

Anomalous Diffusion in Driven Electrolytes due to Hydrodynamic Fluctuations

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The stochastic dynamics of tracers arising from hydrodynamic fluctuations in a driven electrolyte is studied using a self-consistent field theory framework in all dimensions. A plethora of scaling behaviour including two distinct regimes of anomalous diffusion is found, and the crossovers between them are characterized in terms of the different tuning parameters. A short-time ballistic regime is found to be accessible beyond two dimensions, whereas a long-time diffusive regime is found to be present only at four dimensions and above. The results showcase how long-ranged hydrodynamic interactions can dominate the dynamics of non-equilibrium steady-states in ionic suspensions and produce strong fluctuations despite the presence of Debye screening.

Introduction.—Since the pioneering works of Faraday [1] and Nernst [2], driven electrolytes have been at the forefront of many scientific and technological developments to date, most notably, in energy storage and conversion applications. Moreover, evolution has selected them as the physical medium of operation for the most sophisticated signal processing and cognitive biological systems. In recent years, understanding and controlling the physical characteristics of driven electrolytes at the nano-scale have enabled the development of molecular sequencing and sensing technologies [3, 4], de novo strategies to build synthetic nanoscale motors [5–7], as well as theoretical proposals to explain complex biological processes such as the gating of wet ion channels via an intrinsically non-equilibrium mechanism [8] and neuro-morphic computing using solid-state nano-pores [9, 10]. On the other hand, driven electrolytes provide a playground for the study of non-equilibrium steady-states that exhibit a rich variety of interesting physical behaviour [11]. Recent examples include the emergence of long-range correlations and fluctuation-induced forces between boundaries immersed in driven electrolytes [12–15] and related generalizations [16], as well as probes of the so-called $1/f$ -noise in ionic currents in nano-pores [17–21].

The role of the solvent on the stochastic dynamics of mobile ions and tracers in electrolytes has so far received relatively little attention. Hydrodynamic interactions induce long-time tails on tracers in neutral solvents [22], and introduce strong large-scale fluctuations in the phenomenon of sedimentation, where body forces that scale with the system size are exerted externally on the colloidal particles in the suspension [23, 24]. In electrolytes, the body forces appear in pairs due to the overall neutrality of the suspension, and this might introduce additional subtleties, not unrelated to the force-free nature of interfacial transport processes that are relevant to active matter systems [25, 26]. In this Letter, the aim is to study the effect of hydrodynamic fluctuations that arise from the non-equilibrium electrical force-dipoles that stir up a suspension in an electrolyte driven by an external electric field; see Fig. 1. Using a self-consistent field the-

ory framework, the stochastic dynamics of tracers in a driven electrolyte is studied as driven by the fluctuations in the hydrodynamic flow field, covering all dimensions.

I now present a summary of the main results on the way the mean-squared displacement (MSD) of tracers in the electrolyte denoted as $\Delta L^2(t)$ depends on time t in dimension d , covering all dimensions (see Fig. 2). The electrolyte comprises a suspension of monovalent ions with charges $\pm Q$ and mean concentration C_0 in a solvent with viscosity η and dielectric constant ϵ . The diffusion coefficient D of the individual ions is taken to be equal for both positive and negative species. An electric field $\mathbf{E} = E\hat{\mathbf{e}}$ is applied globally on the system (see Fig. 1). Electrostatic effects are subject to Debye screening due to the thermal fluctuations of the ions in the medium with inverse temperature scale $\beta = 1/(k_B T)$, and the Debye screening length κ^{-1} is defined via $\kappa^2 = 2S_d C_0 \beta Q^2 / \epsilon$ in d dimensions (with $S_d = 2\pi^{d/2}/\Gamma(d/2)$, e.g. $S_3 = 4\pi$). An important inverse time-scale in the system is given by $\lambda_e = \epsilon E^2 / (2S_d \eta)$, which represents the typical scale of

