Code Swendsen-Wang Dynamics

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An important open question about Markov chains for preparing quantum Gibbs states is proving rapid mixing. However, rapid mixing at low temperatures has only been proven for Gibbs states with no thermally stable phases, e.g., the 2D toric code. Inspired by Swendsen-Wang dynamics, in this work we introduce a simple Markov chain, *Code Swendsen-Wang dynamics*, for preparing Gibbs states of commuting Hamiltonians. We prove rapid mixing of this chain for classes of quantum and classical Hamiltonians with thermally stable phases, including the 4D toric code, at any temperature. We conjecture its efficiency for all code Hamiltonians away from first-order phase transition points.

Contents

1 Introduction

Recently, there have been tremendous advancements in algorithms for quantum Gibbs sampling—the task of preparing quantum Gibbs states. The main open problem is establishing rapid mixing of these chains, thereby establishing efficiency of the Markov chain.

At high temperatures, recently proposed algorithms based on (quasi-)local dynamics do rapidly mix to the Gibbs state [KB16; BK19; YL23; BLMT24; RFA25]. However, as the temperature decreases, many systems cross a phase transition at a critical temperature, below which local dynamics mix only after exponentially long times. This slowdown can be attributed to the emergence of thermally stable phases, or modes of the Gibbs state, separated by *energy barriers* which local dynamics cannot traverse (except with exponentially small probabilities) [AFH09; GKZ24; RPBK24].

In this paper, we introduce a quantum Markov chain for Gibbs states of commuting Hamiltonians that *do* exhibit energy barriers. Furthermore, we show that it mixes quickly for Hamiltonians with an approximate "graphic" or "cographic" representation, such as code Hamiltonians of certain good LDPC codes and 2D and 4D toric codes. This gives a simpler, discrete alternative to the 2D toric code Markov chain in [DLLZ24] and is the first Markov chain that rapidly mixes to the 4D toric code from any initial configuration.

Our starting point is the Swendsen-Wang (SW) dynamics for the 2D Ising model. The Ising model has two ground states: one with all $+1$-spins and one with all -1-spins. When one runs local (Glauber) dynamics on this model at low temperatures, one is met with a severe bottleneck: it takes an exponential amount of time to traverse from predominantly $+1$ spins to predominantly -1 spins and vice-versa [BCK+99]. In 1987, Swendsen and Wang [SW87] introduced a Markov chain for the 2D Ising model which departed significantly from local updates and instead updated *clusters*. Clusters are connected subgraphs of the 2D lattice and capture the correlations between the spins. Every spin configuration σ defines a subgraph $E(\sigma)$ of the 2D lattice given by those edges of the lattice on which the spins align, defining clusters. The key idea of the Swendsen-Wang algorithm is to alternate between updates of spin configurations and updates of cluster configurations:

- *Cluster formation:* Given a spin configuration σ, clusters are formed by iid. removing edges in $E(\sigma)$ with probability $e^{-2\beta}$. The connected components of the resulting sublattice are the clusters.

- *Cluster update:* Given a cluster configuration, the new spin configuration is sampled by flipping an unbiased coin for every cluster and aligning the spins in each cluster accordingly.

Remarkably, this Markov chain mixes to the Gibbs distribution of the Ising model *and* faces no obstruction to crossing energy barriers. At low temperatures, clusters grow rapidly. Since all spins within each cluster are resampled in every step of the algorithm (cluster update step), once large clusters emerge, the barrier between the modes of the distribution centered at the all $+1$-spins configuration and the all -1-spins configuration is traversable in a *single* step. Consequently, the SW algorithm mixes rapidly for Ising models on any graph [GJ17; Ull14].

The focus of this paper is the formulation of a natural generalization of the SW dynamics to commuting Pauli Hamiltonians. In what follows, we will principally describe this chain—the *code Swendsen-Wang (SW)* dynamics—for quantum CSS codes. The goal of the algorithm is then to prepare the Gibbs state $\rho_\beta \propto e^{-\beta H}$ for the Hamiltonian

$$H = -\sum_{A \in \text{X checks}} X_A - \sum_{A \in \text{Z checks}} Z_A, \tag{1}$$

where X_A denotes the Pauli-X operator supported on the qubits in A, and likewise for Z_A. The key observation of the quantum chain is that a measurement of the stabilizers projects the system into an eigenstate of H. What remains is to switch into an eigenstate whose eigenvalue is distributed according to the correct (Gibbs) distribution. To achieve this, in each step of the quantum code SW chain, we measure X and Z stabilizers and apply complementary Z and X errors drawn from two classical Markov chains whose stationary distributions are Gibbs distributions

$$\pi(x) \propto e^{-\beta H(x)} \tag{2}$$

of the classical X and Z code, each with energy function

$$H(x) = -\sum_{A \in \text{checks}} (-1)^{\sum_{i \in A} x_i}. \tag{3}$$

Because the two sectors evolve independently through the course of the quantum Markov chain, the mixing time of the quantum chain is determined by the slower of the two classical chains. It thus remains to find classical Markov chains for the classical code Hamiltonians that mix rapidly for systems of interest. This is achieved by a generalization of the Ising SW dynamics to binary linear codes.

(Classical) code SW dynamics The code SW dynamics iterates updates of error configurations and (the code analogue of) cluster configurations.

- *Cluster formation:* Given an error configuration x, let $E(x)$ be the checks satisfied by x. Remove checks from $E(x)$ iid. with probability $e^{-2\beta}$, resulting in $E(x)'$.
- *Cluster update:* Sample a new error configuration x' by sampling a random codeword from the code defined by the checks $E(x)'$, i.e., sample x' uniformly such that $\langle A, x' \rangle = 0$ for all $A \in E(x)'$.

We show that the stationary distribution of this chain is given by the Gibbs distribution (2). In fact, the SW dynamics for the Ising model is a special case of the code SW dynamics where the interactions are restricted to be pairwise: in Eq. (3), let $H(x)$ be the Ising model Hamiltonian, where each check is a pairwise check between adjacent vertices in the interaction graph. Then in the cluster formation step of the original SW dynamics, a cluster is now just one dimension of the (repetition) codespace defined by $E(x)'$, and the cluster update step outputs a random linear combinations of these dimensions.

To prove polynomial mixing of this dynamics, we extend the approach of Guo and Jerrum [GJ17] and Ullrich [Ull14] for rapid mixing of the original SW dynamics on the Ising model. Their approach for solving this longstanding open problem relies on a series of three closely related Markov chains, propagating the proof of rapid mixing from the last of these chains to the first:

1. *SW dynamics on the RC model*: Consider the original SW chain where we focus our attention not on the spin configurations but on the *edge* configurations. The stationary distribution of this chain is a distribution over subgraphs called the *random cluster* (RC) model with probability distribution $\phi(S) \propto p^{|S|}(1-p)^{|E \setminus S|} 2^{k(S)}$ for $S \subset E$ and $p = 1 - e^{-2\beta}$, where $k(S)$ is the number of connected components of the subgraph S.
2. *Single-bond dynamics on the RC model*: The Markov chain which picks a random edge and resamples it according to the Metropolis rule for ϕ. This can be seen as a "slowed down" version of SW dynamics which updates a single, random edge rather than all edges.

3. *JS chain on the even-subgraph model*: The Jerrum-Sinclair (JS) chain efficiently samples from the distribution supported on *even subgraphs* $B \subset E$ (subgraphs where every vertex has even degree) proportional to $\xi(B) \propto p^{|B|}(1-p)^{|E\setminus B|}$. This chain works by expanding the state space to include subgraphs with two defects (vertices with odd degree).

Ullrich [Ull13] showed that rapid mixing for single-bond dynamics implies rapid mixing for the SW dynamics on the RC model and therefore also the Ising model. Guo and Jerrum [GJ17] gave an argument in which they "lift" the canonical paths from the rapid mixing proof of the JS chain [JS93] to show rapid mixing of the single-bond dynamics.

We can directly follow this proof for linear codes which have a *graphic* form [Oxl11] to show rapid mixing in this case. This covers the 2D toric code (which has no energy barrier) and certain LDPC codes without phase transitions (including some good LDPC codes). However, the 4D toric does not have such a graphic form and the analogue of the JS chain acts on more complex, high-dimensional objects for which the canonical path argument fundamentally breaks down. To show rapid mixing in this case, we use a different coupling that couples the RC model to the distribution of syndromes of the Gibbs state. The syndromes are themselves codewords of another code with a *dual* parity check matrix. For the 4D toric code this dual parity check is sufficiently *close* to being graphic and we can again lift the JS paths to show rapid mixing.

Altogether, using the correspondence between the quantum and classical chains, and the series of classical Markov chains just described, we can prove that our Gibbs sampling algorithm is efficient at all temperatures for commuting Pauli Hamiltonians with positive coefficients whose classical checks are close to being graphic or cographic (i.e., the dual is close to graphic). In fact, we conjecture that it is efficient at all temperatures for all classical and quantum stabilizer codes other than around first-order phase transition points. This is because the only known obstacles to rapid mixing of the SW algorithm are first-order phase transitions (in the Potts model) [GJ99; BCT12; GŠV19; GLP18].

Discussion and related work Sampling from commuting Hamiltonians, and in particular from the 2D toric code, has already been investigated in recent, as well as older work. Already in 2009, Alicki, Fannes, and Horodecki [AFH09] showed that a local Davies generator for the 2D toric code thermalizes quickly in a time upper bounded by $e^{O(\beta)}$, a scaling that matches the results for Glauber dynamics of the 1D Ising model [LPW09]. Improving the scaling in β, recently, Ding et al. [DLLZ24] showed that a nonlocal Davies generator for the 2D toric code on n qubits thermalizes in time at most $\min\{e^{O(\beta)}, \mathsf{poly}(n)\}$, leading to an efficient, nonlocal Gibbs sampler at arbitrary low temperatures. With an entirely different algorithmic approach based on a reduction to classical Gibbs sampling for 2-local Hamiltonians—that is different from our approach—Hwang and Jiang [HJ25] showed that the Gibbs states of certain commuting Hamiltonians, including the 2D toric code can be prepared efficiently at arbitrary temperatures. However, none of these works have been able to generalize their results to the 4D toric code.

Recently, Bergamaschi, Gheissari, and Liu [BGL25] showed that certain quasi-local block dynamics mix quickly *within* a logical sector of the 4D toric code, but require exponential time to traverse its energy barrier. When applied to a specific input state—the maximally mixed state on the code space—this algorithm can be used to efficiently prepare the full Gibbs state of the 4D toric code. In contrast, the code SW algorithm rapidly converges to the global Gibbs distribution from any initial state.

The code SW algorithm is also closely related to the study of approximation algorithms for the Tutte polynomial of binary matroids [JS09; Oxl11]. The mixing-time proof of Guo and Jerrum [GJ17] implies approximation algorithms for graphic as well as cographic matroids using duality properties of the Tutte polynomial. But while this algorithm just used the original SW

algorithm for graphs, our algorithm shows how to explicitly sample from the random cluster and Gibbs distributions associated with a matroid. Our mixing-time bounds further imply extensions of those approximation results to matroids that are close to being graphic or cographic. Conversely, Goldberg and Jerrum [GJ13] show that the approximation problem for the Tutte polynomial can be solved for so-called regular matroids, leading us to conjecture that the code SW algorithm also mixes quickly in this case.

2 Technical overview

In this section, we give a more detailed overview of the key ideas in our proof of rapid mixing for the code SW dynamics. We will then give the full proofs of correctness and rapid mixing in the subsequent sections, Section 3 and Section 4, respectively.

