

POSITIVE HERMITIAN CURVATURE FLOW ON 2-STEP NILPOTENT LIE GROUPS

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ABSTRACT. We study the positive Hermitian curvature flow for left-invariant metrics on 2-step nilpotent Lie groups with a left-invariant complex structure J . We describe the long-time behavior of the flow under the assumption that $J[\mathfrak{g}, \mathfrak{g}]$ is contained in the center of $\mathfrak{g}$. We show that under our assumption the flow g_t exists for all positive t and $(G, (1+t)^{-1}g_t)$ converges, in the Cheeger-Gromov topology, to a 2-step nilpotent Lie group with a non flat semi-algebraic soliton. Moreover, we prove that, in our class of Lie groups, there exists at most one semi-algebraic soliton solution, up to homothety. Similar results were proved by M. Pujia and J. Stanfield for nilpotent complex Lie groups [21, 23]. In the last part of the paper we study the Hermitian curvature flow for the same class of Lie groups.

1. INTRODUCTION

In [24], J. Streets and G. Tian introduced a new family of geometric flows called *Hermitian curvature flows* (HCFs) which generalize the *Kähler-Ricci flow* to the Hermitian setting.

Let (M, J) be a complex manifold, then the evolution of a Hermitian metric g on M under a Hermitian curvature flow is the following

$$\partial_t g_t = -(S(g_t) - Q(g_t)), \quad g_t|_{t=0} = g,$$

where $S(g)$ denotes the second Chern-Ricci curvature tensor of g and $Q(g)$ is a $(1,1)$ -symmetric tensor which is quadratic in the torsion T of the Chern connection of g . We recall that $S(g)$ is the $(1,1)$ -symmetric tensor defined by

$$S_{j\bar{k}} = g^{\bar{r}s} \Omega_{s\bar{r}j\bar{k}},$$

where Ω is the curvature tensor of the Chern connection.

Since the tensor Q does not affect the parabolicity of the flow, it can be chosen to preserve different geometric properties. Originally, in [24], the tensor Q was chosen to obtain, in the compact case, a gradient flow stable near Kähler-Einstein metrics with non-positive scalar curvature.

In [25], the tensor Q is chosen in order to preserve the *pluriclosed* condition $\partial\bar{\partial}\omega = 0$, where ω denotes the fundamental form of the Hermitian metric g . In [26], Y. Ustinovskiy considered the *positive Hermitian curvature flow* (HCF₊)

$$(1) \quad \partial_t g_t = -\Theta(g_t), \quad g_t|_{t=0} = g,$$

where

$$(2) \quad \Theta(g) \doteq S(g) + \frac{1}{2}Q^2(g), \quad Q_{j\bar{k}}^2 \doteq g^{\bar{p}q} g^{\bar{r}s} T_{sq\bar{k}} T_{\bar{r}p j},$$

and $T_{j\bar{s}\bar{p}} \doteq g_{l\bar{p}} T_{js}^l$. The coefficients T_{js}^l denote the components of the torsion of the Chern connection of g . This flow preserves the Griffiths-positivity and the dual Nakano-positivity of the tangent bundle.

One of the main motivation to study the positive Hermitian curvature flow is that it is conformally equivalent to the *type IIB flow* introduced by D. H. Phong, S. Picard and Z. Zhang in [18]. We recall that,

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if we suppose that (M, J) is equipped also with a holomorphic volume form Ψ , the *type IIB* flow is the geometric flow of Hermitian metrics

$$(3) \quad \partial_t(\|\Psi\|_{\omega_t} \omega_t^{n-1}) = i\partial\bar{\partial}\omega_t^{n-2}, \quad \omega_t|_{t=0} = \omega.$$

This flow preserves the conformally balanced condition

$$d(\|\Psi\|_{\omega_t} \omega_t^{n-1}) = 0$$

and under this assumption its stationary points are Calabi-Yau metrics. On complex 3-folds the flow is a special case of the *anomaly flow* [19, 20]. By [9], if ω_t is a conformally balanced solution to the anomaly flow, then the Hermitian metric induced by

$$\eta_t = \|\Psi\|_{\omega_t} \omega_t$$

solves the HCF_+ .

The purpose of the present paper is to study the behavior of the positive Hermitian curvature flow (1) on Lie groups equipped with a left-invariant complex structure. In this setting, the flow preserves the property of the initial metric to be left-invariant and reduces to an ODE. Our first main result is the following

Theorem 1.1. *Let (G, J, g_0) be a simply-connected, 2-step nilpotent Lie group equipped with a left-invariant Hermitian structure (J, g_0) . Assume that the Lie algebra $(\mathfrak{g}, \mu)$ of G is such that $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$. Then, the HCF_+ starting from g_0 , has a long-time solution g_t such that $(G, (1+t)^{-1}g_t)$ converges to a non-flat semi-algebraic soliton $(\bar{G}, \bar{h})$ as $t \rightarrow \infty$ in the Cheeger-Gromov topology.*

The theorem is proved by using the *bracket flow technique* introduced by J. Lauret in [16] to study the Ricci flow on Lie groups. The technique allows us to regard the flow as a flow in the space of brackets instead of in the space of the inner products. This is a convenient point of view for several reasons (especially for the study of the limits in Cheeger-Gromov topology) and it was adopted in many papers (see e.g. [3, 8, 11, 12, 14, 21, 23] and the references therein).

We mention that, to prove Theorem 1.1, we show that $\Theta(g)$ can be viewed as a moment map for the action of $GL(\xi, J)$, where ξ is the center of the Lie algebra of (G, J) , on the vector space of Lie brackets $\mathfrak{N}$ defined in (18) (see Section 3). We recall that, a result similar to Theorem 1.1 was proved in [21, 23] for complex nilpotent Lie groups.

Notice that the technical assumption $J\mu(\mathfrak{g}, \mathfrak{g})$ contained in the center of $\mathfrak{g}$ in Theorem 1.1 is in particular satisfied if J preserves the center of $\mathfrak{g}$, but it is in fact more general (see e.g. [22, Example 3.9]). Moreover, also the Kodaira-Thurston surface, equipped with the complex structure as in [10, Example 4.5], satisfies this assumption. We mention that, in [4], the complex structures that satisfy $J\mu(\mathfrak{g}, \mathfrak{g})$ contained in the center of $\mathfrak{g}$ are called 2-step nilpotent complex structure, in the sense of [7].

We recall that a sequence of homogeneous manifolds (M_k, g_k) converges to a homogeneous manifold $(\bar{M}, \bar{g})$ in the Cheeger-Gromov topology if there exist a sequence $\Omega_k \subseteq \bar{M}$ of open neighborhoods of a base point $p \in \bar{M}$ and a sequence of embeddings $\psi_k : \Omega_k \rightarrow M_k$ such that $\psi_k^* g_{t_k}$ converge to $\bar{g}$ smoothly as $k \rightarrow \infty$ and Ω_k eventually contains every compact subset of $\bar{M}$. We also recall that, a Hermitian metric g is called a HCF_+ soliton if it satisfies

$$(4) \quad \Theta(g) = cg + \mathcal{L}_Z g,$$

where $c \in \mathbb{R}$, $\mathcal{L}$ denotes the Lie derivative, Z is a complete holomorphic vector field and Θ is defined as in (2). A soliton is called *shrinking* if $c > 0$, *steady* if $c = 0$ or *expanding* if $c < 0$. The HCF_+ starting from a soliton evolves as

$$g_t = k(t) \varphi_t^* g,$$

where $k(t) > 0$ and $\varphi_t : G \rightarrow G$ are biholomorphisms. Furthermore, if φ_t is a family of Lie group automorphisms and g is left-invariant, then we call g a *semi-algebraic HCF_+ soliton*.

By using [9], Theorem 1.1 can be applied in order to study the type IIB flow on Lie groups. We have the following

Corollary 1.2. *Let (G, J, Ψ) be a simply-connected, 2-step nilpotent Lie group equipped with a left-invariant complex structure J and a nowhere vanishing, left-invariant, holomorphic $(n, 0)$ -form Ψ . Assume that the Lie algebra $(\mathfrak{g}, \mu)$ of G is such that $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$, then any left-invariant, balanced solution ω_t to*

$$(5) \quad \partial_t(\|\Psi\|_{\omega_t} \omega_t^{n-1}) = i\partial\bar{\partial}\omega_t^{n-2}$$

is immortal. Moreover $(G, (1+t)^{-1}\omega_t)$ converges to a non-flat, left-invariant, semi-algebraic soliton $(\bar{G}, \bar{\omega})$ in the Cheeger-Gromov topology.

In the present paper, we say that ω is a *soliton to the type IIB flow* if the metric induced by $\|\Psi\|_{\omega}$ is a HCF_+ soliton. It is simple to observe that ω is a soliton to the type IIB equation if and only if

$$\Theta(\|\Psi\|_{\omega}g) = (c\|\Psi\|_{\omega} + \mathcal{L}_Z(\|\Psi\|_{\omega}))g + \|\Psi\|_{\omega}\mathcal{L}_Zg,$$

where $c \in \mathbb{R}$, Z is a complete holomorphic vector field and g is the metric induced by ω . We say that a soliton to the type IIB flow is semi-algebraic if the metric induced by $\|\Psi\|_{\omega}\omega$ is a semi-algebraic HCF_+ soliton (see Definition 2.4).

Note that, by [5, Theorem 2.7], a nilmanifold, i.e., a compact quotient of a connected, simply connected nilpotent Lie group by a lattice, equipped with a left-invariant complex structure, always admits a nowhere vanishing, left-invariant, holomorphic $(n, 0)$ -form.