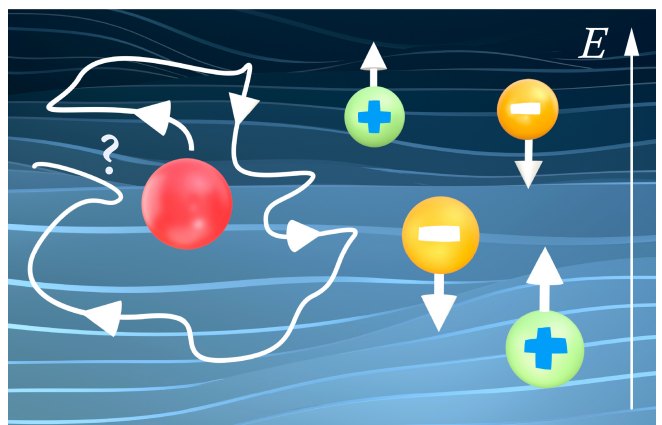


FIG. 1. An electrolyte driven by an externally applied electric field E produces body forces on the positive and negative ions in the fluid. These fluctuating body forces lead to the generation of long-ranged flow fields that can be measured through the stochastic trajectories of tracer particles.

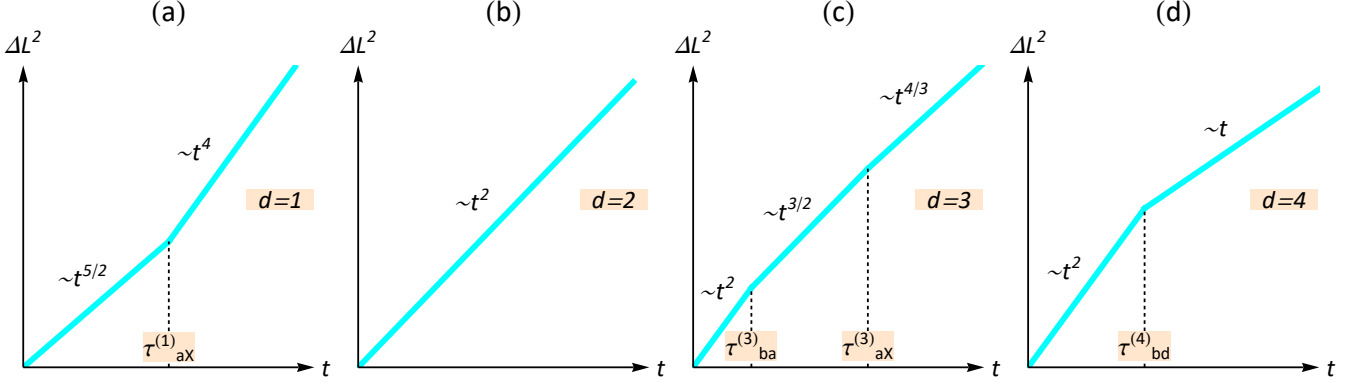


FIG. 2. Summary of the different regimes of the dynamics in different dimensions. (a) In $d = 1$, there is a crossover between two super-ballistic anomalous regimes occurring at $\tau_{ax}^{(1)} = C_0^{2/3} D^{1/3} \lambda_e^{-4/3}$. (b) In $d = 2$, there is only one ballistic regime observed at all times. (c) In $d = 3$, the dynamics shows a crossover from ballistic to a first anomalous regime at the time-scale $\tau_{ba}^{(3)} = a^2 D^{-1}$, followed by another crossover to a second anomalous regime at the time-scale $\tau_{ax}^{(3)} = C_0^2 D^3 \lambda_e^{-4}$. (d) In $d = 4$, there is a crossover from a ballistic regime to a diffusive regime at the time-scale $\tau_{bd}^{(4)} = C_0^{1/2} \lambda_e^{-1} a^2 \sqrt{\ln(L/a)}$. In $d > 4$, the same crossover occurs from ballistic to diffusive behaviour at the time-scale $\tau_{bd}^{(d)} = C_0^{1/2} \lambda_e^{-1} a^{d/2}$.