Consider the series of three Markov chains used in the proof of rapid mixing of the original SW dynamics for the Ising model described above. What changes when we generalize the dynamics to arbitrary linear codes via the code SW dynamics? Suppose we have a parity check matrix $h \in \{0,1\}^{c \times n}$ and run the SW dynamics on h as defined in the introduction. In the case of the code SW dynamics, we can define three related Markov chains on the space of subsets of the checks indexed by $E := [c]$ as follows:

1. *Code SW dynamics on the RC model*: Consider the code SW chain where we focus our attention not on the spin configurations but on the *check* configurations. The stationary distribution of this chain is a *random cluster* model on subsets of checks $S \subset E$ distributed as $\phi(S) \propto p^{|S|}(1-p)^{|E \setminus S|}2^{k(S)}$ with $p = 1 - e^{-2\beta}$. Here, $k(S)$ is the dimension of the code defined by the checks in S.

2. *Single-check dynamics on the RC model*: The Markov chain which picks a random check and resamples it according to the Metropolis rule for ϕ.

3. *"Code JS" chain on the even cover model*: Consider the chain that attempts to sample from the distribution $\xi(B) \propto p^{|B|}(1-p)^{|E \setminus B|}$ supported on *even covers* $B \subset E$, that is, subsets of checks where every variable is incident to an even number of checks. This chain works by expanding the state space to include covers with the minimal number of defects needed to traverse between any two even covers.

Ullrich's proof for the Ising model [Ull14] directly extends to our setting and implies that the code SW dynamics on the RC model mixes faster than the single-check dynamics. Furthermore, if there is a good set of canonical paths for the even-cover model, then, by Guo and Jerrum's proof [GJ17], the single-check dynamics mixes rapidly. Thus, to assess when this proof strategy works we must examine the rapid mixing of the code JS chain.

The original JS chain on a graph with edges E works as follows: Given any state $B \subset E$ with at most two defects, the chain randomly flips single edges while maintaining the invariant of at most *two* defects in the intermediate subgraphs. Because every pair of even subgraphs can be connected via disjoint loops, canonical paths can be defined by unwinding and creating loops with at most two defects. Crucially, the number of subgraphs with only two defects is polynomially related to the number of even subgraphs, so this Markov chain is rapidly mixing.

Unfortunately, when considering even covers of arbitrary codes, the higher-dimensional connectivity poses a fundamental obstacle: the state space may need to be expanded by an exponential factor. As an example, consider the 4D toric code, where bits lie on faces of the 4D torus and checks are associated to edges, i.e., each edge checks its four incident faces. In this setting, the even covers are unions of 3D volumes and to traverse from one even cover to another

using single-check flips we must unwind these volumes. In doing so, we necessarily pass through states with an extensive number of defects which means that the state space must expand by an exponential factor, thereby invalidating the proof strategy.

Our insight is that although there is a coupling between the RC model and the even-cover model (chains 2 and 3 above), which we call the *primal* view, there is yet another *dual* view which couples the RC model to the *syndrome distribution* of the Gibbs state. The cluster formation step of the SW dynamics defines this coupling: In this step, we sample a set of violated checks (syndromes) according to the Gibbs distribution, then add more violated checks independently with probability $e^{-2\beta}$ and output the set of satisfied checks. This is a sample from the RC model.

The set of syndromes form another linear subspace, captured by a dual parity check matrix. Replacing the code JS chain on even covers with an analogous code JS chain on the syndrome distribution, yields a dual code JS chain.

For the 4D toric code, in this dual view, it turns out that even covers can be unwound efficiently, giving a good set of canonical paths. To see this, consider the edge-vertex incidence matrix g of the 4D torus and let h be the (edges by faces) parity check matrix for the 4D toric code. Then $g^T h = 0$ because g^T and h are boundary maps from $1 \to 0$- and $2 \to 1$-dimensional objects, respectively, and the boundary of a boundary is zero. But this condition also implies that $\mathrm{col}(h)$, or the set of syndromes of the code, is contained in the set of all even subgraphs of the 4D torus. As long as the difference between the dimension of the syndrome space $\mathrm{col}(h)$ and the dimension of the even-subgraph space $\ker(g^T)$ is not too big, which is the case for the 4D toric code, we obtain a good set of canonical paths.

We can then lift these canonical paths on the syndrome distribution to a flow on the RC model, analogously to Guo and Jerrum [GJ17]. This flow then implies rapid mixing of the single-check dynamics [JS93]. Then, the code SW dynamics also mix rapidly by an analogous argument to Ullrich [Ull14].

Theorem 1. *The code SW algorithm for the 4D toric code mixes rapidly at any temperature.*

In fact, we are able to show rapid mixing of the code SW chain for a larger class of codes, using the notion of a *graphic* or *cographic* parity check matrix. A parity check matrix is graphic if the linear dependencies of its rows are captured by a graph, i.e., $\ker(h^T) = \ker(g^T)$ for the edge-vertex incidence matrix g of a graph. It is cographic if there is a graphic (column-)generator matrix for the orthogonal complement $\mathrm{col}(h)^\perp$ of the column space $\mathrm{col}(h)$ of h. We can relax this notion to parity checks that are *close* to being graphic or cographic. We say that h is Δ-(co)graphic if at most Δ vectors need to be added to $\ker(h^T)$ ($\mathrm{col}(h)$) to make the linear dependencies graphic.

Theorem 2 (Rapid mixing for Δ-graphic or Δ-cographic codes)**.** *Given a parity check matrix h, the code SW algorithm for the Gibbs distribution of h mixes in time $2^\Delta \cdot \mathsf{poly}(n)$ at any temperature if h is Δ-graphic or Δ-cographic.*

Further examples of quantum codes that satisfy the conditions of Theorem 2 are:

- the 2D toric code. The graph defined by their parity check is a cycle graph.
- codes that arise from any chain complex on a two-dimensional surface, i.e. parity check h which has the property that $g^T h = 0$ for the edge-vertex incidence matrix g of a graph, e.g., the X and Z checks of the 4D toric code.
- codes whose parity checks have no linear dependencies such as certain good LDPC codes [HGL25; PRBK24]. The graph defined by their parity check is a line graph.

3 Quantum Markov chain for code Hamiltonians

In this section, we describe a reduction from quantum to classical Gibbs sampling for Hamiltonians composed of commuting Pauli terms with positive coefficients (i.e., stabilizer Hamiltonians). Specifically, we show how Gibbs state preparation can be achieved by running a classical Markov chain for the Gibbs distribution of a corresponding linear code. Furthermore, we prove that the mixing time of the quantum chain is upper bounded by the mixing time of the classical chain.

3.1 Preliminaries: Codes and their Gibbs states

3.1.1 Linear codes and stabilizer codes

The *linear code* defined by the parity-check matrix $h \in \{0,1\}^{c\times n}$ is given by $C_h = \ker(h)$, the set of vectors x such that $hx = 0$. Let $P_1, \ldots, P_c \in \mathcal{P}_n$ be a set of commuting Pauli operators in the n-qubit Pauli group $\mathcal{P}_n$, i.e., $[P_i, P_j] = 0$ for all i, j, and let $\mathcal{S} = \langle P_1, \ldots, P_c\rangle \leq \mathcal{P}_n$ be the stabilizer group generated by $P_1, \ldots, P_c$ with $\dim(\mathcal{S}) = n - k$. The *stabilizer code* associated with $\mathcal{S}$ is given by

$$C_{\mathcal{S}} = \{|\psi\rangle \in (\mathbb{C}_2)^{\otimes n} : S\,|\psi\rangle = |\psi\rangle\,, \forall S \in \mathcal{S}\}. \tag{4}$$

A convenient representation of Pauli operators is the *stabilizer matrix representation*. Let X, Y, Z be single-qubit Pauli operators and X_i be the Pauli-X operator acting on qubit i. Let $X(x) = \prod_{i\in[n]} X_i^{x_i}$ and likewise for $Z(z)$ with $x, z \in \{0,1\}^n$. We can thus represent every Pauli operator by a string $p = (x, z) \in \{0,1\}^{2n}$ as $P(p) = i^{x\cdot z}X(x)Z(z)$. For example, for a single qubit, $P(0,0) = \mathbb{1}$, $P(1,0) = X$, $P(0,1) = Z$ and $P(1,1) = Y$. The (multiplicative) Pauli group $\mathcal{P}_n$ without phases is therefore equivalent to the (additive) group $\mathbb{Z}_2^{2n}$ in the sense that for $x, y \in \{0,1\}^{2n}$, $P(x + y) \propto P(x)Q(y)$. The commutativity of Pauli operators is captured by the symplectic form ω defined as follows

$$[P(x), P(y)] = 0 \Leftrightarrow \omega(x, y) \coloneqq x^T\omega y = 0, \quad \omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}. \tag{5}$$

Given a stabilizer $\mathcal{S}$ generated by Pauli operators $P_1 = P(g_1), \ldots, P_c = P(g_c)$ for $g_i \in \{0,1\}^{2n}$, we can therefore represent $\mathcal{S}$ by the *stabilizer matrix* g with rows g_i as $\mathcal{S} = \mathcal{P}(g) \coloneqq \langle P(g_1), \ldots, P(g_c)\rangle$. In fact, by linearity, only the row space of g matters and we have $\mathcal{S} = \mathcal{P}(\mathrm{row}(g))$. Let $h = g\omega$. The commutant of $\mathcal{S}$ is given by $\mathcal{C} = \mathcal{P}(\ker(h))$ and contains both the stabilizer itself and the logical operators $\mathcal{L} = \mathcal{C}/\mathcal{S}$ of the code $C_{\mathcal{S}}$.

Letting $k = \dim(C_{\mathcal{S}})$, we can index a complete set of logical Pauli operators $\overline{P(l)}$ by $l \in \{0,1\}^{2k}$ such that $\mathcal{L} = \langle \overline{P(l)}, l \in \{0,1\}^{2k}\rangle$. The *syndrome* of $C_{\mathcal{S}}$ is the column range $S_g = \mathrm{col}(g)$. It is the set of possible measurement outcomes when measuring the Pauli operators corresponding to the rows of g. For every $s \in S$, let $h^{-1}s$ be the set of preimages of s under h. We can associate a representative Pauli error $e_s = \min(h^{-1}s)$ given by the lexicographically first preimage and denote $E(s) = P(e_s)$. It will be important that for any $e \in h^{-1}s$, $e_s + e \in \ker(h)$.

We write the projector onto the code space as $\Pi(g, 0) = 2^{-(n-k)}\sum_{S\in\mathcal{S}} S$, and the projector onto the syndrome-s space as $\Pi(g, s) = E(s)\Pi(g, 0)E(s)$.

3.1.2 Gibbs states of quantum and classical codes

Given a classical code with parity-check matrix $h \in \{0,1\}^{c\times n}$, we associate the energy function

$$H(x) = 2|hx|, \tag{6}$$

where $|x| = \sum_i x_i$ is the Hamming weight of x. Note that this is equivalent to the form given in the introduction, Eq. (3), up to an additive shift. The corresponding Gibbs distribution is the probability distribution

$$\pi(x) \propto e^{-\beta H(x)}. \tag{7}$$

Likewise, given a quantum stabilizer code with stabilizer matrix $g \in \{0,1\}^{c\times 2n}$, we associate the Hamiltonian

$$H = -\sum_{i\in[c]} P(g_i), \tag{8}$$

with Gibbs state

$$\rho_\beta \propto e^{-\beta H}. \tag{9}$$

3.2 Lifting classical code sampling to quantum code sampling

We now describe a quantum Markov chain for preparing Gibbs states of stabilizer codes (as in Eq. (9)). The key idea is to lift a classical Markov chain Q, which samples errors from the Gibbs distribution of a corresponding classical code (of the form Eq. (7)), to a quantum chain whose fixed point is the Gibbs state.

For CSS code, the formulation in the introduction—with two independent underlying chains—follows directly: the Pauli-X and Z terms decouple, so the Gibbs distribution factorizes into a product over Z and X errors. Hence, we can sample them independently. In contrast, for general stabilizer codes, the X and Z components are coupled, and we must consider the joint distribution over both types of errors.

