal if and only if

$$\begin{cases} 0 \leq \eta'_i < d_i, & i = 0, \dots, n, \\ \eta \leq u\eta, & u \in (\mathbb{Z}/\mu\mathbb{Z})^\times. \end{cases}$$

Proof. By Theorem 2.1, a torsion vector ζ of order μ for w is equivalent to η if and only if $\zeta = u\eta + kw$ for $u \in (\mathbb{Z}/\mu\mathbb{Z})^\times$, $k \in \mathbb{Z}$. By Bézout's identity there exist $a_0, b_0 \in \mathbb{Z}$ such that

$$0 \leq \eta'_0 + a_0\mu + b_0w_0 < \gcd(\mu, w_0) = d_0.$$

Hence, the minimal torsion vector equivalent to η must also satisfy this condition. Furthermore, adding kw to η preserves the inequalities above if and only if μ divides kw_0 . By Bézout's identity, there exist $a_1, b_1 \in \mathbb{Z}$ such that

$$\leq \eta'_0 + a_1\mu + b_1w_0 < \gcd(\mu, w_0) = d_0,$$

$$0 \leq \eta'_1 + a_1\mu + b_1w_1 < \frac{\mu \gcd(\mu, w_0, w_1)}{\gcd(\mu, w_0)} = d_1.$$

Iterating the argument for $i = 2, \dots, n$, we prove that the conditions of the assertion are necessary for η to be minimal. Since only one torsion vector equivalent to η satisfies the conditions, they are also sufficient. $\square$

The following statement provides an efficient method for computing all minimal Gorenstein torsion vectors for a weight vector w .

Procedure 4.10. Let $w = [w_0, \dots, w_n]$ be a Gorenstein weight vector, and consider any pair $(a, b) \in \mathbb{Z}_{\geq 1}^2$ such that

$$a \mid L, \quad b \mid \frac{S}{L}, \quad \gcd\left(ab, \frac{w_i w_j}{\gcd(\frac{L}{b}, w_i w_j)}; i, j = 0, \dots, n\right) = 1.$$

For $i = 0, \dots, n$ set

$$c_i := \frac{w_i}{\gcd(\frac{L}{b}, w_i)}, \quad d_i := \frac{ab \gcd(ab, w_0, \dots, w_i)}{\gcd(ab, w_0, \dots, w_{i-1})}, \quad K_i := \left\{0, \dots, \left\lceil \frac{d_i}{c_i} \right\rceil - 1\right\}.$$

Let K be the set of all $[k_0, \dots, k_n] \in \prod_i K_i$ such that

$$\begin{cases} c_n k_n \equiv -(c_0 k_0 + \dots + c_{n-1} k_{n-1}) \pmod{ab}, \\ \gcd\left(ab, c_i k_i w_j - c_j k_j w_i; i, j = 0, \dots, \hat{\ell}, \dots, n\right) = 1, \ell = 0, \dots, n, \\ \gcd\left(ab, \frac{L c_0 k_0}{w_0}, \dots, \frac{L c_n k_n}{w_n}\right) = a. \end{cases}$$

Then $[\eta_0, \dots, \eta_n]$ is a minimal Gorenstein torsion vector of order ab for w if and only if $\eta \leq u\eta$ for all $u \in (\mathbb{Z}/ab\mathbb{Z})^\times$ and there exists $[k_0, \dots, k_n] \in K$ such that $c_i k_i$ is a representative integer of η_i for $i = 0, \dots, n$.

Proof. The assertion follows from Proposition 4.1, Lemma 4.8 and Lemma 4.9. $\square$

We now determine when the degree matrix of a fwps can be extended by a torsion row.

Proposition 4.11. *Let Z be a fwps with $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ and degree matrix*

$$Q = [\omega_0 \quad \cdots \quad \omega_n] = \begin{bmatrix} w_0 & \cdots & w_n \\ \eta_0 & \cdots & \eta_n \end{bmatrix}.$$

Consider a torsion vector $\zeta = [\zeta_0, \dots, \zeta_n]$ of order $\mu \in \mathbb{Z}_{\geq 2}$ for $w = [w_0, \dots, w_n]$, and set

$$Q_\zeta := \begin{bmatrix} \omega_0 & \cdots & \omega_n \\ \zeta_0 & \cdots & \zeta_n \end{bmatrix}, \quad Q_{\zeta,i} := \begin{bmatrix} \omega_0 & \cdots & \hat{\omega}_i & \cdots & \omega_n \\ \zeta_0 & \cdots & \hat{\zeta}_i & \cdots & \zeta_n \end{bmatrix}.$$

Then, Q_ζ is the degree matrix of a fwps $Z_\zeta = Z(Q_\zeta)$ if and only if μ divides μ_r and the maximal minors of $Q_{\zeta,i}$ generate $\mathbb{Z}/\mu\mathbb{Z}$ for $i = 0, \dots, n$.

