THE 3-STATE POTTS MODEL ON PLANAR TRIANGULATIONS: EXPLICIT ALGEBRAIC SOLUTION

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Abstract. We consider the 3-state Potts generating function $T(\nu, w)$ of planar triangulations; that is, the bivariate series that counts planar triangulations with vertices coloured in 3 colours, weighted by their size (number of vertices, recorded by the variable w) and by the number of monochromatic edges (variable ν).

This series was proved to be algebraic 15 years ago by Bernardi and the first author: this follows from its link with the solution of a discrete differential equation (DDE), and from general algebraicity results on such equations. However, despite recent progresses on the effective solution of DDEs, the exact value of $T(\nu, w)$ has remained unknown so far — except in the case $\nu = 0$, corresponding to *proper* colourings and solved by Tutte in the sixties. We determine here this exact value, proving that $T(\nu, w)$ satisfies a polynomial equation of degree 11 in T and genus 1 in w and T. We prove that the critical value of ν is $\nu_c = 1+3/\sqrt{47}$, with a critical exponent $6/5$ in the series $T(\nu_c, \cdot)$, while the other values of ν yield the usual map exponent $3/2$.

By duality of the planar Potts model, our results also characterize the 3-state Potts generating function of planar *cubic* maps, in which all vertices have degree 3. In particular, the annihilating polynomial, still of degree 11, that we obtain for *properly* 3-coloured cubic maps proves a conjecture by Bruno Salvy from 2009.

1. Introduction and main results

A planar map is a connected planar (multi)graph embedded in the sphere, taken up to orientation preserving homeomorphism (Figure 1). The enumeration of planar maps is a venerable topic in combinatorics, born in the early sixties with the pioneering work of William Tutte [61, 62]. Fifteen years later the topic started a second, independent, life in theoretical physics, where planar maps provide a discrete model of *quantum gravity* [22, 10]. The enumeration of maps also has connections with factorizations of permutations, and hence representations of the symmetric group [36, 37]. As a result, many techniques have been invented to count families of maps, from the early recursive approaches [62] to more and more combinatorial and finally bijective techniques, which rely on a much better understanding of maps [58, 18, 19, 20, 8]. Moreover, 40 years after the first enumerative results of Tutte, planar maps crossed the border between combinatorics and probability theory, where they are studied as random metric spaces [2, 25, 44, 46]. The limit behaviour of large planar random maps is now well understood, and gave birth to a variety of limiting objects, either continuous like the Brownian map [45, 48], or discrete like the UIPQ (uniform infinite planar quadrangulation) [2, 24, 28, 47].

The enumeration of maps equipped with some additional structure (a spanning tree, a proper colouring, a self-avoiding-walk, a configuration of the Ising model...) has attracted the interest of both combinatorialists and theoretical physicists since the early days of this study [30, 41, 49,

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64, 63]. At the moment, a challenge is to understand the limiting behaviour of maps equipped with one such structure [1, 11, 40, 42, 56, 59].

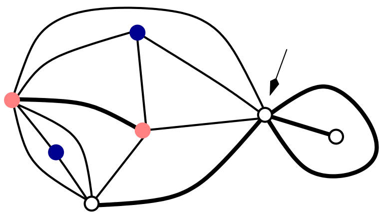


FIGURE 1. A 3-coloured rooted near-triangulation having 7 vertices, 4 monochromatic edges (in thick lines) and outer degree 4. It contributes $w^7\nu^4y^4$ in the series $T(\nu, w; y)$ counting such maps by vertices (w), monochromatic edges (ν) and outer degree (y).

The Potts model. An especially interesting structure is a configuration of the *q-state Potts model*. Given a graph G, the *partition function* of this model counts all colourings of the vertices of G with colours taken in $\{1, 2, \dots, q\}$, with a weight ν for each *monochromatic* edge (that is, an edge having both endpoints of the same colour); see Figure 1. This partition function is in fact a polynomial in ν and q, henceforth called *Potts polynomial* of G. Up to a change of variables, it is equivalent to the *Tutte polynomial* of G. It admits many interesting specializations, like the number of spanning trees or spanning forests of G. Of course, when $\nu = 0$, one recovers the *chromatic polynomial* of G, counting all *proper* q-colourings (with no monochromatic edges). We refer to [66] for details.

Studying a family of maps equipped with this model means summing these partition functions over all maps of fixed size in the family. In combinatorial terms, this boils down to counting q-coloured maps by their size and the number of monochromatic edges (and possibly additional statistics). The corresponding generating function is the *Potts generating function* of the family of maps under study. This question has already been considered for several families of planar maps, both in physics papers [29, 32, 67, 12, 34], and in combinatorics papers [63, 5, 6]. For general planar maps first, a natural starting point, already known to Tutte [63], is a combinatorially founded functional equation that characterizes the Potts generating function, but requires to introduce two additional statistics on maps, and the corresponding variables in the generating function; these statistics and variables are sometimes called *catalytic*. A similar equation was established for planar triangulations in [5], and is recalled in (6) below. Based on these equations, Bernardi and the first author proved in [5, 6] that the Potts generating functions of general planar maps and of planar triangulations both satisfy a polynomial differential equation in the size variable. This equation depends polynomially on q and ν.

This long proof was directly inspired by Tutte's enumeration of triangulations weighted by their chromatic polynomial, which took him 10 years and 10 papers; see [65] for a survey. An important, and remarkable, intermediate step establishes that for $q \neq 0, 4$ of the form $q = 4\cos^2(k\pi/m)$ (with k and m two integers), one can also write an equation involving a *single* catalytic variable, both for general maps and for triangulations; this equation is reported in (14) for triangulations in the case $q = 3$. Such equations are much better understood than those with two catalytic variables [55, 16], and it was proved in [5] that for these values of q, the Potts generating function had to be *algebraic*, that is, to satisfy a polynomial equation in ν and the size variable.

Three states. These special values of q include $q = 2$ (the Ising model), and $q = 3$, which is the setting of the present paper. For $q = 2$, the minimal polynomial of the Potts/Ising generating function was derived from the 1-catalytic equation in [5], both for general maps and for triangulations. But the case $q = 3$ resisted, except in the special case $\nu = 0$. And it has resisted until this date, despite significant progresses on the effective solution of 1-catalytic equations [13, 14, 52, 53]. All effective techniques lead to a polynomial system from which one must extract a single polynomial equation satisfied by the main generating function, but all systems that were obtained so far for this problem turned out to be too big to be solved.

In this paper, we use yet another system, for triangulations, which we manage to solve. The maps that we consider are *rooted* by choosing a corner at a vertex (Figure 1). This is a classical choice, which prevents symmetries. The face containing this corner is the *root face*, or *outer face*. Beyond triangulations, we consider *near-triangulations*, in which the root face has any degree, while all other faces are triangles (Figure 1; precise definitions are given in Section 2.1). In a companion paper [17], we address the case of general planar maps, where an analogous system exists. Its size, however, is bigger, and the solution technique is different.

The main theorem of the present paper reads as follows.

Theorem 1. *For each $i \geq 1$, the generating function $T_i \equiv T_i(\nu, w)$ that counts 3-coloured near-triangulations of outer degree i by vertices (variable w) and monochromatic edges (variable ν) is algebraic of degree 11. All series T_i belong to the same extension of degree 11 of $\mathbb{Q}(\nu, w)$.*

The smallest minimal polynomial that we obtain is for the series $\dot{T}_1 := \partial_w T_1$. This polynomial, given below, has degree 2 in w, 7 in ν, and of course 11 in $\dot{T}_1$. The genus of the underlying curve in w and $\dot{T}_1$ is 1, so the degree 2 in w is optimal (the genus would be 0 if there was an equation of degree 1 in w). This minimal polynomial contains "only" 117 monomials in ν, w and $\dot{T}_1$:

$$
\begin{aligned}
&276480\dot{T}_1^{11}\nu^7 - 27648\nu^6\,(31\nu+24)\,\dot{T}_1^{10} + 1152\nu^5\left(1021\nu^2+1678\nu+541\right)\dot{T}_1^{9}\\
&\quad - 18\nu^4\left(46080\nu^3 w + 51935\nu^3 + 138243\nu^2 + 92253\nu + 17089\right)\dot{T}_1^{8}\\
&\quad + 72\nu^3\left(1920\nu^3\,(17\nu+7)\,w + 6545\nu^4 + 25755\nu^3 + 26863\nu^2 + 10253\nu + 1144\right)\dot{T}_1^{7}\\
&-4\nu^2\left(1008\nu^3\left(727\nu^2+586\nu+127\right)w + 38596\nu^5 + 219355\nu^4 + 322318\nu^3 + 190022\nu^2 + 43274\nu + 2915\right)\dot{T}_1^{6}\\
&+4\nu\left(216\nu^3\left(2433\nu^3+2879\nu^2+1255\nu+153\right)w + 8027\nu^6 + 67626\nu^5 + 134820\nu^4 + 109109\nu^3 + 38007\nu^2\right.\\
&\left.+5103\nu+188\right)\dot{T}_1^{5} + \left(41472\nu^6\,(\nu-1)\,w^2 - 12\nu^3\left(78871\nu^4 + 122456\nu^3 + 80010\nu^2 + 19688\nu + 1375\right)w\right.\\
&\left.-3876\nu^7 - 53138\nu^6 - 145202\nu^5 - 151460\nu^4 - 71656\nu^3 - 14332\nu^2 - 958\nu - 18\right)\dot{T}_1^{4} + \left(13824\nu^5\,(5\nu+1)\,(1-\nu)\,w^2\right.\\
&+8\nu^2\,(5\nu+1)\left(6823\nu^4 + 11843\nu^3 + 9045\nu^2 + 2429\nu + 100\right)w + 208\nu^7 + 6088\nu^6 + 24600\nu^5 + 31836\nu^4 + 19256\nu^3\\
&\left.+5040\nu^2 + 440\nu + 12\right)\dot{T}_1^{3} + \left(1728\nu^4\,(\nu-1)\,(5\nu+1)^2\,w^2 - 12\nu\,(3\nu+1)\left(1358\nu^5 + 2771\nu^4 + 2504\nu^3 + 868\nu^2\right.\right.\\
&\left.\left.+58\nu+1\right)w - 312\nu^6 - 2401\nu^5 - 3747\nu^4 - 2821\nu^3 - 899\nu^2 - 78\nu - 2\right)\dot{T}_1^{2} + \nu\left(-96\nu^2\,(\nu-1)\,(5\nu+1)^3\,w^2\right.\\
&\left.+4\,(\nu+1)\left(1229\nu^5 + 2390\nu^4 + 2114\nu^3 + 697\nu^2 + 49\nu + 1\right)w + \left(104\nu^4 + 189\nu^3 + 177\nu^2 + 67\nu + 3\right)\right)\dot{T}_1\\
&\quad + 2\nu^2\,(\nu-1)\,(5\nu+1)^4\,w^2 - 2\nu^2\,(\nu+2)\left(104\nu^4 + 189\nu^3 + 177\nu^2 + 67\nu + 3\right)w = 0. \qquad (1)
\end{aligned}
$$

For comparison, the minimal polynomial of T_1 itself contains 1304 monomials.

Special values of $\boldsymbol{\nu}$. When $\nu = 0$, we have $T_1 = 0$ (since there is a loop at the root vertex), but for $i > 1$ we recover known results on the enumeration of properly 3-coloured near-triangulations of outer degree i, with a minimal polynomial of degree only 2 for T_i. In particular, $T_2 = T_1/\nu$ for general ν, hence the above equation implies that, at $\nu = 0$,

$$2\dot{T}_2^2 - (4w+3)\,\dot{T}_2 + 2w\,(w+6) = 0.$$

The first result of this type was obtained by Tutte already in 1963, formulated in terms of the number of *bicubic* (bipartite and cubic) maps [62, p. 269]. Indeed, the duals of bicubic maps are the *Eulerian* triangulations (those in which every vertex has even degree). These are the only triangulations that admit a proper 3-colouring, and each of them admits exactly 6 colourings.

The case $\nu = 1$ is also simple and well-known, as T_i then counts near-triangulations by vertices (with a weight $3w$ per vertex); it has degree 3 over $\mathbb{Q}(w)$ [50]. In particular, the minimal polynomial of $\dot{T}_1$ factors for $\nu = 1$, and the factor that vanishes gives

$$2\dot{T}_1^3 - 3\dot{T}_1^2 + \dot{T}_1 - 6w = 0.$$

The genus is 0 in these two cases.

Duality. A duality property of the (planar) Potts polynomial, recalled in Section 2.2, allows us to translate our results in terms of the 3-Potts generating function of *near-cubic* maps (those in which all vertices have degree 3, except possibly the root vertex); see (5) and Corollary 5. In particular, we prove for *properly* 3-coloured near-cubic maps the following result, equivalent to a conjecture of Bruno Salvy that dates back to 2009 (see [5, Conj. 27]).

Corollary 2. *The generating function* $K_1 = 4w^3 + 84w^4 + 1872w^5 + \mathcal{O}(w^6)$ *of properly* 3*-coloured near-cubic maps with root degree* 1*, counted by faces, is algebraic of degree* 11*. Its derivative satisfies*

$$324w^2 = 655360\dot{K}_1^{11} + 1245184\dot{K}_1^{10} + 866304\dot{K}_1^{9} - 80\,(8192w - 1995)\,\dot{K}_1^{8} - 2880\,(512w + 49)\,\dot{K}_1^{7} \\ - 504\,(2944w + 219)\,\dot{K}_1^{6} - 24\,(36640w + 1383)\,\dot{K}_1^{5} - \left(16384w^2 + 334416w + 3033\right)\dot{K}_1^{4} \\ - 6\left(4096w^2 + 13584w - 153\right)\dot{K}_1^{3} - 9\left(1536w^2 + 1300w - 33\right)\dot{K}_1^{2} - 27\,(4w+1)\,(32w-1)\,\dot{K}_1.$$

Singularities and asymptotics. We also study the singularities of the series T_i, as functions of w depending on the parameter $\nu \geq 0$. We find for every ν the usual asymptotic behaviour of uncoloured planar maps, namely $[w^n]T_i \sim \rho_\nu^{-n} n^{-5/2}$, except at the critical point $\nu_c = 1 + 3/\sqrt{47}$, where $[w^n]T_i \sim \rho_\nu^{-n} n^{-11/5}$ (up to a multiplicative constant in both cases). This study requires some care, and occupies a significant part of the paper.

Outline of the paper. We begin in Section 2 with definitions on maps and the Potts model in q colours. We introduce the *Potts generating function* $T(y) \equiv T(q, \nu, w; y)$ of near-triangulations, which counts these maps, equipped with a q-colouring of their vertices, by the size (variable w), the number of monochromatic edges (variable ν) and the outer degree (y). In Section 3 we give three characterizations of $T(y)$, which were established successively in earlier papers [5, 6]. The first one is valid for any q, but it involves an additional variable x and a series that is more general than $T(y)$. The other two characterizations only hold for $q = 3$. One involves $T(y)$, while the other does not involve the variable y any more. It is a polynomial system relating several series in w and ν, among which T_1, T_3, T_5 and T_7. In Section 4, we derive from this system the minimal polynomials of T_1, T_3, T_5 and T_7, and prove Theorem 1. Section 5 is devoted to the singular analysis of the series T_i, culminating in Proposition 6.

Many parts of this paper require using a computer algebra system. The paper is thus accompanied by a Maple session available on the web pages of the authors. We also use `msolve`, a C library for solving in arbitrary precision multivariate polynomial systems [9].

2. Preliminaries

2.1. Planar maps

A *planar map* is a proper embedding of a connected planar graph in the oriented sphere, considered up to orientation preserving homeomorphism. Loops and multiple edges are allowed (Figure 2, left). The *faces* of a map are the connected components of its complement. The numbers of vertices, edges and faces of a planar map M, denoted by $\mathrm{v}(M)$, $\mathrm{e}(M)$ and $\mathrm{f}(M)$, are related by Euler's relation $\mathrm{v}(M) + \mathrm{f}(M) = \mathrm{e}(M) + 2$. The *degree* of a vertex or face is the number of edges incident to it, counted with multiplicity. A *corner* is a sector delimited by two consecutive edges around a vertex; hence a vertex or face of degree k is incident to k corners.

For counting purposes it is convenient to consider *rooted* maps. A map is rooted by choosing a corner. The incident vertex and face are called root vertex and root face. The edge that follows the root corner in counterclockwise order around the root vertex is called the root edge. In figures, we usually choose the root face as the infinite face (Figure 2). This explains why we often call the root face the *outer face*, and its degree the *outer degree* (denoted $\mathrm{drf}(M)$).

The *dual* of a map M, denoted M^*, is the map obtained by placing a vertex of M^* in each face of M and an edge of M^* across each edge of M; see Figure 2, right. The dual of a rooted map is rooted canonically at the dual corner: that is, the root face (resp. root vertex) of M^* is dual to the root vertex (resp. root face) of M. A *triangulation* is a map in which all faces have degree 3. A *near-triangulation* is a map in which all non-root faces have degree 3. We denote by $\mathcal{T}_i$ the set of near-triangulations of outer degree i. Duality transforms triangulations (resp. near-triangulations) into *cubic* (resp. *near-cubic*) maps, that is, maps in which every vertex (resp. every non-root vertex) has degree 3. We denote by $\mathcal{K}_i$ the set of near-cubic maps of root degree i. Observe that in a near-triangulation M with outer degree i, a double counting of edges gives

$$2\,\mathrm{e}(M) = i + 3(\mathrm{f}(M) - 1).$$

Combined with Euler's relation, this yields

$$\mathrm{f}(M) = 2\,\mathrm{v}(M) - i - 1, \qquad \mathrm{e}(M) = 3\,\mathrm{v}(M) - i - 3. \tag{2}$$

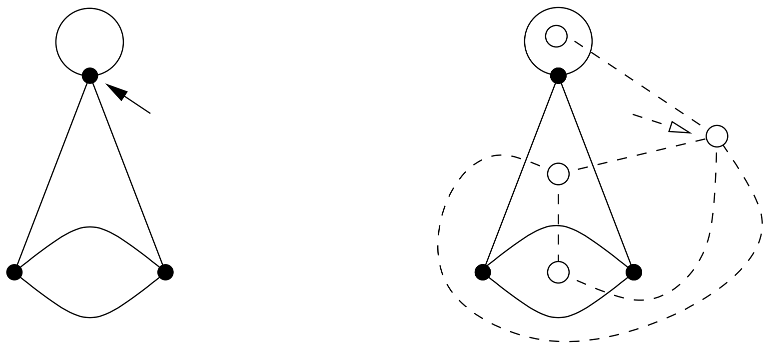


FIGURE 2. *Left:* a rooted planar map with 3 vertices and 4 faces, having outer degree 4. *Right:* the dual map, in dashed edges.

From now on, every map is *planar* and *rooted*, and these adjectives will often be omitted. We include among rooted planar maps the *atomic map* having one vertex and no edge.