To this end, let the stabilizer code (of dimension k) be specified by the stabilizer matrix $g \in \{0,1\}^{c\times 2n}$. Up to an additive shift, the Hamiltonian (8) can be rewritten as

$$H = \sum_{s\in S_g} 2|s|\Pi(g,s), \tag{10}$$

which is diagonal in the syndrome-subspace projectors $\Pi(g,s)$. The Gibbs state is therefore given by

$$e^{-\beta H} \propto \sum_{s\in S_g} e^{-2\beta|s|}\Pi(g,s). \tag{11}$$

Let π be the Gibbs distribution associated with $h \coloneqq g\omega$ at inverse temperature β. We will lift a classical Markov chain Q with stationary distribution π to a quantum Markov chain with fixed point ρ_β.

Theorem 3 (Convergence to the Gibbs state)**.** *The outputs $|\phi\rangle$ sampled from Algorithm 1 satisfy*

$$\lim_{T\to\infty} \mathop{\mathbb{E}}_{\phi} |\phi\rangle\langle\phi| \propto e^{-\beta H}, \tag{12}$$

for any initial state $|\phi_0\rangle$.

Proof. To show that Algorithm 1 converges to the correct distribution, observe that as $T \to \infty$, $Q^T(x_0,\cdot) \to \pi$ for any initial state $x_0 \in \{0,1\}^{2n}$. Therefore, to show correctness, we can assume that $y \leftarrow \pi$ is sampled from π.

Algorithm 1 Quantum code Gibbs sampling

Input: Stabilizer matrix g, classical Markov chain Q, initial state $|\phi_0\rangle$, initial error configuration x_0

1: Let $|\phi\rangle \leftarrow |\phi_0\rangle, x \leftarrow x_0$, cutoff time T.

2: Iterate T times:

3: i. Measure the stabilizers $\mathcal{S}$ on $|\phi\rangle$, yielding syndrome $s \in S_g$, and post-measurement state $|\phi_s\rangle$.

ii. Sample $y \leftarrow Q(x, \cdot)$, and apply $P(y)E(s)$, letting $|\phi\rangle \leftarrow P(y)E(s)|\phi_s\rangle$, $x \leftarrow y$.

Output: $|\phi\rangle$

We begin by observing that the post-measurement state is given by an erroneous logical state $E(s)\overline{|\varphi\rangle}$ for some $s \in S_g \subset \{0,1\}^c$, where $\overline{|\varphi\rangle} = \Pi(h,0)\overline{|\varphi\rangle}$ is a code state. Now, observe that a sample $y \leftarrow \pi$ can be decomposed as $y = d + e + f$ in terms of a logical Pauli, a representative error, and a stabilizer represented by $d, e, f \in \{0,1\}^{2n}$, respectively. We can thus write

$$P(y) \propto E(s)\overline{P(l)}S, \tag{13}$$

up to a global phase for a syndrome $s = he$ a logical $l \in \{0,1\}^{2k}$ such that $P(d) = \overline{P(l)}$, and a stabilizer $S = P(f) \in \mathcal{S}$. In particular, observe that the distribution π only depends on the syndrome of the error e and therefore l and f are uniformly random.

Applying $E(s)$ moves the state to the logical subspace, i.e., the 0-syndrome subspace. Applying $P(y)$ then twirls the logical state and it maps to a syndrome subspace sampled according to the Gibbs distribution, giving

$$\rho \propto \sum_y e^{-2\beta|hy|} P(y)E(s)E(s)\overline{|\varphi\rangle\langle\varphi|}E(s)E(s)P(y) \tag{14}$$

$$= \sum_{t \in S_g} e^{-2\beta|t|} E(t) \left[\mathop{\mathbb{E}}_{l \in \{0,1\}^{2k}} \overline{P(l)}\,\overline{|\varphi\rangle\langle\varphi|}\,\overline{P(l)} \right] E(t) \tag{15}$$

$$= \frac{1}{2^k} \sum_{t \in S_g} e^{-2\beta|t|} E(t)\Pi(g,0)E(t) \tag{16}$$

$$= \frac{1}{2^k} \sum_{t \in S_g} e^{-2\beta|t|} \Pi(g,t) \propto e^{-\beta H} \tag{17}$$

Here, we have used that twirling any state over the Pauli group yields a maximally mixed state and therefore

$$\mathop{\mathbb{E}}_{l \in \{0,1\}^{2k}} \overline{P(l)}\,\overline{|\varphi\rangle\langle\varphi|}\,\overline{P(l)} = \frac{1}{\mathrm{tr}[\Pi(g,0)]}\Pi(g,0) = \frac{1}{2^k}\Pi(g,0) \tag{18}$$

for any logical state $\overline{|\varphi\rangle}$.

□

We now show that the mixing time of the classical chain Q governs the mixing time of the quantum chain above. Formally, consider an ergodic (classical) Markov chain on X with transition matrix P and stationary distribution π. We define the mixing time

$$\tau(P) \coloneqq \min_t \left[\max_{x \in X} \sum_y |P^t(x,y) - \pi(y)| \leq e^{-1} \right]. \tag{19}$$

We furthermore define the mixing time of Algorithm 1 which outputs a state $|\phi\rangle$ after time T as

$$\tau_q \coloneqq \min_T \left[\max_{|\phi_0\rangle \in (\mathbb{C}^2)^n} \left\| \mathop{\mathbb{E}}_{\phi} |\phi\rangle\langle\phi| - \rho_\beta \right\|_{\mathrm{tr}} \leq e^{-1} \right]. \tag{20}$$

Lemma 4 (Coupling of quantum and classical chains)**.** *The mixing times of Q and Algorithm 1 satisfy $\tau_q \leq \tau(Q)$.*

Proof. After time t the average state of the algorithm is given by

$$\rho_t = \mathop{\mathbb{E}}_{y \leftarrow Q^t(x_0,\cdot)} P(y) \overline{|\phi_{t-1}\rangle\langle\phi_{t-1}|} P(y) \tag{21}$$

where $|\phi_{t-1}\rangle$ is the state of the algorithm at time $t-1$ and $\overline{|\phi_{t-1}\rangle} \propto \Pi(g,0)|\phi_{t-1}\rangle$ is its projection to the code space. We can further write the equilibrium state $\rho_\beta \propto e^{-\beta H}$ as

$$\rho_\beta = \mathop{\mathbb{E}}_{y \leftarrow \pi} P(y) \overline{|\phi_{t-1}\rangle\langle\phi_{t-1}|} P(y), \tag{22}$$

by applying the proof of Theorem 3 in reverse and substituting $|\varphi\rangle$ with $|\phi_{t-1}\rangle$. Therefore,

$$\|\rho_t - \rho_\beta\|_{\mathrm{tr}} = \left\| \sum_y \left(Q^t(x_0,y) - \frac{1}{Z} e^{-2\beta|hy|} \right) P(y) \overline{|\phi_{t-1}\rangle\langle\phi_{t-1}|} P(y) \right\|_{\mathrm{tr}} \tag{23}$$

$$\leq \sum_y |Q^t(x_0,y) - \frac{1}{Z} e^{-2\beta|hy|}| \leq 1/e, \tag{24}$$

where $Z = \sum_y e^{-2\beta|hy|}$ and we used that the trace norm of a quantum state is equal to 1. □

3.3 Quantum Markov chains beyond code Hamiltonians

We note that Algorithm 1 is not restricted to stabilizer Hamiltonians, but will also be correct for *signed* stabilizer Hamiltonians. Such Hamiltonians are parameterized by a stabilizer matrix $g \in \{0,1\}^{c \times 2n}$ as well as a vector $t \in \{0,1\}^c$ as

$$H = -\sum_{i \in [c]} (-1)^{t_i} P(g_i). \tag{25}$$

For this signed case—analogous to antiferromagnetic Ising Hamiltonians—it is well known, however, that sampling is intractable. In fact, an efficient sampler would be able to solve MAXCUT, an NP-complete problem. Therefore, we also do not expect an efficient quantum sampler for this case.

We can also conceive of analogous dynamics for Hamiltonians which are sums of arbitrary—potentially non-commuting—Pauli operators. Importantly, the distribution π is still well-defined in this case and the energy function just corresponds to the number of Hamiltonian terms an "error" anticommutes with. However, in this case, syndrome extraction cannot be done for all terms simultaneously and the measurement will always project into the joint eigenspace of a subset of the Hamiltonian terms, giving a non-convergent chain. We believe it is an interesting future direction to explore the dynamics that arise from Algorithm 1 for different measurement protocols of the Hamiltonian terms, however. Some examples that could be interesting are: (i) sequentially measure random maximal subsets of commuting terms and (ii) weakly measure the Hamiltonian terms.

4 Code Swendsen-Wang dynamics and mixing time

In this section, we formally define the (classical) code SW chain and show that it converges to the correct Gibbs distribution. We then introduce structural conditions on parity check matrices – Δ-graphic and Δ-cographic – under which the code SW chain mixes rapidly.

4.1 Recap: Swendsen-Wang dynamics for the Ising model

Let $G = (V, E)$ be a graph. We consider spins on the vertices with configuration space $\Sigma = \{-1, +1\}^V$. The (ferromagnetic) Ising model on Σ is described by the energy function

$$H(\sigma) = -\sum_{(u,v)\in E} \sigma_u \sigma_v. \tag{26}$$

Recall the Swendsen-Wang (SW) chain for sampling the Gibbs distribution $\pi(\sigma) \propto e^{-\beta H(\sigma)}$: For a given configuration of spins σ, define the set of satisfied edge constraints

$$E(\sigma) \coloneqq \{\{u, v\} \in E : \sigma(u) = \sigma(v)\}. \tag{27}$$

Then the SW chain iterates the following two steps, starting from a spin configuration $\sigma \in \Sigma$.

1. *Cluster formation:* For each $e \in E(\sigma)$ include it in A with probability $p = 1 - e^{-2\beta}$, giving $A \subset E$.
2. *Cluster update:* For every connected component $C \subset V$ of A let $\tau(C) = +1$ with probability $1/2$ and -1 otherwise, giving a new state $\tau \in \Sigma$.

This chain is motivated by a correspondence, discovered by Fortuin and Kasteleyn [FK72], between the Gibbs measure $\pi(\sigma)$ and the random cluster (RC) measure

$$\phi(A) \propto \left(\frac{p}{1-p}\right)^{|A|} 2^{k(A)}, \tag{28}$$

on subgraphs $A \subset E$, where we denote the number of connected components of A by $k(A)$. In particular, the correspondence is given by the joint distribution

$$\mu(\sigma, A) \propto \left(\frac{p}{1-p}\right)^{|A|} \mathbb{1}(A \subset E(\sigma)), \tag{29}$$

where we write $\mathbb{1}(e)$ for the indicator of an event e.

Claim 5 (Coupling of Ising and RC model)**.** *μ defines a proper coupling between π and ϕ, with $p = 1 - e^{-2\beta}$, i.e.,*

$$\sum_{A\subset E} \mu(\sigma, A) = \pi(\sigma), \qquad \sum_{\sigma\in\Sigma} \mu(\sigma, A) = \phi(A). \tag{30}$$

Proof. To show the claim, we observe that

$$\mu(\sigma, A) \propto \prod_{(i,j)=e\in A} [p\mathbb{1}(\sigma(i) = \sigma(j))] \prod_{e\notin A} (1-p) \tag{31}$$

$$\propto \prod_{e=(i,j)\in E} \left[p\,\mathbb{1}(\sigma(i) = \sigma(j))\mathbb{1}(e \in A) + (1-p)\mathbb{1}(e \notin A)\right]. \tag{32}$$

Setting $p = 1 - e^{-2\beta}$, we then have

$$\sum_{A \subset E} \mu(\sigma, A) \propto \prod_{(i,j) \in E} \left((1 - e^{-2\beta}) \mathbb{1}(\sigma(i) = \sigma(j)) + e^{-2\beta} \right) \tag{33}$$

$$= \prod_{(i,j) \in E \,:\, \sigma(i) \neq \sigma(j)} e^{-2\beta} \tag{34}$$

$$\propto e^{-\beta H(\sigma)}, \tag{35}$$

and

$$\sum_{\sigma \in \Sigma} \mu(\sigma, A) \propto p^{|A|} (1-p)^{|E \setminus A|} \sum_{\sigma} \prod_{(i,j) \in A} \mathbb{1}(\sigma(i) = \sigma(j)) \tag{36}$$

$$= p^{|A|} (1-p)^{|E \setminus A|} 2^{k(A)} \tag{37}$$

$$\propto \left(\frac{p}{1-p} \right)^{|A|} 2^{k(A)}, \tag{38}$$

which completes the proof. □

Each step of the SW chain simply draws conditional samples from this joint measure $\mu(\sigma, A)$. In particular, the cluster formation step (step 1) is equivalent to drawing a sample $A \leftarrow \mu(\sigma, \cdot)$, and the cluster update step (step 2) is equivalent to drawing a sample $\tau \leftarrow \mu(\cdot, A)$.