the shear strain rate in the electrolyte suspension arising from the Maxwell stress associated with the electric field. A microscopic length scale a , which represents the size of the tracers or κ^{-1} (whichever is larger), and the system size L are used in the regularization of the theoretical framework. The system exhibits ballistic, anomalous, and diffusive regimes, as characterized by the time-dependence of the MSD, defined as $\Delta L^2 \equiv \langle [\mathbf{r}(t) - \mathbf{r}(0)]^2 \rangle$, where $\mathbf{r}(t)$ describes the stochastic trajectory of the tracer particle. In $d = 1$, the dynamics at short times obeys $\Delta L^2 \sim t^{5/2}$ for $0 < t < \tau_{ax}^{(1)}$, which is followed by a crossover to $\Delta L^2 \sim t^4$ for $t > \tau_{ax}^{(1)}$. The crossover time scale between the two super-ballistic anomalous regimes is found as $\tau_{ax}^{(1)} = C_0^{2/3} D^{1/3} \lambda_e^{-4/3}$. In $d = 2$, only one ballistic regime is observed with $\Delta L^2 \sim t^2$ at all time scales. In $d = 3$, the dynamics exhibits ballistic behaviour with $\Delta L^2 \sim t^2$ at short times $0 < t < \tau_{ba}^{(3)}$, followed by a crossover beyond the time-scale $\tau_{ba}^{(3)} = a^2 D^{-1}$ to a first anomalous regime described by $\Delta L^2 \sim t^{3/2}$ for $\tau_{ba}^{(3)} < t < \tau_{ax}^{(3)}$, which is subsequently followed by a crossover to a second anomalous regime that obeys $\Delta L^2 \sim t^{4/3}$ for $t > \tau_{ax}^{(3)}$. The crossover time-scale between the two anomalous regimes is obtained as $\tau_{ax}^{(3)} = C_0^2 D^3 \lambda_e^{-4}$. In $d = 4$, the dynamics starts with ballistic behaviour $\Delta L^2 \sim t^2$ at short times $0 < t < \tau_{bd}^{(4)}$, and exhibits a crossover at the time-scale $\tau_{bd}^{(4)} = C_0^{1/2} \lambda_e^{-1} a^2 \sqrt{\ln(L/a)}$ to a diffusive regime $\Delta L^2 \sim t$ for $t > \tau_{bd}^{(4)}$. In $d > 4$, the same crossover from ballistic to diffusive behaviour is observed at the time-scale $\tau_{bd}^{(d)} = C_0^{1/2} \lambda_e^{-1} a^{d/2}$. These results are presented in Fig. 2.

Theoretical Framework.—The dynamics of the ions is probed by using the stochastic concentrations of the two

charged species $C^\pm(\mathbf{r}, t)$ that give rise to a stochastic electrostatic potential field $\phi(\mathbf{r}, t)$ through the Poisson equation $-\nabla^2 \phi = S_d Q (C^+ - C^-)/\epsilon$, which is written in Gaussian units in d dimensions. Using the Dean-Kawasaki approach [27, 28], the dynamics of $C^\pm$ can be described by continuity equations, $\partial_t C^\pm + \nabla \cdot \mathbf{J}^\pm = 0$, where the stochastic currents are given as $\mathbf{J}^\pm = \mathbf{v} - D \nabla C^\pm \pm D \beta Q C^\pm (-\nabla \phi + \mathbf{E}) - \sqrt{2DC^\pm} \boldsymbol{\eta}^\pm$. Here, $\mathbf{v}(\mathbf{r}, t)$ describes the fluid flow and $\boldsymbol{\eta}^\pm$ are uncorrelated Gaussian white noise fields characterized by zero mean and $\langle \eta_i^\pm(\mathbf{r}, t) \eta_j^\pm(\mathbf{r}', t') \rangle = \delta_{ij} \delta^d(\mathbf{r} - \mathbf{r}') \delta(t - t')$. In what follows, averaging will be performed with respect to the different realizations of this noise, e.g. $\langle C^\pm \rangle = C_0$, and it is convenient to write $C^\pm = C_0 + \delta C^\pm$ where the density fluctuation have zero mean, namely, $\langle \delta C^\pm \rangle = 0$.

