

Massive Gauge Theories from Consistency Conditions of Amplitudes

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Based on the general principles of Lorentz symmetry and unitary, we introduce two consistency conditions – on-shell gauge symmetry and strong massive-massless continuation – in constructing amplitudes of massive gauge theory with elementary particles. In particular we argue on-shell gauge symmetry can be understood as a consequence of Lorentz symmetry, through mixture of a vector boson and a scalar with degenerate mass spectrum. Based on the two conditions, combined with the little group transformation and consistent factorization, we construct 3-point and 4-point vector boson/scalar amplitudes, then analyze the underlying physical models. Given the particle masses, almost all possible vertices, including those involving Goldstone modes, are uniquely fixed. The only exceptions are triple and quartic scalar self-couplings. In addition, all particle masses must have the same physical origin. If the number of vector bosons is smaller than 3, the underlying theories for the amplitudes are either massive gauge theories with spontaneous symmetry breaking (S.S.B) or Stueckelberg theory. The necessary condition for the latter is that the scalars have equal masses. We also discuss different models depending on the number of scalars involved. If the number of vector bosons is larger than 3, the underlying theory must be Yang-Mills theory with S.S.B. In both abelian and non-abelian cases, the specific shape of Higgs potential cannot be determined, which explains the fact that scalar self-couplings are undetermined, and the relations between the masses are generally not linear.

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I. INTRODUCTION

The project of constraining viable gauge theories through Lorentz invariance and analytic structure of S-matrix can be traced back to Weinberg in his work of [1–3], which generated many beautiful results and physical insights. For example, by considering S-matrix for a massless spin-1 particle taking part in interactions, Lorentz invariance requires that S-matrix to obey on-shell gauge invariance: $k_\mu \mathcal{M}^\mu = 0$. Moreover, by taking the soft limit of the spin-1 particle, charge conservation can be derived. Thus an interacting theory of massless spin-1 particles has to be a gauge theory. Similar considerations for massless spin-2 particles lead us to linearized Einstein equation and equivalence principle. This project got a new life in the last decade with the development of modern approach of amplitudes, especially spinor-helicity formalism and BCFW recursion relation[4]. In [5–7], it was found that $U(1)$ little group scaling of 3-point amplitudes uniquely fix all possible 3-point amplitudes. Moreover, by demanding consistent factorization of four-point amplitudes at s, t, u channels when they go to 0, massless spin-1 and spin-2 interacting theories are uniquely fixed to be Yang-Mills theory and linearized general relativity respectively. For a fresh review, see also [8].

Similar projects have also been carried out for massive gauge theory. As early as 70s, ref([9–12]) derived gauge invariance and upper bound of the Higgs boson by applying perturbative unitarity on 4-point vector boson scattering amplitudes. In recent year, ref([8]) introduced a new framework that generalizes the spinor formalism to massive amplitudes, bringing new life to this project. In it and [13] before it, the authors constructed 3-point massive amplitudes, but left 4-point untouched. Following [8] there have also been many works that generalize this approach to 4-point amplitudes([14–19]), EFT([20–24]) and others(e.g.[25–29]).

Despite those achievements, the project of constructing massive gauge theories from amplitudes is still not wholly satisfactory. The spinor formalism, though very usefull in constructing 3-point amplitudes, is still essentially making use of unitarity in constructing 4-point amplitudes. Moreover, the perturbative unitarity approach itself neglects some crucial aspects of the underlying physics. First, precise cancellation between diagrams from perturbative unitarity implies an underlying symmetry. This symmetry, however, is not manifest in the amplitudes, making the cancellation seems ad hoc. It's only in the Lagrangian that it's clear the symmetry is gauge symmetry with spontaneous symmetry breaking(S.S.B). The problem then is how to implement gauge symmetry at the level of amplitudes. It would be also optimal to trace the origin of massive gauge symmetry to Lorentz symmtry directly like the massless case, instead of extracting it from Lagrangian.

Second, the perturbative unitarity approach proves that an elementary scalar can restore the self-consistency of scattering amplitudes and the related theory. However, it's not the only possible solution. In both technicolor theory ([30–32]) and Composite Higgs models ([33–36]), unitarity is restored through exchange of many resonances in scattering and ultimately the dissolution of composite particles. This means that we need a clear criteria on the amplitudes to exclude other possibilities in favor of the elementary scalar solution. Finally, perturbative unitarity does not apply to all physical scenarios. In Stueckelberg theory, individual diagrams have no unitarity violating behavior [37]. It's more complete incorporate this case into the project.

The purpose of this paper is to find consistency conditions for amplitudes that solve the above problems, then apply them to construct massive amplitudes and the corresponding theories. For the first problem of gauge symmetry, we notice that gauge symmetry with S.S.B is manifested in Goldstone equivalence theorem and its precise form the massive Ward identity or on-shell gauge symmetry $k^\mu \mathcal{M}_\mu = \pm i m_V \mathcal{M}(\varphi)$ (φ is Goldstone boson)[38–40]. For the second problem of selecting elementary particles, we notice an elementary particle interacts point-like, meaning the coupling is constant at tree level. Based on those analysis we propose the following two conditions in construting massive amplitudes:

- **On-shell gauge symmetry: the amplitude of massive vector boson with the polarization vector replaced with k^μ equals to some corresponding scalar amplitude, i.e. $k^\mu \mathcal{M}_\mu = \pm i m_V \mathcal{M}(\varphi)$. The vector boson and the scalar have a common mass.**
- **Strong massive-massless continuation: the S-matrix remains finite when one of the particles' mass is taken to be 0 continuously.**

The two conditions can be further understood as reflections of the general principle of massive-massless continuation in different aspects. In particular we will show on-shell gauge symmetry can be derived from Lorentz symmetry and unitarity with some reasonable assumptions. In [41], a construction of particles as unitary and irreducible representations of Poincare that satisfies massive-massless continuation was proposed, providing a solid foundation for this principle.

After finding the consistency conditions that satisfy massive gauge symmetry and select elementary particles, we proceed to apply those conditions, along with little group transformation and consistent factorization, to construct scattering amplitudes for massive vector bosons and scalars up to 4-point. We find our approach to be very powerful: we construct all 3-point and 4-point amplitudes with different mass combinations. All couplings, including those related to Goldstone bosons, are uniquely fixed up to an overall factor, with the sole exceptions of triple and quartic scalar

self-couplings. Most importantly, we find that in constructing 4-point amplitudes with $VV \rightarrow VV$, at least one scalar S and the subsequent vertex VVS are needed to make sure gauge symmetry is satisfied. In addition, all particles' masses goes to 0 if the scalar mass goes to 0, implying they all have the same origin. Thus we conclude the only theory that's consistent with the conditions of on-shell gauge symmetry and (strong) massive-massless continuation is Yang-Mills theory with Higgs mechanism, in the general sense that symmetry is spontaneous broken to give masses to all particles. But the shape of the scalar potential is not specified. Moreover, we found Higgs theory is not the only possible theory when constructing other amplitudes. For example, the underlying theory for two scalars and one vector boson is scalar QED with Stueckelberg mechanism, when the two scalars have equal masses. In this case, the vector boson obtains mass without “eating” a dynamic Goldstone mode.

The rest of the paper is organized as following,

In Sec. (II) we details the motivations and justifications of the consistent conditions of on-shell gauge symmetry and strong massive-massless continuation/

In Sec.(III) we review and introduce the framework about polarizations vectors, propagator and amplitudes.

In Sec.(IV), we reconstruct 3-point amplitudes for all possible combination of scalars and vector bosons with arbitrary masses.

In Sec.(V), we construct 4-point amplitudes from 3-point amplitudes, determine the conditions for it to succeed, discuss the underlying theories and models behind.

Finally in Sec.(VI) we have conclusions and discussions.

II. CONSISTENCY CONDITIONS

In this section we will demonstrate in details that the two consistency conditions proposed in Item(I) come from combined consideration of Lorentz symmetry, the general principle of massive-massless continuation and that all particles involved are elementary.

On-shell Gauge Symmetry

To summarized in advance, we will show on-shell gauge symmetry $k^\mu \mathcal{M}_\mu = im_V \mathcal{M}(\varphi)$ can be derived from the principle of massive-massless continuation and particles as unitary representations

of the Poincare group.

First we discuss the most general case for a particle as the vector representation of Lorentz group, which includes a spin-1 block and a spin-0 block respectively. Thus a massive vector boson could naturally mix with an additional scalar through the spin-0 block. After the mixing we have 5 degrees of freedom in total.

Now based on the mixture, we want to construct a massive spin-1 particle with the standard 3 degrees of freedom. The necessary condition for all the degrees of freedom after mixing to be a single particle, is the mass spectrum being degenerate. This can be accomplished by imposing the gauge condition:

$$k^\mu \epsilon_\mu^\sigma = m_V \delta^{\sigma 0}, \quad (1)$$

The degeneracy of mass eigenvalues can be seen by plugging $\epsilon_0^\mu = \frac{k^\mu}{m_V}$ into Eq.(1), giving $k^2 = m_V^2$. The gauge condition (Eq.(1)) in position space can be written as $\partial^\mu V_\mu = m_V \varphi$. Translating back in momentum space, we distinguish between initial and final state for φ by defining $\epsilon^\varphi = \mp i$, with $-i$ being initial and $+i$ being final for $\sigma = 0$. Also $\epsilon^\varphi = 0$ for $\sigma = \pm 1$. The gauge condition then becomes

$$k_\mu \epsilon_\sigma^\mu = \pm i m_V \epsilon_\sigma^\varphi, \quad + \text{ for initial state} \quad - \text{ for final state} \quad (2)$$

It can also be noted that Eq.(2) is invariant under

$$\epsilon^\mu \rightarrow \epsilon^\mu + \xi k^\mu / m_V, \quad \epsilon^\varphi \rightarrow \epsilon^\varphi \mp \xi i, \quad (3)$$

meaning the polarization vector for the vector-boson-scalar mixing block has a redundancy, which is simply gauge symmetry.

Now we have two conditions on the vector boson mixing with a scalar, reducing the number of degrees of freedom(d.o.f) from 5 to 3, which is precisely the number of d.o.f for a massive spin-1 particle required by unitarity. Furthermore, in our construction the amplitude of a massive vector boson can be written as

$$\mathcal{M}(V) = \epsilon_\sigma^\mu \mathcal{M}_\mu(V) + \epsilon_\sigma^\varphi \mathcal{M}(\varphi) \quad (4)$$

To ensure the gauge redundancy in Eq.(3) has no contribution to amplitudes, we must have

$$k^\mu \mathcal{M}_\mu = \pm i m_V \mathcal{M}(\varphi) \quad (5)$$

So we have derived on-shell gauge symmetry in Item(I).

