Loop functions of sunset diagrams in 2+1 space-time dimensions

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Abstract

In these notes the relativistic n-body phase-phase is calculated iteratively in $2+1$ space-time dimensions for all n. The obtained result shows a simple power-law behavior $\alpha_n(\mu - M)^{n-2}/\mu$ with a dependence only on the total mass $M = m_1 + \cdots + m_n$. As a consequence of this feature, the $(n-1)$-loop integrals $J_n(-q^2)$ associated to sunset diagrams with n internal lines can be expressed through elementary (arctangent and logarithmic) functions, modulo polynomial terms in q^2 with regularization-dependent coefficients. An outlook to the analogous situation in $4+1$ space-time dimensions is given by computing the n-body phase-phases for $n = 2, 3, 4, 5$ with their totally symmetric dependence on the involved masses. Moreover, a digression to $1+1$ space-time dimensions reveals that there the three-body phase-space is already proportional to a complete elliptic integral.

1 Introduction and summary

The understanding of the structure behind the master-integrals that occur in the evaluation of multi-loop Feynman diagrams is a subject of high relevance in current research in theoretical high-energy physics. Numerous advanced mathematical topics, such as Picard-Fuchs differential equations, elliptic curves, Calabi-Yau manifolds, Hopf algebras, cluster algebras, etc. are encountered in these sophisticated studies [1]. Moreover, it was found that there exist relations between the sets of master integrals when the space-time dimension d is increased or lowered by two units, $d \to d \pm 2$.

In the present notes the loop functions associated to the sunrise diagrams with n internal lines are analyzed in $2+1$ space-time dimensions. The pertinent imaginary part is the integrated n-body phase-space, which exhibits a remarkably simple form, $\Gamma_n(\mu; m_1, \ldots, m_n) = \alpha_n(\mu - M)^{n-2}/\mu$, with a power-law dependence on the energy μ and a parametric dependence on the masses only through the total sum $M = m_1 + \cdots + m_n$. As a consequence of this feature the loop-integrals $J_n(-q^2)$ (in the euclidean domain) can be expressed in terms of elementary (arctangent and logarithmic) functions, modulo polynomial terms in q^2 with regularization-dependent coefficients. An outlook to the analogous situation in $4+1$ space-time dimensions is also given by computing the n-body phase-phases for $n = 2, 3, 4, 5$ with their dependence on elementary symmetric polynomials of the involved masses $m_1, \ldots, m_n$. One finds $\widetilde{\Gamma}_n(\mu; m_1, \ldots, m_n) = (8\pi)^{3-2n}\mu^{-3}(\mu - M)^{2n-3}P_{n+1}(\mu)$, where the last factor is a polynomial in μ of degree $n+1$. Moreover, a digression to $1+1$ space-time dimensions reveals that there the three-body phase-space is already proportional to a complete elliptic integral. This signals a similar complexity of multi-loop integrals as in the physical $3+1$ space-time dimensions.

2 Integrated two-, three-, and n-body phase space

With the usual normalization factors, the two-body phase-space with final-state particles of masses m_1 and m_2 is calculated in $2+1$ space-time dimensions as follows:

$$\begin{aligned}\Gamma_2(\mu; m_1, m_2) &= \int \frac{d^2k_1}{(2\pi)^2 2\omega_1} \int \frac{d^2k_2}{(2\pi)^2 2\omega_2} (2\pi)^3 \delta^{(3)}\big(\vec{P} - \vec{k}_1 - \vec{k}_2\big) \\ &= \frac{1}{4}\int_0^\infty dk \frac{k}{\sqrt{m_1^2 + k^2}\sqrt{m_2^2 + k^2}} \,\delta\Big(\mu - \sqrt{m_1^2 + k^2} - \sqrt{m_2^2 + k^2}\Big), \qquad (1)\end{aligned}$$

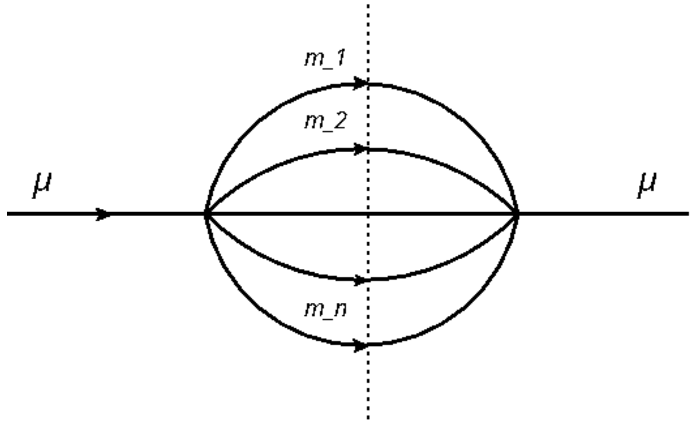

Figure 1: Sunset diagram with n internal lines carrying masses $m_1, m_2, \ldots, m_n$. The vertical dotted line symbolizes the cut, which gives rise to the integrated n-body phase-space.