The last result of the paper characterizes semi-algebraic HCF_+ solitons.

Theorem 1.3. *Let (G, J) be a simply-connected, non abelian, 2-step nilpotent Lie group equipped with a left-invariant complex structure J . Assume that the Lie algebra $(\mathfrak{g}, \mu)$ of G is such that $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$, then, every semi-algebraic HCF_+ soliton is expanding and unique up to homotheties.*

Let us mention that by Corollary 4.2 every semi-algebraic soliton to the HCF_+ is algebraic (see Section 2.2). Hence, the limit solitons in Theorem 1.1 and Corollary 1.2 are in fact algebraic.

This paper is organized as follows. In Section 2 we establish the notation used throughout this article and compute the tensor Θ defined in (2) in terms of the structure constants of the Lie algebra. Furthermore, we briefly recall the bracket flow technique. In Section 3, we prove Theorem 1.1. Meanwhile, in Section 4 we prove Theorem 1.3 and we provide some examples. Finally, in Section 5, we study the Hermitian curvature flow considered in [24] under the assumption that $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$.

Notation and conventions. Throughout this paper, we will adopt the Einstein summation convention for sums over repeated indices unless otherwise stated.

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2. PRELIMINARIES

Given a Hermitian manifold (M, J, g) , we denote by $T_{\mathbb{C}}M \doteq TM \otimes \mathbb{C}$ its complexified tangent bundle. $T_{\mathbb{C}}M$ has the natural splitting $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ and the bundle $\Lambda_{\mathbb{C}}^r M$ of complex r -forms splits consequently as

$$\Lambda_{\mathbb{C}}^r M = \bigoplus_{p+q=r} \Lambda^{p,q} M,$$

where $\Lambda^{p,q} M \doteq \Lambda^p(T^{1,0}M)^* \otimes \Lambda^q(T^{0,1}M)^*$. The Hermitian metric g specifies a canonical connection, called the *Chern connection*, which is the unique affine connection ∇ which preserves both g and J and whose $(1, 1)$ -part of the torsion tensor T vanishes.

Let us consider a Lie group G with a left-invariant Hermitian structure (J, g) . We denote by $(\mathfrak{g}, \mu)$ the Lie algebra of G . Let $\{Z_1, \dots, Z_n\}$ be a g -unitary, left-invariant $(1, 0)$ -frame on G . The Christoffel symbols of the Chern connection can be easily described in terms of the components of μ .

From $T(Z_i, Z_{\bar{j}}) = 0$ we deduce

$$\nabla_{Z_i} Z_{\bar{j}} = \mu(Z_i, Z_{\bar{j}})^{1,0}, \quad \nabla_{Z_{\bar{j}}} Z_i = \mu(Z_i, Z_{\bar{j}})^{0,1}$$

which imply

$$\Gamma_{i\bar{j}}^{\bar{s}} = \mu_{i\bar{j}}^{\bar{s}}, \quad \Gamma_{\bar{j}i}^s = \mu_{\bar{j}i}^s.$$

Furthermore, since

$$g(\nabla_{Z_i} Z_j, Z_{\bar{l}}) = -g(Z_j, \nabla_{Z_i} Z_{\bar{l}}) = -g(Z_j, \mu(Z_i, Z_{\bar{l}})),$$

we infer

$$\Gamma_{ij}^l = -\mu_{i\bar{l}}^{\bar{j}}.$$

Next, we compute the components of the second Chern-Ricci curvature tensor of g in terms of the components of μ . Let Ω denotes the curvature tensor of the Chern connection ∇ . We have that

$$\begin{aligned} \Omega_{i\bar{l}j\bar{k}} &= -g(\nabla_{Z_{\bar{l}}} Z_j, \nabla_{Z_i} Z_{\bar{k}}) + g(\nabla_{Z_i} Z_j, \nabla_{Z_{\bar{l}}} Z_{\bar{k}}) - g(\nabla_{(\mu_{i\bar{l}}^r Z_r + \mu_{i\bar{l}}^{\bar{r}} Z_{\bar{r}})} Z_j, Z_{\bar{k}}) \\ &= -\mu_{i\bar{l}j}^r \mu_{i\bar{k}}^{\bar{r}} + \mu_{i\bar{r}}^{\bar{j}} \mu_{l\bar{r}}^k + \mu_{i\bar{l}}^r \mu_{r\bar{k}}^{\bar{j}} - \mu_{i\bar{l}}^{\bar{r}} \mu_{r\bar{j}}^k, \end{aligned}$$

and, consequently,

$$(6) \quad S_{j\bar{k}} = -\mu_{s\bar{j}}^r \mu_{s\bar{k}}^{\bar{r}} + \mu_{s\bar{r}}^{\bar{j}} \mu_{sr}^k + \mu_{s\bar{s}}^r \mu_{r\bar{k}}^{\bar{j}} - \mu_{s\bar{s}}^{\bar{r}} \mu_{r\bar{j}}^k.$$

The $(2, 0)$ -component of the torsion tensor is given by

$$T(Z_r, Z_s) = \nabla_{Z_r} Z_s - \nabla_{Z_s} Z_r - \mu(Z_r, Z_s),$$

so, by using the relations between the Christoffel symbols and the structure constants expressed above, we get

$$T_{rs}^v = -\mu_{r\bar{v}}^{\bar{s}} + \mu_{s\bar{v}}^{\bar{r}} - \mu_{rs}^v,$$

and by contracting with the metric, we get

$$T_{rs\bar{k}} = -\mu_{r\bar{k}}^{\bar{s}} + \mu_{s\bar{k}}^{\bar{r}} - \mu_{rs}^k.$$

Proposition 2.1. *Let (G, J, g) be a 2-step nilpotent Lie group equipped with a left-invariant Hermitian structure (J, g) . Assume that $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$. Then, with respect to a left-invariant, g -unitary $(1, 0)$ -frame $\{Z_1, \dots, Z_n\}$ on G , we have*

$$(7) \quad \Theta(g)(Z_j, Z_{\bar{k}}) = g(\mu(Z_s, Z_{\bar{r}})^{0,1}, Z_j) g(\mu(Z_{\bar{s}}, Z_r)^{1,0}, Z_{\bar{k}}) + \frac{1}{2} g(\mu(Z_{\bar{s}}, Z_{\bar{r}}), Z_j) g(\mu(Z_s, Z_r), Z_{\bar{k}}).$$

Proof. Let $\{Z_1, \dots, Z_n\}$ be a left-invariant, g -unitary $(1, 0)$ -frame on G , from (6) we have

$$S_{j\bar{k}} = -\mu_{s\bar{j}}^r \mu_{s\bar{k}}^{\bar{r}} + \mu_{s\bar{r}}^{\bar{j}} \mu_{sr}^k + \mu_{s\bar{s}}^r \mu_{r\bar{k}}^{\bar{j}} - \mu_{s\bar{s}}^{\bar{r}} \mu_{r\bar{j}}^k.$$

Since G is 2-step nilpotent and $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$, we have that $\mu_{s\bar{s}}^r \mu_{r\bar{k}}^{\bar{j}} = 0$. Indeed

$$\mu_{s\bar{s}}^r \mu_{r\bar{k}}^{\bar{j}} = g(\mu(\mu_{s\bar{s}}^r Z_r, Z_{\bar{k}}), Z_j) = g(\mu(\mu(Z_s, Z_{\bar{s}})^{1,0}, Z_{\bar{k}}), Z_j) = 0.$$

In the same way $\mu_{s\bar{s}}^{\bar{r}} \mu_{r\bar{j}}^k = 0$, hence

$$S_{j\bar{k}} = -\mu_{s\bar{j}}^r \mu_{s\bar{k}}^{\bar{r}} + \mu_{s\bar{r}}^{\bar{j}} \mu_{sr}^k.$$

Furthermore, with respect to $\{Z_1, \dots, Z_n\}$, we get

$$\begin{aligned} Q_{j\bar{k}}^2 &= T_{rs\bar{k}} T_{\bar{r}s\bar{j}} = (-\mu_{r\bar{k}}^{\bar{s}} + \mu_{s\bar{k}}^{\bar{r}} - \mu_{rs}^k)(-\mu_{\bar{r}j}^s + \mu_{s\bar{j}}^{\bar{r}} - \mu_{\bar{r}s}^{\bar{j}}) \\ &= \mu_{\bar{r}j}^s \mu_{r\bar{k}}^{\bar{s}} - \mu_{s\bar{j}}^r \mu_{r\bar{k}}^{\bar{s}} + \mu_{\bar{r}s}^{\bar{j}} \mu_{r\bar{k}}^{\bar{s}} - \mu_{s\bar{j}}^{\bar{r}} \mu_{r\bar{k}}^{\bar{s}} + \mu_{s\bar{j}}^r \mu_{s\bar{k}}^{\bar{r}} - \mu_{\bar{r}s}^{\bar{j}} \mu_{s\bar{k}}^{\bar{r}} + \mu_{rs}^k \mu_{\bar{r}j}^s - \mu_{rs}^k \mu_{s\bar{j}}^{\bar{r}} + \mu_{rs}^k \mu_{\bar{r}s}^{\bar{j}}, \end{aligned}$$

but, since G is 2-step nilpotent and $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$, we get

$$Q_{j\bar{k}}^2 = \mu_{\bar{r}j}^s \mu_{r\bar{k}}^{\bar{s}} + \mu_{\bar{s}j}^r \mu_{s\bar{k}}^{\bar{r}} + \mu_{rs}^k \mu_{\bar{r}\bar{s}}^{\bar{j}} = 2\mu_{\bar{r}j}^s \mu_{r\bar{k}}^{\bar{s}} + \mu_{rs}^k \mu_{\bar{r}\bar{s}}^{\bar{j}}.$$