Based on the condition Eq.(2), the kinetic Lagrangian for this mixing can be derived by making use of Euler-Lagrangian equations repetitively. The results are

$$\begin{aligned}\mathcal{L}_{V^2} &= -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \frac{1}{2}m_V^2 V_\mu^2 \\ \mathcal{L}_{V-\varphi} &= m_V V_\mu \partial^\mu \varphi \\ \mathcal{L}_{\varphi^2} &= \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi\end{aligned}\tag{6}$$

This is exactly the Lagrangian of gauge-Goldstone fields in S.S.B, with the gauge-Goldstone mixing term $\mathcal{L}_{V-\varphi}$. Moreover, the mass term for the scalar to be zero, which mean it has zero mass without the mixing, thus justifying calling it Goldstone boson. The details of the derivation are in the appendix A.

The main advantage of our construction is that it has a manifestly continous massless limit. By taking $m_V \rightarrow 0$, the gauge condition in Eq.(2) becomes $k^\mu \epsilon_\mu = 0$. In position space it's written as $\partial^\mu V_\mu = 0$. The gauge symmetry also gets back to the massless case: $\epsilon^\mu \rightarrow \epsilon^\mu + \xi k^\mu$. So we still have two conditions for massless spin-1 particle. Furthermore, because the Goldstone boson decouples in the massless limit, the number of degrees of freedom is now $4 - 2 = 2$, exactly that of massless spin-1 particle.

Finally, from the analysis above it's also clear that there can be one scalar at most to mix with the vector boson, because there is only one index on the vector boson (ϵ^μ/V^μ) to contract with the k^μ or derivative of the Goldstone boson, either in the gauge condition or in the mixing Lagrangian term.

Strong Massive-Massless Continuation

On-shell gauge symmetry originates from Lorentz symmetry generally, thus doesn't distinguish between elementary and composite particles, but our purpose is to select theories of elementary particles only. This means that we need another consistent condition.

Particles are elementary in the sense that they remain point-like in interactions at arbitrary energy scalar. This is in contrast with composite particles, of which form factors change drastically with increased energy. When the energy is high enough, a composite particle would eventually fragment into more elementary components. From the perspective of effective field theory, the theory based on the composite particle is only effective up to certain energies. When the energy increases to a certain point, the effective theory becomes invalid, there needs to be another theory that's based on more elementary degrees of freedom to replace the former. In amplitudes, the

process of effective theory becoming invalid is reflected in the energy increasing behaviors that violate perturbative unitarity. That fact that amplitudes eventually blows up to infinity as $E \rightarrow \infty$, reflects the process of the composite particle dissolving into its elementary components.

The above analysis can be captured by the partial wave analysis of amplitudes. However, we would like to have an approach that has the above property directly reflected in the analyticity of S-matrix as function of momenta, polarization vectors, as well as masses and couplings. Noticing for longitudinal vector boson, $E \rightarrow \infty$ is equivalent to $m_V \rightarrow 0$. The condition of unitarity for amplitudes can be expressed as following: amplitudes remain finite when the vector boson mass goes to 0. We further generalize the condition from vector boson's mass to any particle's mass in the process. This finally gives the condition of strong massive-massless continuation in Item I. In mathematical form, it can be written as

$$\lim_{m_i \rightarrow 0} \mathcal{M}(\{p_i, \epsilon_i, m_i\}, \lambda_j) = \text{finite} \quad (7)$$

i being any massive particle.

III. FRAMEWORK OF CONSTRUCTING AMPLITUDES

5-component Formalism

Since in our construction, a vector boson (V) obtains mass by mixing with a scalar (φ) to form a single physical object, it's natural and convenient to define a unified 5-component vector boson for this object as $V^M = (V^\mu, \varphi)$, with $M = 4$ corresponding to the Goldstone component ([42]). The amplitudes for vector bosons, stripping off energy-momentum conservation, are then written as

$$\mathcal{M}(p_1, p_2, \dots) = \epsilon_{s_1}^{M_1}(p_1) \epsilon_{s_2}^{M_2}(p_2) \dots \mathcal{M}_{M_1 M_2 \dots}(p_1, p_2, \dots), \quad (8)$$

in which $\epsilon_{s_i}^{M_i}(p_i)$ ($i = 1, 2, \dots$) is polarization vector with spin s_i and momentum p_i .

Subsequently, polarization vectors can also written in 5-component form: $\epsilon_\sigma^M = (\epsilon_\sigma^\mu, \epsilon_\sigma^\varphi)$. We can also define the 5-component momentum k^M : $k^M = (k^\mu, \pm i m_V)$. To ensure on-shell condition ($k^2 = m_V^2$) and the transverse condition ($k^\mu \epsilon_\sigma^\mu = \pm i \epsilon_\sigma^\varphi$) are satisfied in 5-component form, we introduce 5-component ‘‘matric’’ $g_{MN} = \text{diag}(1, -1, -1, -1, -1)$ for related calculations. Accordingly, the

polarization vectors and 5-component momentum are redefined as

$$\begin{aligned}
\text{initial: } \epsilon_{\pm}^M &= (\epsilon_{\pm}^{\mu}, 0)^T & \text{final: } \epsilon_{\pm}^{*M} &= (\epsilon_{\pm}^{*\mu}, 0)^T \\
\text{initial: } \epsilon_L^M &= (\epsilon_n^{\mu}, +i)^T & \text{final: } \epsilon_L^{*M} &= (\epsilon_n^{\mu}, -i)^T \\
\text{initial: } k^M &= (k^{\mu}, -im_V) & \text{final: } k^{*M} &= (k^{\mu}, +im_V)
\end{aligned} \tag{9}$$

with $\epsilon_n^{\mu}(k) = -\frac{m}{n \cdot k} n^{\mu}$, n being the null vector that imposes light-cone gauge on ϵ^{μ} :

$$n \cdot \epsilon(k, r) = 0 \quad \text{with } n^2 = 0 \tag{10}$$

If we choose $n^{\mu} = (1, -\frac{\vec{k}}{|\vec{k}|})$, the corresponding polarization vectors are precisely the standard ones in textbooks.

On-shell condition, transverse condition and orthogonal conditions can be written in 5-component form as

$$k^{*M} k_M = 0 \quad k^{*M} \epsilon_M^{\sigma} = 0 \quad \epsilon_{\sigma}^{*M} \epsilon_M^{\sigma'} = -\delta_{\sigma\sigma'} \tag{11}$$

On-shell gauge symmetry can also be written in 5-component form as

$$k^M \mathcal{M}_M = 0 \tag{12}$$

Finally, we discuss how crossing symmetry acts on polarization vectors in 5-component form. It is known S-matrix is invariant under crossing symmetry, which takes the initial/final wave function with momentum k and helicity s to the final/initial wave function with momentum $-k$ and helicity $-s$, i.e. $u_s(k)/\bar{u}_s(k) \rightarrow \bar{v}_{-s}(-k)/v_{-s}(-k)$. Acting on the 5-component polarization vectors, a transverse polarization vector simply takes the hermition conjugate: $\epsilon_{\mp}^M(k) \rightarrow \epsilon_{\mp}^{*M}(-k) = \epsilon_{\mp}^{*M}(k)$, since they don't depend on momentum. The situation is different for the longitudinal, $\epsilon_L^M(k) \rightarrow \epsilon_L^{*M}(-k)$. From Eq.(9), we have $\epsilon^{\mu}(-k) = -\epsilon^{\mu}(k)$, which means that $\epsilon_L^{*M}(-k) = (-\epsilon^{\mu}(k), -\epsilon^{\varphi}) = -\epsilon_L^M(k)$. Equivalently, $\epsilon_L^{*M}(k) = -\epsilon_L^{*M}(-k)$. Thus, the complex conjugate for longitudinal is equivalent to crossing symmetry up to an unphysical, overall minus sign. This result can be useful in doing concrete, analytic calculations in 5-component formalism.

Approach of Constructing Amplitudes

Having setting up the consistent conditions, we are now ready to construct massive amplitudes with them and other fundamental principles. Here we give a sketch of the method, conveying the basic idea of our approach. In this paper we focus on constructing 4-point amplitudes, which can

give us enough information to understand the underlying theory. Our method can be divided into two steps: first we construct 3-point amplitudes from their massless limit by making use of the consistent conditions; then we construct 4-point amplitudes from 3-point amplitudes by consistent factorization.

First we analyze 3-point amplitudes. Taking the massless limit of a massive vector amplitude, its transverse and longitudinal parts decouple from each other and form Lorentz invariant blocks separately. Taking an amplitude with one vector boson as an example,

$$\mathcal{M}_s(V) = \epsilon_{\pm,L}^M \mathcal{M}_M(p, m_V) \xrightarrow{m_V \rightarrow 0} \begin{cases} \epsilon_s^\mu \mathcal{M}_\mu(p) & s = \pm \\ \mathcal{M}^\varphi(p) & s = L \end{cases} \quad (13)$$

Massless amplitudes can be constructed from Lorentz symmetry. Moreover, the massless vector amplitude can be extended naturally back to massive case to include the ϵ_n part (0 in massless limit) of the longitudinal amplitude according to Eq.(9). Now we have all the elements ready to construct the massive amplitude. Based on above analysis, we propose that a massive vector amplitudes can be written as a linear combination of massless amplitudes:

$$\mathcal{M}_s^V(p, m_V) = a \epsilon_s^\mu \mathcal{M}_\mu^V(p) + b \mathcal{M}^\varphi(p) \quad (14)$$

a, b are coefficients to be fixed by the two consistent conditions.

It should be noted, however, that there is a potential problem when applying on-shell gauge symmetry on 3-point amplitudes. Generally it's impossible to make all 3 particles to be on-shell simultaneously, except for the special case of one particle mass larger than the sum of the two other masses. In other words, one of them has to be off-shell (in general kinematics). This seems to imply on-shell gauge symmetry cannot apply on 3-point amplitudes.

However, after careful consideration, we show O.G.S is still applicable to a 3-point amplitude when one particle is off-shell. As will be clear in subsequent sections, when applying on-shell gauge symmetry by replacing ϵ^M with k^M , we first make use of energy-momentum conservation to substitute k with other momenta in the amplitude, then we apply on-shell condition, transverse condition and orthogonal conditions (Eq.(11)) for the other particles. In other words, we don't make use of any information that relevant to the on-shellness of the particle we apply O.G.S on. Therefore, on-shell gauge symmetry applies to an amplitude with one particle being off-shell, if O.G.S is being applied on this off-shell particle. This result also eliminates a motivation for complexifying the momenta. Since there is no need, we will keep the momenta real without complexification through out the paper.

After constructing 3-point amplitudes, we then proceed to construct 4-point amplitudes. According consistent factorization, when an internal line is put on-shell, amplitudes of n-point amplitudes factorize into amplitudes of fewer number of external points:

$$\mathcal{M}_4 \xrightarrow{k_a^2 \rightarrow m_a^2} \sum_{s,a} \mathcal{M}_{3,a}^s \frac{i}{k_a^2 - m_a^2} \mathcal{M}_{3,a}^{-s} \quad (15)$$

s is spin indice. a signals the intermediate state of which the full 4-point amplitude requires summing over all possible number.