From this, we can immediately see that the SW dynamics has the distribution π as its unique fixed point, since its transition matrix $P(\sigma, \tau) = \sum_A P(\sigma, A) P(A, \tau)$ satisfies detailed balance:

$$\frac{P(\sigma, \tau)}{P(\tau, \sigma)} = \frac{\sum_A \mu(A|\sigma) \mu(\tau|A)}{\sum_A \mu(A|\tau) \mu(\sigma|A)} = \frac{\pi(\tau)}{\pi(\sigma)}, \tag{39}$$

where we have simply used the law of conditional probabilities $\mu(A|\sigma) = \mu(\sigma, A)/\pi(\sigma)$ and likewise for $\mu(\sigma|A)$. To complete the argument, observe that the chain is aperiodic and ergodic since after step 1 not a single cluster survives with finite probability, and therefore, in step 2, an arbitrary state can be reached.

4.2 Code Swendsen-Wang dynamics

To generalize the SW dynamics to classical codes, we observe that the Ising model on a graph $G = (V, E)$ defines a particular linear code. To see this, notice that every edge of the graph defines a pairwise parity check constraint. If we interpret the edge-vertex incidence matrix $h \in \{0,1\}^{|E| \times |V|}$ of G as a parity check matrix, then the corresponding code C_h of this parity check matrix is the repetition code, $\{0^n, 1^n\}$, if the graph is fully connected. Viewing the Ising model this way, there is nothing special about the relationship between the SW chain and the Ising model. In fact, we are free to choose *any* parity check matrix h of a linear code and define an analogous SW dynamics for that code.

More precisely, in the following we consider a code with parity check matrix $h \in \{0,1\}^{c \times n}$, and the configuration spaces of variables $X = \{0,1\}^n$ and (indices of) checks $E = [c]$. Our goal is to sample from the Gibbs distribution $\pi(x) \propto e^{-2\beta |hx|}$, see Eq. (7). Toward this end, let $\mathrm{supp}(x) = \{i \in [n] : x_i = 1\}$ be the support of x and define the set of satisfied checks

$$E(x) \coloneqq E \setminus \mathrm{supp}(hx) \tag{40}$$

and the parity-check matrix h_A as the row-submatrix of h indexed by $A \subset E$. The code Swendsen-Wang (SW) algorithm is given in Algorithm 2.

Algorithm 2 The code SW algorithm

Input: Parity check h, inverse temperature β, initial state x, cutoff time T.

1: Let $x \leftarrow x_0, A \leftarrow \emptyset$.

2: Iterate T times:

i. *Cluster formation:* For every $e \in E(x)$, include it in A with probability $p = 1 - e^{-2\beta}$, giving $A \subset E$.

ii. *Cluster update:* Pick $y \in \ker(h_A)$ uniformly at random and let $x \leftarrow y, A \leftarrow \emptyset$.

Output: x

To see that the code SW algorithm converges to its unique fixed point π, we observe that it, too, works by coupling to the random cluster model on codes, defined for $A \subset E$ as

$$\phi(A) \propto \left(\frac{p}{1-p}\right)^{|A|} 2^{k(A)}, \tag{41}$$

where $k(A)$ is the dimension of $\ker(h_A)$. This is achieved via a generalized FK measure

$$\mu(x, A) \propto \left(\frac{p}{1-p}\right)^{|A|} \mathbb{1}(A \subset E(x)) \tag{42}$$

at $p = 1 - e^{-2\beta}$.

Lemma 6 (Correctness of code SW dynamics)**.** *The code SW algorithm (Algorithm 2) converges to π as $T \to \infty$ for any initial state x_0.*

Proof. Clearly, the chain is aperiodic and ergodic, since after a single iteration of the algorithm any state can be reached since $\Pr(A = \emptyset) > 0$ for $\beta < \infty$. To show that it satisfies detailed balance, we use Eq. (39) and all that remains is to show that the transition probabilities of step 1 and 2 in Algorithm 2 are

$$P(x, A) = \mu(A|x), \qquad P(A, y) = \mu(y|A). \tag{43}$$

To see this, observe that μ is a valid coupling, i.e.,

$$\sum_{x \in X} \mu(x, A) = \phi(A) \quad \text{and} \quad \sum_{A \subset E} \mu(x, A) = \pi(x). \tag{44}$$

This can be shown via a proof identical to Claim 5, observing that

$$\mu(x, A) \propto \prod_{e \in E} \left(p\, \mathbb{1}(h_e x = 0)\mathbb{1}(e \in A) + (1-p)\mathbb{1}(e \notin A)\right). \tag{45}$$

It remains to consider the case of zero temperature. In this case, we can see that the algorithm converges rapidly: In the cluster-formation step all previously satisfied checks are kept. In the cluster-update step, if there is an unsatisfied check prior to the update, this check will be satisfied with probability 1/2 after the update. Therefore the number of unsatisfied checks converges to zero exponentially fast. Once no unsatisfied checks remain, the cluster update step is a sampler from the zero-temperature Gibbs state. □

4.3 Mixing time of the code Swendsen-Wang dynamics

In this subsection, we prove that the code SW dynamics is rapidly mixing for codes thats are Δ-graphic or Δ-cographic (see below for definition). Our proof proceeds in two steps: first, we adapt Ullrich's proof that SW dynamics mixes quicker than single-bond dynamics to the setting of linear codes; second, we show that single-check dynamics – the analogue of single-bond dynamics – is rapidly mixing for Δ-graphic and Δ-cographic codes.

4.3.1 Comparison to single-check updates

For the proof, we will first switch to the RC view of the SW algorithm. To this end, observe that if we halt the algorithm after the cluster formation step (step 1), the sampled subset of checks will be distributed according to the RC model ϕ (as $T \to \infty$).

Therefore, to prove mixing times of the SW algorithm for the Gibbs distribution, it is sufficient to prove a mixing time bound for the convergence to the RC model instead.

We start with a comparison to the standard (lazy) single-check-update Metropolis dynamics in which with probability $1/2$ no change is made and otherwise, an update $B = A \oplus e$ is proposed for uniformly random $e \in E$. This proposal is accepted with probability

$$\Pr(\text{Accept}) = \min\left\{1, \frac{\phi(B)}{\phi(A)}\right\}, \tag{46}$$

and rejected otherwise. Let

$$P_{\text{Metropolis}}(A,B) = \begin{cases} \frac{1}{2|E|}\min\{1, \frac{\phi(B)}{\phi(A)}\} & |A \oplus B| = 1 \\ 1 - \frac{1}{2|E|}\sum_{e\in E}\min\{1, \frac{\phi(A\oplus e)}{\phi(A)}\} & A = B \\ 0 & \text{else} \end{cases} \tag{47}$$

be the corresponding transition matrix, and likewise P_{SW} be the transition matrix of the SW process for the RC model.

Lemma 7 (SW is faster than Metropolis dynamics)**.** *The SW dynamics of the RC model is faster than Metropolis dynamics (up to a constant), i.e.,*

$$\tau(P_{Metropolis}) \geq \frac{1}{2}\tau(P_{SW}). \tag{48}$$

The proof of Lemma 7 is an adaptation of the result of Ullrich [Ull14] to our setting, which we give in Appendix A.

4.3.2 Fast mixing of single-check dynamics

Here we show that single-check dynamics for the RC model of a parity check matrix h mixes rapidly if h has an approximate "graphic" or "co-graphic" representation.

Definition 8. *Let $h \in \{0,1\}^{c\times n}$ be a parity check matrix. We say that h is*

- *Δ-graphic if there exists an edge-vertex incidence matrix $g \in \{0,1\}^{c\times m}$ of a graph on $m \leq O(n)$ vertices such that $\ker(g^T) \supset \ker(h^T)$ and $\dim(\ker(g^T)) - \dim(\ker(h^T)) \leq \Delta$.*
- *Δ-cographic if a column-generator matrix $h^\perp$ of $\text{col}(h)^\perp$ is Δ-graphic.*

In the first case, we call the edge-vertex incidence matrix g the primal coupling *to h. In the second case, we call the edge-vertex incidence matrix g that is the primal coupling to $h^\perp$ the* dual coupling *to h.*

Which codes admit primal or dual couplings?

Claim 9. *The parity check matrix h of any linear code with independent checks is 0-graphic.*

Proof. Let g be the edge-vertex incidence matrix of a line graph. Then $\ker(g^T)$ and $\ker(h^T)$ are both 0-dimensional. □

Claim 10. *The X and Z parity check matrices h of the 2D toric code are* 0*-graphic*

Proof. Consider the edge-vertex incidence matrix $g \in \{0,1\}^{c\times c}$ for a cycle graph. Then $\ker(g^T)$ is spanned by the all 1's vector, which matches exactly with the even covers of the 2D toric code. □

Claim 11. *The X and Z parity check matrices h of the 4D toric code are* 4*-cographic.*

Proof. Let g be the edge-vertex incidence matrix of the 4D torus. Then $g^T h = 0$ because g^T and h are boundary maps of a chain complex. Therefore, $\ker(g^T) \supset \mathrm{col}(h) = \ker((h^\perp)^T)$. On the other hand, a set of generators for the even subgraphs of the 4D lattice are the trivial loops (boundaries of faces) and nontrivial loops (crossing the boundary of one of the dimensions). There are four generators for the latter, corresponding to each dimension, so $\dim(\ker(g^T)) - \dim(\mathrm{col}(h)) = 4$. □

To show rapid mixing of the single-check dynamics of the RC model for these models, we will use the method of *flow congestion*. To introduce it, we first introduce the concept of a path in the state space. Let the state space of our Markov chain with transitions P be Ω, and the *graph* of P be defined by the edges $\mathcal{E}(P) \coloneqq \{(U,V) \in \Omega\times\Omega : P(U,V) > 0\}$. A path

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{L(\gamma)}) \subset \Omega^*, \tag{49}$$

of length $L(\gamma) \in \mathbb{N}$ is a sequence of states $\gamma_i \in \Omega$ which are connected by transitions of the Markov chain, i.e., $(\gamma_i, \gamma_{i+1}) \in \mathcal{E}(P)$. Let $\Gamma(I,F) \coloneqq \{\gamma \in \Omega^* : \gamma_0 = I, \gamma_{-1} = F\}$ be the set of paths from I to F and $\Gamma = \bigcup_{I,F} \Gamma(I,F)$.

Definition 12 (Flows and canonical paths)**.** *A* flow *with respect to a probability distribution p on* Ω *is a function* $f : \Omega^* \to [0,1]$ *assigning a weight* $f(\gamma)$ *to every path such that*

$$\sum_{\gamma\in\Gamma(I,F)} f(\gamma) = p(I)p(F) \tag{50}$$

for all $I \neq F \subset E$*. A set of* canonical paths *for pis a flow for which there is only a single "canonical" path between every pair* (I,F)*, i.e.,* $|\Gamma(I,F)| = 1$*. In this case* $f(\gamma) = p(I)p(F)$ *for* $\gamma \in \Gamma(I,F)$*.*

Rapid mixing is equivalent to the existence of flows with low *congestion*, where the congestion of a set of flows f is given by

$$\rho(f) \coloneqq \max_{(Z,Z')\in\mathcal{E}(P)} \frac{1}{p(Z)P(Z,Z')} \sum_{\gamma\in\Gamma,(Z,Z')\in\gamma} f(\gamma)L(\gamma), \tag{51}$$

A fundamental result in the analysis of mixing times of Markov chains is that there is a flow such that the mixing time is captured by the congestion of that flow.