2.2. The q-state Potts model

For $q \in \mathbb{N} := \{1, 2, \ldots\}$, the *Potts polynomial* of a map M is defined to be

$$\mathrm{P}_M(q,\nu) := \sum_{c:V(M)\to\{1,\ldots,q\}} \nu^{m(c)},$$

where c is a colouring of the vertices of M in q colours taken in $\{1, 2, \ldots, q\}$, and $m(c)$ is the number of *monochromatic* edges (whose endpoints share the same colour). For instance, the map M shown on the left of Figure 2 has Potts polynomial:

$$\mathrm{P}_M(q,\nu) = q\nu\left((q-1)(q-2) + (q-1)\nu^2 + 2(q-1)\nu + \nu^4\right). \tag{3}$$

It is easy to prove that $\mathrm{P}_M(q,\nu)$ is not only a polynomial in ν, but also a polynomial in q [66].

We define the *Potts generating function of near-triangulations* by:

$$T(y) \equiv T(q,\nu,w;y) := \frac{1}{q}\sum_M \mathrm{P}_M(q,\nu) w^{\mathrm{v}(M)} y^{\mathrm{drf}(M)}, \tag{4}$$

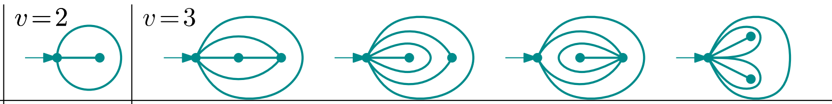
where the sum runs over all planar near-triangulations M (including the atomic map). Since there are finitely many near-triangulations with a given number of vertices, and $\mathrm{P}_M(q,\nu)$ is a multiple of q, the series $T(y)$ is a power series in w with coefficients in $\mathbb{Q}[q,\nu,y]$, the ring of polynomials in q, ν and y with rational coefficients. The expansion of $T(y)$ at order 2 reads

$$T(y) = w + y\,(q - 1 + \nu)\,(\nu + y)\,w^2 + \mathcal{O}(w^3).$$

The generating function of near-triangulations of outer degree 1 is

$$T_1 := [y^1]T(y) = \nu(q-1+\nu)w^2 + \nu\left((q-1)(q-2+2\nu) + \nu^2(q-1+\nu^2)\right.$$
$$\left. +2\nu(q-1+\nu)(q-1+\nu^2) + \nu^2(q-1+\nu)^2\right) w^3 + O(w^4),$$

as illustrated in Figure 3. More generally, we denote by T_i the coefficient of y^i in $T(y)$, that is, the Potts generating function of near-triangulations of outer degree i. In combinatorial terms, $T(y)$ counts q-coloured near-triangulations by vertices (w), monochromatic edges (ν) and outer degree (y), with the convention that the root vertex is coloured in a prescribed colour (this accounts for the division by q).



$\frac{\mathrm{P}_M(q,\nu)}{q}$	$\nu(q-1+\nu)$	$\nu(q-1)(q-2+2\nu)$ $+\nu^3(q-1+\nu^2)$	$\nu^2(q-1+\nu)$ $\times(q-1+\nu^2)$	$\nu^2(q-1+\nu)$ $\times(q-1+\nu^2)$	$\nu^3(q-1+\nu)^2$

FIGURE 3. The rooted near-triangulations of outer degree 1 with $v = 2$ and $v = 3$ vertices, and their Potts polynomials (divided by q).

Duality. It follows from the connection between the Potts polynomial and the Tutte polynomial [66, Sec. 4.4] and from the duality property of the Tutte polynomial [66, Sec. 3.3], that, for a planar map M and its dual M^*,

$$(\nu_* - 1)^{\mathrm{f}(M)-1}\mathrm{P}_M(q,\nu) = (\nu-1)^{\mathrm{f}(M^*)-1}\mathrm{P}_{M^*}(q,\nu_*)$$

where $q = (\nu-1)(\nu_*-1)$. This can be checked for instance on the maps of Figure 2, for which $\mathrm{P}_M(q,\nu)$ is given by (3), while

$$\mathrm{P}_{M^*}(q,\nu_*) = q(q-1+\nu_*)\left((q-1)(q-2) + (q-1)\nu_*^2 + 2(q-1)\nu_* + \nu_*^4\right).$$

If we then define the *Potts generating function* of near-cubic maps by:

$$K(y) \equiv K(q,\nu,w;y) := \frac{1}{q}\sum_M \mathrm{P}_M(q,\nu) w^{\mathrm{f}(M)} y^{\mathrm{drv}(M)},$$

where the sum runs over all planar near-cubic maps and $\mathrm{drv}(\cdot)$ denotes the degree of the root vertex, the above duality relation, combined with (2), yields

$$K(q,\nu,w;y) = \frac{q}{(\nu-1)^3}\, T\left(q,\nu_*,\frac{1}{q}(\nu-1)^3 w; \frac{y}{\nu-1}\right),$$

where as above $q = (\nu-1)(\nu_*-1)$. That is, if $K_i(q,\nu,w)$ is the Potts generating function of near-cubic maps of root degree i, then

$$K_i(q,\nu,w) = \frac{q}{(\nu-1)^{i+3}}\, T_i\left(q,\nu_*,\frac{1}{q}(\nu-1)^3 w\right). \tag{5}$$

In particular, for $q = 3$, $i = 1$ and $\nu = 0$, we obtain that the series K_1 of Corollary 2 is

$$K_1(3,0,w) = 3\,T_1\left(3,-2,-\frac{1}{3}w\right).$$

Corollary 2 then follows from (1).

2.3. Power series

Let A be a commutative ring and x an indeterminate. We denote by $A[x]$ (resp. $A[[x]]$) the ring of polynomials (resp. formal power series) in x with coefficients in A. If A is a field, then $A(x)$ denotes the field of rational functions in x. These notations are generalized to polynomials, fractions and series in several indeterminates. The coefficient of x^n in a power series $F(x)$ is denoted by $[x^n]F(x)$.

Recall that a power series $F(x_1, \dots, x_k) \in \mathbb{K}[[x_1, \dots, x_k]]$, where $\mathbb{K}$ is a field, is *algebraic* (over $\mathbb{K}(x_1, \dots, x_k)$) if it satisfies a non-trivial polynomial equation $P(x_1, \dots, x_k, F(x_1, \dots, x_k)) = 0$.

3. Functional equations

3.1. Two catalytic variables

The first way to compute the coefficients of the series $T(y) \equiv T(q, \nu, w; y)$ defined in (4) (seen as a series in w) relies on a functional equation satisfied by a series $Q(x, y) \equiv Q(q, \nu, w, t; x, y)$ that counts certain Potts-weighted maps, called *quasi-triangulations*, that are more general than near-triangulations and will not be defined here. This series involves an additional variable t counting edges, and an additional "catalytic" variable x. The equation, established in [5, Prop. 2], reads:

$$\begin{aligned} Q(x,y) = 1 + t\,\frac{Q(x,y) - 1 - yQ_1(x)}{y} + xt(Q(x,y) - 1) + xytQ_1(x)Q(x,y) \\ + yt(\nu - 1)Q(x,y)(2xQ_1(x) + Q_2(x)) + y^2wt\left(q + \frac{\nu - 1}{1 - xt\nu}\right)Q(0,y)Q(x,y) \\ + \frac{ywt(\nu-1)}{1 - xt\nu}\,\frac{Q(x,y) - Q(0,y)}{x} \end{aligned} \tag{6}$$

where $Q_1(x) = [y]Q(x, y)$ and $Q_2(x) = [y^2]Q(x, y) = (1 - 2xt\nu)Q_1(x)/(t\nu)$. This equation defines a unique power series Q in t, which has polynomial coefficients in q, ν, w, x and y. It is said to be *catalytic* in x and y, because one cannot derive immediately from it an equation for simpler series in which we could be interested, like $Q(0, y)$ or $Q_1(x)$. Divided differences like $(Q(x, y) - Q(0, y))/x$ are sometimes called *discrete derivatives*, which makes the above equation a *discrete (partial) differential equation.*

The Potts generating function $T(y)$ of near-triangulations is then related to $Q(x, y)$ by:

$$T(q, \nu, w; y) \equiv T(y) = w\,Q(q, \nu, w, 1; 0, y).$$

According to (2), for $n \geq 2$ one needs to know Q up to the coefficient of t^{3n-4} to determine the coefficient of w^n in $T(y)$. In our Maple session we use (6) to compute these coefficients effectively, by induction on n. In [6], the series $T(y)$ is written, alternatively, as $wQ(q, \nu, 1, w^{1/3}; 0, w^{1/3}y)$.

3.2. One catalytic variable

It was proved in [5] that when $q = 3$ (and more generally when $q \neq 0, 4$ is of the form $q = 4\cos^2(k\pi/m)$, for integers k and m), the series $T(y)$ is also characterized by an explicit equation involving only *one* catalytic variable, namely y. Here we write it for $q = 3$, using Proposition 7 in [6] (which is based on [5, Cor. 12]). We introduce the following notation:

- $I(y) \equiv I(\nu, w; y)$ is a variant of the Potts generating function $T(3, \nu, w; y)$:

$$I(y) = 3yT(3, \nu, w; y) - \frac{1}{y} + \frac{1}{y^2}, \tag{7}$$

- $N(y, x)$ and $D(x)$ are the following (Laurent) polynomials, where we write $\beta := \nu - 1$:

$$\begin{aligned} N(y,x) &= \beta/y + 3\nu x + \beta, \\ D(x) &= 3\nu^2x^2 + \beta(4\nu - 1)x - 3\beta\nu w + \beta^2. \end{aligned} \tag{8}$$

We moreover denote by $\mathcal{T}_6$ the 6^{th} Chebyshev polynomial of the first kind:

$$\mathcal{T}_6(u) = 32u^6 - 48u^4 + 18u^2 - 1.$$

The following proposition is then the case $q = 3$ of [6, Prop. 7].

Proposition 3. *There exist 7 formal power series in w with coefficients in $\mathbb{Q}(q,\nu)$, denoted $C_0,\dots, C_6$, such that*

$$D(I(y))^3\ \mathcal{T}_6\left(\frac{N(y,I(y))}{2\sqrt{D(I(y))}}\right) = \sum_{r=0}^{6} C_r I(y)^r. \tag{9}$$

Each series C_r has a rational expression in terms of ν, w, T_1, T_3, T_5 and T_7, where $T_i := [y^i]T(y)$ is the 3-Potts generating function of near-triangulations with outer degree i.

Both sides of (9) expand as polynomials in $T(y)$ and Laurent polynomials in y. We recall from Section 13.2 in [5] (or Lemma 8 in [6]) that the expressions of the series C_r in terms of the T_i's are obtained by expanding (9) around $y = 0$, up to the coefficient of y^0. This expansion also yields $T_0 = w$ and expressions of T_2, T_4 and T_6 in terms of T_1, T_3, T_5. The series C_4, C_5 and C_6 in fact do not involve any T_i:

$$\begin{aligned} C_6 &= -27\nu^6,\\ C_5 &= 27\nu^4(\nu-1)(2\nu-5),\\ C_4 &= \frac{9}{2}\nu^2(\nu-1)(18\nu^3 w + 35\nu^3 - 75\nu^2 + 30\nu + 10). \end{aligned} \tag{10}$$

The series C_3 involves the series $T_1 = [y^1]T(y)$:

$$C_3 = -486\nu^4\,(\nu-1)^2\,T_1 + 135\nu^3\,(\nu-1)^2\,(2+\nu)\,w + (\nu-1)^4\left(136\nu^2 + 43\nu + 1\right). \tag{11}$$

Furthermore, C_2 involves T_1 and T_3, and so on until C_0 which involves T_1, T_3, T_5, T_7. For the series T_i with even index i, one finds:

$$\nu T_2 = T_1, \qquad \nu^2 T_4 = -\left(6\nu^2 w + 1\right)T_1 + \nu\,(\nu+1)\,T_3 + \nu\,w^2\,(2+\nu)\,, \tag{12}$$

$$\begin{aligned} \nu^3 T_6 = 6\nu^2\,(\nu-1)\,T_1^2 + \left(4\nu^3 w + 16\nu^2 w + 7\nu w + \nu + 2\right)T_1 - \nu\left(9\nu^2 w + \nu^2 + 2\nu + 2\right)T_3\\ + \nu^2\,(2\nu+1)\,T_5 - \nu\,w^2\left(9\nu^2 w + \nu^2 + 4\nu + 4\right). \end{aligned} \tag{13}$$

In the end, (9) rewrites as a polynomial equation of degree 5 in $T(y)$, with coefficients in $\mathbb{Q}[\nu, w, T_1, T_3, T_5, T_7, y]$, with the following terms of higher degree:

$$\begin{aligned} 0 = 2916\nu^5 y^{12}\boldsymbol{T(y)}^5 + 27\nu^3 y^9\Big(\beta\,(37\nu+17)\,y^2 - 36\nu\,(3\nu+1)\,y + 144\nu^2\Big)\boldsymbol{T(y)}^4\\ - 2\nu y^6\Big(486\boldsymbol{T_1}\,\nu^4 y^4 - 81\nu^3\,(5\nu+1)\,w\,y^4 + 486\nu^4 w\,y^3 - \left(56\nu^2 + 59\nu + 2\right)\beta^2 y^4\\ +9\nu\beta\,\left(38\nu^2 + 40\nu + 3\right)y^3 - 9\nu^2\left(116\nu^2 + 11\nu - 19\right)y^2 + 486\nu^3\,(3\nu+1)\,y - 972\nu^4\Big)\boldsymbol{T(y)}^3 + \cdots, \end{aligned} \tag{14}$$

where we have written $\beta = \nu - 1$. The complete equation is given in Appendix A, see (32). We also refer to our Maple session where this equation is derived. We use it in Section 4.6 to prove that all series T_i belong to the extension of $\mathbb{Q}(\nu, w)$ generated by T_1.

3.3. A polynomial system

Equation (9), or equivalently (14), is an equation in a single catalytic variable, y. In [5, Sec. 11], it was derived from this equation, using the general results of [16], that $T(y)$ is algebraic over $\mathbb{Q}(\nu, w, y)$. Moreover, [16] also shows that an annihilating polynomial of $T(y)$ can be produced by computing a Gröbner basis for some (big) ideal. However, this approach fails here because of the large size of the polynomials generating this ideal. Beyond the original approach of [16], more and more efficient techniques have been designed to solve such catalytic equations [13, 14, 52]. But, in addition to difficulties due to the sizes of all systems, it appears that a non-degeneracy condition needed to apply these techniques does not hold for this problem [51, Sec. 12.6]. As a result, Equation (14) has so far resisted all attempts, and the minimal polynomial of $T(y)$ (or even T_1) has remained out of reach.

In this paper, we determine the minimal polynomial of T_1 (and in fact of each series T_i for $i \leq 7$) by starting from a polynomial system that is smaller and better structured than those derived from the above general methods. This alternative system was established in [6], in the process of deriving a system of *differential* equations defining T_1 (and valid for any q). We give at the end of the section a few details on the connection between this system and the common basis to the general methods.

Proposition 4. *Let $C(x) = C_0 + \cdots + C_6 x^6$, where the series C_r are those of Proposition 3, and let $D(x)$ be defined by (8). There exist 4 formal power series $X_1, \ldots, X_4$ in w, with coefficients in an algebraic closure of $\mathbb{Q}(\nu)$ and constant terms distinct from $0, -1/4$ and $-1/(2\nu)$, and a pair $\left(\widehat{P}_-(x), \widehat{P}_+(x)\right)$ of polynomials in x with coefficients in $\mathbb{R}(\nu, w)[[t]]$ such that*

$$C(x) - D(x)^3 = \widehat{P}_-(x) \prod_{i=1}^{2} (x - X_i)^2, \tag{15}$$

$$C(x) + D(x)^3 = \widehat{P}_+(x) \prod_{i=3}^{4} (x - X_i)^2. \tag{16}$$

Remark. The above equations form a polynomial system of 8 equations relating $C_0, \ldots, C_6$, $X_1, \ldots, X_4$, once written as

$$C(X_i) = D(X_i)^3, \quad C'(X_i) = 3D'(X_i)D(X_i)^2, \qquad i = 1, 2,$$
$$C(X_i) = -D(X_i)^3, \quad C'(X_i) = -3D'(X_i)D(X_i)^2, \qquad i = 3, 4,$$

where the derivatives are taken with respect to x. Since C_4, C_5 and C_6 are explicit, see (10), there are exactly 8 unknown series. The above system implies the existence of a polynomial $\widehat{Q}(x)$ such that

$$D(x)C'(x) - 3D'(x)C(x) = \widehat{Q}(x) \prod_{i=1}^{4} (x - X_i). \tag{17}$$

This equation occurs as well in [6, Prop. 11].

Proof. The case $m = 6$ of Proposition 11 in [6] shows the existence of 4 series X_i that satisfy (17) and also

$$C(x)^2 - D(x)^6 = \left(C(x) - D(x)^3\right)\left(C(x) + D(x)^3\right) = \widehat{P}(x) \prod_{i=1}^{4} (x - X_i)^2,$$

for some polynomial $\widehat{P}(x)$. The conditions on the constant terms of the X_i arise from [6, Lem. 9]. It remains to refine the above equation into (15) and (16). One way to do this is to dig into the details of the proof of [6, Prop. 11]. Another way is to examine the first coefficients of the roots of the left-hand side of (17), and decide whether they solve (15) or (16). Indeed, the series C_r are explicit in terms of the coefficients T_i of $T(y)$, and we can compute inductively the coefficient of w^n in $T(y)$, so we can also determine the first terms of the roots of $D(x)C'(x) - 3D'(x)C(x)$. This polynomial has degree 6 in x, hence 6 roots, which we find to start as follows (of course, the labelling is chosen so as to satisfy (15) and (16) in a near future):

$$X_1 = \frac{1-\nu}{\nu^2} + \nu\,(2\nu - 1)\,w + \frac{2\nu^3\left(6\nu^3 - 6\nu^2 - 3\nu + 4\right)}{\nu - 1} w^2 + \mathcal{O}(w^3),$$
$$X_2 = \frac{1-\nu}{2\nu} - w - \frac{2\nu\left(\nu^2 + 3\nu - 3\right)}{\nu - 1} w^2 + \mathcal{O}(w^3),$$

$$X_3 = \frac{(1-\nu)\left(8\nu+1+\sqrt{1+16\nu-8\nu^2}\right)}{18\nu^2} + \frac{3}{2}\left(\frac{2\nu+1}{\sqrt{1+16\nu-8\nu^2}}-1\right)w+\mathcal{O}(w^2),$$
$$X_4 = \frac{(1-\nu)\left(8\nu+1-\sqrt{1+16\nu-8\nu^2}\right)}{18\nu^2} - \frac{3}{2}\left(\frac{2\nu+1}{\sqrt{1+16\nu-8\nu^2}}+1\right)w+\mathcal{O}(w^2).$$

The last two roots have constant terms 0 and $-1/(2\nu)$, so they cannot be any of the X_i's.

