

Remoteness, order, size and connectivity constraints in digraphs

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Abstract

Let D be a strongly connected digraph. The average distance of a vertex v in D is defined as the arithmetic mean of the distances from v to all other vertices in D . The remoteness $\rho(D)$ of D is the maximum of the average distances of the vertices in D .

In this paper, we provide a sharp upper bound on the remoteness of a strong digraph with given order, size, and vertex-connectivity. We then characterise the extremal digraphs that maximise remoteness among all strong digraphs of order n , size at least m , and vertex-connectivity κ . Finally, we demonstrate that the upper bounds on the remoteness of a graph given its order, size, and connectivity constraints (see [15]) can be extended to a larger class of digraphs containing all graphs, the Eulerian digraphs.

Keywords: Remoteness; transmission; average distance; size; vertex-connectivity, edge-connectivity, strong digraphs

MSC-class: 05C12

1 Introduction

While distances in graphs have been the subject of extensive study, the investigation of distances in digraphs remains relatively underdeveloped. In particular, the concepts of proximity and remoteness have not been explored as thoroughly in the directed setting as they have in the undirected case. Ai et al. [1] were the first to extend these notions to digraphs, establishing foundational results. This paper aims to contribute to the growing literature on distances in digraphs by providing new results on the maximum remoteness of strong digraphs under certain constraints. To the best of our knowledge,

this is the second paper to focus specifically on the study of remoteness in the context of directed graphs.

Let D be a strongly connected, finite digraph of order n . In a digraph D of order at least two, the average distance $\bar{\sigma}(v, D)$ of a vertex v is defined as the arithmetic mean of the directed distances from v to all other vertices of D , that is,

$$\bar{\sigma}(v, D) = \frac{1}{n-1} \sum_{w \in V(D)} d_D(v, w),$$

where $d_D(v, w)$ denotes the length of the shortest directed path from v to w . The *remoteness* $\rho(D)$ of a strongly connected digraph D is then defined as the maximum of the average distances of its vertices, namely

$$\rho(D) = \max_{v \in V(D)} \bar{\sigma}(v, D),$$

where $V(D)$ is the vertex set of D .

The above definition can be viewed as the directed analogue of the notion of remoteness in graphs.

The term *remoteness* was first appeared in a paper titled *automated comparison of graph invariants* by Aouchiche, Caporossi, and Hansen [3], and has since gained widespread usage. Nevertheless, the concept and related ideas had been explored earlier under different terminology. For instance, Zelinka [29] investigated the *vertex deviation*, defined as $\frac{\sigma(v, G)}{n}$, where $\sigma(v, G)$ is the sum of distances from vertex v to all other vertices in the graph, and n is the total number of vertices. Other researchers have referred to $\sigma(v, G)$ using terms such as *transmission* (e.g., [24]), *total distance*, or simply *distance*.

Bounds on remoteness in terms of order only were given by Zelinka [29] and later, independently, by Aouchiche and Hansen [4].

Theorem 1.1. (Zelinka [29], Aouchiche, Hansen [4])
Let G be a connected graph of order $n \geq 2$. Then

$$\rho(G) \leq \frac{n}{2},$$

with equality if and only if G is a path.

The bound in Theorem 1.1 was extended to digraphs by Ai et al. [1].

Theorem 1.2. (Ai, Gerke, Gutin, Mafunda [1])
Let D be a strong digraph of order $n \geq 3$. Then

$$\rho(D) \leq \frac{n}{2},$$

with equality if and only if D is strong and contains a Hamiltonian dipath $v_1v_2 \dots v_n$ such that no directed edge of the form v_iv_j with $2 \leq i+1 < j \leq n$ is in D .

In [18], Entringer, Jackson, and Snyder strengthen the results of Theorem 1.1 by incorporating the size of a graph.

Theorem 1.3. (Entringer, Jackson, Snyder [18])

Let G be a connected graph of order n and size at least m . Then

$$\rho(G) \leq \frac{n+2}{2} - \frac{m}{n-1}.$$

Recently, Dankelmann et al. [15] demonstrated that the bound in Theorem 1.3 can be significantly improved for κ -connected graphs, where κ is arbitrary, and for λ -edge-connected graphs with $\lambda \in \{2, 3\}$.

Theorem 1.4. (Dankelmann, Mafunda, Mallu [15])

Let G be a κ -connected graph of order n and size at least m with $m \leq \binom{n-1}{2}$. Then

$$\rho(G) \leq \frac{n}{2\kappa} + 2 - \frac{1}{\kappa} - \frac{\kappa-1}{n-1} - \frac{m}{\kappa(n-1)}.$$

The bound is sharp.

Theorem 1.5. (Dankelmann, Mafunda, Mallu [15])

(a) Let G be a 2-edge-connected graph of order n and size m . Then

$$\rho(G) \leq \begin{cases} \frac{n}{3} - \frac{\frac{n}{3}}{3(n-1)} + \frac{5}{3} & \text{if } m < \lceil \frac{5}{3}n \rceil - 2, \\ \frac{n}{3} - \frac{\frac{2m}{3}}{3(n-1)} + \frac{5}{3} & \text{if } m \geq \lceil \frac{5}{3}n \rceil - 2, \end{cases}$$

and this bound is sharp apart from an additive constant.

(b) Let G be a 3-edge-connected graph of order n and size m . Then

$$\rho(G) \leq \begin{cases} \frac{n}{4} - \frac{\frac{n}{4}}{2(n-1)} + \frac{3}{2} & \text{if } m < \lceil \frac{9}{4}n \rceil - 2, \\ \frac{n}{4} - \frac{\frac{m}{2}}{2(n-1)} + \frac{3}{2} & \text{if } m \geq \lceil \frac{9}{4}n \rceil - 2, \end{cases}$$

and this bound is sharp apart from an additive constant.