Theorem 13 ([Sin92; Gur16])**.** *For a lazy, ergodic, reversible Markov chain P, we have*

$$\Omega\left(\inf_f \rho(f)\right) \leq \tau(P) \leq \ln\left(\frac{2e}{\min_{S\in\Omega} p(S)}\right)\rho(f). \tag{52}$$

The mixing time of a Markov chains is therefore equivalent to the congestion of some flow. To show rapid mixing, we will construct a "good" flow for the RC model ϕ.

Theorem 14 (Existence of a good flow for the RC model)**.** *Suppose that $h \in \{0,1\}^{c\times n}$ is Δ-graphic or Δ-cographic. Then there is a flow F for the RC model ϕ with congestion*

$$\rho(F) \le O(c^2 2^{2\Delta+5} n^4). \tag{53}$$

Combining this with Theorem 13, we arrive at a polynomial mixing-time bound.

Corollary 15 (Rapid mixing of code SW dynamics)**.** *Suppose that $h \in \{0,1\}^{c\times n}$ is Δ-graphic or Δ-cographic, for $\Delta \le O(\log n)$. Then the code SW chain for h mixes in $\mathsf{poly}(n)$ time.*

We prove Theorem 14 by coupling the RC model on h to a generalization of the even subgraph model, which we dub the *even cover model*. The even cover model is defined by the distribution

$$\xi_{h,p}(A) \propto \left(\frac{p}{1-p}\right)^{|A|} \mathbb{1}(1_A \in \ker(h^T)), \tag{54}$$

where 1_A is the indicator of $A \subset E$ with $\mathrm{supp}(1_A) = A$. In words, this is a weighted distribution over subsets of checks such that every vertex is incident to an even number of checks. The following lemma makes transparent the connection between the even cover model and the RC model. We will make the dependence of ϕ on p explicit by writing $\phi_p = \phi$ whenever it is needed.

Lemma 16 (Coupling of even covers and RC)**.** *There are two couplings of the even-cover model to the random cluster model. Let $h^\perp$ be a column-generator matrix of* $\mathrm{col}(h)^\perp$.

- *(primal lift) Let $A \sim \xi_{h,p/2}$, and add each $e \in E \setminus A$ iid. with probability $(p/2)/(1-p/2)$ to obtain B. Then $B \sim \phi_p$.*
- *(dual lift) Let $A \sim \xi_{h^\perp,(1-p)/(2-p)}$, and add each $e \in E \setminus A$ iid. with probability $1-p$ to obtain B. Then $E \setminus B \sim \phi_p$.*

We call $\xi_{h,p/2}$ the primal coupling *to ϕ and $\xi_{h^\perp,(1-p)/(2-p)}$ the* dual coupling *to ϕ.*

We prove Lemma 16 in Section B.

Our high-level strategy for constructing flows for the RC model is similar to the strategy of Guo and Jerrum [GJ17]. They begin with a parity check matrix g that is an edge-vertex incidence matrix and consider the primal coupling $\xi_{g,p/2}$. In this case, the primal coupling is a weighted distribution over *even subgraphs* of the original graph, i.e., subgraphs that have even degree on every vertex. Then, they use an idea due to Jerrum and Sinclair [JS93]: they enlarge the state space to also include subgraphs with exactly *two* defects—odd degree vertices—called *worms*. This enlargement allows any two even subgraphs $S, T \subset E$ to be connected by sequences of single edge-flips that sequentially unwind the loops in $S \oplus T$. Any intermediate state of these paths is a worm state, i.e., has at most two defects. These paths can be "lifted" to a good flow for the RC model by *mimicking* edge additions and deletions from unwinding even subgraphs to additions and deletions in the RC model.

Our proof strategy for constructing good flows for Δ-graphic and Δ-cographic h proceeds by adapting the above strategy in two ways. First, instead of just starting from the primal coupling to even covers $\xi_{h,p/2}$ of h, we can also start from the dual coupling $\xi_{h^\perp,(1-p)/(2-p)}$ to even covers of $h^\perp$. Second, when Δ is nonzero, the even covers of h or $h^\perp$ are *strictly contained* in the even subgraphs $\Omega_0 \coloneqq \ker(g^T)$ of a graph defined by the respective parity check g from Definition 8, i.e., $\ker(h^T) \subsetneq \Omega_0$ if using the primal coupling or $\mathrm{col}(h) \subsetneq \Omega_0$ if using the dual coupling. Because we only know how to efficiently sample from *all* even subgraphs, but not a subset thereof, this means that we now need to instead lift paths for this extended space. We show that this difference only leads to an increase in the mixing time of $2^{2\Delta}$.

For the formal argument, let $p_\uparrow = p/2$ ($p_\downarrow = (1-p)/(2-p)$) be the weight parameter for our primal (dual) coupling to ϕ. In what follows we will write $p_{\uparrow/\downarrow}$ to mean $p_\uparrow$ if doing a primal lift and $p_\downarrow$ if doing a dual lift. Furthermore, for graphic (cographic) h, let g be the graphic primal (dual) parity-check matrix from Definition 8.

We start by defining the *worm distribution* on the vector space given by subsets of E. To this end, let $w(S) = (p_{\uparrow/\downarrow}/(1-p_{\uparrow/\downarrow}))^{|S|}$ for $S \subset E$. Define the even-subgraph space $\Omega_0 = \ker(g^T)$, and the worm-space

$$\Omega_2 \coloneqq \bigcup_{u,v\in[m]} \Omega(u,v), \quad \text{with } \Omega(u,v) = \ker((g^T)_{[m]\setminus\{u,v\}}) \setminus \Omega_0 \tag{55}$$

the configurations in which the two vertices u, v have odd degree. Compared to the even covers ($\ker(h^T)$ in the primal case, and $\ker((h^\perp)^T)$ in the dual case) the full space $\Omega_w \coloneqq \Omega_0 \cup \Omega_2$ is now enlarged by the even subgraphs not present in the even covers, and the worm configurations of all even subgraphs. The worm distribution

$$\omega_g(S) \propto w(S)\mathbb{1}(S \in \Omega_0) + \frac{1}{\binom{m}{2}} w(S)\mathbb{1}(S \in \Omega_2) \tag{56}$$

penalizes those near-even-subgraph configurations such that the probability weight on Ω_0 and on Ω_2 is roughly the same.

To lift configurations sampled from the worm distribution to RC configurations, consider now the following lift, which is modified compared to the lift in Lemma 16 only in that we start from configurations sampled from the worm distribution: Sample $S \leftarrow \omega_g$. If performing a primal lift, add each edge $e \in E \setminus A$ iid. with probability $p_\uparrow/(1-p_\uparrow)$ to obtain B. Let the resulting marginal distribution on B be $\phi_\uparrow$. Similarly, if performing a dual lift, add each edge $e \in E \setminus A$ iid. with probability $p_\downarrow/(1-p_\downarrow)$, to obtain B'. Let the resulting marginal distribution on $E \setminus B'$ be $\phi_\downarrow$. Then $\phi_\uparrow$ and $\phi_\downarrow$ are both "close to" the RC model ϕ in the following way.

Lemma 17. *If h is Δ-graphic or Δ-cographic, then*

$$\frac{\phi_\uparrow(B)}{\phi(B)} \le 2^{\Delta+1} \quad \textit{and} \quad \frac{\phi_\downarrow(B)}{\phi(B)} \le 2^{\Delta+1}, \tag{57}$$

respectively, for any B.

We prove Lemma 17 in Section B.

The first step in our proof of rapid mixing for the RC model is to show the existence of a good set of canonical paths for the even cover model *through* the worm space Ω_w. To this end, let us denote the even-cover model from the primal coupling as $\xi_\uparrow = \xi_{h,p/2}$ and the even-cover model from the dual coupling as $\xi_\downarrow = \xi_{h^\perp,(1-p)/(2-p)}$. We further define the primal and dual even-cover spaces as $\Omega_\uparrow = \ker(h^T)$ and $\Omega_\downarrow = \mathrm{col}(h)$.

Definition 18. *A set of canonical paths Γ for the even-cover model $\xi_{\uparrow/\downarrow}$ through the worm space Ω_w is one such that for all $\gamma \in \Gamma$, $\gamma_0, \gamma_{L(\gamma)} \in \Omega_{\uparrow/\downarrow}$, and $\gamma_i \in \Omega_w$ for all $0 < i < L(\gamma)$.*

To construct such paths, we extend the constructions of [JS93; GJ17]. We show that these in fact give us a good set of canonical paths for the even-cover model in the following sense.

Lemma 19 (SC dynamics for the worm model mixes rapidly)**.** *Suppose that h is Δ-graphic or Δ-cographic and coupled to $g \in \{0,1\}^{c\times m}$. Then there is a set of canonical paths Γ with flow function f for the even-cover model $\xi_{\uparrow/\downarrow}$ through Ω_w that satisfies*

$$\sum_{\gamma \ni (W,W')} f(\gamma) \le 2^{\Delta+1} m^4 \omega_g(W) \tag{58}$$

for any $W' = W \oplus e$. In the special case that $W' = W \cup e$, then

$$\sum_{\gamma \ni (W,W')} f(\gamma) \le 2^{\Delta+1} m^4 \omega_g(W) \left(\frac{p_{\uparrow/\downarrow}}{1 - p_{\uparrow/\downarrow}} \right). \tag{59}$$

We prove Lemma 19 in Section C.

Using Lemma 17, we can now lift these canonical paths to a good flow for the RC model for Δ-graphic and Δ-cographic h to prove Theorem 14.

Proof of Theorem 14. Suppose that h is Δ-graphic (Δ-cographic). We show how to construct a good flow for ϕ_p using the canonical paths Γ from Lemma 19. Let $\gamma_{W_0,W_\ell} = (W_0, \dots, W_\ell) \in \Gamma$ with $\ell = L(\gamma)$ be the canonical path from W_0 to W_ℓ through Ω_w. We will construct a flow, i.e., a distribution over paths $(Z_0, \dots Z_{\ell'})$ for the RC model based on γ.

Construction of the flow Given $W_0 \in \Omega_{\uparrow/\downarrow}$, we construct $Z_0 \subset E$ using the coupling of Lemma 16. That is, we add every $e \in E \setminus W_0$ with probability $(p_{\uparrow/\downarrow})/(1 - p_{\uparrow/\downarrow})$ to W_0, and take the complement if taking a dual lift, to obtain Z_0. In other words, for $W \subset Z$, let

$$\delta_\uparrow(W, Z) = \left(\frac{p_\uparrow}{1 - p_\uparrow} \right)^{|Z \setminus W|} \left(1 - \frac{p_\uparrow}{1 - p_\uparrow} \right)^{|E \setminus Z|} \tag{60}$$

and for $Z \subset W^c \coloneqq E \setminus W$,

$$\delta_\downarrow(W, Z) = \left(\frac{p_\downarrow}{1 - p_\downarrow} \right)^{|Z^c \setminus W|} \left(1 - \frac{p_\downarrow}{1 - p_\downarrow} \right)^{|E \setminus Z^c|}. \tag{61}$$

Then $\Pr[Z_0 = Z] = \delta_\uparrow(W_0, Z)$ for any $Z \supset W_0$ if doing a primal lift and $\Pr[Z_0 = Z] = \delta_\downarrow(W_0, Z)$ if doing a dual lift.