Now it suffices to plug the above series X_i in the polynomials $C(x)-D(x)^3$ and $C(x)+D(x)^3$ to see that X_1 and X_2 cannot be roots of $C(x)+D(x)^3$, while X_3 and X_4 cannot be roots of $C(x)-D(x)^3$. This yields the final forms (15) and (16). ■

From now on, we will be only interested in the elementary symmetric functions of X_1 and X_2 on the one hand, and of X_3 and X_4 on the other hand. They have coefficients in $\mathbb{Q}(\nu)$:

$$\begin{aligned}
S_1 &:= X_1+X_2\\
&= -\frac{(\nu+2)(\nu-1)}{2\nu^2} + (2\nu+1)(\nu-1)\,w + 6\,(\nu-1)(\nu+1)\left(2\nu^2+1\right)\nu w^2+\mathcal{O}(w^3),\\
P_1 &:= X_1X_2\\
&= \frac{(\nu-1)^2}{2\nu^3} - \frac{(\nu-1)\left(2\nu^3-\nu^2-2\right)}{2\nu^2}w - \frac{3\left(2\nu^4+2\nu^3+\nu^2+2\nu+2\right)(\nu-1)^2}{\nu}w^2+\mathcal{O}(w^3),\\
S_3 &:= X_3+X_4\\
&= -\frac{(8\nu+1)(\nu-1)}{9\nu^2} - 3w - 6\left(\nu^2+4\nu+1\right)w^2+\mathcal{O}(w^3),\\
P_3 &:= X_3X_4\\
&= \frac{2\,(\nu-1)^2}{9\nu^2} + \frac{(5\nu+1)(\nu-1)}{3\nu^2}w + \frac{(\nu-1)\left(7\nu^3+30\nu^2+24\nu+2\right)}{3\nu^2}w^2+\mathcal{O}(w^3).
\end{aligned}$$

Connection with other approaches. When studying a 1-catalytic equation like (14), written as $\mathrm{Pol}(T(y),y,T_1,T_3,T_5,T_7,w)=0$, the key idea is to examine the series $Y\equiv Y(w)$ such that $\mathrm{Pol}_1'(T(Y),Y,T_1,T_3,T_5,T_7,w)=0$, where Pol_1' denotes the derivative of Pol with respect to its first variable [16]. By the chain rule, this also implies $\mathrm{Pol}_2'(T(Y),Y,T_1,T_3,T_5,T_7,w)=0$. Here, one finds that seven such series exist, say $Y_0,Y_1,\dots,Y_6$. All effective strategies then exploit in one way or another the 3×7 equations $\mathrm{Pol}=\mathrm{Pol}_1'=\mathrm{Pol}_2'=0$, when evaluated at $(T(Y_i),Y_i,T_1,T_3,T_5,T_7,w)$.

One of the series Y starts $Y_0=\nu+2\nu^4w+\mathcal{O}(w^2)$. The other six series have constant terms that are quadratic in ν, and thus go by pairs:

$$Y_{1,2} = \frac{4\nu-1\pm\sqrt{-8\nu^2+16\nu+1}}{4\nu-4}+\mathcal{O}(w),$$
$$Y_{3,4} = \frac{2\nu+1\pm\sqrt{-8\nu^2+16\nu+1}}{2\nu-2}+\mathcal{O}(w),$$
$$Y_{5,6} = \frac{\nu\pm\sqrt{\nu\,(2-\nu)}}{\nu-1}+\mathcal{O}(w).$$

It follows from [6] that the four series X_i of Proposition 4 are the values $I(Y_i)$, for $0\le i\le 6$, where $I(y)$ is the variant of $T(y)$ defined by (7). More precisely,

$$I(Y_0)=X_1,\quad I(Y_1)=I(Y_4)=X_3,\quad I(Y_2)=I(Y_3)=X_4,\quad I(Y_5)=I(Y_6)=X_2.$$

The polynomial system of Proposition 4 gives a set of compact relations between the X_i, not involving any of the Y_i. It seems that handling the four series X_i rather than the seven series Y_i avoids some redundancy.

4. Derivation of the series $T(y)$

We return to the system of Proposition 4.

4.1. Eight polynomial equations obtained from remainders

The first equation can be written

$$\mathrm{rem}\left(C(x)-D(x)^3,(x^2-S_1x+P_1)^2,x\right)=0,$$

where, given two polynomials A and B in x, $\mathrm{rem}(A,B,x)$ is the remainder of A modulo B, and S_1, P_1 are as above the sum and product of X_1 and X_2. The above remainder has degree 3 in x, so its coefficients give four polynomial equations relating the series S_1, P_1, C_0, C_1, C_2, and C_3 (recall from (10) that the other series C_r are explicit). We proceed similarly with the second equation of Proposition 4. This gives a system of $2\times 4=8$ polynomials relating the 8 series S_1, S_3, P_1, P_3 and C_r, for $0\le r\le 3$. Recall from (11) that we are mostly interested in C_3, since it is closely related to T_1.

We will perform a careful, step-by-step elimination procedure based on resultants, in which we exploit the fact that we know the first coefficients of all series involved in the system. In this way, each time we find a polynomial relation between our eight series that factors (and this happens almost systematically), we remove from it the factors that, given the first few terms of the series, cannot vanish.

4.2. Elimination of the series C_r

Our first step is to eliminate the four series C_r. This may seem counter-intuitive, since we want to determine the minimal polynomial of C_3, but turns out to work well. Here we use the additional equation (17) derived from Proposition 4, and take advantage of the fact that it is linear in the C_r. Hence, writing

$$\mathrm{rem}\left(D(x)C'(x)-3D'(x)C(x),(x^2-S_1x+P_1)(x^2-S_3x+P_3),x\right)=0,$$

gives a system of four linear equations in the four series $C_0,\dots,C_3$. We check that its determinant is non-zero, using the first coefficients of the S_i and P_i. Solving this linear system gives expressions of $C_0,\dots,C_3$ as rational functions in S_1, S_3, P_1 and P_3.

We now replace each C_i by its expression in the 8 polynomial equations obtained in Section 4.1. We thus obtain a system of eight equations where the only unknowns are S_1, P_1, S_3 and P_3. The reason why we keep "too many" equations is that this will give us some leeway to choose the smallest ones, when convenient.

4.3. Minimal polynomial of the series $S_1 = X_1 + X_2$

Starting from the system that we have just obtained, we will eliminate first P_3, then P_1, then S_3, to obtain the minimal polynomial of S_1. Let us give a few details. The smallest of the eight equations contains 366 monomials (in ν, w and the four unknown series), has degree 2 in each P_i, and degree 3 in each S_i. We take its resultant, in turn, with each of the other seven polynomials, with respect to P_3. Each of these seven resultants is found to factor, and in each case, we prove using the first coefficients of S_1, P_1 and S_3 that only one factor vanishes: of course this is the only factor that we retain to proceed with further eliminations. Three of these resultants yield the same factor, so at this end of this step, we have five equations between S_1, P_1 and S_3.

Now we repeat the procedure by eliminating P_1 between the smallest of these five equations (170 terms) and each of the others. In each resultant, we only retain the (unique) vanishing factor. This gives a system of four equations between S_1 and S_3. A final elimination of S_3 between two of these equations gives an annihilating polynomial for S_1: again, we decide from the first coefficients of S_1 which of its factors is the minimal polynomial of S_1.

At the end S_1 is found to be algebraic of degree 11 over $\mathbb{Q}(\nu, w)$. Its minimal polynomial contains 394 monomials, when seen as a polynomial in ν, w and S_1, but only 36 as a polynomial in w and S_1. It has degree 21 in ν, 4 in w, and of course 11 in S_1.

4.4. Elliptic parametrization of (w, S_1)

The curve $\mathcal{C} := \{(w, S_1)\}$ (with ν as a parameter) is found to have genus 1, that is, to be elliptic (we use the `algcurves` package in MAPLE). This implies that it can be parametrized by writing w and S_1 as rational functions in some parameter U and the square root of some polynomial in U (with coefficients in $\overline{\mathbb{Q}(\nu)}$). This is the natural counterpart for elliptic curves of the rational parametrization of curves of genus 0. One can then hope that the series S_3, P_1 and P_3 can be expressed in terms of U (and a square root) as well. This will indeed turn out to be the case.

Since the minimal polynomial of S_1 has degree 4 in w, and not 2, the series S_1 itself cannot be used as the parametrizing series U. However, we were able to determine a suitable parametrization using MAPLE. Later we discovered that the series $\partial_w T_1$ could be used as parametrizing series (see its minimal polynomial in (1)), and this is what we will do below. But let us briefly explain how we first constructed a parametrizing series U, since this can be of interest to some readers. Details are available in the MAPLE session accompanying this paper.

Using the command `Weierstrassform`, we first constructed a Weierstrass form of the curve $\mathcal{C}$ for various values of ν, and were then able to conjecture from them a generic form, valid for an indeterminate ν. This conjecture stated that the curve $\mathcal{C}$ was birationally equivalent to the curve

$$U^3 + \frac{(1-\nu)\,V^2}{2} + 3\left(7\nu^2 - 14\nu - 29\right) U + \frac{2\left(5\nu^2 - 10\nu - 13\right)^2}{\nu - 1} = 0. \tag{18}$$

Still with MAPLE, one can also obtain, for a fixed value of ν, rational expressions of w and S_1 in terms of U and V lying on the above curve. These expressions read, respectively,

$$w = \frac{A_9(U)V + A_{10}(U)}{D_{11}(U)}, \qquad S_1 = \frac{A_2(U)V + A_4(U)}{D_4(U)},$$

where the A_i and D_i are polynomials whose degrees are indicated by their subscripts. Having determined them for sufficiently many values of ν, we could derive by rational interpolation (conjectural) expressions of w and S_1 in terms of U and V, valid for any ν. Finally, to prove these conjectured expressions, we just had to replace w and S_1 by these expressions in the minimal polynomial of S_1, and check that this was 0 on the curve (18).

But from now on however, we will use as parametrizing series the only solution of (1) that has constant term 0, denoted by $\dot{T}_1$. Of course at this stage we do not know that this is the w-derivative of T_1 (but we will prove it below). Solving (1) for w gives a rational expression of w in terms of $\dot{T}_1$ and $\sqrt{\Delta}$, where

$$\Delta = \nu\left(144\nu^3\dot{T}_1^4 - 24\nu^2\left(7\nu+5\right)\dot{T}_1^3 + 6\nu\left(13\nu^2+16\nu+7\right)\dot{T}_1^2 \right.$$
$$\left. -2\left(8\nu^3+15\nu^2+12\nu+1\right)\dot{T}_1 + \nu\left(2+\nu\right)^2\right).$$

This is a bit bigger than the square root arising from (18), but in fact the minimal polynomial of U has much more terms than (1). We now replace w by this expression in the minimal polynomial of S_1, and factor the resulting expression over $\mathbb{C}(\nu, \dot{T}_1, \sqrt{\Delta})$: we obtain two factors, and the one that actually vanishes has degree 1 in S_1. This gives the following expression:

$$S_1 = -\frac{3(1-2\dot{T}_1)\sqrt{\Delta}}{2\nu(1+5\nu-12\nu\dot{T}_1)}$$
$$-\frac{72\dot{T}_1^3\nu^3 - 12\nu^2\left(5\nu+4\right)\dot{T}_1^2 + 2\nu\left(4\nu^2+10\nu+13\right)\dot{T}_1 + \left(2+\nu\right)\left(2\nu^2-4\nu-1\right)}{2\nu^2(1+5\nu-12\nu\dot{T}_1)}. \tag{19}$$

4.5. The series T_1

We now make our way backwards in the elimination process that led to the minimal polynomial of S_1. In the smallest equation that we had between S_1 and S_3, we replace w and S_1 by their expressions in terms of $\dot{T}_1$ and Δ, factor the result, observe that the factor that vanishes has degree 1 in S_3, and thus obtain a rational expression of S_3 in terms of $\dot{T}_1$ and Δ, similar to (19). Then we proceed similarly with the series P_1 and finally P_3. Both are found to lie in $\mathbb{Q}(\nu, \dot{T}_1, \sqrt{\Delta})$.

The next step is to return to C_3, which is closely related to T_1 (see (11)) and was expressed as a rational function of S_1, P_1, S_3 and P_3 in Section 4.2. This gives an expression of C_3, then T_1, in $\mathbb{Q}(\nu, \dot{T}_1, \sqrt{\Delta})$.

It remains to derive from this expression that the w-derivative of T_1 is indeed the series $\dot{T}_1$. We proceed as follows:

- using the expression of w in terms of $\dot{T}_1$ and $\sqrt{\Delta}$, we express T_1 rationally in terms of $\dot{T}_1$ and w rather than $\dot{T}_1$ and $\sqrt{\Delta}$,
- we differentiate this in w to obtain an expression of $\partial_w T_1$ in terms of w, $\dot{T}_1$, and $\partial_w \dot{T}_1$,
- we differentiate the minimal polynomial (1) of $\dot{T}_1$ to obtain an expression of $\partial_w \dot{T}_1$ in terms of w and $\dot{T}_1$,
- combining the last two steps, we obtain an expression of $\partial_w T_1$ in terms of w and $\dot{T}_1$,
- we reduce it modulo the minimal polynomial of $\dot{T}_1$ and conclude that $\partial_w T_1 = \dot{T}_1$.

Using the first point above, and the minimal polynomial of $\dot{T}_1$, we also compute the minimal polynomial of T_1, which will be useful later. It has degrees 13 and 27 in w and ν, respectively.

4.6. The series T_i for $i \geq 2$

Recall from Section 4.2 that we have expressed the series $C_0, \ldots, C_3$ as rational functions of S_1, P_1, S_3 and P_3, with coefficients in $\mathbb{Q}(\nu, w)$. In Section 4.5, we have expressed these four series as elements of $\mathbb{Q}(\nu, \dot{T}_1, \sqrt{\Delta})$. In Section 4.4, we had obtained such an expression for w as well. So each C_i can now be written as an element of $\mathbb{Q}(\nu, \dot{T}_1, \sqrt{\Delta})$.

We have already exploited the fact that C_3 is closely related to T_1 to determine T_1. We now proceed similarly for T_3, T_5, and T_7, in this order, using the fact that C_2 (resp. C_1, resp. C_0) is a polynomial in T_1 and T_3 (resp. in T_1, T_3 and T_5, resp. in T_1, T_3, T_5 and T_7), with coefficients in $\mathbb{Q}(\nu, w)$, of degree 1 in T_3 (resp. T_5, resp. T_7). So now each of the series T_3, T_5, T_7 can be written as an element of $\mathbb{Q}(\nu, \dot{T}_1, \sqrt{\Delta})$.

Let us now discuss the series T_2, T_4 and T_6. We recall from (12) and (13) that they have polynomial expressions in terms of T_1, T_3 and T_5. This yields expressions for these series in $\mathbb{Q}(\nu, \dot{T}_1, \sqrt{\Delta})$ as well. Recall also that $T_0 = w$.

Now let us write

$$T(y) = w + T_1 y + T_2 y^2 + \cdots + T_7 y^7 + y^8 S(y),$$

with T_2, T_4, T_6 expressed in terms of T_1, T_3, T_5, for a series $S(y)$ in $\mathbb{Q}(\nu, y)[[w]]$. We inject this expression in the 1-catalytic equation (14) satisfied by $T(y)$. This makes the equation tautological up to the order of y^8, that is, it contains a factor y^8. Removing this factor leaves a polynomial equation (of degree 5) for $S(y)$, with coefficients in $\mathbb{Q}[\nu, w, T_1, T_3, T_5, T_7, y]$. This equation reads

$$36\nu^{15} S(y) + \mathrm{Pol}_0(\nu, w, T_1, T_3, T_5, T_7) + y \times \mathrm{Pol}(\nu, w, T_1, T_3, T_5, T_7, y, S(y)) = 0.$$

This form implies, by induction on $i \geq 0$, that the coefficient of y^i in $S(y)$, that is, the series T_{i+8}, is an element of $\mathbb{Q}(\nu)[w, T_1, T_3, T_5, T_7]$, and thus of $\mathbb{Q}(\nu, \dot{T}_1, \sqrt{\Delta}) = \mathbb{Q}(\nu, w, \dot{T}_1)$. Given that the degree of $\dot{T}_1$ over $\mathbb{Q}(\nu, w)$ is prime, each series T_i is either rational in ν and w, or algebraic of degree 11. The former possibility will be ruled out by an asymptotic argument in Section 5.1; see the proof of Lemma 14. This completes the proof of Theorem 1.

4.7. The Potts model on near-cubic maps

We can now, using the change of variables (5) for $q = 3$, state a result analogous to Theorem 1 for the 3-Potts model on near-cubic maps.

Corollary 5. *For $i \geq 1$, the series $K_i \equiv K_i(\nu, w)$ that counts 3-coloured near-cubic maps with root vertex of degree i (by monochromatic edges and faces) is algebraic of degree 11. All series K_i belong to the same extension of degree 11 of $\mathbb{Q}(\nu, w)$.*

In particular, one derives from (1) the minimal polynomial of $\dot{K}_1 := \partial_w K_1$. It has degree 2 in w again, but degree 15 in ν. The specialization $\nu = 0$ then leads Corollary 2, as already established at the end of Section 2.2.

5. Asymptotic results

5.1. Triangulations

We now study the dominant singularities of the series T_i, seen as power series in w depending on a non-negative parameter ν. The singularity analysis of algebraic series in $\mathbb{R}[[w]]$ has become quasi-automatic [33, Chap. VII.7], but things are more delicate here because of the parameter ν. This is of course a recurrent difficulty in many counting problems. We refer for earlier (and somewhat smaller) instances to [7, 1, 27, 26]. We will use, and sometimes make more systematic, some of the ideas of these papers.

Proposition 6. *Let Δ_1 and Δ_2 be the polynomials in ν and ρ given by (33) and (34) in Appendix B.1. Figure 4 shows, among other curves, a plot of the curves $\Delta_1(\nu,\rho) = 0$ and $\Delta_2(\nu,\rho) = 0$.*

Let $i \geq 1$. Consider $T_i(\nu, w) \equiv T_i$ as a series in w depending on the parameter $\nu > 0$. Let ρ_ν denote its radius of convergence. Then ρ_ν is a continuous non-increasing function of ν for $\nu > 0$, which satisfies

$$\begin{aligned} \Delta_1(\nu,\rho_\nu) &= 0 \quad \text{for} \quad 0 < \nu \leq \nu_c := 1 + 3/\sqrt{47}, \\ \Delta_2(\nu,\rho_\nu) &= 0 \quad \text{for} \quad \nu_c \leq \nu. \end{aligned}$$

More precisely, between 0 and ν_c the radius ρ_ν is the branch of $\Delta_1(\nu,\rho) = 0$ that starts at $\rho_0 := 1/8$ when $\nu = 0$, and beyond ν_c the radius ρ_ν is the highest of the two branches of $\Delta_2(\nu,\rho) = 0$ that start at

$$\rho_{\nu_c} = \frac{1295\sqrt{47} - 7875}{109744}. \tag{20}$$

Moreover, T_i has no singularity other than the radius on its circle of convergence. For $\nu \neq \nu_c$, the behaviour of T_i near $w = \rho_\nu$ is the standard singular behaviour of planar maps series:

$$T_i = \alpha_{i,\nu} + \beta_{i,\nu}(1 - w/\rho_\nu) + \gamma_{i,\nu}(1 - w/\rho_\nu)^{3/2}\,(1 + o(1)), \tag{21}$$

where $\gamma_{i,\nu} \neq 0$. At $\nu = \nu_c$, the nature of the singularity changes:

$$T_i = \alpha_{\nu_c} + \beta_{\nu_c}(1 - w/\rho_{\nu_c}) + \gamma_{\nu_c}(1 - w/\rho_{\nu_c})^{6/5}\,(1 + o(1)). \tag{22}$$

In asymptotic terms,

$$[w^n]T_i \sim \begin{cases} \kappa_{i,\nu}\, \rho_\nu^{-n} n^{-5/2} & \text{for } \nu \neq \nu_c, \\ \kappa_{i,\nu_c}\, \rho_{\nu_c}^{-n} n^{-11/5} & \text{for } \nu = \nu_c, \end{cases} \tag{23}$$

where $\kappa_{i,\nu} > 0$. For $\nu = 0$ and $i > 1$ the series T_i has a unique singularity at $\rho_0 = 1/8$, with a planar map singular behaviour (21), while $T_1 = 0$ when $\nu = 0$.