The literature contains several results on the remoteness of graphs, ranging from bounds on the remoteness of various classes of graphs to the relationships between remoteness and other graph parameters. There are results on remoteness in outerplanar graphs [16], in triangulations and quadrangulations [8, 17], in graphs that forbid certain cycles [12], and in trees [7, 29]. Relations between remoteness and other graph parameters have also been explored for example, girth [5], minimum degree [9], maximum degree [14], and clique number [19]. Differences between remoteness and other graph

parameters have also been studied; see, for example, [4, 10, 13, 17]. A survey on proximity and remoteness in graphs is provided in [6].

This paper is organised as follows. In Section 2, we introduce the terminology and notation used throughout. In Section 3, we establish an upper bound on the remoteness of a strong digraph with given order, size, and vertex-connectivity, and characterise the extremal digraphs that attain this bound. In Section 4, we show that the upper bounds obtained on remoteness for graphs with given order, size, and connectivity constraints in [15] can be generalised to a larger class of digraphs that includes all graphs, in particular the Eulerian digraphs.

2 Terminology and notation

We use the following notation. For a strong digraph D we denote by $V(D)$ and $E(D)$ the *vertex set* and *edge set* (often also called arc set), respectively (D is strong if for every pair u, v of vertices, D contains both a $u - v$ path and a $v - u$ path). The *order* and *size* of D are denoted by $n(D)$ and $m(D)$, respectively. For vertices u and v in D , an arc (u, v) is sometimes denoted by writing $\overrightarrow{uv}$. By an (n, m) -digraph we mean a digraph of order n and size at least m . An Eulerian circuit in a strong digraph D is a closed directed trail. Furthermore, we say that a strong digraph D is an *Eulerian digraph* if it contains an Eulerian circuit.

The *vertex-connectivity* $\kappa(D)$ of a digraph D is the minimum number of vertices whose removal results in a digraph that is not strongly connected. The *edge-connectivity* $\lambda(D)$ of a digraph D is the minimum number of arcs that must be removed to make the digraph not strongly connected.

For a digraph D , the distance $d_D(v, w)$ is defined as the minimum number of arcs on a path from v to w . The *eccentricity* $\text{ecc}_D(v)$ of a vertex v in a digraph D is the distance from v to a vertex farthest from v . The largest of all eccentricities of vertices of D is called the *diameter* and is denoted by $\text{diam}(D)$. For $v \in V(D)$, let $N_i(v) = \{x \in V(D) \mid d_D(v, x) = i\}$ and $|N_i(v)| = n_i(v)$ for $i \in \mathbb{Z}^+$. Clearly, $n_i(v) > 0$ if and only if $0 \leq i \leq \text{ecc}_D(v)$. The *distance degree* of v is the sequence $(n_0(v), n_1(v), n_2(v), \dots, n_d(v))$ where $d \in \mathbb{Z}^+$ and is denoted by $X_D(v)$. Let $N_{\leq i}(v) = \{x \in V(D) \mid d_D(v, x) \leq i\}$ and $N_{\geq i}(v) = \{x \in V(D) \mid d_D(v, x) \geq i\}$.

We denote the complete graph of order n by K_n . If $D_1, D_2, \dots, D_k$ are disjoint digraphs, then the *sequential sum* $D_1 + D_2 + \dots + D_k$ is the digraph obtained from their union by adding an arc from every vertex in D_i to every vertex in D_{i+1} and from every vertex in D_{i+1} to every vertex in D_i .

for $i = 1, 2, \dots, k-1$. The sequential sum of undirected graphs is defined analogously. If G is an undirected graph then by $\overleftrightarrow{G}$ we mean the digraph that is obtained from G by replacing each undirected edge by two arcs in the opposite direction. Hence, by $\overleftrightarrow{K_n}$ we mean a *complete digraph* of order n .

If $t, k \in \mathbb{N}$, then $[K_{a_1} + K_{a_2} + \dots + K_{a_t}]^k$ stands for k repetitions of the pattern $K_{a_1} + K_{a_2} + \dots + K_{a_t}$. If D is a digraph, then the *complement* of D , denoted by $\overline{D}$, is the graph on the same vertex set as D , in which two vertices are adjacent if they are not adjacent in D .

If D and D' are distinct digraphs with the same vertex set, but $A(D) \subset A(D')$, i.e., D' is obtained from D by adding arcs, we will denote this relationship by $D \subsetneq D'$. If (k, b) and (k', b') are distinct pairs of integers, then we write $(k, b) \prec (k', b')$ if (k, b) comes before (k', b') in the lexicographic ordering of pairs of integers, i.e., if either $k < k'$ or $k = k'$ and $b < b'$. With the necessary adjustments, all the above definitions also hold for graphs.

3 Maximum remoteness of a strong digraph with given order and size

In this section, we determine the maximum remoteness of κ -connected strong digraphs for a given order, size, and vertex connectivity. While the proof technique is partially inspired by the work of Dankelmann et al. [15], the digraph setting presents significantly greater challenges. The arguments developed here demand a more detailed analysis, as well as the construction of more sophisticated extremal example. This section begins with the introduction of the necessary definitions.

Definition 3.1. Let $D_1, D_2, \dots, D_k$ be digraphs. We define the digraph

$$D_1 \overleftarrow{+} D_2 \overleftarrow{+} D_3 \overleftarrow{+} \dots \overleftarrow{+} D_k$$

as the digraph obtained from the disjoint union of $D_1, D_2, \dots, D_k$ by adding directed arcs in both directions between each vertex in D_i and each vertex in D_{i+1} for $i = 1, 2, \dots, k-1$, and directed arcs from each vertex in D_i to each vertex in D_j for each $i = 3, 4, \dots, k$ and for all $j \in \{1, 2, \dots, i-2\}$.

Definition 3.2. Let $\ell, a, b \in \mathbb{N}$ with $a \geq \kappa$. A κ -**connected path-complete digraph** is a digraph of the form

$$K_1 \overleftarrow{+} [\overleftrightarrow{K_\kappa}]^\ell \overleftarrow{+} \overleftrightarrow{K_a} \overleftarrow{+} \overleftrightarrow{K_b}.$$

Where $[\overleftrightarrow{K_\kappa}]^\ell$ stands for ℓ repetition of $\overleftrightarrow{K_\kappa}$ such that $[\overleftrightarrow{K_\kappa} \overleftarrow{+} \overleftrightarrow{K_\kappa} \overleftarrow{+} \overleftrightarrow{K_\kappa} \overleftarrow{+} \dots \overleftarrow{+} \overleftrightarrow{K_\kappa}]$.