with $\vec{P} = (\mu, 0, 0)$ in the center-of-mass frame. The derivative of the argument of the δ-function with respect to k, inserted with its reciprocal value, cancels the other factors of the integrand, and only the center-of-mass energy $\mu = \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2}$ remains as a denominator. Thus one obtains the following simple result for the integrated two-body phase-space in $2+1$ dimensions:

$$\Gamma_2(\mu; m_1, m_2) = \frac{1}{4\mu}\,\theta(\mu - m_1 - m_2)\,, \tag{2}$$

which exhibits a non-zero starting value at threshold $\mu_{\rm th} = m_1 + m_2$.

The calculation of the three-body phase-space with final state particles of masses m_1, m_2, m_3 proceeds as follows:

$$\begin{aligned}\Gamma_3(\mu; m_1, m_2, m_3) &= \int \frac{d^2k_1}{(2\pi)^2 2\omega_1} \int \frac{d^2k_2}{(2\pi)^2 2\omega_2} \int \frac{d^2k_3}{(2\pi)^2 2\omega_3}\,(2\pi)^3 \delta^{(3)}\big(\vec{P} - \vec{k}_1 - \vec{k}_2 - \vec{k}_3\big) \\ &= \frac{1}{16\pi^2} \iint\limits_{D(\omega_1,\omega_2)>0} d\omega_1 d\omega_2 \big[D(\omega_1, \omega_2)\big]^{-1/2}\,, \end{aligned} \tag{3}$$

where the angular integration has been performed, and the cubic polynomial $D(\omega_1, \omega_2) = (\mu^2 - 2\mu\,\omega_1 + m_1^2)(\omega_2^+ - \omega_2)(\omega_2 - \omega_2^-) > 0$ demarcates the kinematically allowed Dalitz region in the $\omega_1\omega_2$-plane. Using $\int_{\omega_2^-}^{\omega_2^+} d\omega_2 [(\omega_2^+ - \omega_2)(\omega_2 - \omega_2^-)]^{-1/2} = \pi$ and the upper boundary $\omega_1^{\rm end} = (\mu^2 + m_1^2 - (m_2 + m_3)^2)/2\mu$, one arrives at the elementary integral

$$\Gamma_3(\mu; m_1, m_2, m_3) = \frac{1}{16\pi} \int_{m_1}^{\omega_1^{\rm end}} d\omega_1 (\mu^2 - 2\mu\,\omega_1 + m_1^2)^{-1/2}\,, \tag{4}$$

which has the solution

$$\Gamma_3(\mu; m_1, m_2, m_3) = \frac{1}{16\pi\mu}(\mu - m_1 - m_2 - m_3)\,, \tag{5}$$

and the obvious condition $\mu > m_1 + m_2 + m_3$ must hold. One observes that in $2+1$ dimensions the three-body phase-space grows linearly from threshold, and interestingly it depends only on the total sum $M = m_1 + m_2 + m_3$ of the three masses.

It is well-known that the integrated phase-space for n final state particles (with masses $m_1, \ldots, m_n$) can be recursively built up from the expression for the two-body phase-space. The formula connecting the integrated phase-spaces for n and $n-1$ particles reads (after adaption of 2π factors) from ref.[2]:

$$\Gamma_n(\mu; m_1, \ldots, m_n) = \frac{1}{\pi} \int_{m_1+\cdots+m_{n-1}}^{\mu - m_n} d\mu' \mu'\, \Gamma_2(\mu; \mu', m_n)\, \Gamma_{n-1}(\mu'; m_1, \ldots, m_{n-1})\,, \tag{6}$$

with μ' the total center-of-mass energy of the $(n-1)$-particle subsystem. Using the result in eq.(2) for the two-body phase-space, one reproduces quickly the just derived expression for the integrated three-body phase-space

$$\Gamma_3(\mu; m_1, m_2, m_3) = \frac{1}{16\pi}\int_{m_1+m_2}^{\mu-m_3} d\mu' \frac{\mu'}{\mu\,\mu'} = \frac{1}{16\pi\mu}(\mu - m_1 - m_2 - m_3)\,. \tag{7}$$

