

Spectrum of pure R^2 gravity: full Hamiltonian analysis

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We perform a full Hamiltonian constraint analysis of pure Ricci-scalar-squared (R^2) gravity to clarify recent controversies regarding its particle spectrum. While it is well established that the full theory consistently propagates three degrees of freedom, we confirm that its linearised spectrum around Minkowski spacetime is empty. moreover, we show that this is not a feature unique to Minkowski spacetime, but a generic property of all traceless-Ricci spacetimes that have a vanishing Ricci scalar, such as the Schwarzschild and Kerr black hole spacetimes. The mechanism for this phenomenon is a change in the nature of the constraints upon linearisation: ten second-class constraints of the full theory become first-class, while the three momentum constraints degenerate into a single constraint. Furthermore, we show that higher order perturbation theory around these singular backgrounds reveals no degrees of freedom at any order. This is in conflict with the general analysis and points to the fact that such backgrounds are surfaces of strong coupling in field space, where the dynamics of perturbations becomes nonperturbative. We further show via a cosmological phase-space analysis that the evolving universe is able to penetrate through the singular $R = 0$ surface.

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1 Introduction

Recently some attention has been attracted by works reporting no propagating physical degrees of freedom in the linearised spectrum of perturbations in pure R^2 theory around Minkowski space. This somewhat surprising, though correct result is seemingly at odds with the generally established result that $f(R)$ theories, that includes the pure R^2 theory as a special case, propagate three degrees of freedom — a graviton and a scalar. The absence of propagating degrees of freedom around Minkowski space was first reported in [1]. This work corrected previous work [2] that reported a single scalar perturbative degree of freedom. The origin of the discrepancy between the two results was further elucidated in [3], and attributed to an incorrect implementation of the Stueckelberg trick. The result of no perturbative degrees of freedom was further confirmed in [3, 4], by attributing it to the emergence of an accidental gauge symmetry for linear perturbations around Minkowski space.¹

Despite these efforts, the explanation of the nature of this feature remains unsatisfactory. Firstly, the question of whether any other backgrounds share this property was left open. It was also suggested that there might be no degrees of freedom around Minkowski space even at the nonlinear level [1]. However, the accidental gauge symmetry present for linear perturbations is broken when nonlinearities are considered [4]. It is not clear how to reconcile these observations with the known fact that pure R^2 theory propagates three degrees of freedom, but it is at least clear that the question cannot be addressed by considering linear physics only. For this reason we perform the Hamiltonian constraint analysis in the full nonlinear theory, and investigate how the constraint structure behaves when truncated to (non-)linear perturbations around particular backgrounds.

The Hamiltonian analysis of more general quadratic gravity theories has already been performed a long time ago in [6], and our findings are completely in accord with theirs. However, we take a somewhat different route in performing the analysis, including a more convenient choice of canonical variables which allows for a transparent view of the subtleties arising in considering the theory perturbatively around particular backgrounds. It is especially the latter that allows us to understand the nature of the features reported in [1, 3, 4].

We attribute the feature of disappearing degrees of freedom to the discontinuous change of the nature of constraints when considered perturbatively around specific backgrounds. Moreover, we establish this discontinuity not to be restricted to Minkowski spacetime, but a feature of every traceless-Ricci background, $R=0$, such as Schwarzschild spacetime or Kerr spacetime, a possibility that was anticipated in [3].² We also establish that considering non-linear perturbations around such backgrounds by perturbatively expanding the solutions will never recover any degrees of freedom. However, this is not a physical conclusion, but rather a limitation of the perturbative expansion that is unable to capture physical properties. The non-linear theory without this perturbative expansion reveals three propagating degrees of freedom in the vicinity of traceless-Ricci spacetimes.

We give the canonical formulation of the theory and its constraint analysis in Sec. 2. This is used in Sec. 3 to rederive the Minkowski space result for the linearised perturbations from the perspective of the full Hamiltonian analysis. We find that perturbing around the Minkowski background alters the nature of certain constraints: ten second-class constraints become first-class, while the momentum constraint becomes longitudinal, thereby removing two first-class constraints. These two features together result in the absence of propagating degrees of freedom in the linear spectrum. We also demonstrate how this feature is preserved in perturbation theory to higher orders, implying that the vicinity of Minkowski background should not be explored by perturbative methods. In Sec. 4, we demonstrate that the strong coupling phenomenon identi-

¹Since the exchange of articles in [1–4], the question of what is propagating on Minkowski spacetime in any given tensorial field theory suddenly became very easy to answer, due to the availability of new software [5].