Now we would like to reverse the factorization process and obtain the full amplitude from lower-point ones. A common way is to complexify the momenta and BCFW recursion relation. However, as already stated above, we don't complexify momenta. Instead we look for a natural off-shell continuation for the vector boson propagator. This "off-shell continuation" can be found by writing the summation over polarization vectors[42] as

$$\sum_{s=\pm,0} \epsilon_s^M \epsilon_s^{*N} = -g^{MN} + \frac{k^M n^N + n^M k^{*N}}{n \cdot k} \quad (16)$$

The RH of the above equation has the same form when both on-shell and off-shell, thus can be naturally used for off-shell continuation.

Next notice that k^M terms give 0 to amplitudes by on-shell gauge symmetry, even when off-shell. Thus we can drop the k^M terms for the vector propagator. The off-shell continuation for the propagator of massive vector boson simplifies to:

$$\frac{i \sum_{s=\pm,0} \epsilon_s^M \epsilon_s^{*N}}{k_i^2 - m_i^2} \xrightarrow{k_i^2 \neq m_i^2} \frac{-ig^{MN}}{k_i^2 - m_i^2} \quad (17)$$

To sum up, we can construct an amplitudes from lower points by off-shell continuation in the following way

$$\sum_{s,a} \mathcal{M}_{3,a}^s \frac{i}{k_a^2 - m_a^2} \mathcal{M}_{3,a}^{-s} \xrightarrow{\text{off-shell}} \mathcal{M}_4 = \sum_a \mathcal{M}_{3,a}^M \frac{-ig_{MN}}{k_a^2 - m_a^2} \mathcal{M}_{3,a}^N \quad (18)$$

Finally, the coefficients in constructing 4-point amplitudes should also be fixed by the two consistent conditions.

IV. 3-POINT AMPLITUDES

In this section we set to construct all possible 3-point amplitude involving massive vector boson. We follow the method outlined in Sec.(III): construct massless amplitudes first, then construct massive amplitudes from massless amplitudes by the two consistent conditions.

A. Massless 3-point Amplitudes

Let's first construct massless 3-point amplitudes as basis for massive amplitudes. Lorentz invariance implies that a scattering amplitude should be written as

$$\mathcal{M}(1, 2, \dots, n) = \epsilon_1^{\mu_1} \epsilon_2^{\mu_2} \dots \epsilon_n^{\mu_n} \mathcal{M}(1, 2, \dots, n)_{\mu_1 \mu_2 \dots \mu_n} \quad (19)$$

Polarization vectors ϵ_i^μ undertake little group transformations, which include $U(1)$ scaling under rotation around the reference momentum, as well as the momentum shift $\epsilon^\mu \rightarrow \epsilon^\mu + \xi k^\mu$ [2] under the non-compact members of little group transformation. The latter gives massless Ward identity:

$$k_1^{\mu_1} \epsilon_2^{\mu_2} \dots \epsilon_n^{\mu_n} \mathcal{M}(1, 2, \dots, n)_{\mu_1 \mu_2 \dots \mu_n} = 0. \quad (20)$$

$\mathcal{M}(1, 2, \dots, n)_{\mu_1 \mu_2 \dots \mu_n}$ is a function of momenta $k_i^{\mu_i}$ and their contraction with each other through tensor $g^{\mu\nu}$. We neglect $\epsilon^{\mu\nu\rho\sigma}$ by assuming CP symmetry is conserved. In addition, in this paper we only focus on those corresponding to the tree-level Feynman diagrams, so the energy dimension of the 3-point amplitudes is fixed to 1. Finally, we use S to denote scalar, V to denote vector boson.

Massless SSV

To maintain the dimension to be 1, the amplitude of massless $S_1 S_2 V_3$ must be of the following form

$$\mathcal{M}_0(S_1 S_2 V_3) = (a_1 k_1 + a_2 k_2) \cdot \epsilon_3$$

a_1 and a_2 are dimensionless parameters to be fixed. On-shell gauge symmetry Eq.(20) constrains a_1 and a_2 to be

$$a_1 = -a_2 \quad (21)$$

Thus the amplitude for massless $1^0 2^0 3^T$ is

$$\mathcal{M}_0(S_1 S_2 V_3) = (k_1 - k_2) \cdot \epsilon_3 \quad (22)$$

up to an overall factor.

Furthermore, boson symmetry for the amplitude requires the two scalars not to be identical. Instead, the two must have different quantum numbers, which we can label as a, b , so that the amplitude could be written as

$$\mathcal{M}_0(S_1^a S_2^b V_3) = T_{ab} \mathcal{M}_0(S_1 S_2 V_3) \quad (23)$$

with $T_{ab} = -T_{ba}$. For example, the scalars can be in a $SO(N)$ group, T then is one of its generators. There is also a special case of $N = 2$, giving $SO(2) \cong U(1)$. In this case, the two scalars are to be identified as the anti-particles of each other.

“Massless” SVV

The amplitude of SVV can be generally written as

$$\mathcal{M}_0(S_1 V_2 V_3) = m \epsilon_2 \cdot \epsilon_3 \quad (24)$$

The coefficient m has to be of dimension 1. Notice there isn’t any thing constraining it to be a constant yet. Generally it should be a function of momenta: $m = m(p_1, p_2)$. However, we notice Eq.(24) doesn’t satisfy massless on-shell gauge symmetry (Eq.(20)), which implies Eq.(24) is not really a massless amplitude. In fact, as we will see in Sec.(IV B), Eq.(24) can be only understood as part of the massive SVV amplitude that satisfies massive on-shell gauge symmetry. This also means the amplitude should disappear in the massless limit. Thus following strong massive-massless continuation, we must have $m \rightarrow 0$ as either one of the masses $m_i \rightarrow 0$:

$$\lim_{m_i \rightarrow 0} m = 0 \quad (25)$$

This indicates the dependence of m on particle masses in some form, which further points to common origin of m and particle masses.

Massless VVV

Similar to the derivation of SSV , the massless amplitude of VVV can be written generally as

$$\mathcal{M}_0(V_1 V_2 V_3) = (a_{13} k_1 + a_{23} k_2) \cdot \epsilon_3 \epsilon_1 \cdot \epsilon_2 + \text{cyclic}$$

a_{13}, a_{23} and etc. are dimensionless parameters to be fixed by on-shell gauge symmetry, which gives

$$a_{13} = -a_{23} = a_{32} = -a_{12} = a_{21} = -a_{31} \quad (26)$$

$\mathcal{M}_0(V_1 V_2 V_3)$ becomes

$$\mathcal{M}_0(V_1 V_2 V_3) = (\epsilon_1 \cdot \epsilon_2) (k_1 - k_2) \cdot \epsilon_3 + \text{cyclic} \quad (27)$$

up to an overall factor.

Moreover, boson symmetry requires a totally anti-symmetric factor for the three vector bosons. Labeling them as a, b, c respectively, we arrive at the final form of $M_0(V_1 V_2 V_3)$:

$$\mathcal{M}_0(V_1^a V_2^b V_3^c) = f^{abc} \mathcal{M}_0(V_1 V_2 V_3) = f^{abc} [(\epsilon_1 \cdot \epsilon_2) (k_1 - k_2) \cdot \epsilon_3 + \text{cyclic}] \quad (28)$$

f^{abc} is totally anti-symmetric over a, b, c .

B. Massive 3-point Amplitudes

Having finished constructing the massless amplitudes, we now turn to the massive amplitudes. Apart from the trivial SSS amplitude, the possible amplitudes include SSV , SVV and VVV respectively.

Massive SSV

Let's start with the SSV amplitude, denoted as $\mathcal{M}(S_1 S_2 V_3)$. In the massless limit, $\mathcal{M}(S_1 S_2 V_3)$ decomposes into $\mathcal{M}_0(S_1 S_2 V_3)$ and $\mathcal{M}_0(S_1 S_2 \varphi_3)$ that're Lorentz covariant separately. $\mathcal{M}_0(S_1 S_2 V_3)$ is given by Eq.(22). $\mathcal{M}(S_1 S_2 \varphi_3)$ is the amplitude for 3 scalars, which is trivial.

Up to an overall scaling, the general form of $\mathcal{M}(S_1 S_2 V_3)$ is

$$\mathcal{M}(S_1 S_2 V_3) = (k_1 - k_2) \cdot \epsilon_3 - \lambda_{SS\varphi} \epsilon_3^4 \quad (29)$$

$\lambda_{SS\varphi}$ is a parameter that can be fixed by massive on-shell gauge symmetry:.

$$\mathcal{M}(S_1 S_2 V_3)|_{\epsilon_3^M \rightarrow k_3^M} = 0 \quad (30)$$

which gives

$$\lambda_{S_1 S_2 \varphi} = -i \frac{(m_1^2 - m_2^2)}{m_V} \quad (31)$$

Thus we found the solution. However, the relations between masses can still be constrained by strong massive-massless continuation, which requires that $\lambda_{SS\varphi}$ stays finite as one of the mass goes to 0:

$$\lim_{m_i \rightarrow 0} \lambda_{SS\varphi}(m_1, m_2, m_3) \neq \infty \quad (32)$$

From it we have for m_1 and m_2 :

$$m_1^2 = c^{(0)} + c_1^{(1)} m_V + \mathcal{O}(m_V^2) \quad m_2^2 = c^{(0)} + c_2^{(1)} m_V + \mathcal{O}(m_V^2) \quad (33)$$

This is the most general result we can obtain from the two consistent conditions, indicating at least parts of particles masses have the same physical origin.

Now let's discuss the physical meaning of the solution (Eq.(31) and Eq.(33)) in different limits.

First of all, if $m_V = 0$, vector boson is massless. The $SS\varphi$ amplitude decouples from the gauge part. Applying Ward identity (massless OGS) gives $m_1 = m_2$. So the two scalars must have equal masses, which further implies a $SO(2) \cong U(1)$ symmetry between S_1 and S_2 . In this case, the SSV amplitude simply corresponds to QED with a massive complex scalar field.

Second, if we impose $m_1 = m_2$ while keeping $m_V \neq 0$, we obtain $\lambda_{SS\varphi} = 0$. The equal mass condition is equivalent to imposing a $SO(2) \cong U(1)$ symmetry on (S_1, S_2) . In this case the Goldstone mode is still a component of the vector boson, but never contribute to any amplitudes. In other words, the “Goldstone” component doesn't partake in the interactions. That means we should treat the “Goldstone” component as unphysical that comes from an auxiliary field. This scenario corresponds to Stueckelberg theory.

Third, noticing $\lambda_{SS\varphi}$ has dimension of 1, it's natural to assume it's related to the masses of the particles. Under this assumption we can demand $\lambda_{SS\varphi} \rightarrow 0$ as $m_V \rightarrow 0$, $m_1^2 - m_2^2$ satisfies

$$m_1^2 - m_2^2 = c^{(2)} m_V^2 + \mathcal{O}(m_V^3) \quad (34)$$

The scalar masses (after deducting the constant term) are proportional to the vector mass at the lowest order, but not limited to it.

Finally, the results here only concern 3-point amplitudes. In order to confirm the physical analysis above, we need to complete 4-point amplitudes.