The integrated phase-space for four, five, and more particles in $2+1$ dimensions can be recursively calculated with the help of eq.(6). For n particles the result is given by the power-law expression

$$\Gamma_n(\mu; M) = \alpha_n \frac{(\mu - M)^{n-2}}{\mu}\,, \tag{8}$$

with $M = \sum_{j=1}^n m_j$ the total sum of all masses, and the coefficient α_n is determined by a recursion relation as:

$$\alpha_n = \frac{\alpha_{n-1}}{4\pi}\frac{1}{n-2}\,, \qquad \alpha_2 = \frac{1}{4}\,, \qquad \alpha_n = \big[4(4\pi)^{n-2}(n-2)!\big]^{-1}\,. \tag{9}$$

In summary, this leads to an extraordinarily simple result for the integrated n-body phase-space in $2+1$ dimensions:

$$\Gamma_n(\mu; M) = \frac{(\mu - M)^{n-2}}{4\mu(4\pi)^{n-2}(n-2)!}\,, \qquad M = \sum_{j=1}^{n} m_j\,. \tag{10}$$

3 Loop functions of sunset diagrams

The sunset diagram of order n, shown in Fig. 1, is a $(n-1)$-loop diagram, where the incoming leg (carrying energy-momentum $\vec{P}$) splits at one point into n internal lines, corresponding to the propagators of scalar particles with masses $m_1, \dots, m_n$, and then these n lines recombine at one point to the outgoing leg (carrying again energy-momentum $\vec{P}$). The associated loop function $J_n(s)$ with $s = \vec{P}\cdot\vec{P}$ has an imaginary part for $s > (m_1 + \cdots + m_n)^2 = M^2$ that is equal to 1/2 times the integrated n-body phase-space:

$$\mathrm{Im}J_n(s) = \frac{1}{2}\Gamma_n(\sqrt{s}; M)\,. \tag{11}$$

In the euclidean region $s = -q^2 < 0$ the loop function is conveniently calculated via the dispersion relation

$$J_n(-q^2) = \frac{1}{\pi}\int_M^\infty d\mu\, \frac{\mu\,\Gamma_n(\mu; M)}{\mu^2 + q^2}\,, \tag{12}$$

but subtractions (at $q^2 = 0$) are necessary for $n \geq 3$ due to the polynomial growth of $\Gamma_n(\mu; M)$ with μ. The introduces coefficients of a polynomial subtraction term, that need to be calculated in some regularization scheme, like cutoff or dimensional regularization.

One finds for $n = 2$ the one-loop function

$$J_2(-q^2) = \frac{1}{4\pi q}\arctan\frac{q}{m_1 + m_2}\,. \tag{13}$$

This result is also readily obtained with the help of Feynman parametrization as

$$J_2(-q^2) = \frac{1}{8\pi}\int_0^1 dx \big[m_1^2 x + m_2^2(1-x) + q^2 x(1-x)\big]^{-1/2} = \frac{1}{4\pi\, q}\arctan\frac{q}{m_1+m_2}\,. \tag{14}$$

With little effort one finds for $n=3$ the two-loop function:

$$J_3(-q^2) - J_3(0) = \frac{1}{16\pi^2}\left\{1 - \frac{M}{q}\arctan\frac{q}{M} - \frac{1}{2}\ln\left(1+\frac{q^2}{M^2}\right)\right\}, \tag{15}$$

with $M = m_1 + m_2 + m_3$. The divergent subtraction constant $J_3(0)$ is expected to involve the logarithm of M, and the computation in d dimensions with a regularization scale κ gives:

$$\begin{aligned} J_3(0) &= (4\pi)^{-d}\Gamma(3-d)\int_0^1 dx \int_0^1 dy \frac{y^{1-d/2}H^{d-3}\kappa^{6-2d}}{\left[1+(x-1-x^2)y\right]^{d/2}} \\ &= \frac{1}{64\pi^3}\int_0^1 dx \int_0^1 dy \frac{1}{\sqrt{y}[1+(x-1-x^2)y]^{3/2}}\left\{\frac{1}{3-d} - \gamma_E + \ln 4\pi - 2\ln\frac{M}{\kappa}\right. \\ &\qquad \left. - \ln\frac{H}{M^2} + \frac{1}{2}\ln\left[y+(x-1-x^2)y^2\right]\right\}, \end{aligned} \tag{16}$$