²Traceless Ricci spacetimes do not fall neatly into the Petrov classification [7], which restricts only the Weyl tensor, but Plebański provided the full four-dimensional classification [8, 9].

fied in Minkowski space is, in fact, a general feature of all traceless-Ricci backgrounds within the pure R^2 theory. The same change in the character of constraints observed for Minkowski space also appears in traceless-Ricci backgrounds, signaling the presence of accidental gauge symmetries, that we construct for Ricci-flat backgrounds in Appendix A. We provide several explicit examples in Sec. 4 of other singular spacetimes, before proving the singularity of general traceless-Ricci backgrounds at linear and nonlinear levels, aided by technical details summarized in Appendix B. In Sec. 5 we address spacetimes that are not eternally traceless-Ricci, exploring whether their evolution can bring them through a traceless-Ricci phase. By analyzing the phase space of cosmological spacetimes within the pure R^2 theory, we find that such a transition is indeed possible. This raises an intriguing question regarding the behavior of perturbations as the spacetime evolves through the singular surface. We further discuss this and related issues in Sec. 6, including the potential for $f(R)$ theories to exhibit analogous features.

2 Hamiltonian analysis

Determining the number of degrees of freedom in a given theory by examining the linear spectrum of perturbations around a particular background is generally an unreliable method; it only gives a lower bound on the number of degrees of freedom. The reliable answer is provided by the full Hamiltonian constraint analysis [10]. In this section we perform a full Hamiltonian constraint analysis of the theory defined by the action in (2.1) and obtain agreement with the analysis in [6]. This canonical approach allows for an unambiguous determination of the physical degrees of freedom without relying on any perturbative expansion. The first step is the Arnowitt-Deser-Misner (ADM) decomposition [11], which foliates spacetime and isolates the canonical phase space variables: the spatial metric, the extrinsic curvature, and their conjugate momenta. Following this, the complete set of primary and secondary constraints is systematically uncovered via the Dirac-Bergmann algorithm [12, 13], which requires that all constraints be preserved under time evolution. The resulting constraint algebra, formed by the Poisson brackets between all constraints, allows for their definitive classification into first-class (possibly generating gauge symmetries) and second-class (eliminating pairs of phase-space variables). This rigorous procedure not only provides a definitive count of the propagating modes in the full theory but, as we will show in Sec. 4.3, also precisely identifies the origin of the strong coupling pathology by revealing how the constraint algebra itself degenerates on the singular $R=0$ surfaces.

2.1 Overview of the theory

The pure R^2 theory, a special case of a larger class of $f(R)$ theories (see e.g. [14, 15]) with $f(R) = R^2$, is defined by its action,

$$S[g_{\mu\nu}] = \int d^4x \sqrt{-g} R^2, \quad (2.1)$$

where the Ricci scalar, $R = g^{\mu\nu} R_{\mu\nu}$, is the contraction of the Ricci tensor, $R_{\mu\nu} = \partial_\rho \Gamma_{\mu\nu}^\rho - \partial_\nu \Gamma_{\rho\mu}^\rho + \Gamma_{\mu\nu}^\rho \Gamma_{\sigma\rho}^\sigma - \Gamma_{\mu\sigma}^\rho \Gamma_{\nu\rho}^\sigma$, which in turn is defined in terms of Christoffel symbols, $\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$. Covariance is ensured by the presence of the metric determinant in the measure, $g = \det(g_{\mu\nu})$. Equations of motion for this theory are ³

$$\left(D_\mu D_\nu - g_{\mu\nu} D^\rho D_\rho - R_{\mu\nu} + \frac{1}{4} g_{\mu\nu} R \right) R = 0, \quad (2.2)$$

³We should correct the remark in [1] about the pure R^2 gravity possessing “restricted Weyl symmetry”, i.e. being invariant under a local conformal rescaling of the metric, where the conformal factor is restricted to satisfy a covariant Klein-Gordon equation. This theory, in fact, does not possess this property, since “restricted Weyl symmetry” is neither a local transformation, nor a symmetry, as it does not leave equations of motion invariant [16].

where D_μ denotes the covariant derivative compatible the metric $g_{\mu\nu}$. Note that spacetimes with a vanishing Ricci scalar, $R = 0$, automatically solve these equations. Such spacetimes are known as *traceless-Ricci* spacetimes, and will play a central role in this work, following the Hamiltonian constraint analysis that this section is devoted to. They include *Ricci-flat* spacetimes, $R_{\mu\nu} = 0$, as a special case that corresponds to vacuum solutions of Einstein's general relativity.