Thus,

$$\Theta(g)_{j\bar{k}} = -\mu_{\bar{r}j}^s \mu_{r\bar{k}}^{\bar{s}} + \mu_{\bar{s}r}^{\bar{j}} \mu_{s\bar{k}}^{\bar{r}} + \mu_{\bar{r}j}^s \mu_{r\bar{k}}^{\bar{s}} + \frac{1}{2} \mu_{\bar{r}\bar{s}}^{\bar{j}} \mu_{rs}^k = \mu_{\bar{s}r}^{\bar{j}} \mu_{s\bar{k}}^{\bar{r}} + \frac{1}{2} \mu_{\bar{r}\bar{s}}^{\bar{j}} \mu_{rs}^k,$$

i.e.,

$$\Theta(g)(Z_j, Z_{\bar{k}}) = g\left(\mu(Z_s, Z_{\bar{r}})^{0,1}, Z_j\right)g\left(\mu(Z_{\bar{s}}, Z_r)^{1,0}, Z_{\bar{k}}\right) + \frac{1}{2}g\left(\mu(Z_{\bar{s}}, Z_{\bar{r}}), Z_j\right)g\left(\mu(Z_s, Z_r), Z_{\bar{k}}\right),$$

as required. $\square$

Note that if in the statement of Proposition 2.1 we assume that J is abelian, i.e., if $\mathfrak{g}^{1,0}$ is an abelian Lie algebra (see [1]), then the condition that $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$ is satisfied since the center of $\mathfrak{g}$ is J -invariant and $\Theta(g)$ reduces to

$$(8) \quad \Theta(g)(Z_j, Z_{\bar{k}}) = g\left(\mu(Z_s, Z_{\bar{r}})^{0,1}, Z_j\right)g\left(\mu(Z_{\bar{s}}, Z_r)^{1,0}, Z_{\bar{k}}\right).$$

Remark 2.2. We observe that, if we assume that the Lie group G is complex, then the tensor (7) reduces to the one studied in [21, 23].

2.1. The bracket flow technique. In this section we give a brief outline of the bracket flow approach introduced by J. Lauret in [16]

Let (G, J, g_0) be a simply connected Lie group equipped with a left-invariant Hermitian metric g_0 and a left-invariant complex structure J . Let $\mathfrak{g}$ denote the Lie algebra of G and μ_0 the Lie bracket of $\mathfrak{g}$. Since J and g_0 are left-invariant, their value is determined by $\langle \cdot, \cdot \rangle \doteq g_0(e)$ and $J|_{\mathfrak{g}}$.

The Lie bracket μ_0 can be seen as an element of the algebraic variety of Lie bracket

$$\tilde{\mathfrak{L}} \doteq \{\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} : \mu \text{ satisfies the Jacobi identity and } N_{\mu, J} = 0\},$$

where, $N_{\mu, J}$ is the Nijenhuis tensor associated to μ . The space $\tilde{\mathfrak{L}}$ admits an action of the Lie group $\text{GL}(\mathfrak{g}, J) \doteq \{f \in \text{GL}(\mathfrak{g}) \mid f \circ J = J \circ f\}$. This action is defined by

$$f \cdot \mu \doteq f \circ \mu(f^{-1} \cdot, f^{-1} \cdot).$$

Let us consider the positive Hermitian curvature flow on G starting from g_0

$$(9) \quad \begin{cases} \partial_t g_t = -\Theta(g_t), \\ g(0) = g_0, \end{cases}$$

where Θ is defined as in (2). Since the tensor Θ is equivariant under biholomorphisms, then, if the flow starts from a left-invariant Hermitian metric g_0 , it admits a left-invariant solution, i.e., a solution made by left-invariant Hermitian metrics over an interval $I \subseteq \mathbb{R}$ with $0 \in I$.

Theorem 1.1 in [14] guarantees that the left-invariant solution g_t evaluated at the identity can be rewritten in terms of $\langle \cdot, \cdot \rangle$ and an element of $\text{GL}(\mathfrak{g}, J)$. Specifically, there exists a smooth curve $(f_t)_{t \in I} \in \text{GL}(\mathfrak{g}, J)$ such that $f_0 = \text{Id}_{\mathfrak{g}}$ and

$$g_t(\cdot, \cdot) = \langle f_t \cdot, f_t \cdot \rangle.$$

Simultaneously,

$$\mu_t = f_t \cdot \mu_0$$

is a solution to the bracket flow

$$(10) \quad \begin{cases} \frac{d}{dt} \mu_t = -\pi(\Theta_{\mu_t}) \mu_t, \\ \mu(0) = \mu_0. \end{cases}$$

Here, $\pi : \text{End}(\mathfrak{g}) \rightarrow \text{End}(\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g})$ is defined as

$$(11) \quad \pi(E)\mu(\cdot, \cdot) \doteq E\mu(\cdot, \cdot) - \mu(E\cdot, \cdot) - \mu(\cdot, E\cdot) \quad \forall E \in \text{End}(\mathfrak{g}),$$

furthermore, $\Theta_{\mu_t} \in \text{End}(\mathfrak{g})$ is related to the value of $\Theta(g_t)$ at $e \in G$ by

$$(12) \quad \Theta_{\mu_t} \doteq f_t \Theta_{g_t} f_t^{-1}, \quad g_t(\Theta_{g_t} \cdot, \cdot) = \Theta(g_t)(\cdot, \cdot).$$

In particular, the solutions to (9) and to (10) are defined on the same interval. Moreover, by [14, Theorem 1.1], the Lie group $(G_{\mu_t}, J_{\mu_t}, g_{\mu_t})$, defined by the data $(\mathfrak{g}, J, \mu_t, \langle \cdot, \cdot \rangle)$, is equivariantly, biholomorphically isometric to (G, J, g_t) , where g_t is a solution to the HCF_+ (9) and μ_t is a solution to the bracket flow (10).

2.2. Special solutions to the HCF_+ and to the type IIB flow. Important solutions to the HCF_+ are static metrics and solitons. We recall that a Hermitian metric g on a Lie group (G, J) is called a static metric to the HCF_+ if

$$\Theta(g) = cg, \quad c \in \mathbb{R}.$$

The corresponding solution to the HCF_+ evolves only by scaling of the initial metric.

A left-invariant metric g on a Lie group (G, J) is called a *semi-algebraic soliton* to the HCF_+ if and only if

$$(13) \quad \Theta_g = c \text{Id} + \frac{1}{2}(D + D^t), \quad c \in \mathbb{R}, \quad D \in \text{Der}(\mathfrak{g}, \mu), \quad [D, J] = 0,$$

where $(\mathfrak{g}, \mu)$ denotes the Lie algebra of G and $D \in \text{Der}(\mathfrak{g}, \mu)$ means that D is a derivation of μ .

Moreover, a semi-algebraic soliton is called an *algebraic soliton* to the HCF_+ if

$$(14) \quad \Theta_g = c \text{Id} + D, \quad c \in \mathbb{R}, \quad D \in \text{Der}(\mathfrak{g}, \mu), \quad [D, J] = 0,$$

hence, $D \in \text{Der}(\mathfrak{g}, \mu)$ can be chosen to be self-adjoint.

By using that the type IIB flow is conformally equivalent to the HCF_+ , we can define solitons to the type IIB flow. Let (M, J, g, Ψ) be a n -dimensional complex manifold equipped with a complex structure J , a Hermitian metric g whose fundamental form is ω and a nowhere vanishing holomorphic $(n, 0)$ -form Ψ . Let ω_t be a solution to

$$(15) \quad \partial_t(\|\Psi\|_{\omega_t} \omega_t^{n-1}) = i\partial\bar{\partial}\omega_t^{n-2}, \quad \omega_0 = \omega$$

on $I \subseteq \mathbb{R}$ such that

$$d(\|\Psi\|_{\omega_t} \omega_t^{n-1}) = 0, \quad \forall t \in I.$$

Then, from [9], $\tilde{g}_t \doteq \|\Psi\|_{\omega_t} g_t$, where g_t is the Hermitian metric associated to ω_t , evolves according to

$$(16) \quad \partial_t \tilde{g}_t = -\frac{1}{n-1} \Theta(\tilde{g}_t),$$

where $\Theta(\tilde{g}_t)$ is defined as in (2).

Define $\tilde{g} \doteq \|\Psi\|_{\omega} g$. If $\tilde{g}$ is a soliton to the HCF_+ , then

$$\Theta(\tilde{g}) = c\tilde{g} + \mathcal{L}_Z \tilde{g},$$

where $c \in \mathbb{R}$, $\mathcal{L}$ denotes the Lie derivative and Z is a complete holomorphic vector field. Hence,

$$\begin{aligned} \Theta(\|\Psi\|_{\omega} g) &= c\|\Psi\|_{\omega} g + \mathcal{L}_Z(\|\Psi\|_{\omega} g) \\ &= c\|\Psi\|_{\omega} g + \mathcal{L}_Z(\|\Psi\|_{\omega})g + \|\Psi\|_{\omega} \mathcal{L}_Z g. \end{aligned}$$

This leads to the following definition.