with $H = m_1^2(1-y) + m_2^2(1-x)y + m_3^2 xy$, and one has expanded around $d=3$. The integral $\int_0^1 dy$ for the (four) terms is the second line is performed analytically and it leads to $\int_0^1 dx\, 2(x-x^2)^{-1/2} = 2\pi$. The same contribution, $\int_0^1 dx\, 2(x-x^2)^{-1/2} = 2\pi$, comes as a part after performing analytically the integral $\int_0^1 dy$ for the (two) terms in the third line, while a remainder with a complicated dependence on the masses m_1, m_2, m_3 gives a numerical value very close to zero, at the level of 10^{-14}. When taking it for granted that this remainder depends only on the sum $M = m_1 + m_2 + m_3$, one can set $m_1 = m_2 = 0$ and obtains directly $\int_0^1 dx[x(1-x)]^{-1/2}[\ln(1-x) - \ln x] = 0$. Thus the subtraction constant $J_3(0)$, split off in eq.(15), has in dimensional regularization the value

$$J_3(0) = \frac{1}{32\pi^2}\left\{\frac{1}{3-d} - \gamma_E + \ln 4\pi - 2\ln\frac{M}{\kappa} + 1\right\}. \tag{17}$$

One finds for $n=4$ the three-loop function:

$$J_4(-q^2) - J_4(0) = \frac{1}{2(4\pi)^3}\left\{\frac{M^2-q^2}{q}\arctan\frac{q}{M} - M + M\ln\left(1+\frac{q^2}{M^2}\right)\right\}, \tag{18}$$

with $M = m_1 + m_2 + m_3 + m_4$, and $J_4(0)$ a yet undetermined subtraction constant that will include the logarithmic piece $(4\pi)^{-3}M\ln(M/\kappa)$.

One finds for $n=5$ the four-loop function:

$$\begin{aligned} J_5(-q^2) - J_5(0) + q^2 J_5'(0) &= \frac{1}{6(4\pi)^4}\left\{\frac{M}{q}(3q^2 - M^2)\arctan\frac{q}{M}\right. \\ &\quad \left. + \frac{1}{2}(q^2 - 3M^2)\ln\left(1+\frac{q^2}{M^2}\right) + M^2 - \frac{11}{6}q^2\right\}, \end{aligned} \tag{19}$$

with $M = m_1 + m_2 + m_3 + m_4 + m_5$. Moreover, $J_5(0) = -\frac{1}{2}(4\pi)^{-4}M^2\ln(M/\kappa) + \ldots$ and $J_5'(0) = -\frac{1}{6}(4\pi)^{-4}\ln(M/\kappa) + \ldots$ are two regularization-dependent subtraction constants.

Modulo additive polynomial terms in q^2, the result for the $(n-1)$-loop function associated to the sunset diagram (Fig. 1) reads in $2+1$ dimensions:

$$J_n(-q^2) = \frac{1}{(4\pi)^{n-1}(n-2)!}\left\{A_n(q^2, M)\frac{1}{q}\arctan\frac{q}{M} + L_n(q^2, M)\left[\ln\left(1+\frac{q^2}{M^2}\right) + 2\ln\frac{M}{\kappa}\right] + \text{ polyn}\right\}, \tag{20}$$

where the polynomials in q^2 multiplying the arctangent and logarithmic function are

$$A_n(q^2,M)=\mathrm{Re}\big[(iq-M)^{n-2}\big]\,,\qquad L_n(q^2,M)=-\frac{1}{2q}\mathrm{Im}\big[(iq-M)^{n-2}\big]\,. \tag{21}$$

These expressions are derived from the partial fraction decomposition

$$\frac{(\mu-M)^{n-2}}{\mu^2+q^2}=\frac{(iq-M)^{n-2}}{2iq(\mu-iq)}+\frac{(-iq-M)^{n-2}}{-2iq(\mu+iq)}+\text{polyn}\,, \tag{22}$$

followed by a recombination into terms proportional to $1/(\mu^2+q^2)$ and $-2\mu/(\mu^2+q^2)$, and observing that these two pieces integrate to the basic functions $q^{-1}\arctan(q/M)$ and $\ln(1+q^2/M^2)+2\ln(M/\kappa)$. The extra logarithmic term $\ln(M/\kappa)$ is obtained by treating the scale κ as ultraviolet cutoff in the dispersion integral. The analytical continuation of the $(n-1)$-loop function to positive values of s is given for the real part by

$$\begin{aligned}\mathrm{Re}\,J_n(s) \;&=\; \frac{1}{(4\pi)^{n-1}(n-2)!}\bigg\{A_n(-s,M)\,\frac{1}{2\sqrt{s}}\ln\frac{M+\sqrt{s}}{|M-\sqrt{s}|}\\ &\quad +L_n(-s,M)\bigg[\ln\Big|1-\frac{s}{M^2}\Big|+2\ln\frac{M}{\kappa}\bigg]+\ \text{polyn}\bigg\}\,.\end{aligned} \tag{23}$$