Remarks

1. The above result can be compared to the analogous result for the Ising model on triangulations (the Potts model with 2 colours only), where the critical value of ν is at $1 + 1/\sqrt{7}$, with exponent $4/3$ rather than $6/5$ in the singular expansion of the series at criticality; see [5, Claim 24] or [1, Thm. 2.4].
2. The exponent $-11/5$ occurring in (23) is in agreement with the prediction given by the Knizhnik–Polyakov–Zamolodchikov (KPZ) formula [43]: the 3-state Potts model having *central charge* $c = 4/5$ (see, e.g., [39, Eq. (4.22)]), the KPZ formula gives, at criticality, an asymptotic estimate for the Potts-weighted number of (rooted) maps of size n, of the form

$$\kappa \mu^n n^{\gamma-2},$$

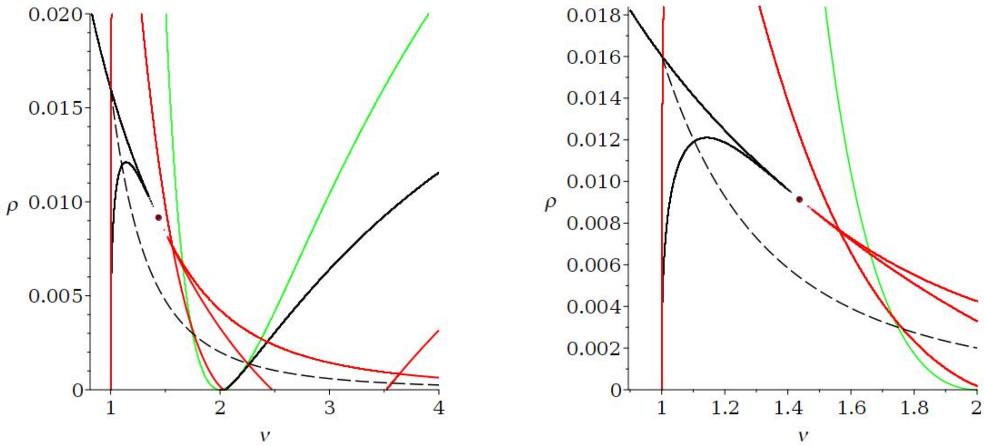


FIGURE 4. The branches of Δ_0 (green/light), Δ_1 (black) and Δ_2 (red). The black dashed curve is the lower bound ρ_1/ν^3 on the radius, for $\nu \geq 1$. The plot on the right zooms on the interval $[1, 2]$. The radius ρ_ν first follows the top black branch, between $\nu = 0$ and $\nu_c \simeq 1.44$, and then the top red branch.

where the *string susceptibility exponent* γ is

$$\gamma = \frac{1}{12}\left(c - 1 - \sqrt{(1-c)(25-c)}\right) = -\frac{1}{5},$$

so that $\gamma - 2 = -11/5$ indeed. For the Ising model, $c = 1/2$ hence $\gamma = -1/3$, in agreement with the exponent $-1 - 4/3 = -7/3$ found in the number of Ising-weighted maps of size n.

The details of the proof require some care, which makes the proof long. We have thus split it into several shorter lemmas. As many parts of this paper, this proof requires using a computer algebra system. Our MAPLE session is available on our web pages. Inside MAPLE, we use the packages `algcurves, plots`, `gfun` [57] and `DA` [54]. We also use `msolve`, a C library for solving in arbitrary precision multivariate polynomial systems [9].

Lemma 7 (**The case $\nu = 0$**). *Proposition 6 holds true for $\nu = 0$: all series T_i with $i > 1$ have radius of convergence $\rho_0 = 1/8$. This is their unique singularity, and they all have a map-like singular expansion of the form (21) near ρ_0.*

For $\nu > 0$, the radius of convergence of T_1 satisfies $\rho_\nu \leq \rho_0 = 1/8$.

Proof. When $\nu = 0$ we have $T_1 = 0$, so we first focus on the series T_2 that counts properly 3-coloured near-triangulations of outer degree 2. Since in general $T_2 = T_1/\nu$, we can derive the minimal polynomial of T_2 from that of T_1, computed in Section 4.5. When $\nu = 0$ we find that T_2 is quadratic only:

$$8T_2^2 - \left(8w^2 + 12w - 1\right) T_2 + 2w^2 \left(w^2 + 11w - 1\right) = 0. \tag{24}$$

This gives:

$$T_2(0, w) = \frac{1}{16}\left((1 - 8w)^{3/2} - 1 + 12w + 8w^2\right),$$

with radius $\rho_0 = 1/8$. There is no other singularity. Near $w = \rho_0$ the singular behaviour of T_2 is of the map-type (21).

Let us now discuss the series T_i for $i > 2$, still with $\nu = 0$. First, by deleting the root edge in a near-triangulation of outer degree 2, we obtain that $T_2 = 2w^2 + T_3$ at $\nu = 0$. In particular, T_3 has the same singularity and the same singular behaviour as T_2. At this point we have reached Tutte's classical result on *bicubic maps*, as discussed in the introduction; see [62, p. 269]. We then return to the 1-catalytic equation (14), where we replace T_1 by νT_2: a factor ν comes out. After dividing by ν, we set $\nu = 0$ and $T_3 = T_2 - 2w^2$. This gives the following equation:

$$4y^5T(y)^3 - y^2\left(8y^2+10y-1\right)T(y)^2 + 2\left(3w\,y^3+2y^2+y-1\right)T(y) \\ + 2\,y^2T_2 - w^2y^2 + 2(y+1)(1-2y)w = 0.$$

Let us now define $S(y)$ by $T(y) = w + y^2T_2 + y^3S(y)$. Then the above equation yields

$$S(y) = T_2 - 2w^2 + y\,\mathrm{Pol}(w, T_2, y),$$

for some polynomial Pol with coefficients in $\mathbb{Q}$. This gives by induction on $i \geq 3$ an expression of $T_i = [y^{i-3}]S(y)$ as a polynomial of $\mathbb{Q}[w, T_2]$. More precisely, in sight of (24), T_i belongs to $\mathbb{Q}[w] + T_2\mathbb{Q}[w]$. In particular, either T_i is a polynomial in w, or it is an algebraic (quadratic) function of w with a unique singularity at $\rho_0 = 1/8$. It is easy to see, by adding a dangling edge in the outer face of a near-triangulation, that $T_{i+2} \geq wT_i$, coefficientwise. Hence $T_{2i} \geq w^{i-1}T_2$ and $T_{2i+1} \geq w^{i-1}T_3$. Given that neither T_2 nor T_3 is a polynomial in w, this proves that none of the T_i, for $i \geq 2$, is a polynomial in w. Since T_i is a polynomial in w and T_2, its singular behaviour near $w = 1/8$ is in $(1-8w)^{3/2+k}$ for k a non-negative integer. This implies that the coefficient of w^n in T_i grows like $8^n n^{-5/2-k}$. However, the above lower bounds on T_{2i} and T_{2i+1} then imply that $k = 0$, so that T_i has a map-type singularity for any $i \geq 2$.

Finally, we note that $T_1(\nu, w) \geq \nu T_2(0, w)$ coefficientwise, so that $\rho_\nu \leq \rho_0 = 1/8$ for $\nu > 0$. ∎

We next focus on the series T_1, and will return to the series T_i for $i > 1$ in Lemma 14. We refer to [33, Chap. VII.7] for generalities on singularities of algebraic series. In particular, given an annihilating polynomial of a series $F(w)$, say $\mathrm{Pol}(F(w)) = 0$ with coefficients in $\mathbb{Q}[w]$, all singularities of F are found among the roots of the leading coefficient and of the discriminant of Pol. Also, for series with non-negative coefficients, the radius of convergence is one of the singularities (Pringsheim's theorem).

Lemma 8 (**The case $\boldsymbol{\nu = 1}$**)**.** *Proposition 6 holds true for $\nu = 1$ and $i = 1$: the series T_1 has radius of convergence $\rho_1 = \sqrt{3}/108$, which is a root of $\Delta_1(1, \cdot)$. This is the unique singularity, and T_1 has a map-like singular expansion of the form (21) near ρ_1.*

For $\nu \geq 1$, the radius of convergence of T_1 satisfies $\rho_\nu \geq \rho_1/\nu^3$.

Proof. As recalled in the introduction, this case is simple, and equivalent to the classical enumeration of near-triangulations counted by vertices (with a weight $3w$ per vertex) [50]. When $\nu = 1$ the minimal polynomial of T_1, determined in Section 4.5, factors. Choosing the correct factor yields:

$$192T_1^3 - (288w-1)\,T_1^2 + w\,(90w-1)\,T_1 - 3w^3\,(81w-1) = 0.$$

The leading coefficient is non-zero, and the discriminant has only two non-zero roots $\rho_1 = \sqrt{3}/108$ and $-\rho_1$, so that ρ_1 is the radius of convergence. A plot of T_1 as a function of w shows that $-\rho_1$ is not a singularity of T_1 (but of its two conjugates). Moreover, a local expansion of T_1 near ρ_1 yields an expansion of the planar map type (21) (this expansion can be computed using for instance the `algeqtoseries` command in the `gfun` package of MAPLE).

We can now derive a lower bound on the radius of T_1 for $\nu \geq 1$: since a near-triangulation of outer degree 1 having n vertices has $3n-4$ edges, each of them getting a weight at most ν, we have $T_1(\nu, w) \leq \nu^{-4}T_1(1, w\nu^3)$ coefficientwise, which implies $\rho_\nu \geq \rho_1/\nu^3$ for $\nu \geq 1$. ∎

We now return to general values of ν. We want to determine the dominant singularities of T_1, and its behaviour near these values. It is sufficient to study the singular behaviour of $\dot{T}_1 = \partial_w T_1$, which as a simpler minimal polynomial (1), and then integrate.

Lemma 9 (**Locating possible singularities**)**.** *Take $\nu > 0$. Let Δ_1 and Δ_2 be defined as in Proposition 6. Moreover, let $\Delta_0 = 16\nu\,(\nu-1)\,w - (\nu-2)^2$. Then any singularity of T_1 is a root of $\Delta_0\Delta_1\Delta_2$.*

Proof. For $\nu = 1$, we refer to Lemma 8.

We now assume $\nu \neq 0, 1$ and return to the minimal polynomial (1) of $\dot{T}_1$, say $\mathrm{Pol}(\nu, w, z)$, of degree 11 in z. Its leading coefficient ν^7 does not vanish, so all singularities of $\dot{T}_1$ will be found among the roots of the discriminant of Pol with respect to z. This discriminant reads:

$$\kappa\, \nu^{45}(\nu-1)^{18}\Delta_0^2\Delta_1\Delta_2\Delta_3^2,$$

where $\kappa \in \mathbb{Z}$, Δ_0, Δ_1 and Δ_2 are as above, and Δ_3 is another polynomial in ν and w, of degree 5 in w. We examine similarly the leading coefficient and the discriminant of the minimal polynomial of T_1, and observe that they do not contain the factor Δ_3. Hence the roots of Δ_3 cannot be singularities of T_1 (unless they are also roots of $\Delta_0\Delta_1\Delta_2$). ∎

Lemma 10 (The radius of convergence). *Let $\nu > 0$. The radius of convergence of T_1, denoted ρ_ν, is a continuous non-increasing function of ν, whose value is given by Proposition 6.*

Proof. The coefficient of w^n in T_1 is a polynomial of $\mathbb{N}[\nu]$ of degree at most $3n - 4$ (this is the number of edges in a map of $\mathcal{T}_1$ having n vertices). By classical arguments, ρ_ν is a continuous, non-increasing, log-convex function of ν on $(0, +\infty)$ (see for instance [35] for a proof in a different context). The graph of this function is obtained by gluing parts of branches of the above polynomials Δ_i, for $0 \leq i \leq 2$. We refer to Figure 4 for plots of real positive branches. In what follows, we often replace arguments based on estimates of branches of algebraic functions at some point in controlled precision, as should be done (see [33, Chap. VII.7], [23]), by discussions on Maple plots of algebraic curves. We hope that our readers will find them convincing enough.

The only branch of $\Delta := \Delta_0\Delta_1\Delta_2$ containing the point $(\nu, w) = (1, \rho_1) = (1, \sqrt{3}/108)$ is the branch of Δ_1 that decreases from $\rho_0 = 1/8$ to ρ_1 as ν increases from 0 to 1 (Figure 4). Let us denote this branch by $\mathcal{B}_1$. We will prove that on the interval $[0, 1]$, this branch does not meet any other branch of Δ (the intersection with the red branch will be shown to occur for $\nu > 1$). An intersection point $(\nu, w) \in \mathcal{B}_1$ would cancel Δ_1 of course, and either $\Delta_0\Delta_2$, or $\partial_w\Delta_1$ (if two branches of Δ_1 meet at this point). We confirm (rigorously) with `msolve` that no such point exists with $\nu \in (0, 1]$ and $w \in [\rho_1, 1/8]$. By continuity, we conclude that $\mathcal{B}_1$ gives the radius of convergence for $\nu \in (0, 1]$.

Following branches. We have just proved that the radius follows the branch $\mathcal{B}_1$ between $\nu = 0$ and $\nu = 1$ (the *antiferromagnetic phase*, in physics terms). This goes on in a neighbourhood of $\nu = 1$ as ν increases. Later the radius may follow another branch $\mathcal{B}_2$ of Δ at a point where $\mathcal{B}_1$ and $\mathcal{B}_2$ meet. We are thus led to determine all points (ν, w) where several positive branches of Δ meet. At these points, either one of the Δ_i's, seen as a polynomial in w, has a multiple root, or two of the Δ_i's vanish. We determine controlled approximations of these points using `msolve`. We naturally restrict our attention to values of (ν, w) such that $\nu \geq 1$ and $w \in [0, \rho_1]$. Moreover, in Table 1 we have also excluded values such that $w < \rho_1/\nu^3$ (Lemma 8). The 10 points listed on this table can be seen on the plots of Figure 4.

Based on this inspection of intersection points of branches, we now return to the radius of convergence of T_1. As ν increases away from 1, the first branch that $\mathcal{B}_1$ meets is a locally increasing branch of Δ_2, at $\nu \simeq 1.0035$. We ignore it because the radius is non-increasing. The next intersection is at the value $\nu_c := 1 + 3/\sqrt{47}$ introduced in the statement of the proposition. There the value of $\mathcal{B}_1$ is found to be the number ρ_{ν_c} given by (20). At the point (ν_c, ρ_{ν_c}), four (real) branches of Δ meet: two branches of Δ_1 (in black in Figure 4), namely $\mathcal{B}_1$ and a lower branch that exists for $\nu < \nu_c$, and two branches of Δ_2 (in red) that start at ν_c and proceed for $\nu > \nu_c$. One is higher than the other: a local expansion reveals that they differ by a sign in the coefficient of $(1 - w/\rho_{\nu_c})^{3/2}$. A similar statement holds for the two branches of Δ_1 at ν_c. The geometry of the branches then implies that ρ_ν is on the top branch of Δ_2 above ν_c (Figure 4): indeed, all non-increasing branches intersecting this branch for some $\nu \geq \nu_c$ reach the value 0 at some point. This concludes the determination of ρ_ν. We refer to our Maple session for details. ∎

$i\backslash j$	0	1	2
0	$\emptyset$	$\emptyset$	$(1.6577, 0.0067)$ $(1.6614, 0.0065)$ $(1.7342, 0.0034)$ $(2.3765, 0.0027)$
1		$(\nu_c, \rho_{\nu_c}) \simeq (1.4375, 0.0091)$	$(1.0035, 0.0159)$ (ν_c, ρ_{ν_c}) $(2.4319, 0.0025)$
2			(ν_c, ρ_{ν_c}) $(1.5629, 0.0076)$ $(1.5653, 0.0075)$ $(3.6393, 0.0008)$

TABLE 1. Intersection points (ν, w) of branches of Δ_i and Δ_j, for $0 \leq i \leq j \leq 2$.

Lemma 11 (**Nature of the dominant singularity**). *Let $\nu > 0$. The singular behaviour of T_1 near its radius ρ_ν is given by (21) if $\nu \neq \nu_c$, and by (22) if $\nu = \nu_c$.*

Proof. We use again the `gfun` package in MAPLE, and more precisely the `algeqtoseries` command [57], which implements the Newton polygon method and computes Puiseux expansions at a given point w_0 of all roots of a polynomial with coefficients in $\mathbb{Q}(w)$ (or $\mathbb{Q}(\nu, w)$). See for instance [33, Sec. VII.7.1] or [60, Sec. 6.1] for generalities on these expansions. We apply this procedure to the minimal polynomial of $\dot{T}_1$, given by (1).

• Let us begin with the critical case $\nu = \nu_c$. At this point we find 6 roots with distinct constant terms, plus 5 other roots that have the same constant term. The only real one expands as

$$\frac{25}{38} - \frac{\sqrt{47}}{19} - \left(\frac{7}{2^4 19^5}\left(8555\sqrt{47} - 57585\right)\right)^{1/5} (1 - w/\rho_{\nu_c})^{1/5} + \mathcal{O}\left((1 - w/\rho_{\nu_c})^{2/5}\right),$$

and the remaining 4 are obtained by multiplying the second coefficient by a non-trivial fifth root of unity. This implies that the first 6 roots are analytic at ρ_{ν_c}, and that the above expansion is that of $\dot{T}_1$ (because $\dot{T}_1$ *is* singular at its radius). By integration, this yields (22).