We now "follow" the underlying path in γ as follows:

- If $W_{k+1} = W_k$, let $Z_{k+1} = Z_k$.
- If $W_{k+1} = W_k \cup e$ for $e \notin W_k$,
  - If primal: let $Z_{k+1} = Z_k \cup e$.
  - If dual: let $Z_{k+1} = Z_k \setminus e$
- If $W_{k+1} = W_k \setminus e$ for $e \in W_k$,
  - If primal: resample the edge, i.e., let $Z_{k+1} = Z_k$ with probability $\frac{p_\uparrow}{1-p_\uparrow}$ and $Z_{k+1} = Z_k \setminus e$ otherwise.
  - If dual: resample the edge, i.e., let $Z_{k+1} = Z_k$ with probability $\frac{p_\downarrow}{1-p_\downarrow}$ and $Z_{k+1} = Z_k \cup e$ otherwise.

One may check that this ensures that $\Pr(Z_k = Z | \gamma) = \delta_\uparrow(W_k, Z)$ for a primal lifting and $\Pr(Z_k = Z|\gamma) = \delta_\downarrow(W_k, Z)$ for a dual lifting.

To finish the construction of the flow, observe that after this procedure, at the end of γ, $Z_{L(\gamma)}$ remains correlated with Z_0. To remove the correlation, we re-randomize the edges $\{e_1, \dots, e_k\} = E \setminus W_\ell$ not in W_ℓ where $k = |E \setminus W_\ell|$. Therefore let

- If primal: $Z_{\ell+i+1} = Z_{\ell+i} \setminus e_i$ with probability $1 - \frac{p_\uparrow}{1-p_\uparrow}$ and $Z_{\ell+i+1} = Z_{\ell+i} \cup e_i$ otherwise.

- If dual: $Z_{\ell+i+1} = Z_{\ell+i} \setminus e_i$ with probability $\frac{p_\downarrow}{1-p_\downarrow}$ and $Z_{\ell+i+1} = Z_{\ell+i} \cup e_i$ otherwise.

We therefore obtain a path $\lambda = (Z_0, Z_1, \ldots, Z_{\ell+k})$ to which we assign the weight

$$F(\lambda) = \sum_{\gamma \in \Gamma} f(\gamma) \Pr(Z = \lambda | \gamma) \tag{62}$$

and observe that $L(\lambda) \leq L(\gamma) + c$.

Let $\Lambda(I, F)$ be the set of paths from I to $F \subset E$. We check that F is a valid flow for ϕ. To this end for a path $\lambda = (Z_1, \ldots, Z_{L(\lambda)})$ let $Z_{-1} = Z_{L(\lambda)}$.

$$\sum_{\lambda \in \Lambda(I,F)} F(\lambda) = \sum_{\substack{U,V \\ U,V \in \Omega_{\uparrow/\downarrow}}} f(\gamma_{U,V}) \Pr(Z_0 = I, Z_{-1} = F | \gamma_{U,V}) \tag{63}$$

$$= \sum_{\substack{U,V \\ U,V \in \Omega_{\uparrow/\downarrow}}} \xi_{\uparrow/\downarrow}(U) \xi_{\uparrow/\downarrow}(V) \delta_{\uparrow/\downarrow}(U, I) \delta_{\uparrow/\downarrow}(V, F) \tag{64}$$

$$= \phi(I)\phi(F), \tag{65}$$

where we have used that $\Pr(Z_0 = I, Z_{-1} = F | \gamma) = \delta_{\uparrow/\downarrow}(U, I)\delta_{\uparrow/\downarrow}(V, F)$, that f is a flow for $\xi_{\uparrow/\downarrow}$, and that

$$\phi(S) = \sum_{W \in \Omega_{\uparrow/\downarrow}, S \subset E} \xi_{\uparrow/\downarrow}(W) \delta_{\uparrow/\downarrow}(W, S). \tag{66}$$

Bounding the flow through a transition We now bound the flow through any edge (Z, Z') in the primal lift. The proof for the dual lift will follow analogously. First, let us define $i(\gamma, W)$ to be the index of state W in path γ, and let $k(W, e)$ be the index of edge e in $E \setminus W_\ell$. Also define $r(p) = \frac{p}{1-p}$. We need to separately consider the three cases (1) $Z' = Z \cup e$, (2) $Z' = Z \setminus e$, and (3) $Z' = Z$.

Case 1 If $Z' = Z \cup e$,

$$\sum_{\lambda \ni (Z,Z')} F(\lambda) = \sum_{W \subset Z} \Bigg(\sum_{\gamma \ni (W, W \cup e)} f(\gamma) \Pr\Big(Z_{i(\gamma,W)} = Z, Z_{i(\gamma,W)+1} = Z' | \gamma\Big) \tag{67}$$

$$+ \sum_{\gamma : \gamma_{-1} = W} f(\gamma) \Pr\Big(Z_{\ell+k(W,e)} = Z, Z_{\ell+k(W,e)+1} = Z' | \gamma\Big) \Bigg) \tag{68}$$

$$= \sum_{W \subset Z} \left(\sum_{\gamma \ni (W, W \cup e)} f(\gamma) \delta_\uparrow(W, Z) + \sum_{\gamma : \gamma_{-1} = W} f(\gamma) \delta_\uparrow(W, Z) r(p_\uparrow) \right) \tag{69}$$

$$= \sum_{W \subset Z} \delta_\uparrow(W, Z) \left(\sum_{\gamma \ni (W, W \cup e)} f(\gamma) + \sum_{\gamma : \gamma_{-1} = W} f(\gamma) r(p_\uparrow) \right) \tag{70}$$

$$\leq \sum_{W \subset Z} \delta_\uparrow(W, Z) \left(2^{\Delta+1} n^4 \omega_g(W) r(p_\uparrow) + \xi_\uparrow(W) r(p_\uparrow) \right) \tag{71}$$

$$= \left[2^{\Delta+1} m^4 \phi_\uparrow(Z) + \phi(Z) \right] r(p_\uparrow) \tag{72}$$

$$\leq (2^{2\Delta+2} m^4 + 1) r(p_\uparrow) \phi(Z) \tag{73}$$

$$\leq 2^{2\Delta+3} m^4 \phi(Z) r(p_\uparrow), \tag{74}$$

where we used Lemma 19 for the first inequality and Lemma 17 for the second inequality, and observed that $\phi_\uparrow(Z) = \sum_{W\subset Z} \omega_g(W)\delta_\uparrow(W,Z)$ and $\phi(Z) = \sum_{W\subset Z} \xi_\uparrow(W)\delta_\uparrow(W,Z)$ by Lemma 16. Using the same arguments, we can bound the flow for cases 2 and 3.

Case 2 If $Z' = Z \setminus e$,

$$\sum_{\lambda\ni(Z,Z')} F(\lambda) = \sum_{W\subset Z}\left(\sum_{\gamma\ni(W,W\cup e)} f(\gamma)\Pr\Big(Z_{i(\gamma,W)} = Z, Z_{i(\gamma,W)+1} = Z'|\gamma\Big)\right. \tag{75}$$

$$\left. + \sum_{\gamma:\gamma_{-1}=W} f(\gamma)\Pr\Big(Z_{\ell+k(W,e)} = Z, Z_{\ell+k(W,e)+1} = Z'|\gamma\Big)\right) \tag{76}$$

$$= \sum_{W\subset Z}\left(\sum_{\gamma\ni(W,W\setminus e)} f(\gamma)\delta_\uparrow(W,Z)(1-r(p_\uparrow)) + \sum_{\gamma:\gamma_{-1}=W} f(\gamma)\delta_\uparrow(W,Z)(1-r(p_\uparrow))\right) \tag{77}$$

$$= \sum_{W\subset Z}\delta_\uparrow(W,Z)\left(\sum_{\gamma\ni(W,W\setminus e)} f(\gamma)(1-r(p_\uparrow)) + \sum_{\gamma:\gamma_{-1}=W} f(\gamma)(1-r(p_\uparrow))\right) \tag{78}$$

$$\le \sum_{W\subset Z}\delta_\uparrow(W,Z)\Big(2^{\Delta+1}m^4\omega_g(W)(1-r(p_\uparrow)) + \xi_\uparrow(W)(1-r(p_\uparrow))\Big) \tag{79}$$

$$= 2^{\Delta+1}m^4(1-r(p_\uparrow))\phi_\uparrow(Z) + (1-r(p_\uparrow))\phi(Z) \tag{80}$$

$$\le 2^{2\Delta+3}m^4\phi(Z)(1-r(p_\uparrow)) \tag{81}$$

Case 3 If $Z' = Z$,

$$\sum_{\lambda\ni(Z,Z')} F(\lambda) = \sum_{W\subset Z}\left(\sum_{\gamma\ni W} f(\gamma)\Pr\Big(Z_{i(\gamma,W)} = Z, Z_{i(\gamma,W)+1} = Z|\gamma\Big)\right. \tag{82}$$

$$\left. + \sum_{\gamma:\gamma_{-1}=W}\sum_{i=1}^{|E\setminus W|} f(\gamma)\Pr(Z_{\ell+i} = Z, Z_{\ell+i+1} = Z|\gamma)\right) \tag{83}$$

$$\le \sum_{W\subset Z}\left(\sum_{\gamma\ni W} f(\gamma)\delta_\uparrow(W,Z) + c\sum_{\gamma:\gamma_{-1}=W} f(\gamma)\delta_\uparrow(W,Z)\right) \tag{84}$$

$$= \sum_{W\subset Z}\delta_\uparrow(W,Z)\left(\sum_{\gamma\ni W} f(\gamma) + c\sum_{\gamma:\gamma_{-1}=W} f(\gamma)\right) \tag{85}$$

$$\le \sum_{W\subset Z}\delta_\uparrow(W,Z)\Big(2^{\Delta+1}m^4\omega_g(W) + \xi_\uparrow(W)c\Big) \tag{86}$$

$$= 2^{\Delta+1}m^4\phi_\uparrow(Z) + c\phi(Z) \tag{87}$$

$$\le 2^{2\Delta+3}m^4\phi(Z)c \tag{88}$$

Bounding congestion To bound the congestion, we again go through the three cases.

Case 1: If $Z' = Z \cup e$,

$$\frac{1}{\phi(Z)P(Z,Z')}\sum_{\gamma,(Z,Z')\in\gamma} F(\gamma)L(\gamma) \le \frac{c}{\phi(Z)P(Z,Z')}2^{2\Delta+3}m^4\phi(Z)r(p_\uparrow) \tag{89}$$

$$\le \frac{c^2}{\phi(Z)\min(1,\frac{p}{2(1-p)})}2^{2\Delta+4}m^4\phi(Z)r(p_\uparrow) \tag{90}$$

$$\le c^2 2^{2\Delta+5}m^4 \tag{91}$$

Case 2: If $Z' = Z \setminus e$,

$$\frac{1}{\phi(Z)P(Z,Z')} \sum_{\gamma,(Z,Z')\in\gamma} F(\gamma)L(\gamma) \leq \frac{c}{\phi(Z)P(Z,Z')} 2^{2\Delta+3} m^4 \phi(Z)(1-r(p_\uparrow)) \tag{92}$$

$$\leq \frac{c^2}{\phi(Z)\min(1, \frac{1-p}{p})} 2^{2\Delta+4} m^4 \phi(Z)(1-r(p_\uparrow)) \tag{93}$$

$$\leq c^2 2^{2\Delta+5} m^4 \tag{94}$$

Case 3: If $Z' = Z$,

$$\frac{1}{\phi(Z)P(Z,Z')} \sum_{\gamma,(Z,Z')\in\gamma} F(\gamma)L(\gamma) \leq \frac{c}{\phi(Z)P(Z,Z')} 2^{2\Delta+3} m^4 \phi_p(Z) c \tag{95}$$

$$\leq c^2 2^{2\Delta+4} m^4 \tag{96}$$

Altogether, the congestion over any edge (Z, Z') is therefore upper-bounded by

$$\rho(F) \leq c^2 2^{2\Delta+5} m^4 = O(c^2 2^{2\Delta+5} n^4), \tag{97}$$

since by Definition 8 $m = O(n)$

The proof for dual lifting congestion follows analogously, except with $p_\downarrow$ in place of $p_\uparrow$ and the analyses for $Z' = Z \cup e$ and $Z' = Z \setminus e$ flipped. □

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A Code Swendsen-Wang dynamics is faster than single-check dynamics

In this section, we prove Lemma 7 by directly following the proof of Ullrich [Ull14].