4 Extension to 4+1 space-time dimensions

For curiosity, let us investigate the analogous situation with n-body phase-spaces in $4+1$ space-time dimensions. The two-body phase-space with final-state particles of masses m_1 and m_2 is now calculated as follows:

$$\begin{aligned}\widetilde{\Gamma}_2(\mu;m_1,m_2) \;&=\; \int\frac{d^4k_1}{(2\pi)^4 2\omega_1}\int\frac{d^4k_2}{(2\pi)^4 2\omega_2}\,(2\pi)^5\delta^{(5)}\big(\vec{P}-\vec{k}_1-\vec{k}_2\big)\\ &=\; \frac{1}{16\pi}\int_0^\infty dk\frac{k^3}{\sqrt{m_1^2+k^2}\sqrt{m_2^2+k^2}}\,\delta\Big(\mu-\sqrt{m_1^2+k^2}-\sqrt{m_2^2+k^2}\,\Big)\,,\end{aligned} \tag{24}$$

with $\vec{P}=(\mu,0,0,0,0)$ in the center-of-mass frame. Dividing by the derivative of the argument of the δ-function with respect to k leaves as an integrand k^2/μ, and one inserts the solution (i.e. the center-of-mass-momentum)

$$k(\mu)=\frac{1}{2\mu}\sqrt{\lambda(\mu^2,m_1^2,m_2^2)}\,, \tag{25}$$

with $\lambda(a,b,c)=a^2+b^2+c^2-2ab-2ac-2bc$ the conventional Källen function. As a result the integrated two-body phase-space in $4+1$ space-time dimensions is given by the rational expression:

$$\widetilde{\Gamma}_2(\mu;m_1,m_2)=\frac{\lambda(\mu^2,m_1^2,m_2^2)}{64\pi\mu^3}\,, \tag{26}$$

which grows linearly ($\sim\mu-m_1-m_2$) from threshold onwards. Note that $\lambda(\mu^2,m_1^2,m_2^2)=(\mu-m_1-m_2)(\mu-m_1+m_2)(\mu+m_1-m_2)(\mu+m_1+m_2)$ depends separately on the sum m_1+m_2 and the difference $|m_1-m_2|$ of the two involved masses. Using the previous result as twice the imaginary part, one finds for the one-loop function in the euclidean domain $s=-q^2$:

$$\begin{aligned}\widetilde{J}_2(-q^2) \;&=\; \frac{1}{64\pi^2}\bigg\{(m_1+m_2)\bigg[\frac{(m_1-m_2)^2}{q^2}-1\bigg]\\ &\quad -\frac{1}{q^3}\big[(m_1+m_2)^2+q^2\big]\big[(m_1-m_2)^2+q^2\big]\arctan\frac{q}{m_1+m_2}\bigg\}\,.\end{aligned} \tag{27}$$

Here, a linear divergence has been dropped, and this leads to the value at $q^2 = 0$:

$$\widetilde{J}_2(0) = -\frac{m_1^2 + m_1 m_2 + m_2^2}{24\pi^2(m_1 + m_2)}\,, \tag{28}$$

that agrees perfectly with the result of dimensional regularization

$$\begin{aligned} \widetilde{J}_2(0) &= (4\pi)^{-d/2}\Gamma\Big(2 - \frac{d}{2}\Big)\int_0^1 dx\big[m_1^2 x + m_2^2(1-x)\big]^{d/2-2}\Big|_{d=5} \\ &= -\frac{1}{16\pi^2}\int_0^1 dx\sqrt{m_1^2 x + m_2^2(1-x)} = -\frac{m_1^3 - m_2^3}{24\pi^2(m_1^2 - m_2^2)}\,. \end{aligned} \tag{29}$$