• We go on with $\nu < \nu_c$, in which case ρ_ν is a root of Δ_1. In our analysis we first consider ν as an indeterminate. We first need to determine the possible values of $\dot{T}_1$ at the point $w = \rho_\nu$. We take the minimal polynomial (1) of $\dot{T}_1$, say $\mathrm{Pol}(\nu, w, z)$, specialize it at $w = \rho_\nu$ and factor it over $\mathbb{Q}(\nu, \rho_\nu)$. We find two irreducible factors: the first one, say $\mathrm{Pol}_1(\nu, \rho_\nu, z)$, has degree 9 in z, with (generically) 9 distinct roots; the other one is the square of a polynomial of degree 1 in z, with an explicit (double) root $c_\nu \in \mathbb{Q}(\nu)[\rho_\nu]$. Since $\dot{T}_1(\rho)$ must be a multiple root of $\mathrm{Pol}(\nu, \rho, \cdot)$, we conclude that $\dot{T}_1(\rho) = c_\nu$, generically. By elimination of ρ_ν in the expression of c_ν, we find its minimal polynomial:

$$\begin{aligned} 20736\nu^4 c_\nu^5 - 432\nu^3 (77\nu + 43) c_\nu^4 + 48\nu^2 \left(451\nu^2 + 484\nu + 109\right) c_\nu^3 \\ - 36\nu \left(203\nu^3 + 293\nu^2 + 143\nu + 9\right) c_\nu^2 + 4 \left(317\nu^4 + 559\nu^3 + 363\nu^2 + 55\nu + 2\right) c_\nu \\ - \nu \left(89\nu^3 + 189\nu^2 + 147\nu + 7\right) = 0. \end{aligned} \quad (25)$$

We now expand the solutions (in z) of $\mathrm{Pol}(\nu, w, z)$ in the vicinity of $(w, z) = (\rho_\nu, c_\nu)$ in powers of $1 - w/\rho_\nu$, and find, using again the `algeqtoseries` command, two series with a square root singularity (in the generic case). The non-decreasing branch must be $\dot{T}_1$. We thus obtain:

$$\dot{T}_1 = c_\nu - (d_\nu)^{1/2} (1 - w/\rho_\nu)^{1/2} + \mathcal{O}(1 - w/\rho_\nu), \quad (26)$$

where $d_\nu \in \mathbb{Q}(\nu, \rho_\nu)$ is explicit. The other singular branch is obtained by changing the sign in the second term. This gives (21) by integration, and this holds for any $\nu \in (0, \nu_c)$ under the following two conditions:

– the polynomial $\mathrm{Pol}(\nu, \rho_\nu, z)$ has indeed 10 distinct roots, with c_ν as its unique double root,
– the above number d_ν is well-defined and non-zero.

These conditions hold except for finitely many (algebraic) values of ν.

We will now prove that the singular behaviour (26) holds in fact in the whole interval $(0, \nu_c)$. First, the value of $\dot{T}_1$ at its radius, namely $\dot{T}_1(\nu, \rho_\nu)$, is continuous at every point ν as a function taking its values in $\mathbb{R} \cup \{\infty\}$ (because the coefficients of $\dot{T}_1$ are non-negative). Then, we observe that when $\nu \to 0^+$, only one solution of (25) is real: this solution must coincide with c_ν near 0 (except possibly at finitely many values of ν). Its expansion near 0 is $c_\nu = 7\nu/9 + \mathcal{O}(\nu^2)$. Moreover, this solution is continuous and increasing on $(0, \nu_c)$, reaching at ν_c the value $\dot{T}_1(\nu_c, \rho_{\nu_c}) = \frac{25}{38} - \frac{\sqrt{47}}{19}$ (Figure 5). By continuity, c_ν coincides with $\dot{T}_1(\rho_\nu)$ on the whole interval $(0, \nu_c)$, including at possibly non-generic points. So we just have to expand the roots (in z) of $\mathrm{Pol}(\nu, w, z)$ near $w = \rho_\nu, z = c_\nu$. This is what we have done in (26), for ν generic. But now we examine the Newton polygon procedure step by step, to see what could go wrong for certain specific values of ν. This is inspired from [7, Prop. 3.4] and [1, Lem. 2.7].

Recall that c_ν belongs to $\mathbb{Q}(\nu)[\rho_\nu]$. Starting from the minimal polynomial (1) of $\dot{T}_1$, we form by elimination of ρ_ν (which is a root of Δ_1) a polynomial $\overline{\mathrm{Pol}}(Y, \varepsilon)$ with coefficients in $\mathbb{Q}[\nu]$ that vanishes for $\varepsilon := 1 - w/\rho_\nu$ and $Y := \dot{T}_1(\nu, w) - \dot{T}_1(\nu, \rho_\nu) = \dot{T}_1 - c_\nu$. This calculation is done for a generic value of ν. However, by continuity in ν, this polynomial vanishes at the above values of ε and Y for any $\nu \in (0, \nu_c)$. It has degree 55 in Y, and, by construction, vanishes when $Y = \varepsilon = 0$. We now apply to it the Newton polygon method; see, e.g., [33, Sec. VII.7.1]. All monomials $Y^i\varepsilon^j$ that occur in $\overline{\mathrm{Pol}}$ satisfy $i + 2j \geq 10$, and, generically, the coefficients of $Y^0\varepsilon^5$ and $Y^{10}\varepsilon^0$ are non-zero: the only negative slope in the Newton polygon being $-1/2$, this explains the square root behaviour found above. We next examine for which values of $\nu \in (0, \nu_c)$ one of these coefficients (or both) vanish: we only find two suspicious values, namely $\nu = 1$ (for which we know that $\dot{T}_1$ has a square root singularity, see Lemma 8), and $\nu = \nu_b := 1 - 3/\sqrt{47}$, the conjugate of ν_c.

So it remains to study this case. Returning to the value of Δ_1, we find that ρ_ν is cubic over $\mathbb{Q}(\sqrt{47})$. The same holds for $\dot{T}_1(\rho_\nu)$. For this value of ν, the polynomial $\overline{\mathrm{Pol}}(Y, \varepsilon)$ factors over $\mathbb{Q}(\sqrt{47})$ as $\overline{\mathrm{Pol}}_1(Y, \varepsilon)\overline{\mathrm{Pol}}_2(Y, \varepsilon)$, with $\overline{\mathrm{Pol}}_1$ (resp. $\overline{\mathrm{Pol}}_2$) of degree 33 (resp. 22) in Y. Applying the Newton polygon method shows that the solutions of $\overline{\mathrm{Pol}}_1$ that vanish at $\varepsilon = 0$ will have a square root singularity, but those of $\overline{\mathrm{Pol}}_2$ will have a singularity in $\varepsilon^{1/5}$. So it remains to check that $\overline{\mathrm{Pol}}_2(\dot{T}_1(w) - \dot{T}_1(\rho_\nu), 1 - w/\rho_\nu) \neq 0$ for $\nu = \nu_b$. To do this we evaluate the above expression at $w = 0$: and indeed, $\overline{\mathrm{Pol}}_2(-\dot{T}_1(\rho_\nu), 1)$ does not reduce to 0 modulo the minimal polynomial (over $\mathbb{Q}(\sqrt{47})$) of $\dot{T}_1(\rho_\nu)$.

• We finally address the case $\nu > \nu_c$, where ρ_ν is a root of Δ_2. The analysis parallels completely the case $\nu < \nu_c$. This time the value c_ν of $\dot{T}_1(\nu, \rho_\nu)$ at its radius of convergence is algebraic of degree 9 rather than 5 (as ρ_ν itself), and is a decreasing function of ν (Figure 5). Computations are heavier in this case because the radius ρ_ν has now degree 9 instead of 5. When applying the Newton polygon method to prove the square root behaviour, we obtain a polynomial $\overline{\mathrm{Pol}}(Y, \varepsilon)$ of degree 99 in Y, with coefficients in $\mathbb{Q}[\nu]$, that vanishes when $\varepsilon = 1 - w/\rho_\nu$, and $Y = \dot{T}_1 - \dot{T}_1(\rho_\nu)$. All monomials $Y^i\varepsilon^j$ that occur in it satisfy $i + 2j \geq 18$, and, generically, the coefficients of $Y^0\varepsilon^9$ and of $Y^{18}\varepsilon^0$ are non-zero: this proves the square root behaviour, except for seven values of ν for which one of these two coefficient vanishes (or both).

These values are $\nu = 2$, $\nu = 1 + 3/\sqrt{2}$, one value of degree 6 over $\mathbb{Q}$, three (conjugate) values of degree 10, and finally one of degree 16. For the first six values, the specialized polynomial $\overline{\mathrm{Pol}}(Y, \varepsilon)$ (or each of its factors) has a unique negative slope $-1/2$ in its Newton polygon again. We thus conclude to a square root singularity in $\dot{T}_1$. We refer to our Maple session for details.

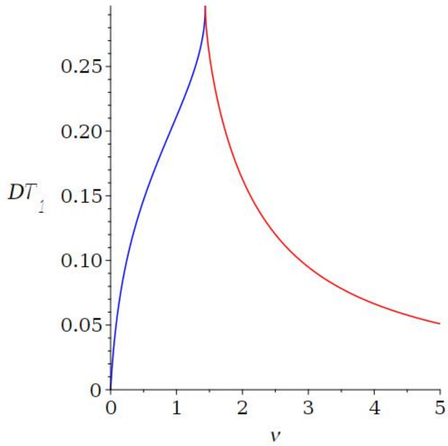


FIGURE 5. The value c_ν of $\dot{T}_1$ at its radius ρ_ν increases up to ν_c and decreases afterwards.

The final value of ν, of degree 16, is more tricky as one segment in the polygon has slope -1. This is analogous to the difficulty raised by $\nu = \nu_b$ in the case $\nu < \nu_c$, where one found the slopes $-1/2$ and $-1/5$ in the Newton polygon. In principle, we could apply here the same strategy as for $\nu = \nu_b$: for this value of ν, denoted ν_{16}, the polynomial Δ_2 factors over $\mathbb{Q}(\nu)$, and one finds ρ_ν and $\dot{T}_1(\rho_\nu)$ to be of degree 8 over $\mathbb{Q}(\nu)$. We expect $\overline{\text{Pol}}$ to factor into a term of degree 88 and one of degree 11. But this factorization just did not finish on our laptops, and there is in this case a more theoretical argument, which we now explain. The relevant part of the set of points (i, j) such that $Y^i\varepsilon^j$ occurs in $\overline{\text{Pol}}$, for $\nu = \nu_{16}$, is shown in Figure 6. Analysing this diagram first tells us that exactly 19 solutions Y of $\overline{\text{Pol}}(Y, \varepsilon) = 0$ vanish at $\varepsilon = 0$. General results on the Newton polygon (see e.g. [3, Thm. p. 424]) imply that exactly one of them will start with a term of the order of ε. This means in particular that this solution does not have a further branching, so it is analytic near $\varepsilon = 0$: this cannot be $\dot{T}_1(w) - \dot{T}_1(\rho_\nu)$, which we know to be singular. So $\dot{T}_1(w) - \dot{T}_1(\rho_\nu)$ must be one of the 18 other solutions, and all of them have a square root singularity. This concludes the proof of the lemma. ■

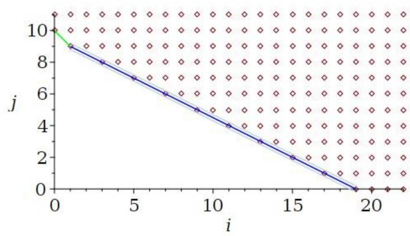


FIGURE 6. The south-west part of the Newton polygon of $\overline{\text{Pol}}(Y, \varepsilon)$ for $\nu = \nu_{16}$.

Let us now address the (non-)existence of other singularities of minimal modulus, called *dominant* singularities. With a parameter ν, this is a difficult task, and we believe to provide new tools for it. We actually give two proofs: the first one rules out the existence of multiple dominant singularities by examining the algebraic conditions and inequalities that ν, ρ_ν, $\dot{T}_1(\rho_\nu)$ and other algebraic quantities should satisfy, and proving, using `msolve`, that there is no solution. These ideas can be applied in many contexts. The second idea was suggested to us by Andrew Elvey Price. Roughly speaking, it says that a series that is obtained as the composition of two series with non-negative coefficients cannot have multiple dominant singularities. This idea is used in one of his recent papers, with Nessmann and Raschel [31, Lem. 10]. It gives a more combinatorial explanation of the uniqueness of the dominant singularity. One difference with [31] is that in the latter paper, the series under study, say $U(z)$ is expressed as $V(\mathrm{Rat}(z))$ from some explicit rational function $\mathrm{Rat}(z)$, while the result that we use here relies on an implicit function schema, $U(z) = G(z, U(z))$ (see Proposition 13).

Lemma 12 (Uniqueness of the dominant singularity). *Let $\nu > 0$. The series T_1 has a unique dominant singularity, which is its radius of convergence ρ_ν. Hence the estimates (23) hold for $i = 1$.*

First proof of Lemma 12. We begin by studying separately the cases $\nu = 1$, $\nu = 2$ and $\nu = \nu_c$. For $\nu = 1$ the result is stated in Lemma 8. For $\nu = 2$, we check (numerically) that the only root of $\Delta := \Delta_0\Delta_1\Delta_2$ that has modulus ρ_2 is ρ_2 itself. The same holds for $\nu = \nu_c$. For instance, the 12 roots of Δ for $\nu = \nu_c$ are plotted in Figure 7, together with the circle of radius ρ_{ν_c}. Observe that Δ has generically $1 + 5 + 9 = 15$ roots, but at ν_c there is a root ρ_{ν_c} of multiplicity 4.

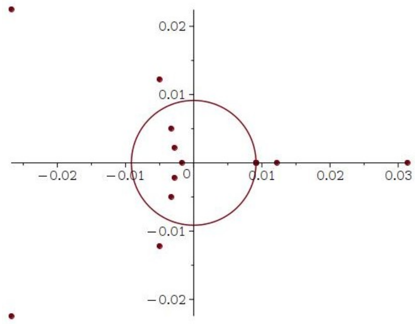


FIGURE 7. The roots of $\Delta(\nu_c, \cdot)$.

We now take $\nu > 0$, with $\nu \notin \{1, 2, \nu_c\}$. Let $s = x + iy$, with x and y real, be a dominant singularity of T_1, or equivalently $\dot{T}_1$, distinct from ρ_ν. By Lemma 9, it satisfies $\Delta_j(\nu, s) = 0$, for some $j \in \{0, 1, 2\}$. The series $\dot{T}_1$ converges at this point (since it converges at ρ_ν by Lemma 11), and, since it is an aperiodic series with non-negative coefficients, we have $\left|\dot{T}_1(\nu, s)\right| < \dot{T}_1(\nu, \rho_\nu)$.

Let us again denote by $\mathrm{Pol}(\nu, w, z)$ the minimal polynomial of $\dot{T}_1$, see (1). Since s is a singularity of $\dot{T}_1$, the value $z = \dot{T}_1(\nu, s)$ must be a multiple root of $\mathrm{Pol}(\nu, w, z)$. Hence we have the following system of 6 equations relating the 5 values ν, $\rho \equiv \rho_\nu$, s, $\dot{T}_1(\rho) \equiv \dot{T}_1(\nu, \rho)$, and $\dot{T}_1(s) \equiv \dot{T}_1(\nu, s)$:

$$\begin{cases} \Delta_i(\nu, \rho) &= 0, \\ \dot{T}_1(\rho) &= c_\nu, \\ |s|^2 &= \rho^2, \end{cases} \qquad \begin{cases} \Delta_j(\nu, s) &= 0, \\ \mathrm{Pol}\left(\nu, s, \dot{T}_1(s)\right) &= 0, \\ \partial_z \mathrm{Pol}\left(\nu, s, \dot{T}_1(s)\right) &= 0, \end{cases} \tag{27}$$

where $i = 1$ if $\nu < \nu_c$ and $i = 2$ otherwise, and c_ν is a fraction in ν and ρ, determined in the proof of Lemma 11, which takes two different values depending on whether $\nu < \nu_c$ or $\nu > \nu_c$; see Figure 5.

If we look for *real* dominant singularities, corresponding to $s = -\rho$, the system simplifies to 5 real equations in 4 real unknowns:

$$\begin{cases} \Delta_i(\nu, \rho) &= 0, \\ \dot{T}_1(\rho) &= c_\nu, \end{cases} \qquad \begin{cases} \Delta_j(\nu, -\rho) &= 0, \\ \mathrm{Pol}\left(\nu, -\rho, \dot{T}_1(-\rho)\right) &= 0, \\ \partial_z \mathrm{Pol}\left(\nu, -\rho, \dot{T}_1(-\rho)\right) &= 0. \end{cases}$$

This can be turned into a *polynomial* system by taking the numerator of $\dot{T}_1(\rho) - c_\nu$ rather than $\dot{T}_1(\rho) - c_\nu$ itself.

If we look for non-real dominant singularities, $s = x + iy$ with $y \neq 0$, each of the 3 equations on the right-hand side of (27) splits into 2 real equations, relating x, y, $U := \Re(\dot{T}_1(s))$ and $V := \Im(\dot{T}_1(s))$, giving a total of 9 real equations for 7 real unknowns. For instance, $\Delta_1(x+iy) = 0$ with $y \neq 0$ splits into two real polynomials in x and $z := y^2$, namely

$$\Delta_1(x+iy) + \Delta_1(x-iy) = 0, \qquad \text{and} \qquad \frac{\Delta_1(x+iy) - \Delta_1(x-iy) = 0}{iy} = 0.$$

The quantities ν, ρ, s, $\dot{T}_1(\rho) \equiv \dot{T}_1(\nu, \rho)$ and $\dot{T}_1(s) \equiv \dot{T}_1(\nu, s)$ are moreover constrained by the following inequalities:

$$\begin{cases} 0 < \nu < \nu_c, & \rho_{\nu_c} < \rho < \rho_0, \quad \text{if } i = 1, \\ \nu_c < \nu, & \rho_1/\nu^3 < \rho < \rho_{\nu_c}, \quad \text{if } i = 2, \end{cases} \tag{28}$$

and

$$0 < \dot{T}_1(\rho) < \dot{T}_1(\nu_c, \rho_{\nu_c}), \qquad \left|\dot{T}_1(s)\right| < \dot{T}_1(\rho), \tag{29}$$

where

$$\rho_0 = 1/8, \qquad \rho_1 = \sqrt{3}/108, \qquad \rho_{\nu_c} = \frac{1295\sqrt{47} - 7875}{109744}, \qquad \dot{T}_1(\nu_c, \rho_{\nu_c}) = \frac{25}{38} - \frac{\sqrt{47}}{19}.$$

We now take the 6 real polynomial systems obtained for $(i, j) \in \{1, 2\} \times \{0, 1, 2\}$, each system being declined in two versions, the real case $s = -\rho$ (5 equations) and the non-real case (9 equations). We use `msolve` to approximate their real solutions (rigorously, up to arbitrary precision). We then examine which of these (finitely many) solutions satisfy (28) and (29). In most cases, no solution remains.

For instance, when $i = j = 1$ and $s = -\rho$, `msolve` returns 23 candidates for $(\nu, \rho, \dot{T}_1(\rho), \dot{T}_1(-\rho))$. Only 7 of them satisfy $0 < \nu < \nu_c$. Among them, only 3 satisfy $\rho_{\nu_c} < \rho < 1/8$. Each of these 3 would be such that $\left|\dot{T}_1(s)\right| > \dot{T}_1(\rho)$, and this case is solved.

In some cases we use an additional sieve: we observe that the value found for ρ, supposed to be ρ_ν, is not on the correct branch of Δ_i (Figure 4) and thus cannot be the radius. This allows us to exclude more points. For instance, when $i = 2$ and $j = 1$, `msolve` returns in the non-real case 39 solutions. The conditions on ν and ρ allow us to restrict our attention to 7 of them. We can exclude 4 more because the value of ρ is not on the correct branch of Δ_2. For the remaining ones we find that $\left|\dot{T}_1(s)\right| > \dot{T}_1(\rho)$.

We must mention two difficulties. First, when $j = 2$, that is $\nu > \nu_c$, the denominator of c_ν contains factors $(\nu - 1)$ and $(\nu - 2)$, and this seems to make some of our systems positive dimensional. So when $j = 2$, we add a new variable c and complete our systems with an equation $c(\nu - 1)(\nu - 2) = 1$ to remedy this problem.