Massive: SVV

Next we go on to construct the scalar-vector-vector amplitude, denoted as $\mathcal{M}(S_3 V_1 V_2)$. In massless limit, it decomposes into $\mathcal{M}_0(S_3 V_1 V_2)$, $\mathcal{M}_0(S_3 \varphi_1 V_2)$, $\mathcal{M}_0(S_3 V_1 \varphi_2)$ and $\mathcal{M}_0(S_3 \varphi_1 \varphi_2)$. The first one is given by Eq. (24), the second and the third are given by Eq. (22), the final one is trivially a three scalar amplitude.

Thus we conclude $\mathcal{M}(S_3 V_1 V_2)$ has the following general form

$$\mathcal{M}(S_3 V_1 V_2) = m(\epsilon_1 \cdot \epsilon_2) - a_1(k_1 - k_3) \cdot \epsilon_2 \epsilon_1^4 - a_2(k_2 - k_3) \cdot \epsilon_1 \epsilon_2^4 + a_3 \epsilon_1^4 \epsilon_2^4 \quad (35)$$

m, a_1, a_2, a_3 are coefficients that can be determined by on-shell gauge symmetry in terms of particle masses up to an overall scaling.

To simplify the calculation, we first apply on-shell gauge symmetry on V_1 by choosing $\epsilon_1^M \rightarrow k_1^M$ and setting $s_2 = T$, this fixes a_1 to be $a_1 = -\frac{im}{2m_1}$; then exchange the label of particle 1 and particle 2 and repeat the last step, we then get $a_2 = -\frac{im}{2m_2}$; finally, we still apply on-shell gauge symmetry on V_1 and set $s_2 = L$, then a_3 is also fixed: $a_3 = \frac{mm_3^2}{2m_1m_2}$. To summarize, a_1, a_2, a_3 are fixed by on-shell gauge symmetry to be

$$a_1 = -\frac{im}{2m_1} \quad a_2 = -\frac{im}{2m_2} \quad a_3 = \frac{mm_3^2}{2m_1m_2} \quad (36)$$

We can further extract an overall factor $g_{VVS} \equiv \frac{m}{m_1}$, after which the amplitude of SVV becomes

$$\mathcal{M}(S_3V_1V_2) = m_1(\epsilon_1 \cdot \epsilon_2) - a_1(k_1 - k_3) \cdot \epsilon_2 \epsilon_1^4 - a_2(k_2 - k_3) \cdot \epsilon_1 \epsilon_2^4 + a_3 \epsilon_1^4 \epsilon_2^4 \quad (37)$$

with

$$a_1 = -\frac{i}{2} \quad a_2 = -\frac{im_1}{2m_2} \quad a_3 = \frac{m_3^2}{2m_2} \quad (38)$$

Thus the massive amplitudes are uniquely fixed given the particle masses. Similar to SSV , massive-massless continuation gives constraints on the mass parameters.

$$m = c_{1m}m_1 + \mathcal{O}(m_1^2) \quad m_2 = c_{12}m_1 + \mathcal{O}(m_1^2) \quad m_3 = c_{31}m_1 + \mathcal{O}(m_1^2) \quad (39)$$

Massive: VVV

Next we turn to the amplitude of three spin-1 particles. We denote it as $\mathcal{M}(V_1V_2V_3)$. In the massless limit, the amplitude $\mathcal{M}(V_1V_2V_3)$ decomposes into

$$\mathcal{M}(V_1V_2V_3) \longrightarrow \begin{cases} \mathcal{M}_0(V_1^T V_2^T V_3^T) \\ \mathcal{M}_0(V_1^0 V_2^T V_3^T) + \text{cyclic} \\ \mathcal{M}_0(V_1^0 V_2^0 V_3^T) + \text{cyclic} \\ \mathcal{M}_0(V_1^0 V_2^0 V_3^0) \end{cases} \quad (40)$$

As in the case of SSV and SVV , we can obtain the general form of massive amplitude $\mathcal{M}(V_1V_2V_3)$ by making continuation from the massless limit:

$$\begin{aligned} \mathcal{M}(V_1V_2V_3) = & [(\epsilon_1 \cdot \epsilon_2 + a_{12}\epsilon_1^4\epsilon_2^4)(k_1 - k_2) \cdot \epsilon_3 + \text{cyclic}] \\ & + [b_{12}m_3\epsilon_1 \cdot \epsilon_2\epsilon_3^4 + \text{cyclic}] + c_{123}\epsilon_1^4\epsilon_2^4\epsilon_3^4 \end{aligned} \quad (41)$$

c_{123} has dimension 1, a_{12}, a_{23}, a_{31} and b_{12}, b_{23}, b_{31} are dimensionless coefficients. m_1, m_2, m_3 are taken to be physical masses of the corresponding particles.

Again we will apply on-shell gauge symmetry to fix the parameters. We apply it on k_3 by the replacement: $\epsilon_3^M \rightarrow k_3^M$ and evaluate the amplitude for different helicity combinations of k_1 and k_2 .

First, choosing $s_1 = T$ and $s_2 = T$ gives

$$\begin{aligned} b_{12} &= -i \frac{m_1^2 - m_2^2}{m_3^2} \quad \text{if } m_3 \neq 0 \\ b_{12} &= 0 \quad \text{and } m_1 = m_2 \quad \text{if } m_3 = 0 \end{aligned} \quad (42)$$

Next, choosing $s_1 = T$, $s_2 = L$ gives

$$\begin{aligned} a_{23} &= -\frac{m_2^2 + m_3^2 - m_1^2}{2m_2m_3} \quad \text{if } m_2 \neq 0 \text{ and } m_3 \neq 0 \\ a_{23} &= 0 \quad \text{if } m_3 = 0 \end{aligned} \quad (43)$$

Finally, choosing $s_1 = L$, $s_2 = L$ fixes c_{123} to be

$$c_{123} = 0 \quad (44)$$

We have fixed b_{12}, a_{23}, c_{124} . Because of cyclic invariance, we can simply rotate the indices on mass parameters and fix all the remain parameters, which are summarized as following:

If $m_1 \neq 0, m_2 \neq 0, m_3 \neq 0$:

$$\begin{aligned} b_{12} &= -i \frac{m_1^2 - m_2^2}{m_3^2} & b_{23} &= -i \frac{m_2^2 - m_3^2}{m_1^2} & b_{31} &= -i \frac{m_3^2 - m_1^2}{m_2^2} \\ a_{12} &= -\frac{m_1^2 + m_2^2 - m_3^2}{2m_1m_2} & a_{23} &= -\frac{m_2^2 + m_3^2 - m_1^2}{2m_2m_3} & a_{31} &= -\frac{m_3^2 + m_1^2 - m_2^2}{2m_3m_1} \\ c_{123} &= 0 \end{aligned} \quad (45)$$

If $m_1 = 0, m_2 \neq 0, m_3 \neq 0$:

$$\begin{aligned} b_{12} &= i & b_{23} &= 0 & b_{31} &= -i \\ a_{12} &= 0 & a_{23} &= -1 & a_{31} &= 0 \\ c_{123} &= 0 \\ m_2 &= m_3 \end{aligned} \quad (46)$$

Notice the additional condition of $m_2 = m_3$ has not been derived in the literature before. It seems to imply a remnant symmetry between particle 2 and 3 in the case of particle 1 being massless.

If $m_1 = 0, m_2 = 0, m_3 \neq 0$, then there is no non-trivial solution. This is commonly known as Yang's theorem. Now we have derived the couplings the VVV amplitude with three massive particles (Eq.(45)), two massive particles (Eq.(46)) and one massive particle (no solution) respectively.

We can further obtain relations between the mass parameters by applying the strong massive-massless continuation on the couplings. This method is only usefull in the case of all three being massive, in which the couplings depend on masses. Taking $m_1/m_2/m_3$ to 0 and requiring the couplings in Eq.(45) to be finite, we deduce that $m_1 \propto m_2 \propto m_3$ at the leading order. We can define them as functions of a common varibale v and have

$$m_1 = c_1 v + \mathcal{O}(v^2) \quad m_2 = c_2 v + \mathcal{O}(v^2) \quad m_3 = c_3 v + \mathcal{O}(v^3) \quad (47)$$

Again, this is a signal that all particle masses have the same origin.

Before ending the VVV amplitude, we discuss two special cases of masses being equal.

First, let's take $m_1 = m_2 = m_3$, we then have $b_{12} = b_{23} = b_{31} = 0$ and $a_{12} = a_{23} = a_{31} = -\frac{1}{2}$. In this case the amplitude has $SO(3)$ symmetry for the three particles.

Second, we take two of the particles to have equal masses. Let's take $m_2 = m_3 \neq m_1$, we have

$$\begin{aligned} b_{12} &= i \left(1 - \frac{m_1^2}{m_3^2} \right) & b_{23} &= 0 & b_{31} &= -b_{12} \\ a_{12} &= -\frac{m_1}{2m_3} & a_{23} &= -\left(1 - \frac{m_1^2}{m_3^2} \right) & a_{31} &= -a_{12} \end{aligned} \quad (48)$$

In addition we have $m_2 = m_3$ automatically for $m_1 = 0$ in Eq.(46).

For $m_1 > m_3$, the above two solutions correspond to WWZ and WWA vertices in the SM respectively. This can be shown by defining the weak angle as $\cos \theta_W \equiv \frac{m_3}{m_1}$ and Eq.(48) as

$$\begin{aligned} b_{12} &= i \frac{\sin^2 \theta_W}{\cos^2 \theta_W} & b_{23} &= 0 & b_{31} &= -i \frac{\sin^2 \theta_W}{\cos^2 \theta_W} \\ a_{12} &= -\frac{1}{2 \cos \theta_W} & a_{23} &= -\frac{\cos 2\theta_W}{2 \cos \theta_W} & a_{31} &= -\frac{1}{2 \cos \theta_W} \end{aligned} \quad (49)$$

This is indeed the couplings of WWZ , including $\varphi\varphi V$ and φVV vertices.

V. MASSIVE 4-POINT AMPLITUDES

As discussed before, 3-point amplitudes are not generally viable by kinematics. The minimal number for n -point amplitudes to be kinematically allowed generically is $n = 4$. Thus in order to have consistent full theories, it is essential to construct up to 4-point amplitudes at least, which is what we are doing in this section. The approach of constructing 4-point amplitudes is explained in Sec.(III), we mainly make use of consistent factorization in addition to the two conditions of on-shell gauge symmetry and massive-massless continuation.

According to the principle of quantum mechanics, to construct a 4-point amplitude, we need to sum over all possible 3-point amplitudes with different intermediate states and 4-point contact terms. Moreover, bosonic symmetry (for the massless limit) constrains the number of types of particles (with different quantum numbers) for 3-point amplitudes. For SSV , the two scalar are required to be of different types; for VVV , the three vector bosons also must be of different types. As a result, we can then classify different theories according to the number of vector bosons n_V and the number of scalars n_S , which fix the possible 3-point amplitudes contributing to the final 4-point amplitude:

- $n_V < 3$
 - $n_V = 1, n_S \geq 2$: SSV
 - $n_V \leq 2, n_S = 1$: VVS
 - $n_V \leq 2, n_S \geq 2$: VVS, SSV
- $n_V \geq 3$
 - $n_V \geq 3, n_S = 0$: VVV
 - $n_V \geq 3, n_S = 1$: VVV, VVS
 - $n_V \geq 3, n_S \geq 2$: VVV, VVS and SSV

For every case with at least one scalar, we can also add SSS .