The theory in (2.1) is formulated in the Jordan frame, in which its equations of motion (2.2) contain higher derivatives of the metric. However, $f(R)$ theories, including pure R^2 as a special case, are more frequently considered in the Einstein frame, in which higher derivatives are traded for a scalar field, and where the metric dynamics is that of general relativity coupled to the extra scalar. But the conformal transformation connecting these two frames is singular precisely for $f'(R) = 0$, which will turn out to be points of particular interest to us. Furthermore, Einstein frame $f(R)$ theories, even though locally equivalent to their Jordan frame counterparts, are known not to be globally equivalent on account of this singularity in the transformation between them [17–19]. For this reasons we refrain from considering the theory in the Einstein frame, and work directly in the Jordan frame throughout.

2.2 ADM decomposition

The first step towards the canonical formulation of the theory in (2.1) is the ADM decomposition [11] of the metric,

$$g^{00} = -\frac{1}{N^2}, \quad g_{0i} = N_i, \quad g_{ij} = h_{ij}, \quad (2.3)$$

where ADM variables are comprised of the lapse scalar N , the shift vector N_i , and the spatial metric tensor h_{ij} induced on equal-time slices. The inverse components of the spacetime metric are then decomposed as,

$$g_{00} = -N^2 + N_i N^i, \quad g^{0i} = \frac{N^i}{N^2}, \quad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}. \quad (2.4)$$

where h^{ij} is the inverse spatial metric, $h_{ij} h^{jk} = \delta_i^k$, that is henceforth used to raise indices on ADM variables, e.g. $N^i = h^{ij} N_j$. The metric determinant also has a simple ADM decomposition, $\sqrt{-g} = N\sqrt{h}$. Note that lapse is not allowed to vanish, $N \neq 0$, in order to respect invertibility of the metric $g_{\mu\nu}$.

Apart from the metric we also need convenient variables for its time derivatives. For the first time derivative this is provided by the extrinsic curvature tensor,

$$K_{ij} = -\frac{1}{2N} \left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i \right). \quad (2.5)$$

where ∇_i is the three-dimensional covariant derivative with respect to the spatial metric h_{ij} and its corresponding Christoffel symbol, $\gamma_{ij}^k = \frac{1}{2} h^{kl} (\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij})$. Since pure R^2 theory is a higher derivative theory, we also need a convenient variable for the second time derivative of the metric. This is provided by a quantity introduced in [6],⁴

$$F_{ij} = -\frac{1}{N} \dot{K}_{ij} - K_{ik} K^k_j + \frac{N^k}{N} \nabla_k K_{ij} + \frac{2}{N} K_{k(i} \nabla_{j)} N^k - \frac{1}{N} \nabla_i \nabla_j N, \quad (2.6)$$

with the shorthand notation for its contraction, $F = h^{ij} F_{ij}$.

After some work, the ADM decomposition of the Ricci scalar follows [20, 21],⁵

$$R = 2F + K^2 - K^{ij} K_{ij} + \mathcal{R}, \quad (2.7)$$

⁴We use a shifted definition for F_{ij} compared to [6], as we find it more convenient, but this is inessential.

⁵Deriving ADM decomposition of various curvature tensors, and scalar invariants is greatly facilitated by the use of *Cadabra* [22–24], as was done in [25].

where $K = K^i_i$, and where $\mathcal{R} = h^{ij}\mathcal{R}_{ij}$ is the Ricci scalar induced on spatial slices, that is formed as a contraction of the induced Ricci tensor, $\mathcal{R}_{ij} = \partial_k \gamma_{ij}^k - \partial_j \gamma_{ki}^k + \gamma_{ij}^k \gamma_{lk}^l - \gamma_{il}^k \gamma_{jk}^l$. The well-known identity in (2.7) finally allows us to write the action (2.1) in terms of ADM variables,

$$S[N, N_i, h_{ij}] = \int d^4x N \sqrt{h} \left(2F + K^2 - K^{ij} K_{ij} + \mathcal{R} \right)^2. \quad (2.8)$$