Definition 2.3. Let (M, J, g, Ψ) be as above. We say that g is a soliton to the type IIB flow if

$$\Theta(\|\Psi\|_{\omega} g) = (c\|\Psi\|_{\omega} + \mathcal{L}_Z(\|\Psi\|_{\omega}))g + \|\Psi\|_{\omega} \mathcal{L}_Z g,$$

where $c \in \mathbb{R}$, $\mathcal{L}$ denotes the Lie derivative, Z is a complete holomorphic vector field and Θ is defined as in (2).

Definition 2.4. Let (G, J, g, Ψ) be a Lie group equipped with a left-invariant Hermitian structure (J, g) and a left-invariant nowhere vanishing holomorphic $(n, 0)$ -form Ψ . We say that g is a semi-algebraic soliton to the type IIB flow if

$$\Theta(\|\Psi\|_\omega g)(\cdot, \cdot) = c \|\Psi\|_\omega g(\cdot, \cdot) + \frac{1}{2} \|\Psi\|_\omega \left(g(D\cdot, \cdot) + g(\cdot, D\cdot) \right), \quad c \in \mathbb{R}, \quad D \in \text{Der}(\mathfrak{g}, \mu), \quad [D, J] = 0,$$

where $(\mathfrak{g}, \mu)$ denotes the Lie algebra of G and $D \in \text{Der}(\mathfrak{g}, \mu)$.

If D can be chosen to be self-adjoint, we call g an algebraic soliton to the type IIB flow.

3. PROOF OF THEOREM 1.1

According to the assumptions of Theorem 1.1, let (G, J, g_0) be a simply connected, 2-step nilpotent Lie group equipped with a left-invariant Hermitian structure. Let $(\mathfrak{g}, \mu_0)$ denote the Lie algebra of G and suppose that $J\mu_0(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$. Let $\{Z_1, \dots, Z_n\}$ be a g_0 -unitary $(1, 0)$ -frame on $(\mathfrak{g}, \mu_0)$. From (7) and (12) we get

$$\Theta_{g_0}(Z_j) = \mu_{s\bar{r}}^{\bar{j}} \mu_{s\bar{r}}^l Z_l + \frac{1}{2} \mu_{s\bar{r}}^{\bar{j}} \mu_{s\bar{r}}^l Z_l, \quad \Theta_{g_0}(Z_{\bar{j}}) = \mu_{s\bar{r}}^j \mu_{s\bar{r}}^{\bar{l}} Z_{\bar{l}} + \frac{1}{2} \mu_{s\bar{r}}^j \mu_{s\bar{r}}^{\bar{l}} Z_{\bar{l}}.$$

Let ξ be the center of $(\mathfrak{g}, \mu_0)$. Note that if $X \in \xi^\perp \doteq \{Z \in \mathfrak{g} : g_0(Z, Y) = 0, \forall Y \in \xi\}$, then $\Theta_{g_0}(X) = 0$. Furthermore, Θ_{g_0} preserves the center ξ , i.e., if $X \in \xi$, then $\Theta_{g_0}(X) \in \xi$. Thus, with respect to the block representation $\mathfrak{g} = \xi^\perp \oplus \xi$, the endomorphism Θ_{g_0} has the following form

$$(17) \quad \Theta_{g_0} = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}.$$

It can be easily seen that, since (17) holds, then the HCF_+ starting from a left-invariant metric on (G, J) preserves the splitting $\mathfrak{g} = \xi \oplus \xi^\perp$.

Let us consider the following space

$$(18) \quad \mathfrak{N} \doteq \{\mu \in \tilde{\mathfrak{L}} : \mu \text{ is 2-step nilpotent and } J\mu(\mathfrak{g}, \mathfrak{g}) \subseteq \xi\},$$

and consider the group $\text{GL}(\xi, J) \doteq \{f \in \text{GL}(\xi) \mid f \circ J = J \circ f\} \subseteq \text{GL}(\mathfrak{g}, J)$ on $\mathfrak{N}$. This group is a subgroup of $\text{GL}(\mathfrak{g}, J)$ via the embedding $f \mapsto \begin{pmatrix} \text{Id} & 0 \\ 0 & f \end{pmatrix}$ and the Lie algebra of $\text{GL}(\xi, J)$ is denoted by $\mathfrak{gl}(\xi, J)$. Moreover, define $\mathfrak{p}(\xi, J) \doteq \mathfrak{gl}(\xi, J) \cap \text{Sym}(\xi, J)$, where $\text{Sym}(\xi, J)$ is the set of endomorphisms that commute with J and that are symmetric with respect to the inner product induced by g_0 on the Lie algebra.

Remark 3.1. We recall that on the Lie algebra $\mathfrak{g}$ we can associate a Hermitian structure that is given by the value at the identity of the Lie group G of g_0 and J . We can also associate a Hermitian product to any tensor product of $\mathfrak{g}$ and every tensor product of $\mathfrak{g}^*$.

Moreover, let g be another left-invariant Hermitian metric on $(\mathfrak{g}, J)$ such that $g(\cdot, \cdot) = g_0(A\cdot, A\cdot)$, where $A \in \text{GL}(\mathfrak{g}, J)$ and set, following the approach described in Section 2.1, $\mu \doteq A \cdot \mu_0$ and

$$\Theta_\mu = A\Theta_g A^{-1}, \quad g(\Theta_g \cdot, \cdot) = \Theta(g)(\cdot, \cdot).$$

Then, by extending any operator to the complexified tangent bundle and considering a g_0 -unitary, left-invariant frame $\{Z_1, \dots, Z_n\}$, we get

$$(19) \quad g_0(\Theta_\mu Z_j, Z_{\bar{k}}) = g(\Theta_g A^{-1} Z_j, A^{-1} Z_{\bar{k}}) = \Theta(g)(A^{-1} Z_j, A^{-1} Z_{\bar{k}}) = \Theta(g)(\tilde{Z}_j, \tilde{Z}_{\bar{k}}),$$

where $\{\tilde{Z}_1 = A^{-1} Z_1, \dots, \tilde{Z}_n = A^{-1} Z_n\}$ is a unitary frame with respect to g . Furthermore,

$$(20) \quad \begin{aligned} \Theta(g)(\tilde{Z}_j, \tilde{Z}_{\bar{k}}) &= g\left(\mu_0(\tilde{Z}_s, \tilde{Z}_{\bar{r}})^{0,1}, \tilde{Z}_j\right) g\left(\mu_0(\tilde{Z}_{\bar{s}}, \tilde{Z}_r)^{1,0}, \tilde{Z}_{\bar{k}}\right) + \frac{1}{2} g\left(\mu_0(\tilde{Z}_{\bar{s}}, \tilde{Z}_{\bar{r}}), \tilde{Z}_j\right) g\left(\mu_0(\tilde{Z}_s, \tilde{Z}_r), \tilde{Z}_{\bar{k}}\right) \\ &= g_0\left(\mu(Z_s, Z_{\bar{r}})^{0,1}, Z_j\right) g_0\left(\mu(Z_{\bar{s}}, Z_r)^{1,0}, Z_{\bar{k}}\right) + \frac{1}{2} g_0\left(\mu(Z_{\bar{s}}, Z_{\bar{r}}), Z_j\right) g_0\left(\mu(Z_s, Z_r), Z_{\bar{k}}\right). \end{aligned}$$

Lemma 3.2. *Let*

$$(21) \quad M : \mathfrak{N} \setminus \{0\} \rightarrow \mathfrak{p}(\xi, J), \quad \mu \mapsto \frac{2}{\|\mu\|^2} \Theta_\mu,$$

then M is a moment map for the linear action of $GL(\xi, J)$ on $\mathfrak{N} \setminus \{0\}$, i.e.,

$$\langle \Theta_\mu, E \rangle = \frac{1}{2} \langle \pi(E) \mu, \mu \rangle, \quad \forall E \in \mathfrak{p}(\xi, J), \quad \mu \in \mathfrak{N} \setminus \{0\}.$$

Proof. Let us consider $\{Z_1, \dots, Z_n\}$ a unitary $(1, 0)$ -basis of $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$. From (19) and (20), we have that

$$(\Theta_\mu)_j^l = \mu_{s\bar{r}}^{\bar{j}} \mu_{s\bar{r}}^l + \frac{1}{2} \mu_{s\bar{r}}^{\bar{j}} \mu_{s\bar{r}}^l, \quad (\Theta_\mu)_{\bar{j}}^{\bar{l}} = \mu_{s\bar{r}}^j \mu_{s\bar{r}}^{\bar{l}} + \frac{1}{2} \mu_{s\bar{r}}^j \mu_{s\bar{r}}^{\bar{l}},$$

where $\mu_{s\bar{r}}^{\bar{j}} \mu_{s\bar{r}}^l, \mu_{s\bar{r}}^j \mu_{s\bar{r}}^{\bar{l}}$ denote the structure constants of μ with respect to the $\langle \cdot, \cdot \rangle$ -unitary basis $\{Z_1, \dots, Z_n\}$. Note that, since (17) holds, then $\Theta_\mu \in \mathfrak{p}(\xi, J)$. Furthermore, since $E \in \mathfrak{p}(\xi, J)$, hence, it preserves the elements of $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$, then

$$\langle E, \Theta_\mu \rangle = 2\text{Re}(E_j^l (\Theta_\mu)_{\bar{j}}^{\bar{l}}) = 2\text{Re}\{\mu_{s\bar{r}}^j \mu_{s\bar{r}}^{\bar{l}} E_j^l + \frac{1}{2} \mu_{s\bar{r}}^j \mu_{s\bar{r}}^{\bar{l}} E_j^l\}.$$

By using (11) and $E \in \mathfrak{p}(\xi, J)$, we get that

$$(22) \quad \begin{aligned} \langle \pi(E) \mu(\cdot, \cdot), \mu(\cdot, \cdot) \rangle &= \langle E \circ \mu(\cdot, \cdot), \mu(\cdot, \cdot) \rangle - \langle \mu(E \cdot, \cdot), \mu(\cdot, \cdot) \rangle - \langle \mu(\cdot, E \cdot), \mu(\cdot, \cdot) \rangle \\ &= \langle E \circ \mu(\cdot, \cdot), \mu(\cdot, \cdot) \rangle, \end{aligned}$$

where the last equality holds true because $E \in \mathfrak{p}(\xi, J)$.