See for an example κ -connected path-complete digraph with $\ell = 2$ in Figure 1.

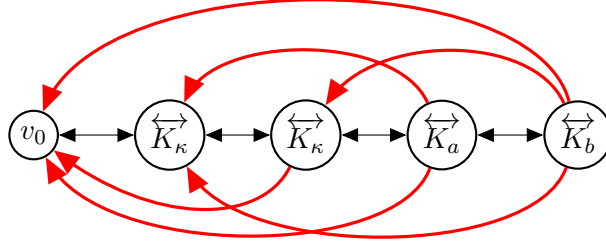


Figure 1: The κ -connected path-complete digraph with $\ell = 2$.

Clearly, this κ -connected path-complete digraph is strong and has diameter at least 3. Furthermore, we refer to a 1-connected path-complete digraph simply as a **path-complete digraph**.

Lemma 3.1. (a) Let H be a κ -connected path-complete digraph, where $\kappa \in \mathbb{N}$. Then $\rho(H + \overrightarrow{uv}) < \rho(H)$ for any arc $\overrightarrow{uv} \in E(\overline{H})$.
(b) Let H_1, H_2 be two distinct κ -connected path-complete strong digraphs of order n . Then either $m(H_1) < m(H_2)$ and $\rho(H_1) > \rho(H_2)$, or $m(H_1) > m(H_2)$ and $\rho(H_1) < \rho(H_2)$.
(c) Given n, κ . Then there exists a κ -connected path-complete strong digraph of order n and size m if and only if $m \equiv (n^2 - 2n - 1) \pmod{\kappa}$ and $\frac{n}{2}(3\kappa + n) - n - \kappa^2 - b(\kappa - b) \leq m \leq n^2 - 2n - 1$, where $b \in \{1, 2, \dots, \kappa\}$ with $b \equiv n - 1 \pmod{\kappa}$.

Proof. (a) Let $a, b, \kappa \in \mathbb{N}$ with $H = K_1 \overleftarrow{\overrightarrow{K_\kappa}}^\ell \overleftarrow{\overrightarrow{K_a}} \overleftarrow{\overrightarrow{K_b}}$ and let v_0 be a vertex in the leftmost K_1 . Then, clearly v_0 uniquely attains the remoteness of H . Now, adding any new arc to H reduces the distance from v_0 to some vertex, and so it reduces $\sigma(v_0, H)$ and thus the remoteness, thereby establishing part (a) of the lemma.

(b) Let H_1 and H_2 be two distinct κ -connected path-complete strong digraphs of order n . It suffices to show that

$$H_2 \not\leq H_1 \quad \text{or} \quad H_1 \not\leq H_2, \quad (1)$$

To see this observe that if $H_2 \leq H_1$, then $m(H_2) < m(H_1)$, and by part (a), we have $\rho(H_2) > \rho(H_1)$. Similarly, if $H_1 \leq H_2$, then $m(H_1) < m(H_2)$, and $\rho(H_1) > \rho(H_2)$. In both cases part (b) of the lemma holds.

To prove (1), let $\ell, \ell', a, a', b, b' \in \mathbb{N}$ with $a, a' \geq \kappa$ such that

$$H_1 = K_1 \xleftrightarrow{\leftarrow} [K_\kappa]^\ell \xleftrightarrow{\leftarrow} K_a \xleftrightarrow{\leftarrow} K_b, \quad H_2 = K_1 \xleftrightarrow{\leftarrow} [K_\kappa]^{\ell'} \xleftrightarrow{\leftarrow} K_{a'} \xleftrightarrow{\leftarrow} K_{b'},$$

with ℓ and ℓ' repetitions of $\xleftrightarrow{\leftarrow}_{\kappa}$ for H and H' , respectively.

Since $H_1 \neq H_2$, it follows that $(\ell, b) \neq (\ell', b')$. We have either $(\ell, b) \prec (\ell', b')$ or $(\ell', b') \prec (\ell, b)$. Without loss of generality we may assume the former. First assume that $\ell = \ell'$ and $b < b'$. Then H_1 is obtained from H_2 by adding arcs from the rightmost complete digraph $\xleftrightarrow{\leftarrow}_{\kappa}$ to $b' - b$ vertices of the complete digraph $\xleftrightarrow{\leftarrow}_{\kappa}$, and so we have that $H_2 \not\leq H_1$. Now assume that $\ell < \ell'$. We obtain H_1 from H_2 by adding arcs as follows: first, add arcs from all vertices of the first rightmost complete digraph $\xleftrightarrow{\leftarrow}_{\kappa}$ to all vertices of the complete digraph $\xleftrightarrow{\leftarrow}_{\kappa}$ in H_2 , resulting in the κ -connected strong digraph $K_1 \xleftrightarrow{\leftarrow} [K_\kappa]^{\ell'} \xleftrightarrow{\leftarrow} K_{a'+b'}$. From this point onwards, we successively add arcs from all vertices of the second rightmost complete digraph $\xleftrightarrow{\leftarrow}_{\kappa}$ to all vertices of the last rightmost complete digraph $\xleftrightarrow{\leftarrow}_{\kappa}$, where $q_i = a' + b' + i\kappa$ for $i \in \{1, 2, \dots, \ell' - \ell - 1\}$ and i denote the i^{th} iteration. This yields the κ -connected path-complete digraphs $K_1 \xleftrightarrow{\leftarrow} [K_\kappa]^{\ell'-1} \xleftrightarrow{\leftarrow} K_{q_1}$, $K_1 \xleftrightarrow{\leftarrow} [K_\kappa]^{\ell'-2} \xleftrightarrow{\leftarrow} K_{q_2}$, and so on until after repeating this process $\ell' - \ell - 1$ we obtain the κ -connected path-complete digraph $K_1 \xleftrightarrow{\leftarrow} [K_\kappa]^{\ell+1} \xleftrightarrow{\leftarrow} K_{q_{\ell'-\ell-1}}$. Note that $a' + b' + (\ell' - \ell - 1)\kappa = a + b - \kappa$.