2.3 Canonical action

The action in (2.8) is still a higher derivative action, even though it is expressed in terms of ADM variables. In order to derive the first-order (i.e. canonical) formulation, we proceed by first constructing the extended action [26]. Here time derivatives are promoted to independent velocity fields,

$$K_{ij} \longrightarrow \mathcal{K}_{ij}, \quad F_{ij} \longrightarrow \mathcal{F}_{ij}, \quad (2.9)$$

and accompanying Lagrange multipliers π_{ij} and ρ_{ij} are introduced to ensure on-shell equivalence,

$$\begin{aligned} \mathcal{S}[N, N_i, h_{ij}, \mathcal{K}_{ij}, \mathcal{F}_{ij}, \pi^{ij}, \rho^{ij}] = \int d^4x \left[N \sqrt{h} \left(2\mathcal{F} + \mathcal{K}^2 - \mathcal{K}^{ij} \mathcal{K}_{ij} + \mathcal{R} \right)^2 + \pi^{ij} \left(\dot{h}_{ij} - 2\nabla_{(i} N_{j)} \right. \right. \\ \left. \left. + 2N \mathcal{K}_{ij} \right) + \rho^{ij} \left(\dot{\mathcal{K}}_{ij} + N \mathcal{K}_{ik} \mathcal{K}_j^k - N^k \nabla_k \mathcal{K}_{ij} - 2\mathcal{K}_{k(i} \nabla_{j)} N^k + \nabla_i \nabla_j N + N \mathcal{F}_{ij} \right) \right]. \end{aligned} \quad (2.10)$$

The canonical action is now constructed from the extended one above by solving on-shell for as many components of $\mathcal{F}_{ij}$ as possible, and plugging these back into the extended action (2.10) as off-shell equalities. Here it is possible to solve only for the trace,

$$\begin{aligned} \frac{\delta \mathcal{S}}{\delta \mathcal{F}_{ij}} = 4N \sqrt{h} \left(2\mathcal{F} + \mathcal{K}^2 - \mathcal{K}^{ij} \mathcal{K}_{ij} + \mathcal{R} \right) h^{ij} + N \rho^{ij} \approx 0 \\ \implies \mathcal{F} \approx \bar{\mathcal{F}} = -\frac{1}{2} \left(\mathcal{K}^2 - \mathcal{K}^{ij} \mathcal{K}_{ij} + \mathcal{R} \right) - \frac{1}{24} \frac{\rho}{\sqrt{h}}, \end{aligned} \quad (2.11)$$

while the transverse part $\lambda_{ij} \equiv -\mathcal{F}_{ij} + \frac{1}{3} h_{ij} \mathcal{F}$ remains undetermined, and plays the role of the Lagrange multiplier. The canonical action is then written in the standard form,

$$\begin{aligned} \mathcal{S}[N, N_i, \lambda_{ij}, h_{ij}, \pi^{ij}, \mathcal{K}_{ij}, \rho^{ij}] \equiv \mathcal{S}[N, N_i, h_{ij}, \mathcal{K}_{ij}, \mathcal{F}_{ij} \rightarrow \frac{1}{3} h_{ij} \bar{\mathcal{F}} - \lambda_{ij}, \pi^{ij}, \rho^{ij}] \\ = \int d^4x \left[\pi^{ij} \dot{h}_{ij} + \rho^{ij} \dot{\mathcal{K}}_{ij} - N (\mathcal{H} + \lambda_{ij} \Phi^{ij}) - N_i \mathcal{H}^i \right], \end{aligned} \quad (2.12)$$

where the Hamiltonian and momentum constraints are, respectively,

$$\mathcal{H} = \sqrt{h} \left[\frac{1}{144} \left(\frac{\rho}{\sqrt{h}} \right)^2 - 2\mathcal{K}_{ij} \frac{\pi^{ij}}{\sqrt{h}} + \frac{1}{6} \left(\mathcal{K}^2 - \mathcal{K}^{ij} \mathcal{K}_{ij} + \mathcal{R} \right) \frac{\rho}{\sqrt{h}} - \mathcal{K}_{ik} \mathcal{K}_j^k \frac{\rho^{ij}}{\sqrt{h}} - \nabla_i \nabla_j \left(\frac{\rho^{ij}}{\sqrt{h}} \right) \right], \quad (2.13)$$

$$\mathcal{H}^i = \sqrt{h} \left[-2\nabla_j \left(\frac{\pi^{ij}}{\sqrt{h}} \right) + \frac{\rho^{kl}}{\sqrt{h}} \nabla^i \mathcal{K}_{kl} - 2\nabla^k \left(\mathcal{K}^{il} \frac{\rho_{kl}}{\sqrt{h}} \right) \right], \quad (2.14)$$

and where the primary traceless constraint,

$$\Phi^{ij} = \rho^{ij} - \frac{1}{3} h^{ij} \rho, \quad (2.15)$$

appears multiplied by its traceless Lagrange multiplier λ_{ij} . Note that this multiplier can simplify the Hamiltonian and momentum constraints by absorbing traceless parts of ρ^{ij} ,