Next, one calculates the integrated three-body phase space in $4+1$ dimensions recursively from the two-body phase-space as

$$\widetilde{\Gamma}_3(\mu; m_1, m_2, m_3) = \frac{1}{\pi}\int_{m_1+m_2}^{\mu - m_3} d\mu'\,\mu'\,\frac{\lambda(\mu^2, \mu'^2, m_3^2)}{64\pi\mu^3}\,\frac{\lambda(\mu'^2, m_1^2, m_2^2)}{64\pi\mu'^3}\,, \tag{30}$$

and obtains the result

$$\widetilde{\Gamma}_3(\mu; m_1, m_2, m_3) = \frac{(\mu - \sigma_1)^3}{105(8\pi\mu)^3}\Big\{\mu^4 + 3\mu^3\sigma_1 + 4\mu^2(7\sigma_2 - 2\sigma_1^2) + (3\mu + \sigma_1)(\sigma_1^3 - 7\sigma_1\sigma_2 + 35\sigma_3)\Big\}\,, \tag{31}$$

with

$$\sigma_1 = m_1 + m_2 + m_3\,, \qquad \sigma_2 = m_1 m_2 + m_1 m_3 + m_2 m_3\,, \qquad \sigma_3 = m_1 m_2 m_3\,, \tag{32}$$

the elementary symmetric polynomials in the three masses m_1, m_2, m_3. The same result as in eq.(31) can be obtained by following eq.(3), performing angular integrals, and arriving at an integration over the Dalitz region in the $\omega_1\omega_2$-plane:

$$\widetilde{\Gamma}_3(\mu; m_1, m_2, m_3) = \frac{1}{128\pi^4}\iint\limits_{D(\omega_1,\omega_2)>0} d\omega_1 d\omega_2\sqrt{D(\omega_1, \omega_2)}\,. \tag{33}$$

In the next step the intermediate result $\int_{\omega_2^-}^{\omega_2^+} d\omega_2\sqrt{(\omega_2^+ - \omega_2)(\omega_2 - \omega_2^-)} = \pi(\omega_2^+ - \omega_2^-)^2/8$ is used and with this the integral $\int_{m_1}^{\omega_1^{\rm end}} d\omega_1\sqrt{\mu^2 - 2\mu\omega_1 + m_1^2}(\omega_2^+ - \omega_2^-)^2$ can be performed to reproduce the expression in eq.(31).

The partial fraction decomposition of $\mu\widetilde{\Gamma}_3(\mu; m_1, m_2, m_3)/(\mu^2 + q^2)$ into pieces proportional to $1/(\mu^2 + q^2)$ and $-2\mu/(\mu^2 + q^2)$ plus a polynomial allows to infer the two-loop function $\widetilde{J}_3(-q^2)$ as a linear combination of $q^{-1}\arctan(q/\sigma_1)$ and $\ln(1 + q^2/\sigma_1^2) + 2\ln(\sigma_1/\kappa)$. Modulo additive polynomial terms ($\sim 1, q^2, q^4$), this two-loop function reads

$$\begin{aligned} \widetilde{J}_3(-q^2) &= \frac{1}{6(4\pi)^4}\bigg\{\bigg[\frac{\sigma_1^4}{q^2}\Big(\frac{\sigma_1^3}{35} - \frac{\sigma_1\sigma_2}{5} + \sigma_3\Big) + 2\sigma_1^2\Big(\frac{\sigma_1^3}{5} - \sigma_1\sigma_2 + 3\sigma_3\Big) + (3\sigma_1\sigma_2 - \sigma_1^3 - 3\sigma_3)q^2\bigg]\frac{1}{q}\arctan\frac{q}{\sigma_1} \\ &+ \bigg[\sigma_1\Big(\frac{\sigma_1^3}{2} - 2\sigma_1\sigma_2 + 4\sigma_3\Big) + \frac{q^2}{5}(2\sigma_2 - \sigma_1^2) - \frac{q^4}{70}\bigg]\bigg[\ln\Big(1 + \frac{q^2}{\sigma_1^2}\Big) + 2\ln\frac{\sigma_1}{\kappa}\bigg] \\ &+ \frac{\sigma_1^3}{q^2}\Big(\frac{\sigma_1\sigma_2}{5} - \sigma_3 - \frac{\sigma_1^3}{35}\Big) + \text{polyn}\bigg\}\,. \end{aligned} \tag{34}$$

Note the extra $1/q^2$-term in third line, which makes the whole expression singularity-free at $q^2 = 0$.