Suppose that $\eta \in \mathfrak{N}$, then

$$\langle \mu, \eta \rangle = 2\text{Re}\{\mu_{s\bar{r}}^j \eta_{s\bar{r}}^{\bar{j}}\} + 4\text{Re}\{\mu_{s\bar{r}}^j \eta_{s\bar{r}}^{\bar{j}}\},$$

so, from (22), we get

$$\langle \pi(E) \mu, \mu \rangle = 2\text{Re}\{E_j^l \mu_{s\bar{r}}^j \mu_{s\bar{r}}^{\bar{l}}\} + 4\text{Re}\{E_j^l \mu_{s\bar{r}}^j \mu_{s\bar{r}}^{\bar{l}}\}.$$

Thus

$$\langle \pi(E) \mu, \mu \rangle = 2 \langle E, \Theta_\mu \rangle,$$

as required. $\square$

Now we are ready to prove Theorem 1.1. The proof is analogous to [3, Theorem 3.3]. For the sake of completeness, we provide the proof in our case.

Proof of Theorem 1.1. The first part of the proof focuses on the long-time existence of the solution, the second part on the convergence to a non-flat semi-algebraic soliton. Since

$$\frac{d}{dt} \|\mu_t\|^2 = 2 \langle \frac{d}{dt} \mu_t, \mu_t \rangle = -2 \langle \pi(\Theta_{\mu_t}) \mu_t, \mu_t \rangle = -4 \langle \Theta_{\mu_t}, \Theta_{\mu_t} \rangle = -4 \|\Theta_{\mu_t}\|^2 \leq 0,$$

then, the bracket flow has a long-time solution according to the standard theory of ordinary differential equations. Given that both the bracket flow's solution and the HCF_+ 's solution are defined over the same interval, we can conclude that the HCF_+ also admits a long-time solution.

Let us denote by ν_t the rescaled solution by a factor of $\|\mu_t\|$, i.e. $\nu_t \doteq \frac{\mu_t}{\|\mu_t\|}$, then $\|\nu_t\| = 1$. By [3, Lemma 2.3] we get that ν_t is a solution to

$$(23) \quad \frac{d}{dt} \nu_t = -\pi(\Theta_{\nu_t} + r_{\nu_t} Id_{\mathfrak{g}}) \nu_t,$$

where $r_{\nu_t} \doteq \langle \pi(\Theta_{\nu_t}) \nu_t, \nu_t \rangle = 2 \|\Theta_{\nu_t}\|^2$. Equation (23) is called the normalized bracket flow equation.

By means of [6, Lemma 7.2], since (21) is a moment map, it turns out that (23) is the negative gradient flow of the following real-analytic functional

$$F : \mathfrak{N} \setminus 0 \rightarrow \mathbb{R}, \quad \mu \mapsto \frac{\|\Theta_\mu\|^2}{\|\mu\|^4}.$$

Since the space of unitary brackets is compact, and ν_t exists $\forall t \in [0, \infty)$, then there must exist an accumulation point $\bar{\nu}$. Thus, the Theorem of Łojasiewicz ([17]) ensures that $\nu_t \rightarrow \bar{\nu}$ for $t \rightarrow \infty$. Consequently, $\bar{\nu}$

is a stationary point of the flow and $\Theta_{\bar{\nu}} + r_{\bar{\nu}} \text{Id}_{\mathfrak{g}}$ is a derivation of $\mathfrak{g}$. Thus, the corresponding metric is a algebraic soliton to the HCF_+ .

A direct computation yields

$$(24) \quad \text{Tr}(\Theta_{\mu}) = \frac{1}{2} \neq 0,$$

hence, the soliton is not flat.

The last part of the proof follows from the fact that the asymptotic behaviour of $\|\mu_t\|$ is $t^{-\frac{1}{2}}$. Specifically, R. Arroyo and R. Lafuente, in the proof of [3, Theorem 3.3], demonstrate that the evolution equation of $\|\mu_t\|^2$ is bounded in both directions by an ODE of the form $\frac{d}{dt}y = \tilde{c}y^2$, with $\tilde{c} > 0$.

According to [15, Section 2.1], scaling the metric by a factor of $k > 0$ is equivalently reflected in scaling the corresponding bracket by a factor of $k^{-\frac{1}{2}}$. Consequently, the asymptotic behavior of the metrics corresponding to $\frac{\mu_t}{\|\mu_t\|}$, which are solutions to (23), is $(1+t)^{-1}g_t$.

Finally, by [13, Corollary 6.20], we know that the convergence of the bracket flow implies the convergence in Cheeger-Gromov topology for the corresponding family of left-invariant metrics. Thus, $(1+t)^{-1}g_t$ subconverges to a non-flat semi-algebraic soliton. $\square$

4. PROOF OF THEOREM 1.3 AND EXAMPLES

This section is devoted to the proof of Theorem 1.3 and to the construction of some examples.

Proposition 4.1. *Let (G, J) be a simply-connected, non-abelian, 2-step nilpotent Lie group equipped with a left-invariant complex structure J . Assume that the Lie algebra $(\mathfrak{g}, \mu)$ of G is such that $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$. Then, there are no left-invariant HCF_+ static metrics.*

Proof. Let us suppose that g is a left-invariant Hermitian static metric to the HCF_+ . Since G is 2-step nilpotent, then we can choose a g -unitary, $(1, 0)$ -left-invariant frame $\{Z_1, \dots, Z_n\}$, such that $\langle Z_1 \rangle \perp \xi \otimes \mathbb{C}$, where ξ denotes the center of $(\mathfrak{g}, \mu)$. By Proposition 2.1, we get

$$\Theta(g)(Z_1, Z_{\bar{1}}) = 0,$$

hence $c = 0$. Furthermore, we also obtain that

$$nc = \text{Tr}_g \Theta(g) = g^{i\bar{j}} \Theta(g)(Z_i, Z_{\bar{j}}) = \mu_{s\bar{r}}^{\bar{j}} \mu_{s\bar{r}}^j + \frac{1}{2} \mu_{s\bar{r}}^{\bar{j}} \mu_{s\bar{r}}^j = \frac{1}{4} \|\mu\|^2.$$

Hence $c > 0$, which is absurd. $\square$

The proof of Theorem 1.3 is analogous to the proof of [21, Theorem B]. For the sake of completeness, we outline a part of the proof in our case.

Proof of Theorem 1.3. According to the assumptions of Theorem 1.3, let (G, J) be a simply-connected, non-abelian, 2-step nilpotent Lie group equipped with a left-invariant complex structure J . Suppose that (G, J) is equipped with a metric g which is a semi-algebraic soliton to the HCF_+ . Denote by $(\mathfrak{g}, \mu)$ the Lie algebra of G and suppose that $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$. Moreover, consider the g -orthogonal splitting of the Lie algebra $\mathfrak{g} = \xi^\perp \oplus \xi$, where ξ denotes the center of $(\mathfrak{g}, \mu)$. Then

$$(25) \quad \Theta_{\mu} = c \text{Id}_{\mathfrak{g}} + \frac{1}{2}(D + D^t),$$

where the transpose is calculated with respect to the Hermitian product induced on $\mathfrak{g}$. In the first part of the proof we show that D^t is a derivation. Indeed, since D is a derivation, we get that, if $x \in \xi$, then $D(x) \in \xi$, indeed

$$D(\mu(x, y)) = \mu(D(x), y) + \mu(x, D(y)), \quad \forall y \in \mathfrak{g}.$$

So $\mu(x, D(y)) = 0$, for all $y \in \mathfrak{g}$, hence

$$D = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

with respect to the decomposition of the Lie algebra above. The morphism π defined in (11) is a Lie algebra homomorphism such that $\pi(E^t) = \pi(E)^t$. Thus, from Lemma 3.2

$$\begin{aligned} 2\text{Tr } \Theta_\mu[D, D^t] &= \langle \pi([D, D^t])\mu, \mu \rangle = \langle [\pi(D), \pi(D^t)]\mu, \mu \rangle \\ &= \langle (\pi(D)\pi(D)^t - \pi(D)^t\pi(D))\mu, \mu \rangle \\ &= \|\pi(D)^t\mu\|^2 - \|\pi(D)\mu\|^2 = \|\pi(D^t)\mu\|^2. \end{aligned}$$

The last equality holds because D is a derivation. Furthermore, from (25)

$$\text{Tr } \Theta_\mu[D, D^t] = c \text{Tr } [D, D^t] + \frac{1}{2} \text{Tr } D[D, D^t] + \frac{1}{2} \text{Tr } D^t[D, D^t],$$

thus,

$$\text{Tr } \Theta_\mu[D, D^t] = 0,$$

so D^t is a derivation. We already know that there cannot be a static invariant metric, which implies that $D + D^t \neq 0$. Furthermore, a soliton cannot be shrinking, i.e., $c > 0$, because the solution would develop a finite-time singularity ([14]). Hence, $c \leq 0$.