For convenience of the proof, we define the (lazy) *single-check (SC) dynamics* of the random cluster model via the following update rule.

1. With probability $1/2$ let $B = A$.
2. Otherwise, choose a uniformly random check $e \in E$.
   - If $k(A) = k(A \cup e)$, let $B = A \cup e$ with probability p, and $B = A \setminus e$ with probqability $1 - p$.
   - If $k(A) \neq k(A \cup e)$, let $B = A \cup e$ with probability $p/2$, and $B = A$ with probability $1 - p/2$.
3. Output B.

We use the standard equivalences between two Markov chains, and apply it to the SC update compared to the Metropolis update. To do this, we define the *gap* of a Markov chain with transition matrix P on state space Ω as

$$\Delta(P) = 1 - \max\{|\lambda| : \lambda \text{ is an eigenvalue of } P, \lambda \neq 1\}. \tag{98}$$

The spectral gap is an equivalent characterization of the mixing time as [LPW09, Theorem 12.3 & 12.4], stated here, following [Ull14]

$$\Delta(P)^{-1} - 1 \leq \tau(P) \leq \log\left(\frac{2e}{\min_{S\in\Omega} p(S)}\right)\Delta(P)^{-1} \tag{99}$$

Theorem 20 ([DS93])**.** *Let P and Q be the transition matrices of two Markov chains over a state space X with the same unique fixed point π, satisfying*

$$P(x,y) \leq Q(x,y) \leq cP(x,y) \tag{100}$$

for all $x \neq y \in X$ and some $c > 0$. Then

$$\Delta(P) \leq \Delta(Q) \leq c\Delta(P). \tag{101}$$

We now observe that

$$P_{\mathrm{SC}}(A,B) \leq P_{\mathrm{Metropolis}}(A,B) \leq 2P_{\mathrm{SC}}(A,B). \tag{102}$$

and subsequently follow the proof of Ullrich [Ull14] for the SC dynamcis. To this end, let us introduce some notation. Define the function space $L_2(\pi) \coloneqq (\mathbb{R}^\Omega, \pi)$ in which the inner product is given by

$$\langle f, g\rangle_\pi = \sum_{x\in\Omega} f(x)f(g)\pi(x), \tag{103}$$

and $\|f\|_\pi \coloneqq \langle f, f\rangle_\pi$. Define the operator P as

$$P : L_2(\pi) \to L_2(\pi) \tag{104}$$

$$Pf(x) \coloneqq \sum_{y\in\Omega} P(x,y)f(y). \tag{105}$$

Furthermore we have the operator norm of P, $\|P\|_\pi \coloneqq \|P\|_{L_2(\pi)\to L_2(\pi)} = \max_{\|f\|_\pi \leq 1} \|Pf\|_\pi$.

To compare SW and SC dynamics, we define two mappings. The first one, $M : \Omega \to X \times \Omega$ lifts a RC configuration to a FK configuration, the second one, T_e updates the FK model. Let the set of configurations satisfying a set of checks $A \subset E$ be given by

$$X(A) \coloneqq \ker(h_A). \tag{106}$$

Then the two mappings are given by

$$M(B,(x,A)) \coloneqq 2^{-k(A)}\mathbb{1}(A = B)\mathbb{1}(x \in X(A)), \tag{107}$$

$$T_e((x,A),(y,B)) \coloneqq \mathbb{1}(x = y)\begin{cases} p, & B = A \cup e \text{ and } e \in E(x) \\ 1-p, & B = A \setminus e \text{ and } e \in E(x) \\ 1, & B = A \text{ and } e \notin E(x) \\ 0, & \text{otherwise.} \end{cases} \tag{108}$$

As above, these define operators $M : L_2(\mu) \to L_2(\phi)$, and $T_e : L_2(\mu) \to L_2(\mu)$. The adjoint of

M is given by $M^*((x,A),B) = \mathbb{1}(A=B)$, since

$$\langle f, Mg\rangle_\phi = \sum_{B\in\Omega} f(B)Mg(B)\phi(B) \tag{109}$$

$$\propto \sum_{B\in\Omega} f(B)\left(\sum_{(x,A)\in X\times\Omega} 2^{-k(A)}\mathbb{1}(A=B)\mathbb{1}(x\in X(A))\right)\left(\frac{p}{1-p}\right)^{|B|}2^{k(B)} \tag{110}$$

$$= \sum_{(x,A)\in X\times\Omega}\left(\sum_{B\in\Omega}\mathbb{1}(A=B)f(B)\right)g(x,A)\left(\frac{p}{1-p}\right)^{|A|}\mathbb{1}(A\subset E(x)) \tag{111}$$

$$= \sum_{(x,A)\in X\times\Omega}\left(\sum_{B\in\Omega}\mathbb{1}(A=B)f(B)\right)g(x,A)\mu(x,A) = \langle M^*f, g\rangle_\mu \tag{112}$$

.

Lemma 21. *Let M, M^* and T_e be the operators defined above. We have*

i. M^*M and T_e are self-adjoint in $L_2(\mu)$.

ii. $MM^*(A,B) = \mathbb{1}(A=B)$ and thus $M^*MM^*M = M^*M$.

iii. $T_eT_e = T_e$ and $T_eT_{e'} = T_{e'}T_e$ for all $e, e' \in E$.

iv. $\|T_e\|_\mu = 1$ and $\|M^*M\|_\mu = 1$.

Proof. (i) Self-adjointness of M: $\langle f, M^*Mg\rangle_\mu = \langle M^*Mf, g\rangle_\mu$.

To show self-adjointness of T_e, we can use that Hence, we can use that

$$\mu(x, B\setminus e)\mathbb{1}(e\in B)\mathbb{1}(e\in E(x)) = \left(\frac{1-p}{p}\right)\mu(x,B)\mathbb{1}(e\in E(x)) \tag{113}$$

$$\mu(x, B\cup e)\mathbb{1}(e\notin B)\mathbb{1}(e\in E(x)) = \left(\frac{p}{1-p}\right)\mu(x,B)\mathbb{1}(e\in E(x)) \tag{114}$$

to find

$$\sum_A f(x,A)pg(x,A\cup e)\mathbb{1}(e\notin A)\mathbb{1}(e\in E(x))\mu(x,A) \tag{115}$$

$$= \sum_B f(x,B\setminus e)pg(x,B)\mathbb{1}(e\in B)\mathbb{1}(e\in E(x))\mu(x,B\setminus e) \tag{116}$$

$$= \sum_B f(x,B\setminus e)(1-p)g(x,B)\mu(x,B), \tag{117}$$

and likewise

$$\sum_A f(x,A)(1-p)g(x,A\setminus e)\mathbb{1}(e\in A)\mathbb{1}(e\in E(x))\mu(x,A) \tag{118}$$

$$= \sum_B f(x,B\cup e)(1-p)g(x,B)\mathbb{1}(e\notin B)\mathbb{1}(e\in E(x))\mu(x,B\cup e) \tag{119}$$

$$= \sum_B f(x,B\cup e)pg(x,B)\mu(x,B)\mathbb{1}(e\in E(x)). \tag{120}$$

(iii) This follows from the fact that the transition probabilities depend only on the coordinates x of a FK configuration, which are not changed by the update.

(iv) Follows from (i-iii) since $\|T_e\|_\mu = \|T_e^2\|_\mu$ by (iii) and $\|T_e^2\|_\mu = \|T_e\|_\mu^2$ by self-adjointness of T_e.

□

Next, we express the Swendsen-Wang and single-check updates in terms of M and T_e.

Lemma 22 ([Ull14])**.** *Let T_e, M, M^* be the operators defined above. Then*

(i) $P_{SW} = M\left(\prod_{e\in E} T_e\right) M^*$

(ii) $P_{SC} = \frac{I}{2} + \frac{1}{2|E|} M\left(\sum_{e\in E} T_e\right) M^*$

Proof. To see (i), we observe that M and T_e generate the update steps of the SW algorithm via left-multiplication. Given B applying M from the right chooses a uniformly random bit string x compatible with B. Likewise, T_e updates the cluster configuration by generating a new cluster configuration in which a check $e \in E(x)$ is kept/added to the cluster with probability p or removed with probability $1-p$.

For $B \subset E(x)$, we have that

$$\left(\prod_{e\in E} T_e\right)((x,B),(x,A)) = \sum_C \underbrace{\left(\prod_{e\notin E(x)} T_e\right)((x,B),(x,C))}_{\mathbb{1}(B=C)} \left(\prod_{e\notin E(x)} T_e\right)((x,C),(x,A)) \quad (121)$$

$$= p^{|A|}(1-p)^{|E(x)|-|A|}\mathbb{1}(A \subset E(x)) \quad (122)$$

which gives, using that M^* acts trivially,

$$\sum_x M(B,(x,B)) \left(\prod_{e\in E} T_e\right)((x,B),(x,A))$$

$$= \sum_x \mathbb{1}(x \in X(A) \cap X(B)) p^{|A|}(1-p)^{|E(x)|-|A|} 2^{-k(B)} \equiv P_{SW}(B,A) \quad (123)$$

To see (ii), we need to show that the transition depends on whether or not the endpoints of e are connected through A.

For a pair (x,B) with $x \in X(B)$ we have

$$T_e((x,B),(x,A)) = p\mathbb{1}(e \in E(x))(\mathbb{1}(A = B \cup e)) - \mathbb{1}(A = B \setminus e)) \quad (124)$$

$$+ \mathbb{1}(e \notin E(x))\mathbb{1}(A = B) + \mathbb{1}(e \in E(x))\mathbb{1}(A = B \setminus e) \quad (125)$$

$$= p\mathbb{1}(e \in E(x))(\mathbb{1}(A = B \cup e)) - \mathbb{1}(A = B \setminus e)) + \mathbb{1}(A = B \setminus e) \quad (126)$$

Now, let us compute

$$MT_eM^*(A,B) = \sum_{x\in X(A)} 2^{-k(A)} T_e((x,B),(x,A)) \quad (127)$$

$$= \mathbb{1}(B = A \setminus e) + p(\mathbb{1}(A = B \cup e)) - \mathbb{1}(A = B \setminus e)) \sum_{x\in X(A)} 2^{-k(A)}\mathbb{1}(e \in E(x)), \quad (128)$$

where we find with $\mathbb{1}_e(A) = \mathbb{1}(\ker(h_A) = \ker(h_{A\cup e}))$

$$\sum_{x\in X(A)} 2^{-k(A)}\mathbb{1}(e \in E(x)) = \mathbb{1}_e(A) + \frac{1}{2}(1 - \mathbb{1}_e(A)) = \frac{1}{2}(1 + \mathbb{1}_e(A)). \quad (129)$$

This gives exactly the transition matrix P_e from above. □

Proof of Lemma 7. The remainder of the proof follows from Sec. 6.1 of Ullrich [Ull14]. □

B Coupling to syndromes and even covers

We will now show that the RC measure is coupled both to the syndrome/satisfied check Gibbs distribution and the even-cover model. We will need the following lemma.

Lemma 23. *Let h, g be matrices.*

$$\dim(\ker(A) \cap \mathrm{col}(B)) = \dim(\ker(AB)) - \dim(\ker(B)) \tag{130}$$

Proof of Lemma 16. (dual lift) Note that the distribution of syndromes S is given by

$$\zeta(S) \propto e^{-2\beta|S|}\mathbb{1}(S \in \mathrm{col}(h)) = e^{-2\beta|S|}\mathbb{1}(S \in \ker(g^T)) \propto \xi_{g,(1-p)/(2-p)}, \tag{131}$$

where g is such that $\mathrm{col}(g) = \mathrm{col}(h)^\perp$.