The velocity field $\mathbf{v}(\mathbf{r}, t)$ in the background medium is subject to the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ as well as momentum conservation, which is enforced via the Stokes equation $-\eta \nabla^2 \mathbf{v} = -\nabla p + \mathbf{f}$ that describes local and instantaneous stochastic force balance. Here, $p(\mathbf{r}, t)$ is the pressure field and $\mathbf{f}(\mathbf{r}, t) = Q \mathbf{E} \rho(\mathbf{r}, t)$ represents the body-force density experienced by the ions due to the electric field and non-vanishing stochastic charge density $\rho = (C^+ - C^-)$ (defined in units of Q). The Stokes equation can be solved exactly in terms of the charge density. The solution can be formally represented as

$$\mathbf{v}(\mathbf{r}, t) = \frac{QE}{\eta} \left(\frac{1}{-\nabla^2} \right) \left(\hat{\mathbf{e}} - \frac{(\hat{\mathbf{e}} \cdot \nabla) \nabla}{\nabla^2} \right) \rho, \quad (1)$$

and the result can be used to derive an expression for the velocity fluctuations of the fluid medium in Fourier space

as follows

$$\begin{aligned} \langle v_i(\mathbf{q}, \omega) v_j(-\mathbf{q}, -\omega) \rangle &= [\hat{e}_i - \hat{q}_i(\hat{\mathbf{q}} \cdot \hat{\mathbf{e}})] [\hat{e}_j - \hat{q}_j(\hat{\mathbf{q}} \cdot \hat{\mathbf{e}})] \\ &\times \frac{Q^2 E^2}{\eta^2 q^4} \langle |\rho(\mathbf{q}, \omega)|^2 \rangle. \end{aligned} \quad (2)$$

In the limit of high concentrations corresponding to strong electrolytes, the stochastic densities are relatively weak $\delta C^\pm \ll C_0$, and therefore the stochastic density equations can be expanded around the background mean. The total number density fluctuations $c(\mathbf{r}, t) = \delta C^+ + \delta C^-$ and the charge density fluctuations $\rho(\mathbf{r}, t) = \delta C^+ - \delta C^-$ satisfy the following stochastic equations

$$\partial_t c + \mathbf{v} \cdot \nabla c = D \nabla^2 c - D \beta Q E \hat{\mathbf{e}} \cdot \nabla \rho + \sqrt{4 D C_0} \eta_c, \quad (3)$$

and $\partial_t \rho + \mathbf{v} \cdot \nabla \rho = D \nabla^2 \rho - D \beta Q E \hat{\mathbf{e}} \cdot \nabla c - D \kappa^2 \rho + \sqrt{4 D C_0} \eta_\rho$, where the noise correlations are given as $\langle \eta_\rho(\mathbf{r}, t) \eta_\rho(\mathbf{r}', t') \rangle = \langle \eta_c(\mathbf{r}, t) \eta_c(\mathbf{r}', t') \rangle = -\nabla^2 \delta^d(\mathbf{r} - \mathbf{r}') \delta(t - t')$, with η_ρ and η_c having zero averages and being uncorrelated. The simplification has been used in a variety of different contexts such as a dense population of soft particles [29], conductivity of strong electrolytes [30], fluctuations of ionic currents across nano-pores [19], and driven binary mixtures [31]. The stochastic equation for ρ provides a strong constraint between the two fields in the long time and large length scale limit