$$\mathcal{H} \longrightarrow \sqrt{h} \left[\frac{1}{144} \left(\frac{\rho}{\sqrt{h}} \right)^2 - 2\mathcal{K}_{ij} \frac{\pi^{ij}}{\sqrt{h}} + \frac{1}{6} \left(\mathcal{K}^2 - 3\mathcal{K}^{ij} \mathcal{K}_{ij} + \mathcal{R} \right) \frac{\rho}{\sqrt{h}} - \frac{1}{3} \nabla^i \nabla_i \left(\frac{\rho}{\sqrt{h}} \right) \right], \quad (2.16)$$

$$\mathcal{H}^i \longrightarrow \sqrt{h} \left[-2\nabla_j \left(\frac{\pi^{ij}}{\sqrt{h}} \right) + \frac{1}{3} \frac{\rho}{\sqrt{h}} \nabla^i \mathcal{K} - \frac{2}{3} \nabla_j \left(\mathcal{K}^{ij} \frac{\rho}{\sqrt{h}} \right) \right]. \quad (2.17)$$

The action in (2.12) with the constraints in (2.15)–(2.17) is now the canonical formulation of the theory in (2.1).

Varying the canonical action (2.12) with respect to variables h_{ij} , π^{ij} , $\mathcal{K}_{ij}$, and ρ^{ij} generates their equations of motion,⁶

$$\dot{h}_{ij} \approx -2N\mathcal{K}_{ij} + 2\nabla_{(i}N_{j)}, \quad (2.18)$$

$$\begin{aligned} \dot{\pi}^{ij} \approx & -\frac{N}{864} \frac{\rho^2}{\sqrt{h}} h^{ij} - N\rho \left(h^{ik} h^{jl} - \frac{1}{6} h^{ij} h^{kl} \right) \left(\mathcal{K}_{km} \mathcal{K}_l^m - \frac{1}{3} \mathcal{K}_{kl} \mathcal{K} \right) + \frac{N\rho}{6} \left(\mathcal{R}^{ij} - \frac{1}{3} \mathcal{R} h^{ij} \right) \\ & - \frac{1}{6} \sqrt{h} \left(N \nabla^i \nabla^j - N h^{ij} \nabla^k \nabla_k - h^{ij} (\nabla^k N) \nabla_k \right) \frac{\rho}{\sqrt{h}} - \frac{\rho}{6} \left(\nabla^i \nabla^j - \frac{2}{3} h^{ij} \nabla^k \nabla_k \right) N \\ & + \sqrt{h} \nabla_k \left(N^k \frac{\pi^{ij}}{\sqrt{h}} - 2N^{(i} \frac{\pi^{j)k}}{\sqrt{h}} \right) + \frac{\rho}{3} \left(N^k \nabla_k \mathcal{K}^{ij} + 2\mathcal{K}^{k(i} \nabla^{j)} N_k - \frac{2}{3} h^{ij} \mathcal{K}^{kl} \nabla_k N_l \right) \\ & - \frac{2}{3} \sqrt{h} N^{(i} \nabla_k \left(\mathcal{K}^{j)k} \frac{\rho}{\sqrt{h}} \right) + \frac{\rho}{3} \left(N^{(i} \nabla^{j)} - \frac{1}{3} h^{ij} N^k \nabla_k \right) \mathcal{K} - \frac{N\rho}{3} \lambda^{ij}, \end{aligned} \quad (2.19)$$

$$\dot{\mathcal{K}}_{ij} \approx \frac{h_{ij}}{3} \left[\frac{N}{2} \left(\frac{1}{12} \frac{\rho}{\sqrt{h}} + \mathcal{K}^2 - 3\mathcal{K}^{kl} \mathcal{K}_{kl} + \mathcal{R} \right) - \nabla_k \nabla^k N + N_k \nabla^k \mathcal{K} + 2\mathcal{K}_{kl} \nabla^k N^l \right] + N\lambda_{ij}, \quad (2.20)$$

$$\dot{\rho}^{ij} \approx 2N\pi^{ij} + N\rho \left(\mathcal{K}^{ij} - \frac{h^{ij}}{3} \mathcal{K} \right) + \frac{h^{ij}}{3} \sqrt{h} \nabla_k \left(N^k \frac{\rho}{\sqrt{h}} \right) - \frac{2}{3} \rho \nabla_{(i} N_{j)}. \quad (2.21)$$