This yields

$$\begin{aligned}
\Pr(B) &= \sum_S \zeta(S)(1-p)^{|B\setminus S|}p^{|E\setminus B|}\mathbb{1}(S \subset B) && (132)\\
&= \sum_S (1-p)^{|S|}(1-p)^{|B\setminus S|}p^{|E\setminus B|}\mathbb{1}(1_S \in \mathrm{col}(h) \wedge S \subset B) && (133)\\
&= \left(\frac{1-p}{p}\right)^{|B|}\sum_S \mathbb{1}(1_S \in (\mathrm{col}(h) \cap \ker(\mathbb{1}_{E\setminus B})) && (134)\\
&= \left(\frac{p}{1-p}\right)^{|E\setminus B|} 2^{\dim(\ker(h_{E\setminus B}))-\dim(\ker(h))} \propto \phi(E\setminus B), && (135)
\end{aligned}$$

where we have used that $S \subset B \Leftrightarrow 1_S \in \ker(\mathbb{1}_{E\setminus B})$, where $\mathbb{1}_{E\setminus B}$ is the projector onto the rows indexed by $E \setminus B$ in Eq. (134), and Lemma 23 in Eq. (135).

(primal lift) We have

$$\begin{aligned}
\Pr(B) &= \sum_S \xi(S)\left(\frac{p}{1-p}\right)^{|B\setminus S|}\left(1-\frac{p}{1-p}\right)^{|E\setminus B|}\mathbb{1}(S \subset B \wedge S \text{ is even}) && (136)\\
&= \sum_S \left(\frac{p}{1-p}\right)^{|S|}\left(\frac{p}{1-p}\right)^{|B\setminus S|}\left(1-\frac{p}{1-p}\right)^{|E\setminus B|}\mathbb{1}(1_S \in \ker(\mathbb{1}_{E\setminus B}) \cap \ker(h^T)) && (137)\\
&= \left(\frac{p}{1-2p}\right)^{|B|}\sum_S \mathbb{1}(1_S \in \underbrace{\ker(\mathbb{1}_{E\setminus B})}_{=\mathrm{col}(\mathbb{1}_B)} \cap \ker(h^T)) && (138)\\
&= \left(\frac{p}{1-2p}\right)^{|B|} 2^{\dim(\ker(h_B^T))-\dim(\ker \mathbb{1}_B)} && (139)\\
&= \left(\frac{p}{1-2p}\right)^{|B|} 2^{\dim(\ker(h_B))+e-v-(e-|B|)} \propto \phi_{2p}(B), && (140)
\end{aligned}$$

where in Eq. (139) we used Lemma 23 and in the last step we used the dimension formula

$$\dim(\mathrm{col}(h)) = v - \dim(\ker(h)) = e - \dim(\ker(h^T)) = \dim(\mathrm{col}(h^T)). \tag{141}$$

□

Proof of Lemma 17. We follow the same argument as in the proof of Lemma 16.

(dual lift) Let $H = (h|c_1, \dots, c_\Delta) \in \{0,1\}^{c\times(n+\Delta)}$ be a column generator matrix of $\ker(g^T)$ where we have added linearly-independent columns $c_1, \dots, c_\ell \notin \mathrm{col}(h)$ to h, and let $p' = (1-p)/(2-p)$. We follow a calculation analogous to [GJ17, Lemma 3.1]

$$\phi_\uparrow(E \setminus B) = \sum_S \omega_{g,p'}(S)(1-p)^{|B\setminus S|}p^{|E\setminus B|}\mathbb{1}(S \subset B) \tag{142}$$

$$\propto \sum_S (1-p)^{|S|}(1-p)^{|B\setminus S|}p^{|E\setminus B|}\mathbb{1}(S \subset B)(\mathbb{1}(S \in \Omega_0) + \frac{1}{n^2}\mathbb{1}(S \in \Omega_2)) \tag{143}$$

$$\propto \left(\frac{p}{1-p}\right)^{|E\setminus B|} \sum_S \mathbb{1}(S \subset B)(\mathbb{1}(S \in \Omega_0) + \frac{1}{n^2}\mathbb{1}(S \in \Omega_2)) \tag{144}$$

$$= \left(\frac{p}{1-p}\right)^{|E\setminus B|} \sum_S \mathbb{1}(1_S \in (\underbrace{\ker(g^T)}_{=\mathrm{col}(H)} \cap \ker(\mathbb{1}_{E\setminus B}))$$
$$+ \frac{1}{n^2} \sum_{i\neq j\in[n+\Delta]} \mathbb{1}(1_S \in (\underbrace{\ker((g^T)_{V(i,j)})}_{=\mathrm{col}(H(V(i,j)))})) \cap \ker(\mathbb{1}_{E\setminus B})) \tag{145}$$

$$= \left(\frac{p}{1-p}\right)^{|E\setminus B|} \Bigg(2^{\dim(\ker(H_{E\setminus B}))-\dim(\ker(H))}$$
$$+ \frac{1}{n^2} \sum_{i\neq j\in[n+\Delta]} 2^{\dim(\ker(H(V_{i,j})_{E\setminus B}))-\dim(\ker(H(V_{i,j})))}\Bigg) \tag{146}$$

$$\leq \left(\frac{p}{1-p}\right)^{|E\setminus B|} 2^{-\dim(\ker(h))}2^{\Delta}2^{\dim(\ker(h_{E\setminus B}))}\left(1 + \binom{n}{2}/n^2\right) \tag{147}$$

$$\propto \frac{3}{2}2^{\Delta}\phi(E \setminus B) \tag{148}$$

where we have defined $V(i,j) = [n+\Delta] \setminus \{i,j\}$. In line (147), we used that $\dim(\mathrm{col}(H_A)) \geq \dim(\mathrm{col}(h_A))$ for $A \subset E$ and therefore, by the dimension formula, $\dim(\ker(H_A)) \leq \dim(\ker(h_A))+\Delta$. Moreover, $\dim(\ker(H(V(i,j))_A)) \leq \ker(H_A)$. Similarly, since $\dim(\mathrm{col}(H)) = \dim(\mathrm{col}(h)) + \Delta$, we have $\dim(\ker(H)) = \dim(\ker(h))$. This yields the claim for the dual lift.

For the primal lift, we follow the same reasoning. □

C Canonical paths for the even subgraph model

In this section, we construct the flows for Lemma 19. These flows are the *unwinding* flows by Jerrum and Sinclair and the proof follows a standard canonical paths argument [JS93].

Proof of Lemma 19. We give the proof for the dual case. The primal follows analogously.

We construct the canonical paths as follows. For any pair of states $A, B \in \mathrm{col}(h)$, which we can interpret as two even subgraphs in the graph described by g due to the Δ-graphic property, we will construct a path from A to B through the state space Ω_w. Consider the symmetric difference $A \oplus B$, which is also an even subgraph in g. Place a canonical ordering on cycles in g and a canonical ordering of edges within each cycle, so that $A \oplus B$ is a disjoint union of cycles, ordered by this canonical ordering. Use the ordering of edges and cycles to decompose $A \oplus B$ into a sequence of edges $(e_1, \dots, e_\ell)$ for $\ell \leq c$. Then traverse from A to B by taking the path determined by this sequence of edges $\gamma_{A,B} = (A, A \oplus e_1, A \oplus e_1 \oplus e_2, \dots, A \oplus e_1 \oplus \dots e_{\ell-1}, B)$. All states along this path are in Ω_w because at most two vertices have odd degree at all times. Assign this path a weight of $f(\gamma_{AB}) = \xi(A)\xi(B)$.

Now we bound the flow through any transition from W to $W' = W \oplus e$. For configurations $A, B \in \mathrm{col}(h)$, let $\Phi(A,B) = W \oplus A \oplus B$. This is an injective map because given (W, W') and $U = \Phi(A,B)$, we can recover A and B in the following way: Since $U \oplus W = A \oplus B$, then there is a canonical ordering on the edges in $U \oplus W$, including e. For any edge before e, its status is that in B and for any edge after e, its status is that in A. Finally, because $U \oplus W$ gives you the symmetric difference between A and B, then one can infer the remaining parts of A and B.

Recall that $w(A) = p_\downarrow^{|A|}(1-p_\downarrow)^{|E\setminus A|}$ since we are in the dual case. Define $Z_\downarrow = \sum_{A\in\mathrm{col}(h)} w(A)$, $Z_0 = \sum_{A\in\Omega_0} w(A)$, and $Z_2 = \sum_{A\in\Omega_2} w(A)$.

We first bound the ratios between these quantities. First define the unnormalized vector $|T\rangle = (|0\rangle + \frac{p_\downarrow}{1-p_\downarrow}|1\rangle)^{\otimes m}$ and $S = \sum_{A\in\mathrm{col}(h)} |A\rangle$. Note that $p_\downarrow \le 1/2$, so $H|T\rangle$ (the m-qubit Hadamard transform) only has positive coefficients, and similarly $H|S\rangle$. Then $Z_\downarrow = \langle T|S\rangle = \langle T|HH|S\rangle$ is a summation of only positive elements. Letting X_a be the X-Pauli tensor product described by a, then $\langle T|X_a|S\rangle = \langle T|HHX_a|S\rangle = \langle T|HZ_aH|S\rangle \le Z_\downarrow$ because this is now the same sum but with potential minus signs. Then letting D be all 2^Δ affine shifts for $\mathrm{col}(h) \subset \ker(g)$ to cover $\ker(g)$, then $Z_0 = \sum_{a\in D}\langle T|X_a|S\rangle \le \sum_{a\in D} Z_\downarrow \le 2^\Delta Z_\downarrow$. Similarly, Guo and Jerrum [GJ17] showed that $Z_2 \le \binom{m}{2} Z_0$ in their Lemma 8.

This allows us to bound the flow through the transition (W, W') as

$$\sum_{\gamma\ni(W,W')} f(\gamma) = \sum_{\substack{A,B\in\mathrm{col}(h):\\ \gamma_{AB}\ni(W,W')}} \xi_\downarrow(A)\xi_\downarrow(B) \tag{149}$$

$$= \sum_{\substack{A,B\in\mathrm{col}(h):\\ \gamma_{AB}\ni(W,W')}} \frac{w(A)w(B)}{Z_\downarrow^2} \tag{150}$$

$$= \sum_{\substack{A,B\in\mathrm{col}(h):\\ \gamma_{AB}\ni(W,W')}} \frac{w(W)w(\Phi(A,B))}{Z_\downarrow^2} \tag{151}$$

$$\le w(W) \sum_{U\in\Omega_w} \frac{w(U)}{Z_\downarrow^2} \tag{152}$$

$$= w(W)\frac{Z_0+Z_2}{Z_\downarrow^2}, \tag{153}$$

where the third equality uses that in every index in which A and B agree, W and $\Phi(A,B)$ agree with A and B, and the inequality used the fact that Φ is an injection.

Continuing,

$$\sum_{\gamma\ni(W,W')} f(\gamma) \le \frac{w(W)}{Z_\downarrow}\frac{Z_0+Z_2}{Z_\downarrow} \tag{154}$$

$$\le \left(\binom{m}{2}\omega(W)\right)\left(2^\Delta + \binom{m}{2}2^\Delta\right) \tag{155}$$

$$\le 2^{\Delta+1}m^4\omega_g(W). \tag{156}$$

In the special case that $W' = W \cup e$, then let $\Phi(A,B) = W' \oplus A \oplus B$ instead. Following the

same proof, we see

$$\sum_{\gamma \ni (W,W')} f(\gamma) \leq w(W') \frac{Z_0 + Z_2}{Z_\downarrow^2} \tag{157}$$

$$\leq w(W) \frac{Z_0 + Z_2}{Z_\downarrow^2} \frac{p_\downarrow}{1 - p_\downarrow} \tag{158}$$

$$\leq 2^{\Delta+1} m^4 \omega_g(W) \frac{p_\downarrow}{1 - p_\downarrow} \tag{159}$$

□