The next iterative step to the four-body phase space is straightforward by employing eq.(6) and this leads to the following result for the integrated four-body phase-space in $4+1$ dimensions:

$$\widetilde{\Gamma}_4(\mu;m_1,m_2,m_3,m_4) = \frac{(\mu-\Sigma_1)^5}{630(8\pi)^5\mu^3}\bigg\{\frac{\mu^5}{10}+\frac{\mu^4}{2}\Sigma_1+\mu^3(7\Sigma_2-2\Sigma_1^2)$$
$$+\mu^2(2\Sigma_1^3-13\Sigma_1\Sigma_2+48\Sigma_3)+(5\mu+\Sigma_1)\bigg[\Sigma_1^2\Sigma_2-\frac{\Sigma_1^4}{10}-8\Sigma_1\Sigma_3+56\Sigma_4\bigg]\bigg\}\,, \tag{35}$$

with the abbreviations

$$\begin{aligned}
\Sigma_1 &= m_1+m_2+m_3+m_4\,,\\
\Sigma_2 &= m_1m_2+m_1m_3+m_1m_4+m_2m_3+m_2m_4+m_3m_4\,,\\
\Sigma_3 &= m_1m_2m_3+m_1m_2m_4+m_1m_3m_4+m_2m_3m_4\,,\\
\Sigma_4 &= m_1m_2m_3m_4\,,
\end{aligned} \tag{36}$$

for the elementary symmetric polynomials in the four masses m_1,m_2,m_3,m_4. The associated three-loop function $\widetilde{J}_4(-q^2)$ has a form analogous to eq.(34) with the extension that the square bracket in front of the basic function $q^{-1}\arctan(q/\Sigma_1)$ includes now additional polynomial terms proportional to q^4,q^6,q^8.

In the next iterative step, one finds that the five-body phase-space in $4+1$ dimensions takes the following form

$$\begin{aligned}
\widetilde{\Gamma}_5(\mu;m_1,\dots,m_5) \;=\; & \frac{2(\mu-\tilde{\Sigma}_1)^7}{22275(8\pi)^7\mu^3}\bigg\{\frac{\mu^6}{91}+\frac{\mu^5}{13}\tilde{\Sigma}_1+\frac{\mu^4}{7}\bigg(10\tilde{\Sigma}_2-\frac{37}{13}\tilde{\Sigma}_1^2\bigg)\\
&+\frac{\mu^3}{7}\bigg(\frac{58}{13}\tilde{\Sigma}_1^3-29\tilde{\Sigma}_1\tilde{\Sigma}_2+99\tilde{\Sigma}_3\bigg)+\frac{\mu^2}{7}\bigg(880\tilde{\Sigma}_4-187\tilde{\Sigma}_1\tilde{\Sigma}_3+27\tilde{\Sigma}_1^2\tilde{\Sigma}_2-\frac{37}{13}\tilde{\Sigma}_1^4\bigg)\\
&+\bigg(\mu+\frac{\tilde{\Sigma}_1}{7}\bigg)\bigg(\frac{\tilde{\Sigma}_1^5}{13}-\tilde{\Sigma}_1^3\tilde{\Sigma}_2+11\tilde{\Sigma}_1^2\tilde{\Sigma}_3-110\tilde{\Sigma}_1\tilde{\Sigma}_4+990\tilde{\Sigma}_5\bigg)\bigg\}\,,
\end{aligned} \tag{37}$$

with the abbreviations

$$\begin{aligned}
\tilde{\Sigma}_1 &= m_1+m_2+m_3+m_4+m_5\,,\\
\tilde{\Sigma}_2 &= m_1m_2+m_1m_3+m_1m_4+m_1m_5+m_2m_3+m_2m_4+m_2m_5+m_3m_4+m_3m_5+m_4m_5\,,\\
\tilde{\Sigma}_3 &= m_1m_2m_3+m_1m_2m_4+m_1m_2m_5+m_1m_3m_4+m_1m_3m_5\\
&\quad +m_1m_4m_5+m_2m_3m_4+m_2m_3m_5+m_2m_4m_5+m_3m_4m_5\,,\\
\tilde{\Sigma}_4 &= m_1m_2m_3m_4+m_1m_2m_3m_5+m_1m_2m_4m_5+m_1m_3m_4m_5+m_2m_3m_4m_5\,,\\
\tilde{\Sigma}_5 &= m_1m_2m_3m_4m_5\,,
\end{aligned} \tag{38}$$

for the elementary symmetric polynomials in the five masses m_1,m_2,m_3,m_4,m_5. By induction, one can deduce that the n-body phase-space takes the form $\widetilde{\Gamma}_n(\mu;m_1,\dots,m_n) = (8\pi)^{3-2n}\mu^{-3}\big(\mu-\sum_{j=1}^n m_j\big)^{2n-3}P_{n+1}(\mu)$, where the last factor $P_{n+1}(\mu)$ is a polynomial in μ of degree $n+1$, whose coefficients are totally symmetric in the n masses $m_1,\dots,m_n$.