For $n_V < 2$, our strategy is to first construct the amplitudes from VVS , then add SSV . In this way we will be able to cover all three cases. For $n_V \geq 3$, we use a different strategy. We first try to construct 4-point amplitudes from VVV alone; then expand to incorporate VVS . We will also neglect SSV and leave it to future work.

We will go into the massive case directly. Massless 4-point amplitudes can be automatically included in the massive case, by taking masses to 0 and keeping the relevant degrees of freedom only. So they will not be discussed separately.

A. $n_V \leq 2$: 4-point Amplitudes from VVS and SSV

Here we construct 4-point amplitudes from VVS , SSV and SSS . Except for the trivial $SSSS$, there are two possible 4-point amplitudes $VVVV$ and $SSVV$, which we will construct one by one. After that, we will classify and discuss the underlying physics of the solutions.

1. $VVVV$ from VVS

To construct the amplitude of $VVVV$ we need only VVS . To simplify the analysis without losing generality, we focus on studying the case of $n_V = 1, n_S = 1$, so that there is only one independent VVS . Then we will generalize to two scalars in the end.

The total amplitude equals the sum of s,t,u channels and the contact term:

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c \quad (50)$$

$\mathcal{M}_s, \mathcal{M}_t$ and $\mathcal{M}_u$ are s, t, u channels from 3-point vertices. By consistent factorization we can obtain their forms when the intermediate states approach on-shell. Plugging in the 3-point amplitudes for VVS (Eq.(37 and 38)) and SSV (Eq.(29 and 31)), we get

$$\begin{aligned} \mathcal{M}_s(V_1 V_2 V_3 V_4) &\xrightarrow{p_{12}^2 \rightarrow m_S^2} \mathcal{M}(V_1 V_2 S_{12}) \frac{1}{p_{12}^2 - m_S^2} \mathcal{M}(-S_{12} V_3 V_4) \\ \mathcal{M}_t(V_1 V_2 V_3 V_4) &\xrightarrow{p_{13}^2 \rightarrow m_V^2} \sum_{s_{13}=\pm,0} \mathcal{M}(V_1 V_3 S_{13}) \frac{1}{p_{13}^2 - m_V^2} \mathcal{M}(-S_{13} V_2 V_4) \\ \mathcal{M}_u(V_1 V_2 V_3 V_4) &\xrightarrow{p_{14}^2 \rightarrow m_V^2} \sum_{s_{14}=\pm,0} \mathcal{M}(V_1 V_3 S_{14}) \frac{1}{p_{14}^2 - m_V^2} \mathcal{M}(-S_{14} V_2 V_3) \end{aligned} \quad (51)$$

Here $\mathcal{M}(VVS)$ should be understood to be multiplied by an overall coupling g_{VVS} . m_S and m_V are the masses of the scalar and the vector boson respectively. The two vector bosons in VVS are identical, therefore have the same mass. The off-shell continuation for scalar is trivial, so

amplitudes for s, t, u channels are simply

$$\begin{aligned}
\mathcal{M}_s(V_1 V_2 V_3 V_4) &= \mathcal{M}(V_1 V_2 S_{12}) \frac{1}{p_{12}^2 - m_S^2} \mathcal{M}(-S_{12} V_3 V_4) \\
\mathcal{M}_t(V_1 V_2 V_3 V_4) &= \mathcal{M}(V_1 V_3 S_{13}) \frac{1}{p_{13}^2 - m_V^2} \mathcal{M}(-S_{13} V_2 V_4) \\
\mathcal{M}_u(V_1 V_2 V_3 V_4) &= \mathcal{M}(V_1 V_3 S_{14}) \frac{1}{p_{14}^2 - m_V^2} \mathcal{M}(-S_{14} V_2 V_3)
\end{aligned} \tag{52}$$

To have the full amplitude we also need the contact term $\mathcal{M}_c$, which can be generally written as

$$\begin{aligned}
\mathcal{M}_c &= a_1 \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 + a_2 \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 + a_3 \epsilon_1 \cdot \epsilon_4 \epsilon_3 \cdot \epsilon_2 \\
&+ b_{11} \epsilon_1 \cdot \epsilon_2 \epsilon_3^4 \cdot \epsilon_4^4 + b_{12} \epsilon_1^4 \cdot \epsilon_2^4 \epsilon_3 \cdot \epsilon_4 \\
&+ b_{21} \epsilon_1 \cdot \epsilon_3 \epsilon_2^4 \cdot \epsilon_4^4 + b_{22} \epsilon_1^4 \cdot \epsilon_3^4 \epsilon_2 \cdot \epsilon_4 \\
&+ b_{31} \epsilon_1 \cdot \epsilon_4 \epsilon_2^4 \cdot \epsilon_3^4 + b_{32} \epsilon_1^4 \cdot \epsilon_4^4 \epsilon_2 \cdot \epsilon_3 \\
&+ c \epsilon_1^4 \epsilon_2^4 \epsilon_3^4 \epsilon_4^4
\end{aligned} \tag{53}$$

$a_1, a_2, a_3, b_{11}, b_{12}, b_{21}, b_{22}, b_{31}, b_{32}, c$ are dimensionless coefficients that are to be fixed by on-shell gauge symmetry for the total amplitude. c is simply the self-coupling of Goldstone bosons, which we also label as $\lambda_{\varphi^4} \equiv c$. So up to an overall coupling g_{VVS}^2 , there are 10 free parameter in total.

To fix the coefficients we apply on-shell gauge symmetry on ϵ_1^M . For s, t, u channels we get

$$\begin{aligned}
\mathcal{M}_s^{\epsilon_1^M \rightarrow k_1^M} &= \frac{i}{2} g_{VVS}^2 \epsilon_2^4 \left(m_V \epsilon_3 \cdot \epsilon_4 + \frac{i}{2} (2p_3 + p_4) \cdot \epsilon_4 \epsilon_3^4 + \frac{i}{2} (2p_4 + p_2) \cdot \epsilon_3 \epsilon_4^4 + \frac{m_S^2}{2m_V} \epsilon_3^4 \epsilon_4^4 \right) \\
\mathcal{M}_t^{\epsilon_1^M \rightarrow k_1^M} &= \frac{i}{2} g_{VVS}^2 \epsilon_3^4 \left(m_V \epsilon_2 \cdot \epsilon_4 + \frac{i}{2} (2p_2 + p_4) \cdot \epsilon_4 \epsilon_2^4 + \frac{i}{2} (2p_4 + p_3) \cdot \epsilon_2 \epsilon_4^4 + \frac{m_S^2}{2m_V} \epsilon_2^4 \epsilon_4^4 \right) \\
\mathcal{M}_u^{\epsilon_1^M \rightarrow k_1^M} &= \frac{i}{2} g_{VVS}^2 \epsilon_4^4 \left(m_V \epsilon_2 \cdot \epsilon_3 + \frac{i}{2} (2p_2 + p_3) \cdot \epsilon_3 \epsilon_2^4 + \frac{i}{2} (2p_3 + p_4) \cdot \epsilon_2 \epsilon_3^4 + \frac{m_S^2}{2m_V} \epsilon_2^4 \epsilon_3^4 \right)
\end{aligned} \tag{54}$$

For the contact channel we get

$$\begin{aligned}
\mathcal{M}_c^{\epsilon_1^M \rightarrow k_1^M} &= a_1 k_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 + a_2 \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 + a_3 \epsilon_1 \cdot \epsilon_4 \epsilon_3 \cdot \epsilon_2 \\
&+ b_{11} \epsilon_1 \cdot \epsilon_2 \epsilon_3^4 \cdot \epsilon_4^4 + b_{12} \epsilon_1^4 \cdot \epsilon_2^4 \epsilon_3 \cdot \epsilon_4 \\
&+ b_{21} \epsilon_1 \cdot \epsilon_3 \epsilon_2^4 \cdot \epsilon_4^4 + b_{22} \epsilon_1^4 \cdot \epsilon_3^4 \epsilon_2 \cdot \epsilon_4 \\
&+ b_{31} \epsilon_1 \cdot \epsilon_4 \epsilon_2^4 \cdot \epsilon_3^4 + b_{32} \epsilon_1^4 \cdot \epsilon_4^4 \epsilon_2 \cdot \epsilon_3 \\
&+ c \epsilon_1^4 \epsilon_2^4 \epsilon_3^4 \epsilon_4^4
\end{aligned} \tag{55}$$

Adding them up and we obtain the following results:

$$\begin{aligned}
a_1 &= a_2 = a_3 = 0 \\
g_{\varphi\varphi VV} = b_{11} &= b_{12} = b_{21} = b_{22} = b_{31} = b_{32} = \frac{1}{2}g_{VVS}^2 \\
c \equiv \lambda_{\varphi^4} &= \frac{3}{4}g_{VVS}^2 \frac{m_S^2}{m_V^2}
\end{aligned} \tag{56}$$

From Eq.(56), $g_{\varphi\varphi VV}$ (i.e. b_{ij}) and λ_{φ^4} are related by $\lambda_{\varphi^4} = \frac{3}{2} \frac{m_S^2}{m_V^2} g_{\varphi\varphi VV}$. Thus all 4-point couplings are fixed by g_{VVS} , an overall coupling.

If we have more than one scalar, the results can be generalized naturally. For example, let's assume we have two scalars S^a and S^b that couple with V , the solution of on-shell gauge symmetry becomes

$$\begin{aligned}
a_1 &= a_2 = a_3 = 0 \\
b_{11} &= b_{12} = b_{21} = b_{22} = b_{31} = b_{32} = \frac{1}{2} (g_{VVS^a}^2 + g_{VVS^b}^2) \\
c \equiv \lambda_{\varphi^4} &= \frac{3}{4m_V^2} (g_{VVS^a}^2 m_a^2 + g_{VVS^b}^2 m_b^2)
\end{aligned} \tag{57}$$

In order to completely fix the couplings, we need two parameters as input, which can choose as g_{VVS^a} and the ratio of $r_1 \equiv \frac{g_{VVS^a}}{g_{VVS^b}}$.

2. *SSVV from SSV and VVS*

Here we construct *SSVV*, which is from either *VVS* or *SSV*. *SSV* also already implies the existence of *VVS* and *SSS*. Our strategy is to first construct *SSVV* with *SSV*, *VVS* and *SSS*, then take $g_{VVS} = 0$ and $g_{SSV} = 0$ respectively to see the minimal structure for the amplitude.

Again, to avoid unnecessary complication, we focus on the case with minimal number of particles, i.e. with only one type of vector boson and two types of scalars. This implies one *SSV* amplitudes $\mathcal{M}(S^a S^b V)$; as well as two *VVS* amplitudes of $\mathcal{M}(VVS^a)$ and $\mathcal{M}(VVS^b)$. We focus on analyzing the example of $\mathcal{M}(S_1^a S_2^a V_3 V_4)$, as there is no essential difference in other 4-point amplitudes.