$$\rho = -\frac{\beta Q E}{\kappa^2} \hat{\mathbf{e}} \cdot \nabla c, \quad (4)$$

leaving only one soft mode to consider in the effective field theory. In Eq. (3) and (4) all irrelevant nonlinear terms in the sense of Renormalization Group (RG) theory have been omitted. This procedure can be performed by scaling of these equations according to $\mathbf{r} \rightarrow b\mathbf{r}$, $t \rightarrow b^z t$, $\rho \rightarrow b^{\chi_\rho} \rho$, $c \rightarrow b^{\chi_c} c$, which yields $z = 2$, $\chi_c = -d/2$, and $\chi_\rho = -1 - d/2$ for the Gaussian fixed point, and calculating the effective scaling exponent for the nonlinear terms and showing that they will be negative in the region of interest (see Refs. [12, 13] for details). For the advection term $\mathbf{v} \cdot \nabla c$ in Eq. (3), this scaling assessment yields $b^{2-d/2}$, which shows that this key nonlinearity is relevant for $d < 4$.

Instead of resorting to standard perturbative RG calculations, here a self-consistent calculation strategy is adopted to study the coupling between density fluctuations in driven electrolytes and the hydrodynamic fluid flow fluctuation. The starting point is to calculate the velocity fluctuations using the linear stochastic theory for the strong electrolytes, which yields

$$\begin{aligned} \langle v_i(\mathbf{q}, \omega) v_j(\mathbf{q}', \omega') \rangle &= (2\pi)^d \delta^d(\mathbf{q} + \mathbf{q}') (2\pi) \delta(\omega + \omega') \\ &\times \frac{4 D \lambda_e^2}{C_0} (\hat{\mathbf{q}} \cdot \hat{\mathbf{e}})^2 \frac{[\hat{e}_i - \hat{q}_i(\hat{\mathbf{q}} \cdot \hat{\mathbf{e}})] [\hat{e}_j - \hat{q}_j(\hat{\mathbf{q}} \cdot \hat{\mathbf{e}})]}{[i\omega + Dq^2] [-i\omega + Dq^2]} \end{aligned} \quad (5)$$

The resulting stochastic velocity field $\mathbf{v}(\mathbf{r}, t)$ is then treated as a correlated noise whose spectrum is given by

Eq (5) in the Langevin equation for a tracer particle that follows the stochastic trajectory $\mathbf{r}(t)$, namely

$$\frac{d}{dt} \mathbf{r}(t) = \mathbf{v}(\mathbf{r}(t)), \quad (6)$$

which can be used to calculate the MSD as follows

$$\Delta L^2(t) = \int_0^t dt_1 \int_0^t dt_2 \mathcal{A}^L(t_1, t_2), \quad (7)$$

in terms of the Lagrangian velocity auto-correlation function defined as

$$\mathcal{A}^L(t, t') \equiv \langle \mathbf{v}(\mathbf{r}(t), t) \cdot \mathbf{v}(\mathbf{r}(t'), t') \rangle, \quad (8)$$

which yields

$$\begin{aligned} \mathcal{A}^L(t, t') &= \frac{2\lambda_e^2}{C_0} \int_{\mathbf{q}} \left\langle e^{i\mathbf{q} \cdot [\mathbf{r}(t) - \mathbf{r}(t')]} \right\rangle e^{-Dq^2|t-t'|} \\ &\times \frac{1}{q^2} (\hat{\mathbf{q}} \cdot \hat{\mathbf{e}})^2 [1 - (\hat{\mathbf{q}} \cdot \hat{\mathbf{e}})^2], \end{aligned} \quad (9)$$

after frequency integration, where we have used the shorthand notation $\int_{\mathbf{q}} \equiv \int \frac{d^d \mathbf{q}}{(2\pi)^d}$. Since the right-hand side of Eq. (9) depends on the stochastic trajectory itself, the calculation will need to be closed via a self-consistency requirement. Note that the background linear Langevin noise corresponding to the passive diffusion coefficient $\mathcal{D}_t$ of the tracer has been ignored in Eq. (6) and throughout this work for simplicity of the presentation. Let us now consider the different regimes that arise from Eqs. (7) and (9).