5 Digression to 1+1 dimensions

In passing we note that the two-body phase-space in $1+1$ space-time dimensions is given by the expression

$$\bar{\Gamma}_2(\mu;m_1,m_2) = \frac{1}{\sqrt{\lambda(\mu^2,m_1^2,m_2^2)}}\,, \tag{39}$$

which exhibits a singular behavior $\sim (\mu - m_1 - m_2)^{-1/2}$ near threshold. The associated one-loop function in $1+1$ dimensions is then calculated as

$$\bar{J}_2(-q^2) = \frac{1}{\pi W_+ W_-} \ln \frac{W_+ + W_-}{2\sqrt{m_1 m_2}}\,, \qquad W_\pm = \sqrt{q^2 + (m_1 \pm m_2)^2}\,. \tag{40}$$

The iterative procedure (see eq.(6)) leads for the three-body phase-space in $1+1$ dimensions to the integral representation

$$\bar{\Gamma}_3(\mu; m_1, m_2, m_3) = \frac{1}{2\pi} \int_{(m_1+m_2)^2}^{(\mu-m_3)^2} ds' \left[\lambda(\mu^2, s', m_3^2)\, \lambda(s', m_1^2, m_2^2)\right]^{-1/2}, \tag{41}$$

which gives rise to a non-vanishing value at threshold

$$\bar{\Gamma}_3(m_1+m_2+m_3; m_1, m_2, m_3) = \frac{1}{8\sqrt{(m_1+m_2+m_3)m_1 m_2 m_3}} = \frac{1}{8\sqrt{\sigma_1 \sigma_3}}\,. \tag{42}$$

Actually, the integral representation of $\bar{\Gamma}_3(\mu; m_1, m_2, m_3)$ can be mapped by a linear fractional transformation onto that of a complete elliptic integral of the first kind:

$$\begin{aligned} \bar{\Gamma}_3(\mu; m_1, m_2, m_3) &= \frac{1}{8\pi\sqrt{\mu m_1 m_2 m_3}} \int_1^\rho dx \frac{1}{\sqrt{x(x-1)(\rho - x)}} \\ &= \frac{1}{4\pi\sqrt{\mu\sigma_3}} \int_0^{\pi/2} d\varphi \left[1 + (\rho - 1)\sin^2\varphi\right]^{-1/2}, \end{aligned} \tag{43}$$

with the auxiliary parameter

$$\begin{aligned} \rho &= \frac{1}{16\mu m_1 m_2 m_3}\left[(\mu+m_3)^2 - (m_1+m_2)^2\right]\left[(\mu-m_3)^2 - (m_1-m_2)^2\right] \\ &= \frac{1}{16\mu\sigma_3}\left[\mu^4 + 2\mu^2(2\sigma_2 - \sigma_1^2) + 8\mu\sigma_3 + \sigma_1^4 - 4\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3\right] > 1\,, \end{aligned} \tag{44}$$

that is totally symmetric in the three masses m_1, m_2, m_3. The ρ-dependent value of the complete elliptic integral $\int_1^\rho dx[x(x-1)(\rho-x)]^{-1/2}$ in eq.(43) turns out to be equal to the real Weierstraß half-period $\Omega_1/2$ that belongs to the two (real) Weierstraß invariants

$$g_2 = {\sum_{n_1,n_2}}' \frac{60}{(n_1\Omega_1 + in_2\Omega_2)^4} = \frac{\rho^2 - \rho + 1}{12}\,, \qquad g_3 = {\sum_{n_1,n_2}}' \frac{140}{(n_1\Omega_1 + in_2\Omega_2)^6} = \frac{(2-\rho)(1+\rho)(2\rho-1)}{432}\,. \tag{45}$$

Note that the other Weierstraß half-period $i\,\Omega_2/2$ is purely imaginary, and thus the period lattice of the underlying doubly-periodic elliptic function in the complex plane is of rectangular shape. The transformation of the cubic polynomial under the square root in eq.(43) to the Weierstraß normal form $4\wp(z)^3 - g_2\wp(z) - g_3$ is achieved by setting $x = -4\wp(z) + (1+\rho)/3$.

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