These can be written in the form of Hamilton equations,

$$\dot{h}_{ij} \approx \{h_{ij}, H_{\text{tot}}\}, \quad \dot{\pi}^{ij} \approx \{\pi^{ij}, H_{\text{tot}}\}, \quad \dot{\mathcal{K}}_{ij} \approx \{\mathcal{K}_{ij}, H_{\text{tot}}\}, \quad \dot{\rho}^{ij} \approx \{\rho^{ij}, H_{\text{tot}}\}, \quad (2.22)$$

using the total Hamiltonian,

$$H_{\text{tot}} = \int d^3x \left[N(\mathcal{H} + \lambda_{ij} \Phi^{ij}) + N_i \mathcal{H}^i \right], \quad (2.23)$$

and the canonical nonvanishing Poisson brackets,

$$\{h_{ij}(t, \vec{x}), \pi^{kl}(t, \vec{x}')\} = \delta_{(i}^k \delta_{j)}^l \delta^3(\vec{x} - \vec{x}'), \quad \{\mathcal{K}_{ij}(t, \vec{x}), \rho^{kl}(t, \vec{x}')\} = \delta_{(i}^k \delta_{j)}^l \delta^3(\vec{x} - \vec{x}'), \quad (2.24)$$

encoded in the symplectic part of the action. Varying the action with respect to Lagrange multipliers N , N_i , and λ_{ij} generates ten primary constraints,

$$\mathcal{H} \approx 0, \quad \mathcal{H}^i \approx 0, \quad \Phi^{ij} \approx 0. \quad (2.25)$$

The conservation of these constraints as the system evolves in time is the focus of the following section.

2.4 Constraint analysis

Having derived the canonical formulation and identified all the primary constraints in (2.25), we proceed with the Dirac-Bergmann algorithm [10, 12, 13] for performing constraint analysis. This algorithm provides a reliable way of counting the number N_{phy} of propagating physical degrees of freedom, corresponding to half the number of independent initial conditions needed to define the Cauchy problem. The algorithm requires the identification of all generations of

⁶Hamilton equations of motion have recently been derived for quadratic curvature theories in [27], but we cannot compare them to the ones derived here, on account of the presence of the Ricci scalar-squared term, which precludes a smooth limit in the canonical formulation.

constraints in order to identify the total number of first-class constraints N_{1st} , and second-class constraints N_{2nd} . Then the number of physical degrees of freedom is given by the formula

$$N_{\text{phy}} = \frac{1}{2} \left(N_{\text{can}} - 2N_{1st} - N_{2nd} \right), \quad (2.26)$$

where N_{can} is the number of canonical variables, not counting Lagrange multipliers.

Generations of constraints beyond the primary one are identified by requiring constraints to be conserved. For systems without explicit time dependence this is systematically inferred from the constraint algebra, best presented in terms of smeared constraints,

$$\mathcal{H}[f] \equiv \int d^3x f(x) \mathcal{H}(x), \quad \mathcal{H}^i[f_i] \equiv \int d^3x f_i(x) \mathcal{H}^i(x), \quad \Phi^{ij}[f_{ij}] \equiv \int d^3x f_{ij}(x) \Phi^{ij}(x). \quad (2.27)$$

The smearing functions f , f_i , and f_{ij} in the definitions above are strictly assumed to be independent of canonical variables, and consequently they have vanishing Poisson brackets with all quantities. We find the following on-shell algebra for primary constraints,

$$\{\mathcal{H}[f], \mathcal{H}[s]\} \approx 2\Psi^{ij} [s \nabla_i \nabla_j f - f \nabla_i \nabla_j s], \quad (2.28a)$$

$$\{\mathcal{H}[f], \mathcal{H}^i[s_i]\} \approx 4\Psi^{ij} [f \mathcal{K}_i^k \nabla_j s_k] + 2\Psi^{ij} [f s_k \nabla^k \mathcal{K}_{ij}], \quad (2.28b)$$

$$\{\mathcal{H}^i[f_i], \mathcal{H}^j[s_j]\} \approx 0, \quad (2.28c)$$

$$\{\Phi^{ij}[f_{ij}], \mathcal{H}[s]\} \approx 2\Psi^{ij} [s f_{ij}], \quad (2.28d)$$

$$\{\Phi^{ij}[f_{ij}], \mathcal{H}^k[s_k]\} \approx 0, \quad (2.28e)$$

$$\{\Phi^{ij}[f_{ij}], \Phi^{kl}[s_{kl}]\} \approx 0, \quad (2.28f)$$

where the nonvanishing quantity on the right-hand side is

$$\Psi^{ij}[f_{ij}] \equiv \int d^3x f_{ij}(x) \Psi^{ij}(x), \quad \text{where} \quad \Psi^{ij}(x) = \pi^{ij} - \frac{1}{3} h^{ij} \pi + \left(\mathcal{K}^{ij} - \frac{1}{3} h^{ij} \mathcal{K} \right) \frac{\rho}{6}. \quad (2.29)$$