Finally, by adding arcs from the second rightmost complete digraph $\xleftrightarrow{\leftarrow}_{\kappa}$ to a subset of $a - \kappa$ vertices of the complete digraph $\xleftrightarrow{\leftarrow}_{\kappa}$ in $K_1 \xleftrightarrow{\leftarrow} [K_\kappa]^{\ell+1} \xleftrightarrow{\leftarrow} K_{q_{\ell'-\ell-1}}$, we obtain the digraph $K_1 \xleftrightarrow{\leftarrow} [K_\kappa]^\ell \xleftrightarrow{\leftarrow} K_{a+b}$, which is H_1 . Hence, $H_2 \not\leq H_1$, and (1) follows, confirming that part (b) holds.

(c) Fix n and κ . If for $\ell, a, b \in \mathbb{N}$ the strong digraph $K_1 \xleftrightarrow{\leftarrow} [K_\kappa]^\ell \xleftrightarrow{\leftarrow} K_a \xleftrightarrow{\leftarrow} K_b$ has order n and is κ -connected, then $n = 1 + \ell\kappa + a + b$, and $a \geq \kappa$. This implies that $\ell = \frac{n-1-a-b}{\kappa} \leq \frac{n-2-\kappa}{\kappa}$, and $b = n - 1 - \ell\kappa - a \leq n - 1 - (\ell + 1)\kappa$. With respect to the order $\prec$, the smallest and largest pairs (ℓ, b) satisfying these conditions are $(1, 1)$ and (ℓ_0, b_0) , respectively, where $\ell_0 = \lfloor \frac{n-2-\kappa}{\kappa} \rfloor$ and $b_0 = n - 1 - (\ell_0 + 1)\kappa$. It thus follows as in the proof of (1) that the κ -connected path-complete strong digraph $K_1 \xleftrightarrow{\leftarrow} K_\kappa \xleftrightarrow{\leftarrow} K_{n-\kappa-2} \xleftrightarrow{\leftarrow} K_1$, arising from the pair $(1, 1)$, has maximum size among all κ -connected path-complete strong digraphs of order n . Simple calculations show that its size is $n^2 - 2n - 1$. The κ -connected path-complete strong digraph $K_1 \xleftrightarrow{\leftarrow} [K_\kappa]^{\ell_0} \xleftrightarrow{\leftarrow} K_{a_0} \xleftrightarrow{\leftarrow} K_{b_0}$, arising from the pair (ℓ_0, b_0) has minimum size among κ -connected path-complete strong digraphs of order n . Its size is $\frac{n}{2}(3\kappa + n) - n - \kappa^2 - b(\kappa - b)$.

The proof of part (b) shows that if $m(H_1) < m(H_2)$, then H_2 is obtained from H_1 by adding arcs, and the number of arcs added is a multiple of κ . Hence the number of arcs of a κ -connected path-complete strong digraph of order n is congruent $n^2 - 2n - 1$ modulo κ . If H_1 is a κ -connected path-complete strong digraph of order n , $H_1 = K_1 \xleftarrow{\quad} [K_\kappa]^\ell \xleftarrow{\quad} \overleftrightarrow{K_a} \xleftarrow{\quad} \overleftrightarrow{K_b}$, then unless $(\ell, b) = (0, 1)$, there exists a κ -connected path-complete strong digraph of order n with exactly κ more arcs than H_1 : the strong digraph $K_1 \xleftarrow{\quad} [K_\kappa]^\ell \xleftarrow{\quad} \overleftrightarrow{K_{a+1}} \xleftarrow{\quad} \overleftrightarrow{K_{b-1}}$ (if $b > 1$) or the strong digraph $K_1 \xleftarrow{\quad} [K_\kappa]^{\ell-1} \xleftarrow{\quad} \overleftrightarrow{K_{a+1}}$ (if $\ell > 0$ and $b = 1$). This completes the proof of part (c). $\square$

Given $n, m \in \mathbb{N}$ for which there exists a κ -connected path-complete (n, m) -strong digraph, we define $DPK_{n,m,\kappa}$ to be such a graph of minimum size. It follows from Lemma 3.1(a) that there exists at most one κ -connected path-complete strong digraph of given order and size, so $DPK_{n,m,\kappa}$ is well-defined.

Theorem 3.2. (a) Let D be a κ -connected strong digraph of order n and size m with $m \leq n^2 - 2n - 1$. Then

$$\rho(D) \leq \rho(DPK_{n,m,\kappa}). \quad (2)$$

(b) Assume that $m \equiv (n^2 - 2n - 1) \pmod{\kappa}$ and $\frac{n}{2}(3\kappa + n) - n - \kappa^2 - b(\kappa - b) \leq m \leq n^2 - 2n - 1$, where b is the integer in $\{1, 2, \dots, \kappa\}$ with $b \equiv n - 1 \pmod{\kappa}$. Then equality in (a) holds only if $D = DPK_{n,m,\kappa}$.

Proof. We first prove that there exists a κ -connected path-complete (n, m) -strong digraph D' with

$$\rho(D) \leq \rho(D'). \quad (3)$$

We may assume that D has maximum remoteness among all κ -connected (n, m) -strong digraphs, and that among all such strong digraphs with maximum remoteness, D is one with the maximum size. Furthermore, let $v \in V(D)$ with $\bar{\sigma}(v, D) = \rho(D)$, $d = \text{ecc}_D(v)$, $N_i = \{z \in V(D) \mid d_D(v, z) = i\}$, and $|N_i| = n_i$ for $i \in \{0, 1, \dots, d\}$. Clearly, $n_0 = 1$, and $\sum_{i=0}^d n_i = n$.