Velocity Fluctuations.—At the shortest time-scales one can approximate the auto-correlation function in Eq. (7) as $\mathcal{A}^L(t, t) \equiv \langle \mathbf{v}(\mathbf{r}(t), t) \cdot \mathbf{v}(\mathbf{r}(t), t) \rangle$, which amounts to using the local and instantaneous velocity fluctuations, for which one finds

$$\langle \mathbf{v}^2 \rangle = \frac{2(d-1)}{d(d+2)} \frac{\lambda_e^2}{C_0} \mathcal{G}_2^{(d)}, \quad (10)$$

where

$$\mathcal{G}_2^{(d)} = \int_{1/a}^{1/L} \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{1}{q^2} \sim \begin{cases} 1/a^{d-2}, & d > 2, \\ \ln(L/a), & d = 2, \\ L^{2-d}, & d < 2, \end{cases} \quad (11)$$

which reveals strong differences as a function of dimensionality. In particular, it emerges that the ballistic regime defined as $\Delta L^2 = \langle \mathbf{v}^2 \rangle t^2$ at the shortest time scales is only accessible for $d \geq 2$. The anisotropy originating from the symmetry breaking gives rise to different coefficients for the parallel and perpendicular components, namely, $\langle v_{\parallel}^2 \rangle = \frac{3}{d+4} \langle \mathbf{v}^2 \rangle$ and $\langle v_{\perp}^2 \rangle = \frac{d+1}{d+4} \langle \mathbf{v}^2 \rangle$, as defined by the direction of the electric field $\hat{\mathbf{e}}$. Since the anisotropy is expected to only affect numerical prefactors, its effect is not explicitly calculated in the remainder of the paper for simplicity of the presentation.

Anomalous Diffusion.—In the intermediate time regime, a self-consistent treatment of the dynamics is needed. To this end, an *ansatz* of the form $\langle e^{i\mathbf{q} \cdot [\mathbf{r}(t) - \mathbf{r}(t')]} \rangle = e^{-\mathcal{D}_z q^z |t-t'|}$ is used for the anomalous diffusion regime in the intermediate time-scales, where z is the dynamic exponent and $\mathcal{D}_z$ is the corresponding anomalous diffusion coefficient, both of which need to be calculated self-consistently. In the diffusive regime at long times, the form $\langle e^{i\mathbf{q} \cdot [\mathbf{r}(t) - \mathbf{r}(t')]} \rangle = e^{-\mathcal{D}_{\text{sc}} q^2 |t-t'|}$ will be used, where the effective diffusion coefficient $\mathcal{D}_{\text{sc}}$ is to be calculated self-consistently. Using these forms, the double time-integration can be carried out as follows $\int_0^t dt_1 \int_0^t dt_2 e^{-\Lambda t} = \frac{2t}{\Lambda} + \frac{2}{\Lambda^2} (e^{-\Lambda t} - 1)$.

Using $\Delta L^2 = (2d\mathcal{D}_z t)^\alpha$ for intermediate time scales where α is the anomalous diffusion exponent, which is related to the dynamic exponent z via $\alpha = 2/z$, the self-consistency equation reads

$$(\mathcal{D}_z t)^{2/z} = a_d \left(\frac{\lambda_e^2}{C_0} \right) \int_{\mathbf{q}} \frac{e^{-(\mathcal{D}_z q^z + D q^2)t}}{q^2 (\mathcal{D}_z q^z + D q^2)^2}, \quad (12)$$

where a_d is a d -dependent numerical prefactor. Equation (12) gives rise to the emergence of two different scaling regimes, and a crossover between them that occurs at the crossover time

$$\tau_{\text{a}\times}^{(d)} = \left(\frac{C_0 D^{d/2}}{\lambda_e^2} \right)^{\frac{2}{4-d}}, \quad (13)$$