The other difficulty is due to the size of our systems, in the non-real case. For the moment, we have actually only given to `msolve` the equations that do not involve the series $\dot{T}_1$. For each solution (ν, ρ, x, z) (with $z = y^2$), after applying the above sieves on ν and ρ, and checked that ρ is on the correct branch of Δ_i, we compute (not certified) estimates of $\dot{T}_1(\rho)$ and $\dot{T}_1(x + iy)$ (using the final equations of the system) and use the condition on the moduli of these values to rule out the remaining candidates.

In the end, we conclude that the radius of convergence of $\dot{T}_1$ is always the unique dominant singularity. ∎

We now come to an alternative proof, which relies on the existence of a composition equation for the series T_1. Such equations are quite common in the world of maps [4]. The proof of the following proposition is given in Appendix C.

Proposition 13. *Let $U(z) \in \mathbb{R}[[z]]$ be an aperiodic*[1] *power series in z that converges in a neighbourhood of 0, and such that $U(0) = 0$. Assume that*

$$U(z) = G(z, U(z)) \tag{30}$$

for some series $G(z,u) = \sum_{m,n} g_{m,n} z^m u^n \in \mathbb{R}[[z,u]]$ that converges in a neighbourhood of $(0,0)$ and satisfies the following conditions:

$$g_{0,0} = 0, \quad g_{m,n} \geq 0 \ \forall (m,n), \quad g_{0,1} < 1 \quad \text{and} \ \ \exists (m,n) \text{ such that } g_{m,n} > 0 \text{ with } n \geq 2.$$

Let $\mathcal{R}$ be the non-negative region of convergence of G:

$$\mathcal{R} := \left\{ (z,u) \in \mathbb{R}^2_{\geq 0} \text{ such that } G(z,u) < \infty \right\}.$$

Assume moreover that the closure $\overline{\mathcal{R}}$ of $\mathcal{R}$ has no vertical boundary: that is, if $(z,u) \in \overline{\mathcal{R}} \cap \mathbb{R}^2_{>0}$ and $0 < u' < u$, then $(z,u') \in \mathcal{R}^\circ$, the interior of $\mathcal{R}$.

Then $U(z)$ has non-negative coefficients, its radius of convergence ρ is finite, and is the unique singularity of U on its disk of convergence.

Second proof of Lemma 12. We will first write a composition equation for the series T_1, or rather, for T_1/w. Recall that $T_1 = \mathcal{O}(w^2)$.

Let $L(q; a, b) \equiv L(a, b)$ be the generating function of *loopless* q-coloured near-triangulations of outer degree 2, where a (resp. b) records the number of monochromatic (resp. bicoloured) edges. The colour of the root vertex is prescribed, as always.

As already observed, we have $T_1 = \nu T_2$. Given a near-triangulation M of $\mathcal{T}_2$, two cases may occur:

- the boundary of the outer face of M consists of two loops; such maps are counted by T_1^2/w,
- the edges incident to the outer face are not loops.

In the latter case, let us draw M with the root face as the outer face, and consider a *maximal* loop, that is, a loop that is not included (for this drawing) in another loop. Then the outer side of this loop belongs to the boundary of a finite face of degree 3. The two other edges that bound this face share the same endpoints (Figure 8). For every maximal loop, we then merge these two edges (and what lies between them) into a single edge. This yields a loopless near-triangulation of outer degree 2, which we call the *projection* of M.

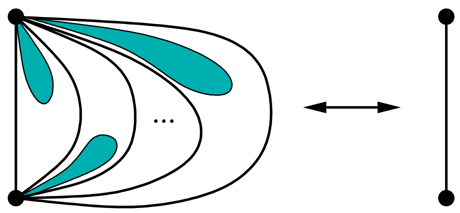


FIGURE 8. Compression of the maximal loops and the faces containing them.

Conversely, starting from a loopless near-triangulation $\overline{M}$ of outer degree 2, we obtain all maps of $\mathcal{T}_2$ that project on $\overline{M}$ by replacing each edge of $\overline{M}$ by a sequence of digons that include a map of $\mathcal{T}_1$ attached to one of the two corners of the digon (Figure 8). If $\overline{M}$ has m monochromatic edges and p bicoloured ones, then the contribution in the series T_2 of maps that project on $\overline{M}$ is thus

$$w^{5/3} \left(\frac{\nu w^{1/3}}{1 - 2\nu T_1/w} \right)^m \left(\frac{w^{1/3}}{1 - 2T_1/w} \right)^p$$

[1]there exist three integers $i < j < k$ such that the corresponding powers of z actually appear in $U(z)$ and $\gcd(j - i, k - i) = 1$.

(recall that a near-triangulation of outer degree 2 having n vertices has $3n-5$ edges). Putting together the above observations finally gives

$$T_1 = \nu T_2 = \nu \frac{T_1^2}{w} + w^{5/3} \nu L \left(\frac{\nu w^{1/3}}{1-2\nu T_1/w}, \frac{w^{1/3}}{1-2T_1/w} \right).$$

Note that $L(a,b) = \sum_{m,p} \ell_{m,p} a^m b^p$ where $p+m+2 \equiv 0 \bmod 3$ as soon as $\ell_{m,p} \neq 0$.

We now apply Proposition 13 with $z=w$, $U(z) = T_1(q,\nu,z)/z$ and

$$G(z,u) = \nu u^2 + z^{2/3} \nu L \left(\frac{\nu z^{1/3}}{1-2\nu u}, \frac{z^{1/3}}{1-2u} \right).$$

The series $U(z)$ is aperiodic, as there exist coloured near-triangulations of outer degree 2 with any number n of vertices, for $n \geq 2$. The conditions on the coefficients of G are easily checked. In particular, $g_{0,1} = 0$. Moreover $G(z,u)$ converges around the origin as the number of near-triangulations with n vertices grows exponentially.

Now let $\mathcal{R}_L$ be the non-negative region of convergence of L:

$$\mathcal{R}_L := \left\{ (a,b) \in \mathbb{R}^2_{\geq 0} \text{ such that } L(a,b) < \infty \right\}.$$

Then L is analytic in the interior of $\mathcal{R}_L$. Moreover, if $(a,b) \in \overline{\mathcal{R}}_L$ and $0 < a' < a$, $0 < b' < b$, then $(a',b') \in \mathcal{R}_L^\circ$. Now, for $z, u \geq 0$, the series $G(z,u)$ converges if and only if

$$u < \frac{1}{2} \min(1, 1/\nu) \quad \text{and} \quad \left(\frac{\nu z^{1/3}}{1-2\nu u}, \frac{z^{1/3}}{1-2u} \right) \in \mathcal{R}_L.$$

These conditions thus define the region $\mathcal{R}$. Now assume that $(z,u) \in \overline{\mathcal{R}} \cap \mathbb{R}^2_{>0}$. Then

$$u \leq \frac{1}{2} \min(1, 1/\nu) \quad \text{and} \quad \left(\frac{\nu z^{1/3}}{1-2\nu u}, \frac{z^{1/3}}{1-2u} \right) \in \overline{\mathcal{R}}_L.$$

Given that $\nu > 0$, then for $0 < u' < u$ we have $(z,u') \in \mathcal{R}^\circ$ by the above observation on $\mathcal{R}_L$, hence $\overline{\mathcal{R}}$ has no vertical boundary as required by Proposition 13. We conclude that $U(w) = T_1(w)/w$ has a unique dominant singularity. ∎

Lemma 14 (The series T_i with $i > 1$). *Let $\nu > 0$. For $i > 1$, the series T_i has radius of convergence ρ_ν. This is its only dominant singularity. The singular behaviour of T_i at this point is (21) or (22), depending on whether $\nu \neq \nu_c$ or $\nu = \nu_c$.*

Proof. We first establish the following bounds, for $\nu > 0$:

$$\nu^{2i-2} [w^{n-i+1}] T_1 \leq [w^n] T_i \leq \frac{1}{\min(1,\nu)^{i-1}} [w^n] T_1.$$

They follow from two basic constructions, illustrated in Figure 9. For the lower bound, we construct a near-triangulation of outer degree i by adding $(i-1)$ vertices and $2(i-1)$ monochromatic edges to a near-triangulation of outer degree 1. For the upper bound, we construct a near-triangulation of outer degree 1 by adding $i-1$ edges, which may be monochromatic or not, to a near-triangulation of outer degree i.

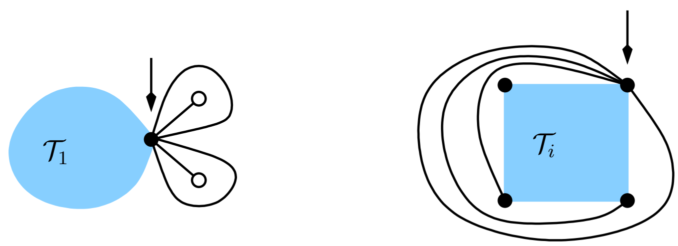


FIGURE 9. *Left.* Construction of a map of $\mathcal{T}_i$ from a map of $\mathcal{T}_1$ (here, $i=3$). *Right.* Construction of a map of $\mathcal{T}_1$ from a map of $\mathcal{T}_i$ (here, $i=4$).

These bounds imply that T_i has the same radius of convergence as T_1. Moreover, by Lemma 12, its coefficients admit lower and upper bounds of the form (23): there exist positive constants κ_1 and κ_2, depending on i and ν, such that

$$\kappa_1 \rho_\nu^{-n} n^{-1-e} \le [w^n]T_i \le \kappa_2 \rho_\nu^{-n} n^{-1-e}, \tag{31}$$

with $e = 3/2$ for $\nu \neq \nu_c$ and $e = 6/5$ otherwise.

We have proved in Section 4.6 that T_i belongs to $\mathbb{Q}(\nu, w, \dot{T}_1)$, or equivalently to $\mathbb{Q}(\nu, w)[\dot{T}_1]$. Thus T_i is either rational in w, or algebraic of degree 11. The former case is impossible because of the bounds (31), so T_i is algebraic of degree 11. This is the asymptotic argument mentioned at the end of Section 4.6.

Let us now discuss the dominant singularities of T_i. We have already argued that T_i has radius $\rho_\nu \equiv \rho$. Assume that T_i has dominant singularities other than the radius. Since T_1 has a unique dominant singularity, and $T_i \in \mathbb{Q}(\nu, w)[\dot{T}_1]$, all dominant singularities of T_i distinct from ρ are poles. Moreover, if s is a pole, its complex conjugate $\bar{s}$ is also one. Isolate these dominant poles by writing

$$T_i = R(\nu, w) + S(\nu, w),$$

where

- $R(\nu, w) \in \mathbb{R}(\nu, w)$ has only poles of modulus ρ, distinct from ρ,
- $S(\nu, w) \in \mathbb{R}(\nu)[[w]]$ has ρ as its unique dominant singularity.

Let $m \ge 1$ be the maximal order of a pole of $R(\nu, w)$. Then as n tends to infinity,

$$[w^n]R(\nu, w) = \rho^{-n} n^{m-1} \left(\alpha_1 \zeta_1^n + \cdots + \alpha_k \zeta_k^n + \overline{\alpha_1} \overline{\zeta_1}^n + \cdots + \overline{\alpha_k} \overline{\zeta_k}^n \right) + \mathcal{O}(\rho^{-n} n^{m-2}),$$

for some complex numbers α_i and $\zeta_i \neq 1$, with $|\zeta_i| = 1$. Now by [21, Lem. 4], there exists a constant $c > 0$ such that, for n large enough,

$$[w^n]R(\nu, w) < -c\rho^{-n} n^{m-1}.$$

Let $\beta \in \mathbb{Q} \setminus \{0, 1, 2, \ldots\}$ be the exponent describing the singular behaviour of T_i, or equivalently $S(\nu, w)$, near $w = \rho$. By this we mean that T_i differs from a function H that is holomorphic at ρ by a term that is equivalent to $(1 - w/\rho)^\beta$. Since T_i converges at ρ by (31), we have $\beta > 0$. Thus for n large enough,

$$[w^n]T_i = [w^n]R(\nu, w) + [w^n]S(\nu, w) \le -c\rho^{-n} n^{m-1} + d\rho^{-n} n^{-1-\beta},$$

with $m - 1 > -1 - \beta$, contradicting the fact that T_i has non-negative coefficients. We conclude that T_i has a unique dominant singularity. The bounds (31) finally imply that $\beta = 3/2$ if $\nu \neq \nu_c$, and $\beta = 6/5$ otherwise. ∎

5.2. Cubic maps

We can study analogously the singularities of the 3-Potts generating function K_i of near-cubic maps with root degree i, which are dual to near-triangulations of outer degree i. Recall that the corresponding Potts generating functions are related by (5), specialized at $q = 3$.

Proposition 15. *Let $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ be the polynomials in ν and ρ given by (35) and (36) in Appendix B.2. Figure 10 shows, among other curves, plots of the curves $\tilde{\Delta}_1(\nu, \rho) = 0$ and $\tilde{\Delta}_2(\nu, \rho) = 0$. These polynomials are related to those of Proposition 6 by*

$$\tilde{\Delta}_1(\nu, \rho) = \frac{(\nu - 1)^{12}}{729} \Delta_1 \left(\frac{\nu + 2}{\nu - 1}, \frac{1}{3}(\nu - 1)^3 \rho \right), \qquad \tilde{\Delta}_2(\nu, \rho) = (\nu - 1)^{17} \Delta_2 \left(\frac{\nu + 2}{\nu - 1}, \frac{1}{3}(\nu - 1)^3 \rho \right).$$

Let $i \ge 1$. Consider $K_i(\nu, w) \equiv K_i$ as a series in w depending on a parameter $\nu \ge 0$. Let $\tilde{\rho}_\nu$ denote its radius of convergence. Then $\tilde{\rho}_\nu$ is a continuous decreasing function of ν for $\nu \ge 0$, which satisfies

$$\tilde{\Delta}_2(\nu, \tilde{\rho}_\nu) = 0 \quad \text{for} \quad 0 \le \nu \le \tilde{\nu}_c := 1 + \sqrt{47},$$
$$\tilde{\Delta}_1(\nu, \tilde{\rho}_\nu) = 0 \quad \text{for} \quad \tilde{\nu}_c \le \nu.$$

More precisely, between 0 *and* $\tilde\nu_c$ *the radius* $\tilde\rho_\nu$ *is the branch of* $\tilde\Delta_2(\nu,\rho)=0$ *that starts at* $\tilde\rho_0\simeq 0.024$ *when* $\nu=0$*, and beyond* $\tilde\nu_c$ *the radius is the highest of the two branches of* $\tilde\Delta_1(\nu,\rho)=0$ *that start at*

$$\tilde\rho_{\tilde\nu_c}=\frac{3885}{5157968}-\frac{23625\sqrt{47}}{242424496}.$$

Moreover, K_i *has no dominant singularity other than its radius of convergence. For* $\nu\neq\nu_c$*, the behaviour of* K_i *near* $w=\tilde\rho_\nu$ *is the standard singular behaviour of planar maps series:*

$$K_i=\tilde\alpha_{i,\nu}+\tilde\beta_{i,\nu}(1-w/\tilde\rho_\nu)+\tilde\gamma_{i,\nu}(1-w/\tilde\rho_\nu)^{3/2}\,(1+o(1)),$$

where $\tilde\gamma_{i,\nu}\neq 0$*. At* $\nu=\tilde\nu_c$*, the nature of the singularity changes:*

$$K_i=\tilde\alpha_{i,\tilde\nu_c}+\tilde\beta_{i,\nu_c}(1-w/\tilde\rho_{i,\tilde\nu_c})+\tilde\gamma_{i,\tilde\nu_c}(1-w/\tilde\rho_{\tilde\nu_c})^{6/5}\,(1+o(1))$$

where $\tilde\gamma_{i,\tilde\nu_c}\neq 0$*. In asymptotic terms:*

$$[w^n]K_i\sim\begin{cases}\tilde\kappa_{i,\nu}\,(\tilde\rho_\nu)^{-n}\,n^{-5/2} & \text{for } \nu\neq\tilde\nu_c,\\ \tilde\kappa_{i,\tilde\nu_c}\,(\tilde\rho_{\tilde\nu_c})^{-n}\,n^{-11/5} & \text{for } \nu=\tilde\nu_c.\end{cases}$$

Remark. The corresponding result for the Ising model on cubic maps, with critical value $1+2\sqrt7$ and exponent $4/3$, can be found in [27]; see also [15].

Proof.
A. The case $\boldsymbol{i=1}$**.** We begin with the case $i=1$. By (5) and Lemma 9, we first observe that, for $\nu\neq1$ (a case that will be studied separately), any singularity of $K_1(\nu,\cdot)$ is a zero of $\tilde\Delta:=\tilde\Delta_0\tilde\Delta_1\tilde\Delta_2$, where $\tilde\Delta_1$ and $\tilde\Delta_2$ are given in the proposition, and

$$\tilde\Delta_0(\nu,w)=(\nu-1)^2\Delta_0\left(\nu_*,\frac13(\nu-1)^3w\right)=16\,(\nu+2)\,(\nu-1)^3\,w-(\nu-4)^2\,,$$

with $\nu_*=(\nu+2)/(\nu-1)$. The real positive branches of these three polynomials are shown in Figure 10.

It also follows from (5) that the series $\partial_wK_1:=\dot K_1$ satisfies

$$\dot K_1(\nu,w)=\frac{1}{\nu-1}\,\dot T_1\left(\nu_*,\frac13(\nu-1)^3w\right).$$

We then derive from (1) the minimal polynomial of $\dot K_1$, again of degree 11 in $\dot K_1$.

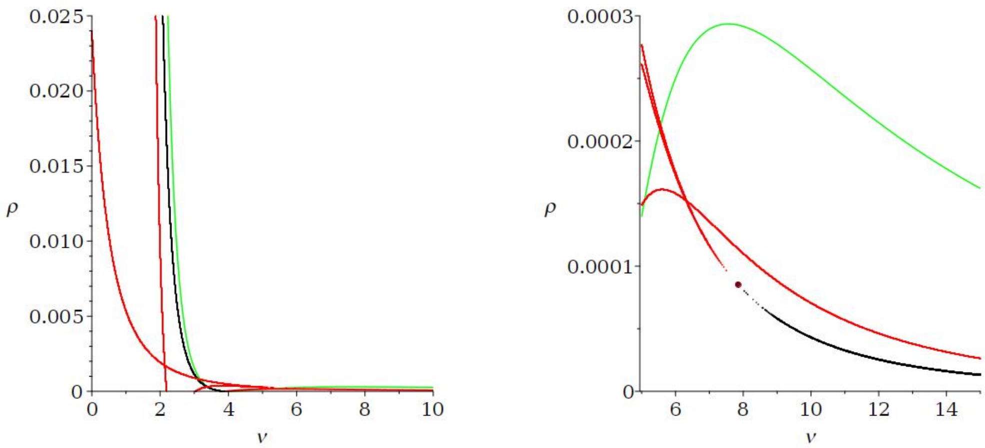


FIGURE 10. The branches of $\tilde\Delta_0$ (green), $\tilde\Delta_1$ (black) and $\tilde\Delta_2$ (red). The plot on the right zooms on the interval $\nu\in[5,15]$. The radius $\tilde\rho_\nu$ first follows the red branch, between $\nu=0$ and $\tilde\nu_c\simeq7.85$, and then the top black branch (since two distinct black branches start at $\nu=\tilde\nu_c$, even if this is not clear on the figure).