Claim 1: $D = K_{n_0} \xleftarrow{\quad} \overleftrightarrow{K_{n_1}} \xleftarrow{\quad} \dots \xleftarrow{\quad} \overleftrightarrow{K_{n_{d-1}}} \xleftarrow{\quad} \overleftrightarrow{K_{n_d}}$.

Recall that D has the maximum size among all strong digraphs of size at least m for which $\sigma(v, D)$ is maximised. Consequently, each N_i induces a complete digraph $\overleftrightarrow{K_{n_i}}$ in the digraph D , otherwise we could add arcs between vertices of N_i without changing the remoteness of D . Additionally, D has arcs from every vertex in N_i to every vertex in N_{i+1} , and arcs from every vertex in N_{i+1} to every vertex in N_i for $i = 0, 1, 2, \dots, d-1$. Furthermore, every vertex in N_j is adjacent to every vertex in N_i for all $j > i$ where $i, j \in \{0, 1, \dots, d\}$. Hence, Claim 1 follows.

Note that $n_i \geq \kappa$ holds for $i = 1, 2, \dots, d-1$, but not for 0.

Claim 2: For all $i \in \{1, 2, \dots, d-3\}$, we have $n_i = \kappa$.

Suppose to the contrary that there exists $j \in \{1, \dots, d-3\}$ with $n_j > \kappa$. Let j be the smallest such value. Then $n_i = \kappa$ for all $i \in \{1, \dots, j-1\}$. Now consider the strong digraph D^* that is obtained from D by moving a vertex from N_j to N_{j+1} , i.e., $D^* = K_{n'_0} \xleftarrow{\quad} \xrightarrow{\quad} K_{n'_1} \xleftarrow{\quad} \xrightarrow{\quad} \dots \xleftarrow{\quad} \xrightarrow{\quad} K_{n'_d}$, where $n'_i = n_i$ for $i \in \{0, 1, \dots, d\} - \{j, j+1\}$, $n'_j = n_j - 1$ and $n'_{j+1} = n_{j+1} + 1$. Then $m(D^*) = m(D) + 2(n_{j+2} - n_{j-1}) \geq m(D)$ since $n_{j+2} \geq \kappa = n_{j-1}$ and D^* is (n, m) -strong digraph with $\sigma(v, D^*) > \sigma(v, D)$, and thus $\rho(D^*) > \rho(D)$. This contradiction to the maximality of $\rho(D)$ proves Claim 2.

Claim 3: $n_{d-2} = \kappa$.

Since D is κ -connected, we have that $n_{d-2} \geq \kappa$. Now, suppose to the contrary that $n_{d-2} \neq \kappa$. Then $n_{d-2} > \kappa$. Consider the strong digraph D^* , which is obtained from D by moving one vertex from N_{d-2} and N_d to N_{d-1} , i.e., $D^* = K_{n'_0} \xleftarrow{\quad} \xrightarrow{\quad} K_{n'_1} \xleftarrow{\quad} \xrightarrow{\quad} \dots \xleftarrow{\quad} \xrightarrow{\quad} K_{n'_d}$, where $n'_i = n_i$ for $i \in \{0, 1, \dots, d-3\} - \{d-2, d-1, d\}$, $n'_{d-2} = n_{d-2} - 1$, $n'_d = n_d - 1$ and $n'_{d-1} = n_{d-1} + 2$. It is easy to verify that $m(D^*) = m(D) + 2n_d - n_{d-3} + n_{d-2} - 1$. Since $n_{d-3} = \kappa$ and $n_{d-2} \geq \kappa + 1$, we have $m(D^*) \geq m(D) + 2n_d - 1 > m(D)$. Moreover, $\rho(D^*) = \rho(D)$. This contradicts our choice of D as a digraph with the maximum size among those of maximum remoteness. Thus, Claim 3 follows.

It follows from Claims 1 to 3 that D is a κ -connected path-complete strong digraph. Letting $D' = D$ proves (3).

By (3), there exists a κ -connected path-complete strong digraph D' of order n and size at least m with $\rho(D) \leq \rho(D')$. By the definition of $DPK_{n,m,\kappa}$, we have $m(D') \geq m(DPK_{n,m,\kappa})$. By Lemma 3.1(a), it follows that $\rho(D) \leq \rho(DPK_{n,m,\kappa})$. Hence

$$\rho(D) \leq \rho(D') \leq \rho(DPK_{n,m,\kappa}),$$

which proves (a).

(b) Now assume that equality holds in (2), i.e., that $\rho(D) = \rho(DPK_{n,m,\kappa})$, and furthermore that $m \equiv (n^2 - 2n - 1) \pmod{\kappa}$ and $\frac{n}{2}(3\kappa + n) - n - \kappa^2 - b(\kappa - b) \leq m \leq n^2 - 2n - 1$, where b is as defined above. It follows from Lemma 3.1(c) that the graph $DPK_{n,m,\kappa}$ has exactly m arcs.

It follows from part (b) that D has maximum remoteness among all κ -connected (n, m) -strong digraphs. We claim that D has maximum size

among all such strong digraphs maximising the remoteness. Suppose not. Then there exists a κ -connected $(n, m+1)$ -strong digraph D'' with $\rho(D) = \rho(D'')$. Applying (2) to D'' we get that

$$\rho(D) = \rho(D'') \leq \rho(DPK_{n,m+1,\kappa}) < \rho(DPK_{n,m,\kappa}),$$

where the last inequality follows from Lemma 3.1(b) and the fact that $m(DPK_{n,m,\kappa}) = m < m(DPK_{n,m+1,\kappa})$. Hence D has maximum size among all κ -connected strong digraphs of order n maximising the remoteness.