The first scaling regime at relatively shorter times is characterized by

$$z_1 = \frac{4}{6-d}, \quad \alpha_1 = 3 - \frac{d}{2}, \quad \mathcal{D}_{z_1} = \left(b_d D^{1-\frac{d}{2}} \lambda_e^2 / C_0 \right)^{\frac{2}{6-d}}, \quad (14)$$

where $b_d = a_d \Gamma(-3 + d/2) / [2^d \pi^{d/2} \Gamma(d/2)]$. The derivation of this first scaling regime is equivalent to using the Eulerian velocity autocorrelation function $\mathcal{A}^{\text{E}}(t, t') \equiv \langle \mathbf{v}(\mathbf{r}, t) \cdot \mathbf{v}(\mathbf{r}, t') \rangle$. The second scaling regime at relatively longer times is characterized by

$$z_2 = \frac{d}{2}, \quad \alpha_1 = \frac{4}{d}, \quad \mathcal{D}_{z_2} = (c_d \lambda_e^2 / C_0)^{\frac{1}{2}}, \quad (15)$$

where $c_d = a_d \Gamma(-4/d) / [2^{d-2} \pi^{d/2} d \Gamma(d/2)]$. The crossover time can be calculated from both self-consistent solutions and they turn out to match exactly.

These results are summarized in Table I. It is interesting to note that $\alpha_1 < \alpha_2$ for $0 < d < 2$ and $\alpha_1 > \alpha_2$ for $2 < d < 4$. For $d = 2$ we have $\alpha_1 = \alpha_2 = 2$ and for $d \geq 4$ we have $\alpha_1 = \alpha_2 = 1$. Figure 2 shows these features (reflected in convexity or concavity of the plots) and the anomalous regimes in different dimensions.

Effective Diffusion.—At long times the system might be able to cross over to a purely diffusive regime. To probe this, a self-consistent equation similar to Eq. (12)

d	$z_1 = \frac{4}{6-d}$	$\alpha_1 = 3 - \frac{d}{2}$	$z_2 = \frac{d}{2}$	$\alpha_2 = \frac{4}{d}$
1	4/5	5/2	1/2	4
2	1	2	1	2
3	4/3	3/2	3/2	4/3
4	2	1	2	1

TABLE I. Exponents corresponding to the two types of anomalous dynamics in the intermediate-time regime for different dimensions. Anomalous diffusion ceases to exist for $d > 4$.

is used, which yields the following equation for the effective diffusion coefficient

$$\mathcal{D}_{\text{sc}} = g'_d \frac{\lambda_e^2}{C_0} \mathcal{G}_2^{(d)} \frac{1}{\mathcal{D}_{\text{sc}} + D}, \quad (16)$$

where

$$\mathcal{G}_4^{(d)} = \int_{1/a}^{1/L} \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{1}{q^4} \sim \begin{cases} 1/a^{d-4}, & d > 4, \\ \ln(L/a), & d = 4, \\ L^{4-d}, & d < 4, \end{cases} \quad (17)$$

and g'_d is a d -dependent numerical prefactor. Due to the structure of $\mathcal{G}_4^{(d)}$, the leading order solution to the self-consistency equation given in Eq. (16) strongly depends on dimensionality. For $d > 4$, $\mathcal{G}_4^{(d)}$ remains finite and therefore $\mathcal{D}_{\text{sc}} \ll D$, which yields

$$\mathcal{D}_{\text{sc}} = \frac{g_d \lambda_e^2}{D C_0 a^{d-4}}, \quad (d > 4) \quad (18)$$

where g_d is a d -dependent numerical prefactor. For $d \leq 4$, $\mathcal{G}_4^{(d)}$ diverges with system size L and therefore $\mathcal{D}_{\text{sc}} \gg D$, which yields

$$\mathcal{D}_{\text{sc}} = \left(\frac{g_d \lambda_e^2}{C_0} \right)^{\frac{1}{2}} L^{2-\frac{d}{2}}, \quad (d < 4) \quad (19)$$

and

$$\mathcal{D}_{\text{sc}} = \left(\frac{g_4 \lambda_e^2}{C_0} \right)^{\frac{1}{2}} \sqrt{\ln(L/a)}, \quad (d = 4) \quad (20)$$

Therefore, the diffusive regime is not expected to be accessible for $d < 4$ at long times (see below).