Proof. If Q_ζ is a degree matrix, then μ divides μ_r because $\text{Cl}(Z_\zeta)$ is in invariant factor form. Since any n of the (ω_i, ζ_i) generate $\text{Cl}(Z_\zeta)$, the maximal minors of $Q_{\zeta,i}$ generate $\mathbb{Z}/\mu\mathbb{Z}$ for $i = 0, \dots, n$ by Proposition 2.8. Viceversa, if μ divides μ_r , then $\text{Cl}(Z) \oplus \mathbb{Z}/\mu\mathbb{Z}$ is in invariant factor form. Since Q is a degree matrix, any n of the ω_i generate $\text{Cl}(Z)$. Hence, by Proposition 2.8, if the maximal minors of $Q_{\zeta,i}$ generate $\mathbb{Z}/\mu\mathbb{Z}$ for $i = 0, \dots, n$ then any n of the (ω_i, ζ_i) generate $\text{Cl}(Z) \oplus \mathbb{Z}/\mu\mathbb{Z}$. $\square$

Corollary 4.12. *Let Z be a fwps with $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ and degree matrix*

$$Q = [\omega_0 \quad \cdots \quad \omega_n] = \begin{bmatrix} w_0 & \cdots & w_n \\ \eta_0 & \cdots & \eta_n \end{bmatrix},$$

and let ζ, θ be two torsion vectors for $w = [w_0, \dots, w_n]$. By Proposition 2.8 and Proposition 4.11, if Q_ζ is a degree matrix and Q_θ is not a degree matrix, then $(Q_\zeta)_\theta$ is also not a degree matrix.

Procedure 4.13. Let $w = [w_0, \dots, w_n]$ be a Gorenstein weight vector, and let H be the set of all pairs (μ, η) , where $\mu \in \mathbb{Z}_{\geq 2}$ and $\eta \in (\mathbb{Z}/\mu\mathbb{Z})^{n+1}$ is a minimal Gorenstein torsion vector of order μ for w . We build the degree matrices of all Gorenstein fwps with weight vector w by progressively adding torsion factors to the class group. Clearly, $\mathbb{P}(w_0, \dots, w_n)$ is the only wps with weight vector w . By Proposition 4.5, all Gorenstein fwps with $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z}$ admit a degree matrix

$$Q_{\eta_1} := \begin{bmatrix} w_0 & \cdots & w_n \\ \eta_{01} & \cdots & \eta_{n1} \end{bmatrix},$$

where $(\mu_1, [\eta_{01}, \dots, \eta_{n1}]) \in H$. Now, for each $(\mu_1, [\eta_{01}, \dots, \eta_{n1}]) \in H$, let H_{η_1} be the set of all $(\mu_2, [\eta_{02}, \dots, \eta_{n2}])$ in H such that $[\eta_{01}, \dots, \eta_{n1}] < [\eta_{02}, \dots, \eta_{n2}]$ if $\mu_1 = \mu_2$ and the matrix

$$Q_{\eta_1, \eta_2} := \begin{bmatrix} w_0 & \cdots & w_n \\ \eta_{01} & \cdots & \eta_{n1} \\ \eta_{02} & \cdots & \eta_{n2} \end{bmatrix}$$

is the degree matrix of a Gorenstein fwps. This is the case if and only if Q_{η_1, η_2} satisfies the conditions of Proposition 4.11. By Proposition 4.5, all Gorenstein fwps with $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \mathbb{Z}/\mu_2\mathbb{Z}$ admit such a degree matrix. Now, for