Furthermore, by means of (17) and (25), we have that

$$D + D^t = \begin{pmatrix} -c\text{Id}_{\xi^\perp} & 0 \\ 0 & (D + D^t)_\xi \end{pmatrix}.$$

Thus, if we suppose that $c = 0$, then $\Theta_\mu = \frac{1}{2}(D + D^t)$, hence

$$\begin{aligned} \text{Tr } (D + D^t)^2 &= 2\text{Tr } \Theta_\mu(D + D^t) = 2\langle \pi(D + D^t)\mu, \mu \rangle \\ &= 2\langle \pi(D)\mu, \mu \rangle + 2\langle \pi(D^t)\mu, \mu \rangle = 0, \end{aligned}$$

but $D + D^t$ is a symmetric operator such that $\text{Tr } (D + D^t)^2 = 0$, hence $D + D^t = 0$, which is absurd. Thus, every semi-algebraic soliton is expanding, i.e. $c < 0$.

The proof of the uniqueness follows the same argument of [21, Theorem B]. $\square$

From the proof of Theorem 1.3, it follows that

Corollary 4.2. *Let g be a semi-algebraic soliton to the HCF_+ on the Lie group (G, J) , then g is an algebraic HCF_+ soliton.*

In the following example, we study the HCF_+ starting from a *balanced* metric on a 6-dimensional, 2-step nilpotent Lie algebra. We recall that a Hermitian metric is called balanced if its fundamental form is coclosed.

Example 4.3. Let $\mathfrak{g}$ be the 6-dimensional, 2-step nilpotent Lie algebra which satisfies the following structure equations

$$\begin{aligned} de^i &= 0 \quad i = 1, \dots, 4, \\ de^5 &= e^{13} - e^{24}, \quad de^6 = e^{14} + e^{23}, \end{aligned}$$

where $e^{ij} \doteq e^i \wedge e^j$.

Let us consider the following complex structure

$$Je^1 = -e^2, \quad Je^3 = e^4, \quad Je^5 = e^6,$$

then, $Je_1 = -e_2$, $Je_2 = e_1$, $Je_3 = e_4$, $Je_4 = -e_3$, $Je_5 = e_6$, $Je_6 = -e_5$. Thus, J is an abelian complex structure.

Let us set

$$Z_1 \doteq \frac{1}{\sqrt{2}}(e_1 - iJe_1), \quad Z_2 \doteq \frac{1}{\sqrt{2}}(e_3 - iJe_3), \quad Z_3 \doteq \frac{1}{\sqrt{2}}(e_5 - iJe_5),$$

then, the only non vanishing Lie bracket are:

$$\mu(Z_1, Z_2) = -\sqrt{2}Z_3, \quad \mu(Z_1, Z_2) = -\sqrt{2}Z_3.$$

Thus, if $\{\zeta^1, \zeta^2, \zeta^3\}$ is dual to the $(1, 0)$ -frame $\{Z_1, Z_2, Z_3\}$, the Lie bracket takes the following expression

$$(26) \quad \mu = -\sqrt{2}\zeta^1 \wedge \zeta^{\bar{2}} \otimes Z_{\bar{3}} - \sqrt{2}\zeta^{\bar{1}} \wedge \zeta^2 \otimes Z_3.$$

Let us call a metric g *diagonal* if it can be written as

$$(27) \quad g \doteq a\zeta^1 \odot \zeta^{\bar{1}} + b\zeta^2 \odot \zeta^{\bar{2}} + c\zeta^3 \odot \zeta^{\bar{3}},$$

where $a, b, c \in \mathbb{R}$ and $a, b, c > 0$. We mention that, it is easy to show that g is a balanced metric. With respect to this metric, the orthonormal frame is

$$\tilde{Z}_1 \doteq \frac{1}{\sqrt{a}}Z_1, \quad \tilde{Z}_2 \doteq \frac{1}{\sqrt{b}}Z_2, \quad \tilde{Z}_3 \doteq \frac{1}{\sqrt{c}}Z_3,$$

hence

$$\mu(\tilde{Z}_1, \tilde{Z}_2) = -\sqrt{\frac{2c}{ab}}\tilde{Z}_3, \quad \mu(\tilde{Z}_1, \tilde{Z}_2) = -\sqrt{\frac{2c}{ab}}\tilde{Z}_3.$$

Let us study the behavior of the HCF_+ starting at the diagonal metric

$$g_0 \doteq \zeta^1 \odot \zeta^{\bar{1}} + \zeta^2 \odot \zeta^{\bar{2}} + \zeta^3 \odot \zeta^{\bar{3}}.$$

From (8), we have that, for an arbitrary diagonal Hermitian metric as in (27)

$$\Theta(g) = \frac{2c^2}{ab}\zeta^3 \odot \zeta^{\bar{3}}.$$

Thus, the flow starting from g_0 is equivalent to

$$\begin{cases} a' = 0, \\ b' = 0, \\ c' = -\frac{2c^2}{ab}, \end{cases}$$

with $a(0) = b(0) = c(0) = 1$.

Hence

$$g_t = \zeta^1 \odot \zeta^{\bar{1}} + \zeta^2 \odot \zeta^{\bar{2}} + \frac{1}{2t+1}\zeta^3 \odot \zeta^{\bar{3}}.$$

Proposition 4.4. *Let G be the simply connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$ as in Example (4.3). Then, every left-invariant Hermitian metric on G is an expanding HCF_+ algebraic soliton.*

Proof. Let g be a left-invariant Hermitian metric on G . Then, we can always find a left-invariant, $(1, 0)$ -unitary frame $\{W_1, W_2, W_3\}$ of g such that

$$W_1 \in \text{Span}\{Z_1, Z_2, Z_3\}, \quad W_2 \in \text{Span}\{Z_2, Z_3\}, \quad W_3 \in \text{Span}\{Z_3\},$$

where $\{Z_1, Z_2, Z_3\}$ is the left-invariant $(1, 0)$ -frame satisfying (26).

With respect to this new frame, we have that

$$\begin{aligned} \mu(W_i, W_j) &= \mu(W_{\bar{i}}, W_{\bar{j}}) = 0, \quad \forall i, j \in \{1, 2, 3\}, \\ \mu(W_1, W_{\bar{3}}) &= \mu(W_2, W_{\bar{3}}) = \mu(W_3, W_{\bar{3}}) = \mu(W_2, W_{\bar{2}}) = 0, \end{aligned}$$

while

$$\mu(W_1, W_{\bar{2}}) = uW_{\bar{3}}, \quad \mu(W_1, W_{\bar{1}}) = vW_3 - \bar{v}W_{\bar{3}},$$

with $u, v \in \mathbb{C}$ and $u, v \neq 0$. Hence

$$\mu = u\alpha^1 \wedge \alpha^{\bar{2}} \otimes W_{\bar{3}} + \bar{u}\alpha^{\bar{1}} \wedge \alpha^2 \otimes W_3 + \alpha^1 \wedge \alpha^{\bar{1}} \otimes (vW_3 - \bar{v}W_{\bar{3}}),$$

where $\{\alpha^1, \alpha^2, \alpha^3\}$ is dual to the left-invariant, g -unitary, $(1, 0)$ -frame $\{W_1, W_2, W_3\}$. From (8), we have

$$\Theta(g) = (\|u\|^2 + \|v\|^2)\alpha^3 \odot \alpha^{\bar{3}}.$$

Hence, the endomorphism Θ_g extended to $\mathfrak{g}^{\mathbb{C}}$ is $\Theta_g = \text{Diag}(0, 0, \|u\|^2 + \|v\|^2, 0, 0, \|u\|^2 + \|v\|^2)$.

Let $D \doteq \Theta_g - c\text{Id}_{\mathfrak{g}^c}$, then $D = \text{Diag}(-c, -c, -c + \|u\|^2 + \|v\|^2, -c, -c, -c + \|u\|^2 + \|v\|^2)$. Furthermore, we have that

$$\begin{aligned} D\mu(W_1, W_2) - \mu(DW_1, W_2) - \mu(W_1, DW_2) &= \\ &= uDW_3 - uD_1^1W_3 - uD_5^5W_3 = \\ &= u(D_6^6 - D_1^1 - D_5^5)W_3, \end{aligned}$$

and

$$\begin{aligned} D\mu(W_1, W_{\bar{1}}) - \mu(DW_1, W_{\bar{1}}) - \mu(W_1, DW_{\bar{1}}) &= \\ &= D(vW_3 - \bar{v}W_{\bar{3}}) - D_1^1(vW_3 - \bar{v}W_{\bar{3}}) - D_4^4(vW_3 - \bar{v}W_{\bar{3}}) = \\ &= v(D_3^3 - D_1^1 - D_4^4)W_3 - \bar{v}(D_6^6 - D_1^1 - D_4^4)W_{\bar{3}}. \end{aligned}$$

Hence, D is a derivation if and only if $c = -\|u\|^2 - \|v\|^2$. $\square$

5. HERMITIAN CURVATURE FLOW ON 2-STEP NILPOTENT LIE GROUPS

Let (M, J, g) be a Hermitian manifold with Chern connection denoted by ∇ . Let T be the torsion tensor of the Chern connection ∇ and Q^j , for $j = 1, \dots, 4$ be the $(1, 1)$ -symmetric tensors, quadratic in the torsion T of the Chern connection ∇ , defined by

$$(28) \quad \begin{aligned} Q_{j\bar{k}}^1 &\doteq g^{\bar{p}q}g^{\bar{r}s}T_{js\bar{p}}T_{\bar{k}r\bar{q}}, & Q_{j\bar{k}}^2 &\doteq g^{\bar{p}q}g^{\bar{r}s}T_{sq\bar{k}}T_{\bar{r}\bar{p}j}, \\ Q_{j\bar{k}}^3 &\doteq g^{\bar{p}q}g^{\bar{r}s}T_{js\bar{r}}T_{\bar{k}p\bar{q}}, & Q_{j\bar{k}}^4 &\doteq \frac{1}{2}g^{\bar{p}q}g^{\bar{r}s}(T_{qs\bar{r}}T_{\bar{p}k\bar{j}} + T_{qj\bar{k}}T_{\bar{p}\bar{r}s}), \end{aligned}$$

where $T_{js\bar{p}} \doteq g_{l\bar{p}}T_{js}^l$ and $T_{j\bar{s}}^l$ denote the components of T .