The proof of (a) shows that, if D has maximum size among all κ -connected path-complete (n, m) -strong digraphs, then D is a path-complete strong digraph. Hence $D = DPK_{n,m',\kappa}$ for some m' with $m' \geq m$. Since by Lemma 3.1 we have $\rho(DPK_{n,m',\kappa}) < \rho(DPK_{n,m,\kappa})$ if $m' > m$, we have $m' = m$, and thus $D = DPK_{n,m,\kappa}$, as desired. $\square$

Evaluating the remoteness of $DPK_{n,m,\kappa}$, yields the following corollary. Assume that $m \equiv (n^2 - 2n - 1) \pmod{\kappa}$ and $\frac{n}{2}(3\kappa + n) - n - \kappa^2 - b(\kappa - b) \leq m \leq n^2 - 2n - 1$, where $b \in \{1, 2, \dots, \kappa\}$ with $b \equiv n - 1 \pmod{\kappa}$.

Corollary 3.3. *Let D be a κ -connected strong digraph of order n and size m , with $\frac{n}{2}(3\kappa + n) - n - \kappa^2 - b(\kappa - b) \leq m \leq n^2 - 2n - 1$, where $b \in \{1, 2, \dots, \kappa\}$ with $b \equiv n - 1 \pmod{\kappa}$. Let m^* be the smallest integer with $m^* \geq m$ and $m^* \equiv (n^2 - 2n - 1) \pmod{\kappa}$. Then*

$$\rho(D) \leq \frac{n}{\kappa} + 2 - \frac{1}{\kappa} - \frac{\kappa - 1}{n - 1} - \frac{m^*}{\kappa(n - 1)},$$

and this bound is sharp.

Proof. Let m^* be as defined above. It follows from Lemma 3.1 that the digraph $DPK_{n,m,\kappa}$ has size m^* . Note that $a + b = n - \ell\kappa - 1$. Let v_0 be the vertex in first K_1 of $DPK_{n,m,\kappa}$, so v_0 realizes the remoteness. Let $H := DPK_{n,m,\kappa} - V(\overleftrightarrow{K_a} \cup \overleftrightarrow{K_b})$. Straightforward calculations show that $n(H) = \ell\kappa + 1$, $\sigma(v_0, H) = \frac{\ell\kappa}{2}(\ell + 1)$ and $m(H) = \frac{1}{2}(\ell\kappa^2)(\ell + 3) - \kappa(\kappa - 1)$. Hence

$$\sigma(v_0, DPK_{n,m,\kappa}) = \sigma(v_0, H) + (\ell + 1)(a + b) + b = \frac{\kappa\ell}{2}(\ell + 1) + (\ell + 1)(a + b) + b,$$

and, since $m(DPK_{n,m,\kappa}) = m^*$,

$$\begin{aligned} m^* &= m(H) + 2\binom{a+b}{2} + 2\kappa a + (a+b)\ell\kappa - (\kappa-1)a + b \\ &= \frac{1}{2}(\ell\kappa^2)(\ell + 3) - \kappa(\kappa - 1) + (a+b)(a+b-1) + 2\kappa a + (a+b)\ell\kappa \\ &\quad - (\kappa - 1)a + b. \end{aligned}$$

Define $\epsilon = \rho(DPK_{n,m,\kappa}) - \left(\frac{n}{\kappa} + 2 - \frac{1}{\kappa} - \frac{\kappa-1}{n-1} - \frac{m^*}{\kappa(n-1)}\right)$. Substituting the above terms for $\rho(DPK_{n,m,\kappa})$ and m^* , it is straightforward to verify that $\epsilon = 0$. This proves that $\rho(DPK_{n,m,\kappa}) = \frac{n}{\kappa} + 2 - \frac{1}{\kappa} - \frac{\kappa-1}{n-1} - \frac{m^*}{\kappa(n-1)}$, and the corollary follows. $\square$

For $\kappa = 1$, Corollary 3.3 yields the following corollary. Note that for $\kappa = 1$ we have $m^* = m$.

Corollary 3.4. *Let D be a strong digraph of order n and size m , with $\frac{(n-1)(n+2)}{2} \leq m \leq n^2 - 2n - 1$. Then*

$$\rho(D) \leq n + 1 - \frac{m}{n-1},$$

and this bound is sharp.

Note that the bound in the above corollary also holds for $m < \frac{(n-1)(n+2)}{2}$, i.e., for smaller values of m , and it is sharp; however, the extremal digraph is not unique.

4 Extending bounds on remoteness of graphs for given size and connectivity constraints to Eulerian digraphs

In this section we give sharp upper bounds on the remoteness of an Eulerian digraph with given order n and size at least m . We do the same for Eulerian digraphs with connectivity constraints such as vertex connectivity, as well as similar bounds when vertex connectivity is replaced by edge-connectivity for $\lambda \in \{2, 3\}$. There are bounds on distances in Eulerian digraphs that improve bounds for strong, and not necessarily Eulerian, digraphs (see [11], [22]).

Dankelmann [11] established an upper bound on the size of an Eulerian digraph for given diameter and order n .

Theorem 4.1. (Dankelmann [11])

Let D be an Eulerian digraph of diameter d , and v be a vertex of eccentricity d . If $X(v) = (n_0, \dots, n_d)$ then

$$m(D) \leq 2m(K_{n_0} + K_{n_1} + \dots + K_{n_d}),$$

equality holds if and only if $D = \overleftrightarrow{K_{n_0}} + \overleftrightarrow{K_{n_1}} + \dots + \overleftrightarrow{K_{n_{d-2}}} + \overleftrightarrow{K_{n_{d-1}}} + \overleftrightarrow{K_{n_d}}$.

We now apply Theorem 4.1 in order to obtain sharp upper bounds on the maximum remoteness for Eulerian digraphs of given order, size and vertex connectivity from sharp bounds for graphs. We first introduce the additional notation required to understand the corresponding results for graphs proved by Dankelmann et al. [15].