The total amplitude of $\mathcal{M}(S_1^a S_2^a V_3 V_4)$ can be constructed from s, t, u channels and the contact term:

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c \tag{58}$$

with

$$\begin{aligned}
\mathcal{M}_s(S_1^a S_2^a V_3 V_4) &= \sum_{i=a,b} \mathcal{M}(S_1^a S_2^a S_{12}^i) \frac{1}{p_{12}^2 - m_i^2} \mathcal{M}(S_{12}^i V_3 V_4) \\
\mathcal{M}_t(S_1^a S_2^a V_3 V_4) &= \mathcal{M}(S_1^a V_3 S_{13}^b) \frac{1}{p_{13}^2 - m_b^2} \mathcal{M}(S_{13}^b S_2^a V_4) \\
&\quad + \mathcal{M}(S_1^a V_3 V_{13}) \frac{M_{13}}{p_{13}^2 - m_V^2} \frac{-g_{M_{13} N_{13}}}{p_{13}^2 - m_V^2} \mathcal{M}^{N_{13}}(V_{13} S_2^a V_4^{s_4}) \\
\mathcal{M}_u(S_1^a S_2^a V_3 V_4) &= \mathcal{M}(S_1^a V_4 S_{14}^b) \frac{1}{p_{14}^2 - m_i^2} \mathcal{M}(S_{14}^b S_2^a V_3) \\
&\quad + \mathcal{M}(S_1^a V_4 V_{14}) \frac{M_{14}}{p_{14}^2 - m_V^2} \frac{-g_{M_{14} N_{14}}}{p_{14}^2 - m_V^2} \mathcal{M}^{N_{14}}(V_{14} S_2^a V_3^{s_3})
\end{aligned} \tag{59}$$

Notice we have to sum over all possible particles in the intermediate particles for every channel. However, there are only 3 of them that are independent, up to an overall scaling.

In addition, the contact term gives:

$$\mathcal{M}_c(S_1^a S_2^a V_3 V_4) = g_{S^a S^a V V} \epsilon_3 \cdot \epsilon_4 + \lambda_{SS\varphi^a \varphi^a} \epsilon_3^4 \epsilon_4^4 \tag{60}$$

Thus the full amplitude of $S_1^a S_2^a V_3 V_4$ has 7 free parameters: $\lambda_{S^a S^a S^a}, \lambda_{S^a S^a S^b}, g_{VV S^a}, g_{VV S^b}, g_{S^a S^b V}, g_{S^a S^a V V}$ and $\lambda_{SS\varphi^a \varphi^a}$. Those parameters are subject to additional constraints from on-shell gauge symmetry. Choosing replacing ϵ_3^M with p_3^M , we have:

$$\mathcal{M}_{\text{tot}}^{\epsilon_3^M \rightarrow p_3^M} = 0 \tag{61}$$

which gives the following solution:

$$\begin{aligned}
g_{S^a S^a V V} &= -2 \left(g_{S^a S^b V}^2 - \frac{g_{VV S^a}^2}{4} \right) \\
\frac{1}{2} (\lambda_{S^a S^a S^a} g_{VV S^a} + \lambda_{S^a S^a S^b} g_{VV S^b}) - \lambda_{S^a S^a \varphi \varphi} m_V &= g_{S^a S^b V}^2 \frac{m_a^2 - m_b^2}{m_V} - \frac{m_a^2}{2m_V} g_{VV S^a}^2
\end{aligned} \tag{62}$$

The first line comes from taking $s_4 = T$, the second line comes from taking $s_4 = L$. Exchange a and b , we further get

$$\begin{aligned}
g_{S^b S^b V V} &= -2 \left(g_{S^a S^b V}^2 - \frac{g_{VV S^b}^2}{4} \right) \\
\frac{1}{2} (\lambda_{S^a S^a S^a} g_{VV S^a} + \lambda_{S^a S^a S^b} g_{VV S^b}) - \lambda_{S^a S^a \varphi \varphi} m_V &= g_{S^a S^b V}^2 \frac{m_a^2 - m_b^2}{m_V} - \frac{m_a^2}{2m_V} g_{VV S^a}^2
\end{aligned} \tag{63}$$

For simplicity, let's analyze the solution in Eq.(62) by taking various limits.

First, if we reduce the number of scalars to one, which means $g_{S^a S^b V} = 0$ and $\lambda_{S^a S^a S^b} = 0$. Setting $S^a = S$, Eq.(62) becomes

$$\begin{aligned}
g_{SSVV} &= \frac{1}{2} g_{VV S}^2 \\
\lambda_{\varphi \varphi SS} &= g_{VV S} \frac{\lambda_{SSS}}{m_V} + \frac{m_S^2}{2m_V^2} g_{VV S}^2
\end{aligned} \tag{64}$$

In this case, the 3-point amplitudes that contribute to the full amplitude are VVS and SSS . There is only one free parameters λ_{SSS} after setting the overall coupling $g_{VVS} = 1$. Considering that λ_{SSSS} also has no constraints, all couplings can be fixed by particle masses except for the scalar self-couplings (λ_{SSS} and λ_{SSSS}). However, strong massive-massless continuation on Eq.(64) (second line) constrains the behavior of λ_{SSS} at the limit of $m_V \rightarrow 0$:

$$\lim_{m_V \rightarrow 0} \lambda_{SSS} = 0 \quad (65)$$

So λ_{SSS} doesn't exist in the massless limit. Also it indicates that λ_{SSS} has the same physical origin as the particle masses.

Next, we take $g_{VVS^a} = g_{VVS^b} = 0$ and $m_a = m_b$. Eq.(62) then reduces to

$$g_{S^a S^a VV} = -2g_{S^a S^b V}^2 \quad \lambda_{S^a S^a \varphi \varphi} = 0 \quad (66)$$

In this case, φ doesn't contribute to the amplitude. Physically it means that we should treat it as an auxiliary mode, instead of a physical, dynamic degree of freedom. Moreover, scalar couplings $\lambda_{S^a S^a S^a/S^b}$ decouple from the amplitude.

Finally, we analyze the general case in Eq.(62). The 3 conditions (Eq.(62) and the freedom of overall scaling) reduce the number of free parameters 7 from to 4, only enough to eliminate the two 4-point couplings and one 3-point coupling. In addition we treat $\lambda_{S^a S^a S^a}$ and $\lambda_{S^a S^a S^b}$ as input. We then have 5 relevant parameters constrained by 3 conditions. In summary, we still need $5 - 3 = 2$ free parameters. To fix the remaining two parameters, we can set the ratios:

$$r_1 \equiv \frac{g_{VVS^a}}{g_{VVS^b}} \quad r_2 \equiv \frac{g_{VVS^a}}{g_{S^a S^b V}} \quad (67)$$

as input, in addition to $\lambda_{S^a S^a S^a}$ and $\lambda_{S^a S^a S^b}$.

3. Underlying Theories and Models

Now having finished constructing 4-point amplitudes for $n_V < 3$, we then set to discuss the underlying physical theories and models for those constructions.

From one vector boson and one scalar, we can construct only one 3-point amplitude of VVS and three 4-point amplitudes of $SSSS$, $SSVV$ and $VVVV$. In constructing those amplitudes, we found all couplings are fixed by the masses, except for the scalar couplings of SSS and $SSSS$. Moreover, by applying strong massive-massless continuation on both 3-point and 4-point amplitudes, we found that when one of the masses goes to 0, other particles' masses and g_{VVS} also go to 0. This implies

a common origin for the all particle masses and couplings of mass dimension, a mark of S.S.B. However, the relations between the masses are generally not linear. Physically, the nonlinearity between masses and scalar couplings being free parameters means the Higgs potential is not limited to the terms of $\dim \leq 4$, therefore can't be fixed by $\dim \leq 4$ physics only. For example, we can add dim-6 SMEFT operators, or consider Coleman-Weinberg potential. In conclusion, the underlying constructible theory with one massive vector boson and one scalar, is massive QED with S.S.B., with the corresponding Lagrangian being composed of a vector field and a complex scalar field. Moreover, the specific shape of the Higgs potential cannot be determined.

From one vector boson and two scalars, we can construct 3-point amplitudes of SVV and SSV , as well as 4-point amplitudes of $SSSS$, $SSVV$ and $VVVV$. In constructing those amplitudes, we need two more parameters in Eq.(67) as input, apart from masses, to completely fix the couplings except for scalar self-couplings. The other aspects are the same as the case of one vector boson and one scalar. Thus the underlying theory is still massive gauge theory with S.S.B., but with more than one Higgs field. For a specific model, we can take the $U(1)$ version of two-Higgs doublet model as an example. The two parameters in Eq.(67) fix the angle between the two vevs(vacuum expectation values) (β) and the mixing angle of the two neutral and CP-even scalars (α). This case can also be basis for analyzing more general cases, such as the non-abelian cases with mixing between Higgs fields and etc.

Finally, there is a special limit for one vector boson and two scalars, in which any couplings involving the Goldstone φ are zero. In the cases we studied, there are two necessary conditions. The first is that the two scalars have equal masses. The second is all VVS couplings are 0. The underlying theory of this scenario is Stueckelberg theory, in which the vector boson has mass without S.S.B. φ is only a pure gauge that doesn't contribute to any amplitude.

B. $n_V \geq 3$: 4-point Amplitudes from VVV , VVS and SSV

For $n_V \geq 3$, we have possible 3-point amplitudes of VVV , VVS and SSV . The possible 4-point amplitudes constructed are $VVVV$, $VVSS$ and $SSSS$. $SSSS$ is not constrained by on-shell gauge symmetry, so we will skip it. Similarly, we also won't construct $VVSS$, which has been constructed in the case of $n_V = 1$. There is no new physical insights for $n_V \geq 3$. So in summary we will only construct $VVVV$ with VVV and VVS . Our strategy is to construct $VVVV$ from the 3-point amplitude VVV alone first (by assuming there is no scalar), to see if it's possible to be compatible with consistent conditions. If it doesn't work, we will then add VVS .

1. VVVV from VVV

Here we set to construct $VVVV$ from VVV and VVS . We start with $n_S = 0$, meaning no VVS first. We also put the “color” explicitly for the amplitudes. For VVV we have $\mathcal{M}(V_1^a V_2^b V_3^c) = f^{abc} \mathcal{M}(V_1 V_2 V_3)$, with f^{abc} to be anti-symmetric tensor. For simplicity, we set all vector bosons to have the same mass.