For $\nu = 1$, the degree of $\dot{K}_1$ drops to 3 (we are then counting cubic maps with a weight 3 on each vertex), and the radius of $\dot{K}_1$ is found to be $\tilde{\rho}_1 = \sqrt{3}/324$, with a square root behaviour. We observe that $\tilde{\rho}_1$ is a root of $\tilde{\Delta}_2$. Moreover, the discriminant of the minimal polynomial of $\dot{K}_1$ has a second root, namely $-\rho_1$ (while the leading coefficient is constant). But this value is only singular for the other two solutions of the minimal polynomial of $\dot{K}_1$. This proves all claims of the proposition for $\nu = 1$.
For $\nu > 1$, we have $\nu_* > 1$, and Eq. (5) offers an extremely convenient shortcut to derive the results of Proposition 15 from those of Proposition 6. In particular, the radius of convergence $\tilde{\rho}_\nu$ of $K_i(\nu, \cdot)$ is then $3\rho_{\nu_*}/(\nu-1)^3$. A transition will occur at the dual value $\tilde{\nu}_c := (\nu_c+2)/(\nu_c-1) = 1 + \sqrt{47}$. Since ν_* is a non-increasing function of ν for $\nu > 1$, the radius of convergence of K_i is a root of $\tilde{\Delta}_1$ above $\tilde{\nu}_c$ and a root of $\tilde{\Delta}_2$ between 1 and $\tilde{\nu}_c$, with continuity at 1^+ as we have just seen. The nature and uniqueness of the dominant singularity transfer from T_1 to K_1. So the proposition is proved for $i = 1$ and $\nu \geq 1$.
For $\nu \in (0, 1]$, the radius $\tilde{\rho}_\nu$ is a continuous function of ν, and we now that it is a root of $\tilde{\Delta}_2$ when $\nu = 1$. The branch of $\tilde{\Delta}_2$ that contains $\tilde{\rho}_1$ decreases continuously between 0 and 1, and meets no other branch of $\tilde{\Delta} := \tilde{\Delta}_0\tilde{\Delta}_1\tilde{\Delta}_1$ on $[0, 1]$: hence it gives the value of the radius for $\nu \in (0, 1]$. In fact, by considering the minimal polynomial of $\dot{K}_1$ for $\nu = 0$, we see that continuity holds on the closed interval $[0, 1]$. This concludes the determination of the radius of K_1.

It remains to study the nature of the singularity of K_1 near $\tilde{\rho}_\nu$ and to prove uniqueness of this dominant singularity, for $\nu \in [0, 1)$.

Nature of the singularity at $\tilde{\rho}_\nu$ for $\nu \in [0, 1)$. We have proved that $\tilde{\rho}_\nu$ is a root of $\tilde{\Delta}_2$ on this interval. Note that $\nu_* = (\nu+2)/(\nu-1)$ is then in $(-\infty, -2]$. By (5), we see that $\dot{T}_1(\nu_*, \cdot)$ has coefficients with alternating signs, and a negative dominant singularity at $s_{\nu_*} := (\nu - 1)^3\tilde{\rho}_\nu/3$. This singularity is a root of $\Delta_2(\nu_*, \cdot)$. Moreover, $\dot{K}_1(\nu, \tilde{\rho}_\nu)$ is continuous in $\nu \in [0, 1)$ by positivity of its coefficients, which implies that $\dot{T}_1(\nu_*, s_{\nu_*})$ is a continuous function of $\nu_* \in (-\infty, -2]$. We can now recycle the ingredients and calculations done at the end of the proof of Lemma 11 when ρ was a root of $\Delta_2(\nu, \cdot)$. In particular, $\dot{T}_1(\nu_*, s_{\nu_*})$ satisfies an equation of degree 9 over $\mathbb{Q}(\nu_*)$, which is also satisfied by $\dot{T}_1(\nu_*, \rho_{\nu_*})$ when $\nu_* > \nu_c$. Moreover, upon replacing ν by ν_*, the polynomial $\overline{\mathrm{Pol}}(Y, \varepsilon)$ obtained at the end of the proof of Lemma 11 vanishes for $Y = \dot{T}_1(\nu_*, w) - \dot{T}_1(\nu_*, s_{\nu_*})$ and $\varepsilon = 1 - w/s_{\nu_*}$. Recall that all points (i, j) in its support satisfy $i + 2j \geq 18$. We happily check that the coefficients of $Y^{18}\varepsilon^0$ and of $Y^0\varepsilon^9$ have no root in $(-\infty, -2]$, and conclude that the expansion of $\dot{T}_1(\nu_*, \cdot)$ at its negative dominant singularity s_{ν_*}, or equivalently of $K_1(\nu, \cdot)$ at its radius of convergence $\tilde{\rho}_\nu$, is of a square root type.

Uniqueness of the dominant singularity for $\nu \in [0, 1)$. This boils down to proving that $\dot{T}_1(\nu_*, \cdot)$ has a unique dominant singularity s_{ν_*} (which is then negative) for $\nu_* \leq -2$. Imagine there is another one, say s. Then the equations of System (27) hold for some $j \in \{0, 1, 2\}$, with $i = 2$, ν replaced by ν_* and ρ replaced by s_{ν_*}. We have already computed with `msolve` estimates of the solutions of these systems in the proof of Lemma 12. Now the solutions we are interested in should satisfy, among other conditions, the following simple ones:

$$\nu_* \leq -2, \qquad s_{\nu_*} < 0.$$

But one checks that there are no such solutions. Hence there is only one dominant singularity in $\dot{K}_1(\nu, \cdot)$.

B. The case $i > 1$. Let us now consider the case $i > 1$, with $\nu \geq 0$. The idea is to adapt the proof of Lemma 14, in a way that includes the case $\nu = 0$. In other words, we should relate the series K_i and K_1 by combinatorial constructions that do not create monochromatic edges. The key is the following bounds, explained by Figure 11:

$$[w^{n-2i+2}]K_1 \leq [w^n]K_i \leq [w^{n+2i-4}]K_1.$$

Based on this, the rest of the proof mimics the proof of Lemma 14. ∎

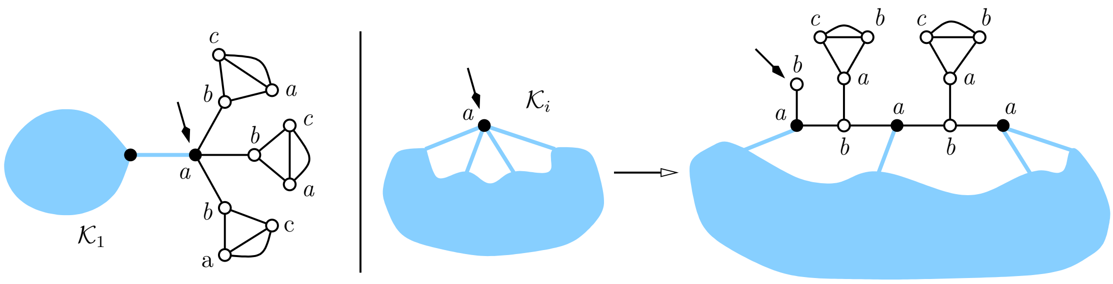


FIGURE 11. *Left.* Construction of a map of $\mathcal{K}_i$ from a map of $\mathcal{K}_1$ (here, $i = 4$). *Right.* Construction of a map of $\mathcal{K}_1$ from a map of $\mathcal{K}_i$ (with $i = 4$ again). The set of colours is $\{a, b, c\}$. No monochromatic edge is created.

5.3. Negative values of ν

It seems to be of interest, in the physics literature, to examine the position and nature of dominant singularities of the 3-Potts series also when $\nu < 0$. Indeed, this has been done on some regular lattices (see e.g. [38]). We briefly return here to the case of near-triangulations, and present, without proof, what we predict in this case, based on the minimal polynomial (1) of $\dot{T}_1$.

Here are some of the results that we have rigorously established so far:

- for $\nu \in [0, \nu_c)$, the radius of convergence ρ_ν of T_1 (or more precisely, of T_1/ν) is given by the branch of Δ_1 that equals $1/8$ at $\nu = 0$, and the series T_1/ν has a singularity in $(1 - w/\rho_\nu)^{3/2}$ near its radius (Section 5.1),
- for $\nu \le -2$, the (unique) dominant singularity s_ν of T_1 is given by the negative branch of Δ_2 that tends to 0 as ν tends to $-\infty$, say $\mathcal{B}_2'$, and the series T_1 has a singularity in $(1 - w/s_\nu)^{3/2}$ near s_ν (from Section 5.2).

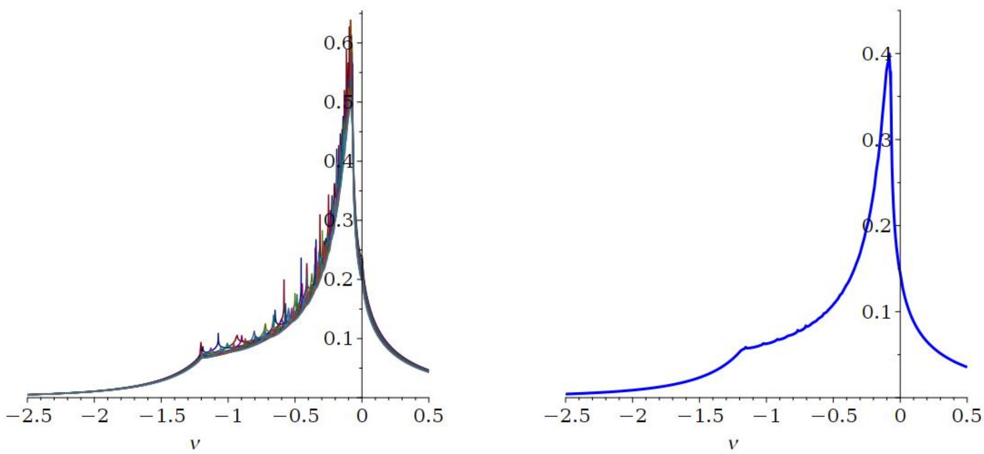


FIGURE 12. Estimates of the radius of convergence of T_1, as a function of ν, by $\liminf |[w^n]T_1|^{-1/n}$. *Left*: for $n \in [\![20, 30]\!]$ already, three phases seem to appear between -2 and 0. *Right*: for $n = 120$.

By combining exact computations for specific values of ν, completed with estimates of the singularities and exponents obtained via differential approximants [54], and studies of the branches of Δ_0, Δ_1 and Δ_2, here is what we predict for values of ν in $(-2, 0)$, starting, say, from the largest of these values; see Figure 12 for a confirmation, and our MAPLE session for details.

- **Continuity at 0**: the (positive) radius of convergence ρ_ν remains a dominant singularity, and lies on the branch $\mathcal{B}_1$ of Δ_1 that goes through $(0, 1/8)$ as long as $\nu > \nu_d := 1 - 12/\sqrt{127}$ (Figure 13, left). The singular exponent remains $3/2$.
- **A negative critical value of $\boldsymbol{\nu}$:** at $\nu_d := 1 - 12/\sqrt{127} \simeq -0.0648$, the branch $\mathcal{B}_1$ meets another branch of Δ_1 at the point (ν_d, ρ_{ν_d}), with
$$\rho_{\nu_d} = \frac{5}{4 \cdot 17^3}\left(3 \cdot 5^2 \cdot 7 + \frac{13 \cdot 43\sqrt{127}}{12}\right) \simeq 0.267.$$
For $\nu = \nu_d$, *the nature of the singularity changes, with an exponent* $4/3$ in T_1, which happens to be the exponent for the critical Ising model on planar maps (this part is rigorously proved).
- **Two dominant singularities.** On the left of the point (ν_d, ρ_{ν_d}), the two real branches of Δ_1 become a pair of complex conjugate branches. They correspond to two dominant singularities s_ν and $\overline{s_\nu}$ of T_1, resulting in oscillating coefficients. The exponent remains $1/2$. This can be checked rigorously for $\nu = -1$ for instance (Figure 13, middle).
- **Reaching the $\boldsymbol{\Delta_2}$-regime.** As ν approaches $\nu_e \simeq -1.1832$ from above, with ν_e an algebraic number of degree 435, the modulus of the complex singularities s_ν and $\overline{s_\nu}$ becomes as large as the modulus of the point of the branch $\mathcal{B}_2'$ lying at abscissa ν. Below this value, one dominant singularity of T_1 lies on the negative branch $\mathcal{B}_2'$ of Δ_2, as is the case for $\nu \leq -2$ (Figure 13, right). We do not know what the exponent is at ν_e.

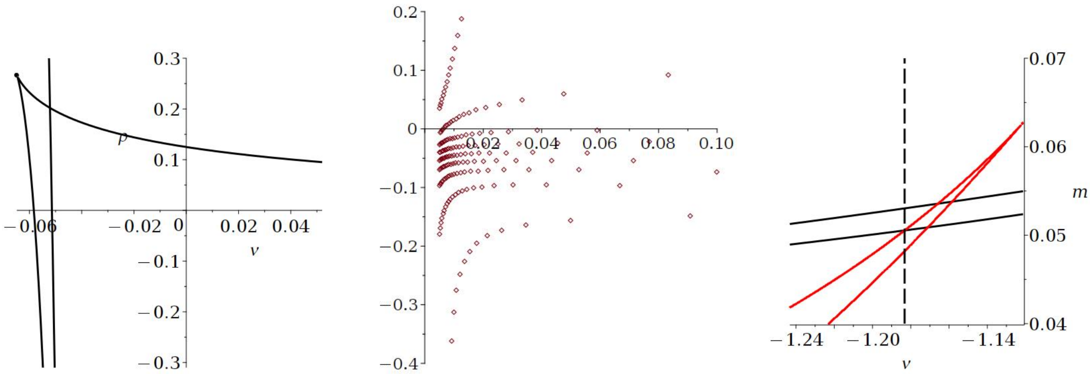


FIGURE 13. *Left.* Above $\nu_d \simeq -0.0648$, the radius (and dominant singularity) remains on the branch of Δ_1 that goes through $(0, 1/8)$. *Middle.* For $\nu = -1$, the ratios between the coefficients of w^n and w^{n+1}, plotted against $1/n$. *Right.* As ν approaches $\nu_e \simeq -1.1832$ from above, the modulus of the complex dominant singularities, which are roots of Δ_1 (bottom black curve), becomes as large as the modulus of the (negative) branch $\mathcal{B}_2'$ (top red curve).

6. Final comments

Exact results: future work. We have thus determined, more than fifteen year after it was proved to be algebraic, the 3-Potts generating function of planar triangulations, or dually of planar cubic maps. Obviously, a more combinatorial proof would be highly desirable. In a forthcoming paper [17], we address the case of general planar maps. The starting point is a counterpart of the polynomial system of Proposition 4. It is however bigger than for triangulations, and we resort to a different approach to solve it, based on evaluation-interpolation. Also, there are more parameters in this model: the number of edges is the natural size, but one can also record the number of vertices, and this gives rise to a richer singular landscape.

Asymptotic results: some missing tools? The singular analysis of the series T_1 (Section 5.1) takes almost a third of the paper. The difficulty lies in the fact that T_1 is a series in w with coefficients that depend (polynomially) on the parameter ν. It is not too hard to determine the radius of convergence ρ_ν of T_1 (Lemma 10), but the next two steps are more delicate, namely:

- determining the nature of the singularity of T_1 near ρ_ν (Lemma 11),
- proving that T_1 has no other dominant singularity (Lemma 12).

We observe that the singular exponent of T_1 at ρ_ν only changes at ν_c, being constant on $(0,+\infty)\setminus\{\nu_c\}$. Could there be a general result that would guarantee, for instance, that $(0,+\infty)$ splits as a finite union of intervals on which the exponent is constant? This would allow one to consider only one value of ν per interval, which would be much easier. For instance, assume that F is an algebraic series in $\mathbb{N}[\nu][[w]]$, with minimal polynomial $\mathrm{Pol}(z)\in\mathbb{Q}[\nu,w][z]$. Assume that the product of the leading coefficient and the discriminant of Pol factors over $\mathbb{Q}[\nu,w]$ as

$$D_0(\nu)^{e_0}D_1(\nu,w)^{e_1}\cdots D_k(\nu,w)^{e_k},$$

with the D_i's distinct and irreducible. Consider a value ν such that $D_0(\nu)\neq 0$, and assume that the radius ρ_ν lies on a branch of D_i: if $\partial_w D_i(\nu,\rho_\nu)\neq 0$ and $D_j(\nu,\rho_\nu)\neq 0$, can we say that the critical exponent is constant in a neighbourhood of ν? That is, is the exponent constant except at the finitely many points where "something special" happens in the singularity landscape?

Regarding uniqueness of the dominant singularity, we believe to have introduced new useful tools in the first proof of Lemma 12 by forming a number of polynomial conditions. It is worth noting that, in a personal communication, Linxiao Chen suggested an additional, non-obvious polynomial relation, inspired from an unpublished preprint [26, Prop. 26]. This remains to be explored. Moreover, the criterion of Proposition 13, suggested by discussions with Andrew Elvey Price, should also be applicable to other generating functions.

Acknowledgements. We are grateful to several colleagues for very informative discussions during the preparation of this paper. In particular, we thank Marie Albenque for live explanations on her paper [1] with Laurent Ménard and Gilles Schaeffer; Alin Bostan, for providing the reference [21]; Linxiao Chen for e-mail discussions on uniqueness of the dominant singularity; Andrew Elvey Prive for suggesting a combinatorial approach to the uniqueness of the dominant singularity; Mohab Safey El Din for his repeated help with `msolve`, and Tony Guttmann for his help with differential approximants; and finally Jesper Jacobsen and Andrea Sportiello for fascinating discussions on the physical (and non-physical) aspects of the Potts model.