The Hermitian curvature flow studied in [24] is defined as

$$(29) \quad \partial_t g_t = -K(g_t), \quad g_t|_{t=0} = g,$$

where

$$K(g) \doteq S(g) - Q(g),$$

where $S(g)$ is the second Chern-Ricci curvature tensor of g and $Q(g)$ is defined as follows

$$(30) \quad Q(g) \doteq \frac{1}{2}Q^1(g) - \frac{1}{4}Q^2(g) - \frac{1}{2}Q^3(g) + Q^4(g).$$

In [12], the authors studied the behavior of (29) on complex unimodular Lie group and on unimodular Lie group equipped with an abelian complex structure. Here, we study the behavior of (29) on 2-step nilpotent Lie group such that $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$.

Let us consider a Lie group G equipped with a left-invariant Hermitian structure (J, g) . We denote by $(\mathfrak{g}, \mu)$ the Lie algebra of G . Suppose that the Lie algebra $(\mathfrak{g}, \mu)$ of G is such that $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$.

Let $\{Z_1, \dots, Z_n\}$ be a g -unitary, left-invariant frame on G . Formulae in Section 2 imply that

$$\begin{aligned} Q_{j\bar{k}}^1 &= \mu_{j\bar{s}}^{\bar{r}}\mu_{\bar{k}s}^{\bar{r}} + \mu_{r\bar{s}}^{\bar{j}}\mu_{\bar{r}s}^k + \mu_{jr}^s\mu_{\bar{k}\bar{r}}^{\bar{s}}, \\ Q_{j\bar{k}}^2 &= 2\mu_{\bar{r}j}^s\mu_{r\bar{k}}^{\bar{s}} + \mu_{rs}^k\mu_{\bar{r}\bar{s}}^{\bar{j}}, \\ Q_{j\bar{k}}^3 &= \mu_{r\bar{r}}^{\bar{j}}\mu_{\bar{s}s}^k, \\ 2Q_{j\bar{k}}^4 &= \mu_{r\bar{r}}^{\bar{s}}\mu_{\bar{k}j}^s + \mu_{\bar{r}r}^s\mu_{j\bar{k}}^{\bar{s}}. \end{aligned}$$

Hence,

$$\begin{aligned} K_{j\bar{k}} &= -\mu_{s\bar{j}}^r\mu_{\bar{s}k}^{\bar{r}} + \mu_{s\bar{r}}^{\bar{j}}\mu_{\bar{s}r}^k - \frac{1}{2}(\mu_{j\bar{s}}^{\bar{r}}\mu_{\bar{k}s}^{\bar{r}} + \mu_{r\bar{s}}^{\bar{j}}\mu_{\bar{r}s}^k + \mu_{jr}^s\mu_{\bar{k}\bar{r}}^{\bar{s}}) \\ &\quad + \frac{1}{4}(2\mu_{\bar{r}j}^s\mu_{r\bar{k}}^{\bar{s}} + \mu_{rs}^k\mu_{\bar{r}\bar{s}}^{\bar{j}}) + \frac{1}{2}\mu_{r\bar{r}}^{\bar{j}}\mu_{\bar{s}s}^k - \frac{1}{2}(\mu_{r\bar{r}}^{\bar{s}}\mu_{\bar{k}j}^s + \mu_{\bar{r}r}^s\mu_{j\bar{k}}^{\bar{s}}) \end{aligned}$$

and a direct computation yields

$$(31) \quad K_{j\bar{k}} = \frac{1}{2}(\mu_{r\bar{s}}^{\bar{j}}\mu_{rs}^k - \mu_{j\bar{s}}^{\bar{r}}\mu_{ks}^r - \mu_{jr}^s\mu_{k\bar{r}}^{\bar{s}}) + \frac{1}{4}(\mu_{rs}^k\mu_{r\bar{s}}^{\bar{j}} - 2\mu_{rj}^s\mu_{r\bar{k}}^{\bar{s}}) + \frac{1}{2}\mu_{r\bar{r}}^{\bar{j}}\mu_{ss}^k - \frac{1}{2}(\mu_{r\bar{r}}^{\bar{s}}\mu_{kj}^s + \mu_{r\bar{r}}^s\mu_{j\bar{k}}^{\bar{s}}).$$

Remark 5.1. We observe that, if we assume that the Lie group G is complex and 2-step nilpotent, then (31) reduces to the one studied in [12]. Moreover, if the Lie group G is nilpotent and it is equipped with an abelian complex structure, then (31) reduces to the one studied in [12, Section 6].

The following theorem provides a proof of a part of [12, Theorem 6.2] for our class of Lie groups. To prove the long-time existence of $\partial_t g_t = -\text{Ric}^{1,1}(g_t)$, using our techniques, we need an additional assumption (see Theorem 5.4).

Theorem 5.2. *Let (G, J) be a simply-connected, 2-step nilpotent Lie group equipped with a left-invariant complex structure J . Assume that the Lie algebra $(\mathfrak{g}, \mu)$ of G is such that $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$. A left-invariant Hermitian metric g is balanced if and only if the trace of the tensor K coincides with the Riemannian scalar curvature. Moreover, if g is balanced, then the tensor K coincides with the $(1, 1)$ -component of the Riemannian Ricci tensor.*

Proof. Let us denote by g a left-invariant Hermitian metric on (G, J) . Moreover, denote by ∇^{LC} the Levi-Civita connection of g and by Γ_{ir}^j the Christoffel symbols of ∇^{LC} . Then, the $(1, 1)$ -component of the Riemannian Ricci tensor is given by

$$\begin{aligned} \text{Ric}_{j\bar{k}} &= \Gamma_{r\bar{r}}^s\Gamma_{js}^k + \Gamma_{r\bar{r}}^{\bar{s}}\Gamma_{j\bar{s}}^k - \Gamma_{j\bar{r}}^s\Gamma_{rs}^k - \Gamma_{j\bar{r}}^{\bar{s}}\Gamma_{r\bar{s}}^k - \mu_{js}^r\Gamma_{r\bar{s}}^k + \Gamma_{r\bar{r}}^s\Gamma_{js}^k + \Gamma_{r\bar{r}}^{\bar{s}}\Gamma_{j\bar{s}}^k - \Gamma_{jr}^s\Gamma_{r\bar{s}}^k - \mu_{j\bar{s}}^r\Gamma_{rs}^k - \mu_{j\bar{s}}^{\bar{r}}\Gamma_{r\bar{s}}^k \\ &= (\Gamma_{r\bar{r}}^s + \Gamma_{r\bar{r}}^{\bar{s}})\Gamma_{js}^k + (\Gamma_{r\bar{r}}^{\bar{s}} + \Gamma_{r\bar{r}}^s)\Gamma_{j\bar{s}}^k - (\Gamma_{j\bar{r}}^s + \mu_{j\bar{s}}^r)\Gamma_{rs}^k - (\Gamma_{jr}^s + \mu_{j\bar{s}}^{\bar{r}})\Gamma_{r\bar{s}}^k - \mu_{js}^r\Gamma_{r\bar{s}}^k - \Gamma_{j\bar{r}}^{\bar{s}}\Gamma_{r\bar{s}}^k. \end{aligned}$$

By using the Koszul's formula, we get

$$\begin{aligned} \Gamma_{kr}^s &= \frac{1}{2}(\mu_{kr}^s - \mu_{r\bar{s}}^{\bar{k}} - \mu_{k\bar{s}}^{\bar{r}}), \quad \Gamma_{kr}^{\bar{s}} = \frac{1}{2}(\mu_{kr}^{\bar{s}} - \mu_{r\bar{s}}^k - \mu_{k\bar{l}}^{\bar{r}}), \quad \Gamma_{kr}^{\bar{s}} = \frac{1}{2}(\mu_{kr}^{\bar{s}} - \mu_{k\bar{s}}^{\bar{r}} - \mu_{rs}^k), \\ \Gamma_{k\bar{r}}^s &= \frac{1}{2}(\mu_{k\bar{r}}^s - \mu_{k\bar{s}}^r - \mu_{r\bar{s}}^{\bar{k}}), \quad \Gamma_{k\bar{r}}^{\bar{s}} = \frac{1}{2}(\mu_{k\bar{r}}^{\bar{s}} - \mu_{k\bar{s}}^{\bar{r}} - \mu_{rs}^{\bar{k}}). \end{aligned}$$