The condition of consistent factorization, combined by a natural off-shell continuation, ensures the total 4-point amplitude $\mathcal{M}(V^a V^b V^c V^d)$ equals the sum of s, t, u channel and the contact channel:

$$\mathcal{M}_{\text{tot}}(V_1^a V_2^b V_3^c V_4^d) = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c \quad (68)$$

with $\mathcal{M}_s, \mathcal{M}_t, \mathcal{M}_u$ being

$$\begin{aligned} \mathcal{M}_s(V_1^a V_2^b V_3^c V_4^d) &= \sum_e \mathcal{M}(V_1^a V_2^b V_{12}^e)^{M_{12}} \frac{-g_{M_{12} N_{12}}}{p_{12}^2 - m_V^2} \mathcal{M}^{N_{12}}(V_{12}^e V_3^c V_4^d) \\ \mathcal{M}_t(V_1^a V_2^b V_3^c V_4^d) &= \sum_e \mathcal{M}(V_1^a V_3^c V_{13}^e)^{M_{13}} \frac{-g_{M_{13} N_{13}}}{p_{13}^2 - m_V^2} \mathcal{M}^{N_{13}}(V_{13}^e V_2^b V_4^d) \end{aligned} \quad (69)$$

$$\mathcal{M}_u(V_1^a V_2^b V_3^c V_4^d) = \sum_e \mathcal{M}(V_1^a V_4^d V_{14}^e)^{M_{14}} \frac{-g_{M_{14} N_{14}}}{p_{14}^2 - m_V^2} \mathcal{M}^{N_{14}}(V_{14}^e V_2^b V_3^c) \quad (70)$$

The contact term $\mathcal{M}_c$ has no pole, therefore should be the linear combination of all possible terms from Lorentz contraction between $\epsilon_i^\mu, \epsilon_i^4$ with $i = 1, 2, 3, 4$.

$$\begin{aligned} \mathcal{M}_c &= a_{12,34} \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 + a_{13,24} \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 + a_{14,23} \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 \\ &+ b_{12,34} \epsilon_1 \cdot \epsilon_2 \epsilon_3^4 \epsilon_4^4 + b_{13,24} \epsilon_1 \cdot \epsilon_3 \epsilon_2^4 \epsilon_4^4 + b_{14,23} \epsilon_1 \cdot \epsilon_4 \epsilon_2^4 \epsilon_3^4 \\ &+ b_{34,12} \epsilon_3 \cdot \epsilon_4 \epsilon_1^4 \epsilon_2^4 + b_{24,13} \epsilon_2 \cdot \epsilon_4 \epsilon_1^4 \epsilon_3^4 + b_{23,14} \epsilon_2 \cdot \epsilon_3 \epsilon_1^4 \epsilon_4^4 \\ &+ \lambda_{\varphi^4} \epsilon_1^4 \epsilon_2^4 \epsilon_3^4 \epsilon_4^4 \end{aligned} \quad (71)$$

The coefficients are to be determined by the additional constraint of on-shell gauge symmetry.

We first apply on-shell gauge symmetry on $\mathcal{M}(V_1 V_2 V_{12})$ with one of the other legs, say $p_{12} = p_1 + p_2$, being off-shell. Choosing the particle for replacing ϵ^M with p^M to be p_1 , we then have:

$$\mathcal{M}_{\epsilon_1^M \rightarrow p_1^M}^{p_{12}^2 \neq m_V^2}(V_1 V_2 V_{12}) = \left(\epsilon_2 \cdot \epsilon_3 - \frac{1}{2} \epsilon_2^4 \epsilon_3^4 \right) (p_{12}^2 - m_V^2) + p_{12}^M \cdot \epsilon_M^{12*} \left(-\frac{i}{2} m_V \epsilon_2^4 - p_3 \cdot \epsilon_2 \right) \quad (72)$$

Making use of it, we can obtain the solution for $\mathcal{M}_{\text{tot}}(V_1^a V_2^b V_3^c V_4^d)_{\epsilon_1^M \rightarrow p_1^M} = 0$. However, because the solution is a little complicated. We simplify it by solving on-shell gauge symmetry with s_2, s_3, s_4 being transverse(T) and longitudinal(L) respectively.

First, taking $s_2 = T, s_3 = T, s_4 = T$. In this case only terms such as $\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4$ contribute to $\mathcal{M}_c$, which can be written as

$$\begin{aligned} & \mathcal{M}_c^{\epsilon_1 \rightarrow p_1}(s_2 = T, s_3 = T, s_4 = T) \\ &= a_{12,34} p_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 + a_{13,24} p_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 + a_{14,23} p_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 \end{aligned} \quad (73)$$

We evaluate $(\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u)_{\epsilon_1^M \rightarrow p_1^M}$ first, which gives

$$\begin{aligned} & (\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u)_{\epsilon_1^M \rightarrow p_1^M} \\ &= -(f^{abe} f^{cde} + f^{ace} f^{dbe} + f^{ade} f^{bce}) [\epsilon_2 \cdot \epsilon_3 (p_1 + 2p_2) \cdot \epsilon_4 - 2\epsilon_2 \cdot \epsilon_4 p_2 \cdot p_3 + 2\epsilon_3 \cdot \epsilon_4 (-p_1 - p_4) \cdot \epsilon_2] \\ & \quad + f^{abe} f^{cde} [p_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - p_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3] \\ & \quad + f^{ace} f^{dbe} [p_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 - p_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4] \\ & \quad + f^{ade} f^{bce} [p_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 - p_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4] \end{aligned} \quad (74)$$

comparing with Eq.(73), we find that there is no solution for on-shell gauge symmetry, unless there is an additional condition:

$$f^{abe} f^{cde} + f^{ace} f^{dbe} + f^{ade} f^{bce} = 0 \quad (75)$$

in which case we have the solution for coefficients:

$$a_{12,34} = f^{ace} f^{bde} - f^{ade} f^{bce}; \quad a_{13,24} = f^{ade} f^{bce} - f^{abe} f^{cde}; \quad a_{14,23} = f^{abe} f^{cde} - f^{ace} f^{dbe} \quad (76)$$

we then obtain $\mathcal{M}_c$ with all particles being transverse:

$$\begin{aligned} \mathcal{M}_c &= -f^{abe} f^{cde} [\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3] \\ & \quad - f^{ace} f^{dbe} [\epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 - \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4] \\ & \quad - f^{ade} f^{bce} [\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4] \end{aligned} \quad (77)$$

Eq.(75) is simply Jacobi identity, indicating that the involved vector bosons with different quantum numbers belong to the same Lie group. Notice all transverse cases reduce to massless vector scattering directly when $m_V \rightarrow 0$, thus we arrive at the conclusion that the only consistent massless theory with vector boson scattering is Yang-Mills theory.

Then we continue to apply on-shell gauge symmetry with one or several particles being longitudinal. We found, however, there is non-trivial solution for on-shell gauge symmetry. In particular, the solutions for $b_{ij,kl}$ by applying O.G.S on $s_2 = L, s_3 = T, s_4 = T$ and $s_2 = L, s_3 = L, s_4 = T$ contradict with each other.

Choosing $s_2 = L$, $s_3 = T$ and $s_4 = T$, on-shell gauge symmetry is satisfied if the coefficients in Eq.(71) satisfy the following conditions:

$$b_{34,12} = 0; \quad b_{24,13} = 0; \quad b_{23,14} = 0 \quad (78)$$

which fixes the $VV\varphi\varphi$ vertices. By permutation symmetry we also obtain all $b_{ij,kl}$ coefficients:

$$b_{12,34} = b_{34,12} = 0; \quad b_{13,24} = b_{24,13} = 0; \quad b_{14,23} = b_{23,14} = 0 \quad (79)$$

On the other hand, the solution of $b_{ij,kl}$ for O.G.S by choosing $s_2 = L, s_3 = L, s_4 = T$ is

$$b_{14,23} = -f^{abe} f^{cde} = f^{ace} f^{dbe} \quad b_{34,12} = 0 \quad b_{24,13} = 0 \quad (80)$$

It's impossible for Eq.(79) and Eq.(80) to be correct at the same time, except for the trivial solution of $f^{abe} f^{cde} = f^{ace} f^{dbe} = 0$. Therefore, it's impossible to satisfy on-shell gauge symmetry for $\mathcal{M}(V^a V^b V^c V^d)$ with vector bosons only.

For the case of elementary particles, the only solution is to add additional degrees of freedom in the theory. For $\mathcal{M}(V^a V^b V^c V^d)$, the only possibility is to add additional scalars, and consequently the 3-point amplitude $\mathcal{M}(VVS)$. For simplicity, we only add one scalar to the theory. This gives 3 additional s, t, u channels contributing to $\mathcal{M}(V^a V^b V^c V^d)$ with the scalar to be the intermediate particle. The total amplitude now becomes

$$\mathcal{M}_{\text{tot}}(V_1^a V_2^b V_3^c V_4^d) = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_s^S + \mathcal{M}_t^S + \mathcal{M}_u^S + \mathcal{M}_c \quad (81)$$

The scalar channels' contribution to on-shell gauge symmetry was obtained in Eq.(54).

Applying on-shell gauge symmetry again for $s_2 = L, s_3 = T, s_4 = T$, the related coefficients are modified to

$$b_{34,12} = \frac{1}{2} g_{V^a V^b S} g_{V^c V^d S} \quad (82)$$

Using permutation, we can obtain all $b_{ij,kl}$ coefficients:

$$\begin{aligned} b_{12,34} &= b_{34,12} = \frac{1}{2} g_{V^a V^b S} g_{V^c V^d S} \\ b_{13,24} &= b_{24,13} = \frac{1}{2} g_{V^a V^c S} g_{V^b V^d S}; \\ b_{14,23} &= b_{23,14} = \frac{1}{2} g_{V^a V^d S} g_{V^b V^c S} \end{aligned} \quad (83)$$

We then apply on-shell gauge symmetry to $s_2 = L, s_3 = L, s_4 = T$, which gives the following

constraint on VVS vertices:

$$\begin{aligned}
0 = & -\frac{1}{4}f^{abe}f^{cde}\epsilon_2^4\epsilon_3^4(2p_2+p_4)\cdot\epsilon_4 + \frac{1}{4}f^{ace}f^{dbe}\epsilon_2^4\epsilon_3^4(2p_3+p_4)\cdot\epsilon_4 \\
& + b_{34,12}\epsilon_3\cdot\epsilon_4(-im_V)\epsilon_2^4 + b_{24,13}\epsilon_3\cdot\epsilon_4(-im_V)\epsilon_3^4 + b_{14,23}p_1\cdot\epsilon_4\epsilon_2^4\epsilon_3^4 \\
& + g_{V^aV^bS}g_{V^cV^dS}[\frac{i}{2}m_V\epsilon_2^4(\epsilon_3\cdot\epsilon_4) + \frac{1}{4}(2p_3+p_4)\cdot\epsilon_4\epsilon_2^4\epsilon_3^4] \\
& + g_{V^aV^cS}g_{V^dV^bS}[\frac{i}{2}m_V\epsilon_3^4(\epsilon_2\cdot\epsilon_4) + \frac{1}{4}(p_2+p_4)\cdot\epsilon_4\epsilon_2^4\epsilon_3^4]
\end{aligned} \tag{84}$$

Plugging in Eq.(83), we obtain the following relations between g_{VVS} and f^{abc} :

$$\begin{aligned}
f^{abe}f^{cde} &= g_{V^aV^cS}g_{V^bV^dS} - g_{V^aV^dS}g_{V^bV^cS} \\
f^{ace}f^{dbe} &= -g_{V^aV^bS}g_{V^cV^dS} + g_{V^aV^dS}g_{V^bV^cS}
\end{aligned} \tag{85}$$

$g_{V^iV^jS}$ is symmetric with $i \leftrightarrow j$ for equal vector boson masses, e.g. $g_{V^bV^aS} = g_{V^aV^bS}$. We can also apply O.G.S on $s_2 = L, s_3 = T, s_4 = L$ and $s_2 = T, s_3 = L, s_4 = L$, but there is no new independent results. Eq.(85) gives us the only constraints between the group structure coefficients f^{abc} and the VVS couplings.