The conservation of primary constraints, according to the algebra above, necessitates us to identify a secondary traceless constraint,

$$\Psi^{ij} \approx 0, \quad (2.30)$$

which implies that all the brackets between primary constraints vanish. Bracket of the secondary traceless constraint with itself vanishes,

$$\{\Psi^{ij}[f_{ij}], \Psi^{kl}[s_{kl}]\} \approx 0, \quad (2.31)$$

but one with the primary traceless constraint does not,

$$\{\Phi^{ij}[f_{ij}], \Psi^{kl}[s_{kl}]\} \approx \int d^3x f_{ij} s_{kl} \left(g^{i(k} g^{l)j} - \frac{1}{3} g^{ij} g^{kl} \right) \frac{\rho}{6}, \quad (2.32)$$

except at singular points where $\rho \approx 0$, that will be discussed in the remainder of the paper. Rather than generating further constraints, the conservation of the secondary traceless constraint determines the Lagrange multiplier on-shell,

$$\begin{aligned} \lambda_{ij} \approx \bar{\lambda}_{ij} \equiv & \left(\delta_{(i}^k \delta_{j)}^l - \frac{1}{3} h_{ij} h^{kl} \right) \left[\mathcal{R}_{kl} - 2\mathcal{K}_k^m \mathcal{K}_{ml} + \frac{2}{3} \mathcal{K} \mathcal{K}_{kl} - \frac{1}{\rho} \left(2\pi \mathcal{K}_{kl} + \nabla_k \nabla_l \rho \right) \right. \\ & \left. + \frac{1}{N} \left(2\mathcal{K}^{m(k} \nabla^{l)} N_m + N^m \nabla_m \mathcal{K}_{kl} - \nabla_k \nabla_l N \right) \right]. \end{aligned} \quad (2.33)$$

The fact that this quantity does not vanish means that the Poisson brackets between the primary Hamiltonian and momentum constraints in (2.16) and (2.17) with the secondary traceless constraint do not vanish. Superficially this might seem as though all constraints are second-class. However, that this is not the case is revealed by shifting the off-shell Lagrange multiplier by the value (2.33) it takes on-shell,

$$\lambda_{ij} \longrightarrow \lambda_{ij} + \bar{\lambda}_{ij}. \quad (2.34)$$

This changes the on-shell value of the multiplier to zero, $\lambda_{ij} \approx 0$, and modifies the Hamiltonian and momentum constraints.

$$\mathcal{H} + \left[\mathcal{R}_{ij} - 2\mathcal{K}_i^k \mathcal{K}_{kj} + \frac{2}{3}\mathcal{K}\mathcal{K}_{ij} - \frac{1}{\rho} \left(2\pi\mathcal{K}_{ij} + \nabla_i \nabla_j \rho \right) - \nabla_i \nabla_j \right] \Phi^{ij} \longrightarrow \mathcal{H}, \quad (2.35)$$

$$\mathcal{H}^i - 2\nabla^k (\mathcal{K}^{ij} \Phi_{jk}) + \Phi^{jk} \nabla^i \mathcal{K}_{jk} \longrightarrow \mathcal{H}^i. \quad (2.36)$$

In general, shifts of Lagrange multipliers effectively define different linear combinations of primary constraints.⁷ This shift in the Hamiltonian and momentum constraint does not change anything about the vanishing brackets in (2.28a)–(2.28f), but it makes the brackets with the secondary traceless constraint vanish,

$$\{\Psi^{ij}[f_{ij}], \mathcal{H}[s]\} \approx 0, \quad \{\Psi^{ij}[f_{ij}], \mathcal{H}^k[s_k]\} \approx 0. \quad (2.37)$$

thereby identifying linear combinations of primary constraints that are first-class. Thus, the total number of first-class constraints is $N_{\text{1st}} = 4$, and the total number of second class constraints is $N_{\text{2nd}} = 10$. Given that the number of canonical variables is $N_{\text{can}} = 24$, the number of physical degrees of freedom is

$$N_{\text{phy}} = \frac{1}{2} (24 - 2 \times 4 - 10) = 3, \quad (2.38)$$

which is the result obtained in [6].