Appendix A. The 1-catalytic equation for $T(y)$

Here is the equation in one catalytic variable (namely, y), satisfied by $T(y)$, derived from Proposition 3 by expansion near $y=0$. It involves the four series T_1,T_3,T_5 and T_7, where $T_i=[y^i]T(y)$. Below, we write $\beta:=\nu-1$:

$$
\begin{aligned}
&2916\nu^5y^{12}\boldsymbol{T(y)}^5+27\nu^3y^9\Big(\beta\left(37\nu+17\right)y^2-36\nu\left(3\nu+1\right)y+144\nu^2\Big)\boldsymbol{T(y)}^4\\
&\quad-2\nu y^6\Big(486\boldsymbol{T_1}\,\nu^4y^4-81\nu^3\left(5\nu+1\right)w\,y^4+486\nu^4w\,y^3-\left(56\nu^2+59\nu+2\right)\beta^2y^4\\
&\quad+9\nu\beta\left(38\nu^2+40\nu+3\right)y^3-9\nu^2\left(116\nu^2+11\nu-19\right)y^2+486\nu^3\left(3\nu+1\right)y-972\nu^4\Big)\boldsymbol{T(y)}^3\\
&\quad-\nu y^3\Big(18\nu^2\left(25\nu^2-23\nu-20\right)\boldsymbol{T_1}\,y^6-972\boldsymbol{T_1}\,\nu^4y^5+972\boldsymbol{T_1}\,\nu^4y^4+324\boldsymbol{T_3}\,\nu^4y^6+972\nu^4w^2y^6\\
&\quad-6\nu\beta\left(37\nu^2+40\nu+4\right)w\,y^6+54\nu^2\left(17\nu^2+3\nu-2\right)w\,y^5-216\nu^3\left(7\nu+2\right)w\,y^4+972\nu^4w\,y^3\\
&\quad-4\left(\nu+2\right)\beta^3y^6+2\left(20\nu^2+53\nu+5\right)\beta^2y^5-\beta\left(226\nu^3+276\nu^2-15\nu-1\right)y^4\\
&\quad+6\nu\left(106\nu^3+35\nu^2-64\nu-5\right)y^3-3\nu^2\left(353\nu^2+104\nu-25\right)y^2+324\nu^3\left(3\nu+1\right)y-432\nu^4\Big)\boldsymbol{T(y)}^2\\
&\quad-2\nu\Big(324\nu^4w^2y^6+108\boldsymbol{T_3}\,\nu^4y^6-324\boldsymbol{T_1}\,\nu^4y^5+2\beta^3y^7+162\boldsymbol{T_1}^2\nu^4y^8+54\boldsymbol{T_5}\,\nu^4y^8-108\boldsymbol{T_3}\,\nu^4y^7\\
&-\left(2\nu^2+10\nu+1\right)\beta^2y^6-\nu\left(37\nu^3+22\nu^2-38\nu-3\right)y^4-\nu^2\left(79\nu^2+31\nu-2\right)y^2+27\nu^3\left(5\nu+19\right)\boldsymbol{T_1}w\,y^8
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\nu\beta\left(34\nu^2+46\nu+1\right) w\,y^7 - \nu\left(17\nu-11\right)\left(13\nu^2+13\nu+1\right) w\,y^6 + 162\boldsymbol{T_1}\,\nu^4 y^4 + 162\nu^4 w\,y^3\\
&+ \left(36\nu^4-69\nu^3+9\nu^2+126\nu+6\right)\boldsymbol{T_1}\,y^8 - 6\nu^2\left(25\nu^2-23\nu-20\right)\boldsymbol{T_1}\,y^7 + 6\nu^2\left(52\nu^2-23\nu-20\right)\boldsymbol{T_1}\,y^6\\
&+ 3\nu^2\left(7\nu^2-41\nu-20\right)\boldsymbol{T_3}\,y^8 + 9\nu^2\left(13\nu^2-32\nu-17\right) w^2y^8 - 27\nu^3\left(11\nu+1\right) w^2y^7\\
&- \left(8\nu^2+28\nu+3\right)\beta^2 w\,y^8 + 6\nu^2\left(65\nu^2+20\nu-4\right) w\,y^5 - 9\nu^3\left(41\nu+13\right) w\,y^4\\
&+ \beta\left(12\nu^3+26\nu^2-10\nu-1\right) y^5 + \nu\left(68\nu^3+33\nu^2-27\nu-2\right) y^3 + 18\nu^3\left(3\nu+1\right) y - 18\nu^4\Bigr)\boldsymbol{T(y)}\\
&+ \Bigl(972\boldsymbol{T_1}\,\nu^5w^2y^7 - 324\boldsymbol{T_1T_3}\,\nu^5y^7 + 108\boldsymbol{T_1}\,\nu^5y^2 - 36\boldsymbol{T_1}\,\nu^5y - 36\nu^5w + 108\boldsymbol{T_1}^2\nu^5y^6 - 36\boldsymbol{T_7}\,\nu^5y^7\\
&- 108\boldsymbol{T_1}^2\nu^5y^5 + 36\boldsymbol{T_5}\,\nu^5y^6 - 36\boldsymbol{T_5}\,\nu^5y^5 + 72\boldsymbol{T_3}\,\nu^5y^4 - 108\nu^5w^2y^3 - 36\boldsymbol{T_3}\,\nu^5y^3\\
&+ 2\nu^2\left(73\nu^3+339\nu^2+768\nu+170\right)\boldsymbol{T_1}w\,y^7 + 18\nu^4\left(5\nu+19\right)\boldsymbol{T_1}w\,y^6 - 18\nu^4\left(5\nu+19\right)\boldsymbol{T_1}\,y^5w\\
&+ 4\nu^3\left(2\nu+1\right)\left(17\nu-20\right)\boldsymbol{T_1}\,y^4 - 18\nu^4\left(29\nu+25\right)\boldsymbol{T_3}w\,y^7 + 3\nu^3\left(71\nu^2+92\nu-55\right)\boldsymbol{T_1}^2y^7\\
&+ \left(-4\nu^5+12\nu^4+34\nu^3+98\nu^2+142\nu+6\right)\boldsymbol{T_1}\,y^7 + 2\nu\left(12\nu^4-23\nu^3+3\nu^2+42\nu+2\right)\boldsymbol{T_1}\,y^6\\
&- 2\nu\left(37\nu^4-46\nu^3-17\nu^2+42\nu+2\right)\boldsymbol{T_1}\,y^5 - 2\nu^3\left(79\nu^2-23\nu-20\right)\boldsymbol{T_1}\,y^3\\
&- 2\nu\left(23\nu^4+47\nu^3+100\nu^2+62\nu+2\right)\boldsymbol{T_3}\,y^7 + 2\nu^3\left(7\nu^2-41\nu-20\right)\boldsymbol{T_3}\,y^6 - 2\nu^3\left(25\nu^2-41\nu-20\right)\boldsymbol{T_3}\,y^5\\
&+ 2\nu^3\left(29\nu^2+59\nu+20\right)\boldsymbol{T_5}\,y^7 - 54\nu^4\left(13\nu+17\right) w^3y^7 - \nu\left(62\nu^4+132\nu^3+387\nu^2+274\nu+9\right) w^2y^7\\
&- 9\nu^3\left(19\nu^2-20\nu-11\right) y^5w^2 + 18\nu^4\left(11\nu+1\right) w^2y^4 + 4\nu\beta^3w\,y^7 - 2\nu^2\left(37\nu^3+22\nu^2-38\nu-3\right) w\,y^4\\
&+ 2\nu^2\left(68\nu^3+33\nu^2-27\nu-2\right) w\,y^3 - 2\nu^3\left(79\nu^2+31\nu-2\right) w\,y^2 + 36\nu^4\left(3\nu+1\right) wy\\
&+ 36\nu^3\left(2\nu+1\right)\left(\nu-3\right) w^2y^6 - 2\nu\left(2\nu^2+10\nu+1\right)\beta^2w\,y^6 + 2\nu\beta\left(12\nu^3+26\nu^2-10\nu-1\right) w\,y^5\Bigr) = 0.
\end{aligned}
\tag{32}
$$

Appendix B. Explicit minimal polynomials of the radii of convergence

B.1. Triangulations

Depending on the value of $\nu \geq 0$, the minimal polynomial of ρ_ν (the radius of $T_i(\nu,\cdot)$) is one of the following two polynomials, where we denote $\beta = \nu - 1$ (see Proposition 6):

$$
\begin{aligned}
\Delta_1(\nu,\rho) ={}& 5135673858195456\nu^{12}\rho^5 + 3869835264\nu^9\beta\left(23311\beta^2-94464\right)\rho^4\\
&- 221184\nu^6\left(13675471\beta^6-31778190\beta^4-2827548\beta^2+5971968\right)\rho^3\\
&- 6912\nu^3\beta\left(16987825\beta^8-94263704\beta^6+178105122\beta^4-125057520\beta^2+22523184\right)\rho^2\\
&- 32\beta^2\left(44998721\beta^{10}-354609900\beta^8+1100056473\beta^6-1675428138\beta^4+1251637596\beta^2-366996096\right)\rho\\
&- \beta(\beta-1)^3\left(6509057\beta^8-44137521\beta^6+110675808\beta^4-118226304\beta^2+45349632\right),
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\Delta_2(\nu,\rho) ={}& 173133498956120064000\nu^{23}\rho^9 - 86566749478060032\nu^{20}\beta\left(2\beta^2-525\right)\rho^8\\
&+ 2348273369088\nu^{17}\left(30157\beta^4+60528\beta^2+1674720\right)\beta^2\rho^7\\
&- 764411904\nu^{14}\beta\left(554491\beta^8-37429344\beta^6+63722112\beta^4-384466944\beta^2+298598400\right)\rho^6\\
&-42467328\nu^{11}\beta^2\left(1797719\beta^{10}-24965736\beta^8+79207569\beta^6-298516608\beta^4+421311240\beta^2-41990400\right)\rho^5\\
&- 884736\nu^8\beta^3\bigl(750262\beta^{12}-5588808\beta^{10}+11945370\beta^8-215187504\beta^6+744324669\beta^4-1061891640\beta^2\\
&+1062357120\bigr)\rho^4 - 2048\nu^5\beta^2\bigl(6476656\beta^{16}-146287584\beta^{14}+1597400568\beta^{12}-9587073168\beta^{10}\\
&+29838912921\beta^8-51034914594\beta^6+52101038457\beta^4-21850334496\beta^2-7652750400\bigr)\rho^3\\
&- 768\nu^2\beta^3\bigl(286992\beta^{18}-7681792\beta^{16}+88324784\beta^{14}-561120928\beta^{12}+2166482658\beta^{10}-5422315320\beta^8\\
&+9146245860\beta^6-9455624856\beta^4+4707143523\beta^2-637729200\bigr)\rho^2\\
&- 96\nu\beta^4(\beta-1)^2\bigl(127552\beta^{12}-3288320\beta^{10}+34397596\beta^8-192023736\beta^6+599850702\beta^4\\
&-880127532\beta^2+449067645\bigr)\rho
\end{aligned}
$$

$$+ \beta^3 (\beta-1)^4 \left(510208\beta^{10} - 11133824\beta^8 + 95834752\beta^6 - 383558976\beta^4 + 649066608\beta^2 - 358722675\right). \tag{34}$$

B.2. Cubic maps

Depending on the value of $\nu \geq 0$, the minimal polynomial of $\tilde{\rho}_\nu$ (the radius of $K_1(\nu, \cdot)$) is one of the following two polynomials, where we write $\beta = \nu - 1$ (see Proposition 15):

$$\begin{aligned}\tilde{\Delta}_1(\nu,\rho) = {} & 28991029248\,(3+\beta)^{12}\,\beta^{15}\rho^5 - 1769472\left(10496\beta^2 - 23311\right)(3+\beta)^9\,\beta^{12}\rho^4 \\ & - 8192\left(8192\beta^6 - 34908\beta^4 - 3530910\beta^2 + 13675471\right)(3+\beta)^6\,\beta^9\rho^3 \\ & - 2304\left(30896\beta^8 - 1543920\beta^6 + 19789458\beta^4 - 94263704\beta^2 + 152890425\right)(3+\beta)^3\,\beta^6\rho^2 \\ & + 864\left(55936\beta^{10} - 1716924\beta^8 + 20684298\beta^6 - 122228497\beta^4 + 354609900\beta^2 - 404988489\right)\beta^3\rho \\ & + 27\left(6912\beta^8 - 162176\beta^6 + 1366368\beta^4 - 4904169\beta^2 + 6509057\right)(-3+\beta)^3, \end{aligned} \tag{35}$$

$$\begin{aligned}\tilde{\Delta}_2(\nu,\rho) = {} & 8796093022208000\beta^{21}\,(3+\beta)^{23}\,\rho^9 + 118747255799808\left(175\beta^2 - 6\right)\beta^{18}\,(3+\beta)^{20}\,\rho^8 \\ & + 86973087744\left(186080\beta^4 + 60528\beta^2 + 271413\right)\beta^{15}\,(3+\beta)^{17}\,\rho^7 \\ & - 84934656\left(11059200\beta^8 - 128155648\beta^6 + 191166336\beta^4 - 1010592288\beta^2 + 134741313\right)\times \\ & \beta^{12}\,(3+\beta)^{14}\,\rho^6 + 382205952\left(172800\beta^{10} - 15604120\beta^8 + 99505536\beta^6 - 237622707\beta^4 + 674074872\beta^2 \right. \\ & \left. -436845717\right)\beta^9\,(3+\beta)^{11}\,\rho^5 - 71663616\left(4371840\beta^{12} - 39329320\beta^{10} + 248108223\beta^8 - 645562512\beta^6 \right. \\ & \left. +322524990\beta^4 - 1358080344\beta^2 + 1640822994\right)\beta^6\,(3+\beta)^8\,\rho^4 + 4478976\left(1166400\beta^{16} + 29973024\beta^{14} \right. \\ & -643222697\beta^{12} + 5670546066\beta^{10} - 29838912921\beta^8 + 86283658512\beta^6 - 129389446008\beta^4 \\ & \left. +106643648736\beta^2 - 42493340016\right)\beta^3\,(3+\beta)^5\,\rho^3 + 5038848\left(291600\beta^{18} - 19370961\beta^{16} + 350208328\beta^{14} \right. \\ & -3048748620\beta^{12} + 16266945960\beta^{10} - 58495031766\beta^8 + 136352385504\beta^6 - 193166302608\beta^4 \\ & \left. +151200711936\beta^2 - 50839771824\right)(3+\beta)^2\,\rho^2 - 1889568\beta\,(3+\beta)\left(616005\beta^{12} - 10865772\beta^{10} \right. \\ & \left. +66650078\beta^8 - 192023736\beta^6 + 309578364\beta^4 - 266353920\beta^2 + 92985408\right)(-3+\beta)^2\,\rho \\ & -19683\left(492075\beta^{10} - 8013168\beta^8 + 42617664\beta^6 - 95834752\beta^4 + 100204416\beta^2 - 41326848\right)(-3+\beta)^4. \end{aligned} \tag{36}$$

Appendix C. Proof of Proposition 13

Proof. First, observe that the conditions $g_{0,0} = 0$ and $g_{0,1} < 1$ imply that $U(z) = \sum_{n\geq 1} u_n z^n$ is completely characterized by (30), once G is fixed: the coefficient u_n is determined uniquely by induction on $n \geq 1$ (recall that $u_0 = 0$ by assumption). Upon replacing U by $(1-g_{0,1})U$, we could even assume that $g_{0,1} = 0$. Note also that U has non-negative coefficients since G has non-negative coefficients and $g_{0,1} < 1$.

Let us prove that the radius of convergence ρ of U is finite. Let $m \geq 0, n \geq 2$ such that $g_{m,n} > 0$, and let $k \geq 0$ such that $g_{k,0} > 0$: such a k exists, otherwise $U(z)$ would be uniformly 0, and in particular periodic. Let $V \equiv V(z)$ be the unique power series satisfying

$$V = g_{k,0} z^k + g_{m,n} z^m V^n.$$

Then V is an algebraic series, which has a finite radius of convergence. Moreover, its coefficients bound from below the coefficients of U. Hence U itself has a finite radius of convergence ρ. By Pringsheim's theorem, ρ is a singularity of U.

Let us prove that $\rho > 0$. Recall that $G(z,u)$ is analytic around $(0,0)$, that $G'_u(0,0) = g_{0,1} < 1$ and $U(0) = 0$. Then the analytic version of the implicit function theorem implies that U is analytic around 0.

Let us now prove that $U(z)$ converges at $z = \rho$. We first note that U is an increasing continuous function of z on $[0,\rho)$, with $U(0) = 0$. Moreover, there exist $m \geq 0$ and $n \geq 2$ such that on the interval $(0,\rho)$, we have $U(z) \geq g_{m,n} z^m U(z)^n$, where $g_{m,n} > 0$. In particular, for

$z > \rho/2$, we have $1 \geq g_{m,n}(\rho/2)^m U(z)^{n-1}$, hence $U(z)$ remains bounded on $[0, \rho)$, and thus converges at ρ (by non-negativity of its coefficients).

We now consider the curve of $\mathbb{R}^2_{\geq 0}$ consisting of the points $(z, U(z))$ for $0 \leq z \leq \rho$. It starts at $(0, 0)$ in $\mathcal{R}$, and then increases continuously to the point $(\rho, U(\rho))$. We study three cases, depending on the relative positions of this point and $\mathcal{R}$. Note that if $(z, u) \in \mathcal{R}$, then $[0, z] \times [0, u] \subset \mathcal{R}$. Also, $G(z, u)$ is analytic at any point of $\mathcal{R}^\circ$. Moreover, if $z_0 > 0$ and $(z_0, \varepsilon) \in \mathcal{R}^\circ$ for some $\varepsilon > 0$, then $G(z, u)$ is analytic at (z_0, u) as well.

Case 1: if $(\rho, U(\rho)) \in \mathcal{R}^\circ$, then it must be that $G'_u(\rho, U(\rho)) = 1$, otherwise the analytic version of the implicit function theorem would provide an analytic continuation of U at ρ. By Theorem VII.3 of [33], the series U has a square root singularity at ρ, and ρ is the unique dominant singularity.

Case 2: if $(\rho, U(\rho)) \in \overline{\mathcal{R}} \setminus \mathcal{R}^\circ$, then it must be that $G'_u(z, U(z)) < 1$ for $z \in [0, \rho)$. Indeed, we have $G'_u(0, U(0)) = G'_u(0, 0) = g_{0,1} < 1$, and if the above inequality did not persist on $[0, \rho)$, Theorem VII.3 of [33] would imply that U has radius of convergence less than ρ. By continuity, and non-negativity of the coefficients of G, we have $G'_u(\rho, U(\rho)) \leq 1$.

Now let $s \neq \rho$ have modulus ρ. We will prove that U has an analytic continuation at s. The aperiodicity assumption implies that $|U(s)| < U(\rho)$. By assumption on the boundary of $\mathcal{R}$, this means that $(|s|, |U(s)|) = (\rho, |U(s)|)$ lies in $\mathcal{R}^\circ \cup \{(\rho, 0)\}$. Hence $G(z, u)$ is analytic in a neighbourhood of $(|s|, |U(s)|)$, and hence, by non-negativity, in a neighbourhood of $(s, U(s))$. Moreover, $|G'_u(s, U(s))| \leq G'_u(\rho, |U(s)|) < |G'_u(\rho, U(\rho)| \leq 1$ (the strict inequality follows from the fact that $|U(s)| < U(\rho)$ and that $G'_u(z, u)$ actually depends on u). Hence, by the implicit function theorem, U has an analytic continuation at the point s, which cannot be a singularity.

Case 3: finally, it is impossible to have $(\rho, U(\rho)) \notin \overline{\mathcal{R}}$, or even $(\rho, U(\rho)) \notin \mathcal{R}$, because the series G *does* converge at $(\rho, U(\rho))$. Indeed, since U converges at ρ and all coefficients are non-negative, we have

$$\begin{aligned} U(\rho) = \sum_{n \geq 1} u_n \rho^n &= \sum_{n \geq 1} \rho^n \left(\sum_{i,j \geq 0} g_{i,j} [z^{n-i}] U(z)^j \right) \qquad \text{by (30)} \\ &= \sum_{i,j \geq 0} g_{i,j} \rho^i \left(\sum_{n \geq i} \rho^{n-i} [z^{n-i}] U(z)^j \right) \\ &= \sum_{i,j \geq 0} g_{i,j} \rho^i U(\rho)^j \qquad \text{since } U \text{ converges at } \rho. \end{aligned}$$

This concludes the proof of the proposition. ■

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