Finally, we choose $s_2 = L, s_3 = L, s_4 = L$ and apply on-shell gauge symmetry on p_1 , which fixes λ_{φ^4} to be

$$\lambda_{\varphi^4} = \frac{1}{4}\frac{m_S^2}{m_V^2}(g_{V^aV^bS}g_{V^cV^dS} + g_{V^aV^cS}g_{V^dV^bS} + g_{V^aV^dS}g_{V^bV^cS}) \tag{86}$$

So we have successfully constructed massive $VVVV$ from VVV , with all vector boson masses being equal. We proved a scalar is needed for the construction to be consistent. We believe this conclusion holds for general cases of vector boson masses. All couplings are constrained by particle masses and the group structure constants f^{abc} in Eq.(83, 85 and 86).

Those solutions can be further simplified if we introduce a mild and reasonable assumption. Because all vector bosons have the same mass and there is only one scalar, it's reasonable to assume they all equal, i.e. $g \equiv g_{V^aV^bS} = g_{V^cV^dS} = g_{V^aV^cS} = g_{V^bV^dS} = g_{V^aV^dS} = g_{V^bV^cS}$. However, in this case, Eq.(85) becomes $f^{abe}f^{cde} = f^{ace}f^{dbe} = f^{ade}f^{bce} = 0$, reducing the solution to the abelian case. Nevertheless, there is a way to save the situation by making one of the VVS coupling equal to 0. For example, we can take

$$\begin{aligned}
g_{V^aV^bS} &= 0 \quad \text{or} \quad g_{V^cV^dS} = 0 \\
g_{VVS} &\equiv g_{V^aV^cS} = g_{V^bV^dS} = g_{V^aV^dS} = g_{V^bV^cS}
\end{aligned} \tag{87}$$

The condition eliminates one scalar mediated channel. Eq.(85) then reduces to

$$f^{abe}f^{cde} = 0 \quad g_{VVS}^2 = f^{ace}f^{dbe} = f^{ade}f^{bce} \tag{88}$$

We see that one of vector boson mediated channel is also eliminated. As a result, the $VVVV$ amplitude has only s and t channels that can be organized as planar diagrams. This can be checked in simplest case of $SU(2)$ group. In the $W^\pm$ basis, the diagrams correspond to exactly Eq.(88) with $f^{abc} = \epsilon^{abc}$ and $a, b, c = \pm, 3$. It's also interesting to point out that Eq.(88) is also the solution for

$$g_{V^a V^b S} g_{V^c V^d S} = g_{V^a V^d S} g_{V^b V^c S} = 0 \quad g_{VVS} \equiv g_{V^a V^c S} = g_{V^b V^d S} \quad (89)$$

In other words, there is only one scalar mediated channel while all relevant VVS couplings are equal.

2. The Underlying Theory with Amplitudes from VVV

For the number of vector bosons equal to or larger than 3, we have the 3-point VVV , from which we try to construct $VVVV$. For all particles being transverse, it leads to Jacobi identity for the couplings of VVV , thus establishing the classic result of massless Yang-Mills theory. For other polarizations, we found that an additional scalar is needed for self-consistency of the amplitude as required on-shell gauge symmetry. After adding the scalar and the subsequent 3-point amplitude VVS , we are able to construct $VVVV$ with all 4-point couplings and VVS couplings fixed by masses and group structure constants (Eq.(83), Eq.(85) and Eq.(86)). Those results, combined with the results of VVV in Sec.(IV B), are enough for us to conclude that, for $VVVV$ with the number of vector bosons larger than or equal to 3 and all particles are elementary, the only possible underlying theory is Yang-Mills theory with S.S.B. In particular, we can conclude that Stukelburge theory cannot be applied on theory with the vertex of VVV , i.e. non-abelian gauge theory. The reason is that the Goldstone mode φ always contributes non-trivially to amplitudes in this case. Therefore, it's impossible to have Stukelburge theory, in which φ is simply pure gauge.

Furthermore, we also discuss the solutions under the reasonable assumption of all VVS couplings being equal. We found that non-trivial solutions require one of the s, t, u channels of both one scalar mediated and vector boson mediated amplitudes can not exist. The couplings are fixed as in Eq.(87) and Eq.(88). The corresponding amplitudes then have only planar diagrams. A specific example is $SU(2)$ group in the basis of $W^\pm$.

VI. CONCLUSION

In this paper we set to construct massive amplitudes with elementary particles. We focus on amplitudes with vector bosons and scalars up to 4-point. For this purpose, we propose two consistent conditions for the amplitudes: on-shell gauge symmetry and strong massive-massless continuation(Item(I)). The latter condition is to ensure the particles are elementary by imposing on the analytic property of amplitudes as functions of masses. The former condition is simply massive Ward identity that becomes Goldstone equivalence theorem in high energy limit. However, we argued in the paper that it can be derived from Lorentz symmetry by noting that a vector boson can naturally mix with a scalar. By requiring that the mass spectrum to be degenerate after mixing, we naturally obtain on-shell gauge symmetry and the correct number of degrees of freedom.

After proposing the two consistent conditions, we continue to construct 3-point and 4-point amplitudes. To construct 3-point, we apply the principle of massive-massless continuation on amplitudes, concluding that massive amplitudes can be written as linear combinations of the massless counter parts(Eq.(14)). So the basic strategy is to construct the massless amplitudes first, then apply the two consistent conditions to construct massive amplitudes, which fix the coefficients and their relations. With this strategy, we successfully construct all possible 3-point massive amplitudes: SSV , SVV and VVV . All couplings, up to an overall scaling, are fixed in terms of masses. For VVV we also discuss the solution with different mass combinations and reconstruct the VVV amplitudes in the SM. Finally, when one of the mass goes to 0, other masses and dimensional couplings also must go to 0, indicating common physical origin.

To construct 4-point amplitudes, we make use of consistent factorization that the 4-point amplitude factorizes into products of 3-point amplitudes when either of s, t, u channels goes on-shell for one of the particles. Starting from 4-point amplitudes when the intermediate states are on-shell, we then extend the propagator off-shell continuously to obtain the amplitudes at general kinematics(Eq.(18)). We classify our construction according to the number of vector bosons n_V : for $n_V < 3$, we construct $VVSS$ and $VVVV$. With only one scalar we found all couplings are fixed in terms of masses, except for scalar self-couplings. The underlying theory is massive scalar QED with S.S.B. With two scalars, we also need the ratio of $\frac{g_{VVSS^a}}{g_{VVS^b}}$ and $\frac{g_{VVSS^a}}{g_{S^a S^b V}}$. Physically, the underlying theory for one vector boson and two scalars are the $U(1)$ version of 2HDM. The two ratios are the mixing angles between two VEVs and between two neutral, CP-even scalars. There is also a special case when the scalars have equal masses. In this case the ‘‘Goldstone’’ mode φ doesn’t contribute the amplitude. This corresponds to Stueckelberg theory, in which there is no

need for symmetry breaking for a vector boson to have mass.

For $n_V \leq 3$, we construct $VVVV$ from VVV . We found that in order to satisfy on-shell gauge symmetry, additional scalars must be added. After adding an additional scalar and vertex VVS , the amplitude of $VVVV$ is successfully constructed. All 4-point couplings except scalar self-couplings are then fixed. VVS couplings are further constrained by the group structure constants in Eq.(85). We therefore conclude the only possible underlying theory for $VVVV$ from VVS is Yang-Mills theory with S.S.B., when all particles are elementary. We also discuss the solutions in some special conditions such as the case of all VVS couplings being equal, the $SU(2)$ group and etc.

In comparison with other similar works in the literature, our approach of constructing amplitudes has a few advantages. First of all, our method has manifest gauge symmetry at the level of amplitudes by making Goldstone bosons part of physical spectrum. Second, we essentially derive on-shell gauge symmetry from Lorentz symmetry and some reasonable assumptions of mixing between vector boson and scalar, thus putting our approach on a firm theoretical foundation. Third, the condition of strong massive-massless continuation allows us to distinguish theories with elementary particles from composite particles through analytic properties of amplitudes. Finally, our results are also more complete compared with other approaches, as demonstrated by the conclusions of Stueckelberg theory, particles masses have the same physical origin, Higgs self-couplings are modified by $\dim > 4$ terms of Higgs potential and etc.

Our results open the door to many new directions, while also leaving a few questions to be answered. To name a few topics for future research: 4-point amplitudes with fermions remain to be constructed; for $VVVV$ from VVV , we have only studied the case of equal masses. The amplitude with general masses are still yet to be fully constructed. Furthermore, it's also interesting to go beyond 4-point amplitudes to see what can be learned.

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Appendix A: Derivation of Vector-Scalar Kinematic Lagrangian

We will derive the kinematic Lagrangian terms for $V - \varphi$ mixing, from the assumptions of both V and φ has mass m_V , and physical condition $\partial^\mu V_\mu = m_V \varphi$.

The kinematic Lagrangian related to the vector boson is

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \frac{1}{2}m_V^2 V_\mu V^\mu + \mathcal{L}_{V-\varphi}$$

$\mathcal{L}_{V-\varphi}$ is unknown. Euler-Lagrangian equation gives

$$-(\partial^2 + m_V^2)V_\mu + \partial_\mu(\partial \cdot V) + (V - \varphi \text{ terms}) = 0$$

Applying the physical condition $\partial^\mu V_\mu = m\varphi$ fixes the $V - \varphi$ term above to be $-m_V \partial_\mu \varphi$, meaning $\mathcal{L}_{V-\varphi}$ is

$$\mathcal{L}_{V-\varphi} = m_V V_\mu \partial^\mu \varphi$$

In similar way we can derive the Lagrangian for φ , of which the general form is

$$\mathcal{L}_\varphi = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m_\varphi^2 \varphi^2$$

The equation of motion for φ is then

$$(\partial^2 + m_\varphi^2)\varphi + m_V \partial^\mu V_\mu = 0$$

Again plugging physical condition eliminates V_μ and gives

$$(\partial^2 + m_\varphi^2 - m_V^2)\varphi = 0$$

This fixes $m_\varphi = 0$.

To sum up, the kinematic Lagrangian for V and φ with mixing is ‘

$$\begin{aligned}\mathcal{L}_{V^2} &= -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \frac{1}{2}m_V^2 V_\mu^2 \\ \mathcal{L}_{V-\varphi} &= m_V V_\mu \partial^\mu \varphi \\ \mathcal{L}_{\varphi^2} &= \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi\end{aligned}\tag{A1}$$

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