Crossover Time-scales.—It is instructive to examine the behaviour of crossover time-scales for the possible scenarios. The crossover from the ballistic regime to the first anomalous regime occurs at a time-scale $\tau_{\text{ba}}^{(d)}$ that can be obtained via $\langle \mathbf{v}^2 \rangle (\tau_{\text{ba}}^{(d)})^2 = \mathcal{D}_{z_1}^{3-d/2} (\tau_{\text{ba}}^{(d)})^{3-d/2}$, where we will use input from Eqs. (10) and (14). While for $d = 2$, this does not yield a result, we find $\tau_{\text{ba}}^{(d)} = a^2/D$ for $d > 2$ and $\tau_{\text{ba}}^{(d)} = L^2/D$ for $d < 2$. This implies that the ballistic regime only exists for $d > 2$, as the velocity fluctuations diverge with the system size in $d < 2$.

The crossover time from ballistic regime to diffusive regime, $\tau_{\text{bd}}^{(d)}$, can be obtained via $\langle v^2 \rangle (\tau_{\text{bd}}^{(d)})^2 = \mathcal{D}_{\text{sc}} \tau_{\text{bd}}^{(d)}$. Note that this crossover exists only for $d \geq 4$. Using Eqs. (10), (18), and (20), the calculation yields $\tau_{\text{bd}}^{(d)} = C_0^{1/2} \lambda_e^{-1} a^{d/2}$ for $d > 4$ and $\tau_{\text{bd}}^{(4)} = C_0^{1/2} \lambda_e^{-1} a^2 \sqrt{\ln(L/a)}$ for $d = 4$. For $d < 4$, a long-time crossover occurs at time $\tau_L^{(d)}$ when the MSD in the second anomalous regime is saturated by the system size L . Using $L^2 = \mathcal{D}_{z_2}^{4/d} (\tau_L^{(d)})^{4/d}$ and Eq. (15), the calculation gives $\tau_L^{(d)} = C_0^{1/2} \lambda_e^{-1} L^{d/2}$ for $d < 4$. Note that a similar calculation by using the divergent self-consistent effective diffusion equation from Eq. (19) yields the same result, namely, $\tau_L^{(d)} = L^2 / \mathcal{D}_{\text{sc}} = C_0^{1/2} \lambda_e^{-1} L^{d/2}$ for $d < 4$. Finally, since in $d = 2$, the anomalous diffusion exponents for both regimes are the same as the ballistic dynamics exponent, i.e. $\alpha_1 = \alpha_2 = 2$, the ballistic regime is expected to govern all time scales in this case (see Fig. 2b).

Concluding Remarks.—The stochastic fluctuations of tracers in a bulk electrolyte have been found to exhibit a wide range of dynamical regimes and crossover scales that depend strongly on dimensionality. The existence of two anomalous regimes is particularly interesting, with the anomalous diffusion exponent of the first regime $\alpha_1 = 3 - d/2$ being related to the anomalous dynamics of catalytically active colloids [26, 32]. Anomalous dynamics can also arise in anisotropic active suspensions [33]. Due to the important role that ionic currents play in molecular sensing technologies [3, 4], it will be important to investigate how such strong and anomalous fluctuations can influence these sensing devices. A starting point can be to incorporate hydrodynamic and ionic effects in simple models of molecular sensing that use polymer translocation [34].

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