each such matrix Q_{η_1, η_2} , let H_{η_1, η_2} be the set of all $(\mu_3, [\eta_{03}, \dots, \eta_{n3}]) \in H_{\eta_1}$ such that $[\eta_{02}, \dots, \eta_{n2}] < [\eta_{03}, \dots, \eta_{n3}]$ if $\mu_2 = \mu_3$ and the matrix

$$Q_{\eta_1, \eta_2, \eta_3} := \begin{bmatrix} w_0 & \dots & w_n \\ \eta_{01} & \dots & \eta_{n1} \\ \eta_{02} & \dots & \eta_{n2} \\ \eta_{03} & \dots & \eta_{n3} \end{bmatrix}$$

is the degree matrix of a Gorenstein fwps. This is the case if and only if $Q_{\eta_1, \eta_2, \eta_3}$ satisfies the conditions of Proposition 4.11. By Proposition 4.5 and Corollary 4.12, all Gorenstein fwps with $\text{Cl}(Z) = \mathbb{Z} \oplus \mathbb{Z}/\mu_1\mathbb{Z} \oplus \mathbb{Z}/\mu_2\mathbb{Z} \oplus \mathbb{Z}/\mu_3\mathbb{Z}$ admit such a degree matrix. We iterate this process until $H_{\eta_1, \dots, \eta_r}$ is the empty set. By Proposition 2.8, the procedure terminates in at most n steps.

In order to detect when two generator matrices yield isomorphic fwps, we introduce a normal form for a generator matrix P . It shares similarities with the PALP normal form presented in [9] and the normal form presented in [5]. We denote by $\text{HNF}(P)$ the Hermite normal form of the matrix P .

Definition 4.14. Let Z be a fwps with generator matrix P and degree matrix Q :

$$P = [v_0 \quad \dots \quad v_n], \quad Q = \begin{bmatrix} w_0 & \dots & w_n \\ \eta_0 & \dots & \eta_n \end{bmatrix},$$

where $v_i \in \mathbb{Z}^n$, $w_i \in \mathbb{Z}_{\geq 1}$ and $\eta_i \in \mathbb{Z}/\mu_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\mu_r\mathbb{Z}$ for $i = 0, \dots, n$. Let S_Z be the set of all permutations $\sigma \in S_{n+1}$ such that $\sigma(i) < \sigma(j)$ if $w_i < w_j$. We define the *normal form* of P as

$$\text{Norm}(P) := \min(\text{HNF}(P_\sigma); \sigma \in S_Z),$$

where $P_\sigma = [v_{\sigma(0)}, \dots, v_{\sigma(n)}]$, and the minimum is taken with respect to the lexicographic order row by row.

Remark 4.15. Two fwps $Z = Z(P)$ and $Z' = Z(P')$ are isomorphic if and only if $\text{Norm}(P) = \text{Norm}(P')$.

Procedure 4.16. *Input:* a positive integer n ; the collection $Q(n)$ of degree matrices produced by Procedure 4.13.

Procedure:

- (1) Initialize $P(n) := \emptyset$.
- (2) For each $D \in Q(n)$:
 - (a) compute an associated generator matrix P ;
 - (b) compute its normal form $\text{Norm}(P)$;
 - (c) if $\text{Norm}(P) \notin P(n)$, set $P(n) := P(n) \cup \{\text{Norm}(P)\}$.
- (3) Return $P(n)$.

Output: a set $P(n)$ of generator matrices containing exactly one representative from each isomorphism class of n -dimensional Gorenstein fwps.

Algorithm 4.17 (Classification of the Gorenstein fwps). *Input:* a positive integer n . *Algorithm:*

- (1) Compute or load the list of all Gorenstein weight vectors $w := [w_0, \dots, w_n]$ according to Remark 4.6;
- (2) For each Gorenstein weight vector w and each $\mu \in \mathbb{Z}_{\geq 2}$, compute all minimal Gorenstein torsion vectors of order μ for w using Procedure 4.10;

- (3) For each Gorenstein weight vector w , compute all degree matrices obtained by suitably combining w with the torsion vectors from (2) using Procedure 4.13;
- (4) Compute the set of associated generator matrices to the degree matrices from (3), then select a subset that contains exactly one generator matrix from each isomorphism class of fwps, using Procedure 4.16.

Output: a list of representatives for the isomorphism classes of n -dimensional Gorenstein fwps, complete according to Proposition 4.5.

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