Since G is a 2-step nilpotent Lie group, it is in particular a unimodular Lie group. We recall that, once a unimodular Lie group is equipped with a left-invariant Hermitian structure, we can read the unimodular condition in terms of a left-invariant unitary frame as

$$\mu_{ir}^r + \mu_{i\bar{r}}^{\bar{r}} = 0, \quad i = 1, \dots, n.$$

Hence, we get that $\Gamma_{r\bar{r}}^s + \Gamma_{r\bar{r}}^{\bar{s}} = 0$ and $\Gamma_{r\bar{r}}^{\bar{s}} + \Gamma_{r\bar{r}}^s = 0$. Moreover,

$$\Gamma_{j\bar{r}}^s + \mu_{j\bar{s}}^r = \frac{1}{2}(\mu_{j\bar{r}}^s + \mu_{j\bar{s}}^r - \mu_{r\bar{s}}^{\bar{j}}), \quad \Gamma_{jr}^s + \mu_{j\bar{s}}^{\bar{r}} = \frac{1}{2}(\mu_{jr}^s - \mu_{r\bar{s}}^{\bar{j}} + \mu_{j\bar{s}}^{\bar{r}}).$$

By using that the Lie group is 2-step nilpotent and $J\mu(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$, we get

$$\text{Ric}_{j\bar{k}} = \frac{1}{4}(\mu_{jr}^s\mu_{r\bar{k}}^{\bar{s}} + \mu_{r\bar{s}}^{\bar{j}}\mu_{rs}^k - \mu_{s\bar{j}}^{\bar{r}}\mu_{s\bar{k}}^r - \mu_{j\bar{r}}^s\mu_{k\bar{r}}^{\bar{s}} + \mu_{j\bar{s}}^{\bar{r}}\mu_{s\bar{k}}^r + \mu_{r\bar{s}}^{\bar{j}}\mu_{rs}^k + \mu_{j\bar{r}}^s\mu_{r\bar{k}}^{\bar{s}} - \mu_{js}^r\mu_{s\bar{k}}^{\bar{r}} + \mu_{r\bar{s}}^{\bar{j}}\mu_{rs}^k) + \frac{1}{2}\mu_{js}^r\mu_{s\bar{k}}^{\bar{r}},$$

and a direct computation yields

$$\text{Ric}_{j\bar{k}} = \frac{1}{2}(\mu_{js}^r\mu_{s\bar{k}}^{\bar{r}} + \mu_{r\bar{s}}^{\bar{j}}\mu_{rs}^k - \mu_{s\bar{j}}^{\bar{r}}\mu_{s\bar{k}}^r - \mu_{j\bar{r}}^s\mu_{k\bar{r}}^{\bar{s}}) + \frac{1}{4}\mu_{r\bar{s}}^{\bar{j}}\mu_{rs}^k.$$

Therefore,

$$K_{j\bar{k}} - \text{Ric}_{j\bar{k}} = \frac{1}{2}(\mu_{r\bar{r}}^{\bar{j}}\mu_{ss}^k - \mu_{r\bar{r}}^{\bar{s}}\mu_{kj}^s - \mu_{r\bar{r}}^s\mu_{j\bar{k}}^{\bar{s}}),$$

and

$$k - \text{Tr}_g \text{Ric} = -\frac{1}{2}\mu_{r\bar{r}}^s\mu_{l\bar{l}}^{\bar{s}}.$$

Since G is unimodular then the metric g is balanced if and only if $\sum_l \mu(Z_l, Z_{\bar{l}}) = 0$. Therefore, the metric g is balanced if and only if $k = \text{Tr}_g \text{Ric}$.

Moreover, if g is balanced, then the tensor K coincides with the $(1,1)$ -component of the Riemannian Ricci tensor, as required. $\square$

Remark 5.3. E. Fusi pointed out to me that, by using [2, Equation 2.8], one can prove that, in general, for nilpotent Lie groups, a left-invariant Hermitian metric g is balanced if and only if the trace of the tensor K coincides with the Riemannian scalar curvature.

Our last result concerns the long-time existence of the solution of $\partial_t g_t = -\text{Ric}^{1,1}(g_t)$ for simply-connected, 2-step nilpotent Lie group G equipped with a left-invariant complex structure J that preserves the commutators of the Lie algebra $(\mathfrak{g}, \mu_0)$ of G . We remark that, since the Lie group is 2-step nilpotent, then $J\mu_0(\mathfrak{g}, \mathfrak{g})$ is contained in the center of $\mathfrak{g}$.

Note that, if $X \in \mu_0(\mathfrak{g}, \mathfrak{g})$, then $P_g(X) \in \mu_0(\mathfrak{g}, \mathfrak{g})$, where P_g is the endomorphism associated to $\text{Ric}^{1,1}(g)$ as in (12). Thus, with respect to the block representation $\mathfrak{g} = \mu_0(\mathfrak{g}, \mathfrak{g})^\perp \oplus \mu_0(\mathfrak{g}, \mathfrak{g})$, the endomorphism P_g has the following form

$$(32) \quad P_g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

Let us consider the following space

$$\mathfrak{V} \doteq \{ \mu \in \tilde{\mathfrak{L}} : \mu \text{ is 2-step nilpotent and } J\mu(\mathfrak{g}, \mathfrak{g}) \subseteq \mu_0(\mathfrak{g}, \mathfrak{g}) \},$$

and consider the action of $N \doteq \{ f \in \text{GL}(\mathfrak{g}, J) \mid f\mu_0(\mathfrak{g}, \mathfrak{g}) \subseteq \mu_0(\mathfrak{g}, \mathfrak{g}) \}$ on $\mathfrak{V}$.

Theorem 5.4. *Let (G, J) be a simply-connected, 2-step nilpotent Lie group equipped with a left-invariant complex structure J . Assume that the Lie algebra $(\mathfrak{g}, \mu)$ of G is such that J preserves the commutators. The parabolic flow $\partial_t g_t = -\text{Ric}^{1,1}(g_t)$ has always a long-time solution for every left-invariant initial Hermitian metric.*

Proof. To prove the theorem we use the bracket flow technique, as was done in the proof of Theorem 1.1. The bracket flow equation associated to this flow is the following

$$\frac{d}{dt}\mu_t = -\pi(P_{\mu_t})\mu_t, \quad \mu|_{t=0} = \mu_0,$$

where, for $\mu \in \mathfrak{V}$, we have

$$(P_\mu)_j^l = \frac{1}{2}(\mu_{js}^r \mu_{\bar{s}l}^{\bar{r}} + \mu_{\bar{r}s}^{\bar{j}} \mu_{rs}^l - \mu_{\bar{s}j}^{\bar{r}} \mu_{sl}^r - \mu_{jr}^s \mu_{\bar{l}r}^{\bar{s}}) + \frac{1}{4}\mu_{\bar{r}s}^{\bar{j}} \mu_{rs}^l.$$

Let us consider a real endomorphism E that commutes with J and such that $E\mu_0(\mathfrak{g}, \mathfrak{g}) \subseteq \mu_0(\mathfrak{g}, \mathfrak{g})$, then

$$\langle E, P_\mu \rangle = 2\text{Re}(E_j^l (P_\mu)_{\bar{j}}^{\bar{l}}) = \text{Re}\{E_j^l (\mu_{\bar{r}s}^j \mu_{rs}^{\bar{l}} - \mu_{\bar{s}j}^r \mu_{sl}^{\bar{r}} - \mu_{\bar{j}r}^{\bar{s}} \mu_{lr}^s + \mu_{js}^{\bar{r}} \mu_{sl}^r)\} + \frac{1}{2}\text{Re}\{E_j^l \mu_{rs}^j \mu_{\bar{r}s}^{\bar{l}}\}.$$

Moreover, if $\mu \in \mathfrak{V}$, then

$$\begin{aligned} \langle \pi(E)\mu(\cdot, \cdot), \mu(\cdot, \cdot) \rangle &= \langle E \circ \mu(\cdot, \cdot), \mu(\cdot, \cdot) \rangle - \langle \mu(E\cdot, \cdot), \mu(\cdot, \cdot) \rangle - \langle \mu(\cdot, E\cdot), \mu(\cdot, \cdot) \rangle \\ &= 2\text{Re}\{E_j^l \mu_{sr}^j \mu_{\bar{s}r}^{\bar{l}}\} + 4\text{Re}\{E_j^l (\mu_{\bar{r}s}^j \mu_{rs}^{\bar{l}} - \mu_{sl}^r \mu_{\bar{s}j}^{\bar{r}} - \mu_{\bar{l}r}^s \mu_{jr}^{\bar{s}} - \mu_{\bar{s}l}^{\bar{r}} \mu_{rs}^s)\}, \end{aligned}$$

hence

$$\langle E, P_\mu \rangle = \frac{1}{4}\langle \pi(E)\mu(\cdot, \cdot), \mu(\cdot, \cdot) \rangle.$$

Thus,

$$\frac{d}{dt}\|\mu_t\|^2 = 2\langle \frac{d}{dt}\mu_t, \mu_t \rangle = -2\langle \pi(P_{\mu_t})\mu_t, \mu_t \rangle = -8\langle P_{\mu_t}, P_{\mu_t} \rangle = -8\|P_{\mu_t}\|^2 \leq 0,$$

and the thesis follows. $\square$

Remark 5.5. We recall that the pluriclosed flow evolves the fundamental form ω of a Strong Kähler with torsion metric g in the direction of the $(1, 1)$ -component of the Ricci form associated with the Bismut connection of g . Meanwhile, in the setting considered in this section, the flow evolves a Hermitian metric in the direction of the $(1, 1)$ -component of the Ricci tensor of the Levi-Civita connection. We would like to point out that, in [8, Theorem 1.1] and [3, Theorem A], the authors establish long-time existence and describe the asymptotic behavior of the solution to the pluriclosed flow on 2-step nilpotent Lie groups endowed with a left-invariant complex structure, without requiring any additional assumptions on the complex structure. Whereas in our case, using our techniques, we need extra assumptions on the complex structure.

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