Definition 4.1. (Dankelmann, Mafunda, Mallu [15])

A graph G is said to be a κ -connected path-complete graph if there exist $\ell, a, b \in \mathbb{N}$, $a \geq \kappa$, with

$$G = K_1 + [K_\kappa]^\ell + K_a + K_b.$$

For given $n, m \in \mathbb{N}$ for which there exists a κ -connected path-complete graph of order n and size at least m , Dankelmann et al. [15] defined $PK_{n,m,\kappa}$ to be such a graph of minimum size. Note that for graphs of diameter greater than 2, Definition 4.1 generalises the path-complete graphs defined by Soltés in [27], which are the 1-connected path-complete graphs.

Theorem 4.2. (Dankelmann, Mafunda, Mallu [15])

Let G be a κ -connected graph of order n and size at least m with $m \leq \binom{n-1}{2}$. Then

$$\rho(G) \leq \rho(PK_{n,m,\kappa}) = \frac{n}{2\kappa} + 2 - \frac{1}{\kappa} - \frac{\kappa-1}{n-1} - \frac{m}{\kappa(n-1)}.$$

Equality holds only if $G = PK_{n,m,\kappa}$ and m is congruent $\binom{n-1}{2} \pmod{\kappa}$ and $\frac{1}{2}[n(3\kappa-1) - 2\kappa^2 - \kappa + 1 - b(\kappa-b)] \leq m \leq \binom{n-1}{2}$, where $b \in \{1, 2, \dots, \kappa\}$ with $b \equiv \binom{n-1}{2} \pmod{\kappa}$.

Theorem 4.3. Let D be a κ -connected Eulerian digraph of order n and size at least $2m_0$, where $m_0 \in \mathbb{N}$. Then

$$\rho(D) \leq \rho(\overleftrightarrow{PK}_{n,m_0,\kappa}) = \frac{n}{2\kappa} + 2 - \frac{1}{\kappa} - \frac{\kappa-1}{n-1} - \frac{m_0}{\kappa(n-1)}.$$

The bound is sharp if $m_0 \equiv \binom{n-1}{2} \pmod{\kappa}$ and $\frac{1}{2}[n(3\kappa-1) - 2\kappa^2 - \kappa + 1 - b(\kappa-b)] \leq m_0 \leq \binom{n-1}{2}$, where b is the integer in $\{1, 2, \dots, \kappa\}$ with $b \equiv \binom{n-1}{2} \pmod{\kappa}$.

Proof. Let D be an arbitrary κ -connected Eulerian digraph of order n and size at least $2m_0$. Assume v is chosen such that $\rho(D) = \overline{\sigma}(v, D)$. Let $X_D(v) = (n_0, n_1, n_2, \dots, n_d)$ be the distance degree of v , where $d \in \mathbb{N}$. Furthermore, note that for all $i \in \{1, 2, \dots, d-1\}$, we have that $n_i \geq \kappa$, since each N_i is a cutset of D .

Let D' be the digraph $\overleftrightarrow{K}_{n_0} + \overleftrightarrow{K}_{n_1} + \dots + \overleftrightarrow{K}_{n_{d-2}} + \overleftrightarrow{K}_{n_{d-1}} + \overleftrightarrow{K}_{n_d}$. Clearly the digraph D' is Eulerian with the same order as D and is κ -connected since $X_D(v) = X_{D'}(v)$. Furthermore, $\rho(D') = \rho(D)$ and by Theorem 4.1 we have that $m(D') \geq m(D) \geq 2m_0$.

Consider the underlying graph G' of D' . Note that the graph G' is κ -connected since $X_G(v) = X_{D'}(v)$ and $\rho(G') = \rho(D')$. Since $m(D') = 2m(G')$, we have that $m(G') \geq m_0$ and by Theorem 4.2, we have that

$$\rho(G') \leq \rho(PK_{n,m_0,\kappa}).$$

Now we consider the digraph $\overleftrightarrow{PK_{n,m_0,\kappa}}$. Note that $\overleftrightarrow{PK_{n,m_0,\kappa}}$ is also κ -connected Eulerian digraph with size at least $2m_0$ since $m(\overleftrightarrow{PK_{n,m_0,\kappa}}) = 2m(PK_{n,m_0,\kappa})$ and further $\rho(PK_{n,m_0,\kappa}) = \rho(\overleftrightarrow{PK_{n,m_0,\kappa}})$. Hence we obtain that

$$\rho(D) = \rho(D') = \rho(G') \leq \rho(PK_{n,m_0,\kappa}) = \rho(\overleftrightarrow{PK_{n,m_0,\kappa}}).$$

If $m_0 \equiv \binom{n-1}{2} \pmod{\kappa}$ and $\frac{1}{2}[n(3\kappa-1) - 2\kappa^2 - \kappa + 1 - b(\kappa-b)] \leq m_0 \leq \binom{n-1}{2}$, the Eulerian digraph $\overleftrightarrow{PK_{n,m_0,\kappa}}$, which has size $2m_0$, shows that the bound is sharp. $\square$

The following corollary is an immediate consequence of Theorem 4.3.

Corollary 4.4. *Let D be an Eulerian digraph of order n and size at least $2m_0$, where $m_0 \in \mathbb{N}$. Then*

$$\rho(D) \leq \rho(\overleftrightarrow{PK_{n,m_0}}) = \frac{n+2}{2} - \frac{m_0}{n-1}.$$

This bound is sharp.

Next we use Theorem 4.1 to obtain a sharp upper bound on the size of Eulerian digraphs of given order, size, and λ -edge-connectivity for $\lambda \in \{2, 3\}$. We first introduce the additional notation needed to understand the corresponding results for graphs proved by Dankelmann et al. [15].

Definition 4.2. (Dankelmann, Mafunda, Mallu [15])

Let $\lambda \in \{2, 3\}$. A graph G is said to be a λ -edge-connected path-complete graph if there exist $k \in \mathbb{N} \cup \{0\}$ and $a, b \in \mathbb{N}$ with

$$G = \begin{cases} [K_1 + K_\lambda]^k + K_a + K_b & \text{if } k \geq 1 \text{ and } ab \geq \lambda, \text{ or} \\ [K_1 + K_\lambda]^k + K_1 + K_a + K_b & \text{if } a \geq \lambda, \text{ or} \\ [K_1 + K_3]^k + K_2 + K_a + K_1 & \text{if } \lambda = 3, k \geq 1 \text{ and } a \geq 3. \end{cases}$$

For given $n, m \in \mathbb{N}$ for which there exists a λ -edge-connected path-complete graph of order n and size at least m , Dankelmann et al. [15] defined $PK_{n,m}^\lambda$ to be such a graph of minimum size, where $\lambda \in \{2, 3\}$.

Theorem 4.5. (Dankelmann, Mafunda, Mallu [15])

Let $\lambda \in \{2, 3\}$ and let G be a λ -edge-connected graph of order n and size at least m . Then

$$\rho(G) \leq \rho(PK_{n,m}^\lambda).$$

Theorem 4.6. (Dankelmann, Mafunda, Mallu [15])

(a) Let G be a 2-edge-connected graph of order n and size m . Then

$$\rho(G) \leq \begin{cases} \frac{n}{3} - \frac{\frac{n}{3}}{2m} + \frac{5}{3} & \text{if } m < \lceil \frac{5}{3}n \rceil - 2, \\ \frac{n}{3} - \frac{\frac{n}{3}}{3(n-1)} + \frac{5}{3} & \text{if } m \geq \lceil \frac{5}{3}n \rceil - 2, \end{cases}$$

and this bound is sharp apart from an additive constant.

(b) Let G be a 3-edge-connected graph of order n and size m . Then

$$\rho(G) \leq \begin{cases} \frac{n}{4} - \frac{\frac{n}{4}}{2m} + \frac{3}{2} & \text{if } m < \lceil \frac{9}{4}n \rceil - 2, \\ \frac{n}{4} - \frac{\frac{n}{4}}{2(n-1)} + \frac{3}{2} & \text{if } m \geq \lceil \frac{9}{4}n \rceil - 2, \end{cases}$$

and this bound is sharp apart from an additive constant.

Theorem 4.7. Let D be a λ -edge-connected Eulerian digraph of order n and size at least $2m_0$, where $m_0 \in \mathbb{N}$ and $\lambda \in \{2, 3\}$. Then

$$\rho(D) \leq \overleftrightarrow{\rho(PK_{n,m_0}^\lambda)}.$$

This bound is sharp.

Proof. Let D be an arbitrary λ -edge-connected Eulerian digraph of order n and size at least $2m_0$ with $\lambda \in \{2, 3\}$. Assume v is chosen such that $\rho(D) = \overline{\sigma}(v, D)$. Let $X_D(v) = (n_0, n_1, n_2, \dots, n_d)$ be the distance degree of v , where $d \in \mathbb{Z}^+$. Furthermore, note that for all $i \in \{0, 1, 2, \dots, d-1\}$, we have that $n_i n_{i+1} \geq \lambda$ since $N_i \cup N_{i+1}$ is an edge-cut.

Let D' be the digraph $\overleftarrow{K}_{n_0} + \overleftarrow{K}_{n_1} + \dots + \overleftarrow{K}_{n_{d-2}} + \overleftarrow{K}_{n_{d-1}} + \overleftarrow{K}_{n_d}$. Clearly the digraph D' is Eulerian with the same order as D and D' is also λ -edge-connected since $X_D(v) = X_{D'}(v)$. Furthermore, $\rho(D') = \rho(D)$ and by Theorem 4.1 we have that $m(D') \geq m(D) \geq 2m_0$.

Consider the underlying graph G' of D' . Note that the graph G' is λ -edge-connected since $X_G(v) = X_{D'}(v)$ and $\rho(G') = \rho(D')$. Since $m(D') = 2m(G')$, we have that the size $m(G') \geq m_0$ and by Theorem 4.5, we have that $\rho(G') \leq \rho(PK_{n,m_0}^\lambda)$.

Now we consider the digraph $\overleftarrow{PK}_{n,m_0}^\lambda$. Note that $\overleftarrow{PK}_{n,m_0}^\lambda$ is also λ -edge-connected Eulerian digraph with size at least $2m_0$ since $m(\overleftarrow{PK}_{n,m_0}^\lambda) = 2m(PK_{n,m_0}^\lambda)$ and further $\rho(PK_{PK_{n,m_0}^\lambda}^\lambda) = \rho(\overleftarrow{PK}_{n,m_0}^\lambda)$. Hence we obtain that

$$\rho(D) = \rho(D') = \rho(G') \leq \rho(PK_{n,m_0}^\lambda) = \rho(\overleftarrow{PK}_{n,m_0}^\lambda),$$

where both D and $\overleftarrow{PK}_{n,m_0}^\lambda$ are λ -edge-connected with size at least $2m_0$. This completes the proof. $\square$

The following corollary is an immediate consequence of Theorem 4.7.

Corollary 4.8. (a) Let D be a 2-edge-connected Eulerian digraph of order n and size at least $2m_0$, where $m_0 \in \mathbb{N}$. Then

$$\rho(D) \leq \begin{cases} \frac{n}{3} - \frac{\frac{n}{3}}{3(n-1)} + \frac{5}{3} & \text{if } m_0 < \lceil \frac{5}{6}n \rceil - 1, \\ \frac{n}{3} - \frac{2m_0}{3(n-1)} + \frac{5}{3} & \text{if } m_0 \geq \lceil \frac{5}{6}n \rceil - 1, \end{cases}$$

and this bound is sharp apart from an additive constant.

(b) Let D be a 3-edge-connected Eulerian digraph of order n and size at least $2m_0$, where $m_0 \in \mathbb{N}$. Then

$$\rho(G) \leq \begin{cases} \frac{n}{4} - \frac{\frac{n}{4}}{2(n-1)} + \frac{3}{2} & \text{if } m_0 < \lceil \frac{9}{8}n \rceil - 1, \\ \frac{n}{4} - \frac{m_0}{2(n-1)} + \frac{3}{2} & \text{if } m_0 \geq \lceil \frac{9}{8}n \rceil - 1, \end{cases}$$

and this bound is sharp apart from an additive constant.

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