It should be noted that the number of physical propagating degrees of freedom does not depend on the choice of field variables, as long as field redefinitions are invertible.⁸ This includes shifting the field variables,

$$\begin{aligned} h_{ij} &\longrightarrow \bar{h}_{ij} + \delta h_{ij}, & \pi^{ij} &\longrightarrow \bar{\pi}^{ij} + \delta \pi^{ij}, & \mathcal{K}_{ij} &\longrightarrow \bar{\mathcal{K}}_{ij} + \delta \mathcal{K}_{ij}, & \rho^{ij} &\longrightarrow \bar{\rho}^{ij} + \delta \rho^{ij}, \\ N &\longrightarrow \bar{N} + \delta N, & N_i &\longrightarrow \bar{N}_i + \delta N_i, & \lambda_{ij} &\longrightarrow \bar{\lambda}_{ij} + \delta \lambda_{ij}. \end{aligned} \quad (2.39)$$

In particular this is true for choosing the barred quantities to be solutions of the equations of motion (2.18)–(2.21) and constraints (2.25). In that case the new dynamical fields are understood to be perturbations around the background given by barred quantities. As long as we do not truncate the action for these perturbations and keep all of the terms, this shift does not change the number of degrees of freedom. However, truncating the action might easily do just that. This is in fact what happens in Minkowski space, that we examine in Sec. 3, and other critical spacetimes we discuss in Sec. 4.

Another aspect we should comment on is the step of reducing the phase space, that is in principle available after identifying and classifying all the constraints. This is accomplished by using the second-class constraints to eliminate some canonical variables off-shell. which is generally a legitimate step that reduces the dimensionality of phase space without changing physics. However, this step is not strictly necessary, and we refrain from taking it because of the subtlety with singular points in the bracket (2.32). We anticipate that this explicit reduction of phase space would introduce similar issues that the transformation between Jordan and Einstein frame does, that would interfere with our analysis of the singular points in the remainder of the paper.

⁷The change in the Hamiltonian and momentum constraints induced by shifting the Lagrange multiplier associated to another constraint is sometimes called *dressing* the Hamiltonian and momentum constraint, and is necessary to correctly identify the first-class constraints, see e.g. [28].

⁸Strictly speaking transformations should be invertible and non-singular, as invertible singular transformations can introduce new degrees of freedom [29].

3 Perturbations around Minkowski space

In this section we examine the degrees of freedom for perturbations around Minkowski space, and how they are embedded into the general picture of the Hamiltonian analysis of the preceding section. These perturbations are defined by shifts (2.39) such that the background field values are those of Minkowski space, for which the only nonvanishing ones are

$$\bar{h}_{ij} = \delta_{ij}, \quad \bar{N} = 1. \quad (3.1)$$

We first establish that *linearised* perturbations, i.e. perturbations defined by truncating the action at quadratic order after the shift in (2.39), exhibit an empty spectrum of degrees of freedom. This is because linear truncation changes the character of second-class constraints in the full theory to first-class, and furthermore removes the transverse part of the momentum constraint.

We then proceed to examine higher order perturbation theory, finding consistently an empty spectrum to an arbitrary order. Nonetheless, the correct interpretation of this observation is that perturbative expansion is not an appropriate method for probing the vicinity of Minkowski space, where three degrees of freedom propagate in a full theory.

3.1 Linear perturbations

Linear perturbations around Minkowski space are given by the quadratic canonical action,

$$\begin{aligned} \mathcal{S}_{(2)} & [\delta N, \delta N_i, \delta \lambda_{ij}, \delta h_{ij}, \delta \pi^{ij}, \delta \mathcal{K}_{ij}, \delta \rho^{ij}] \\ &= \int d^4x \left[\delta \pi^{ij} \delta \dot{h}_{ij} + \delta \rho^{ij} \delta \dot{\mathcal{K}}_{ij} - \mathcal{H}_{(2)} - \delta N \mathcal{H}_{(1)} - \delta N_i \mathcal{H}_{(1)}^i - \delta \lambda_{ij} \Phi_{(1)}^{ij} \right]. \end{aligned} \quad (3.2)$$

with linearised primary constraints given by

$$\mathcal{H}_{(1)} = -\frac{1}{3} \partial_i \partial^i \delta \rho, \quad \mathcal{H}_{(1)}^i = -2 \partial_j \delta \pi^{ij}, \quad \Phi_{(1)}^{ij} = \delta \rho^{ij} - \frac{1}{3} \delta^{ij} \delta \rho, \quad (3.3)$$

where now indices are raised and lowered by the Kronecker delta symbol. The canonical Hamiltonian density for linearised perturbations receives contributions only from the second perturbation of the Hamiltonian constraint,

$$\mathcal{H}_{(2)} = \frac{\delta \rho^2}{144} - 2 \delta \mathcal{K}_{ij} \delta \pi^{ij} + \frac{\delta \rho}{6} (\partial^i \partial^j - \delta^{ij} \partial^k \partial_k) \delta h_{ij}. \quad (3.4)$$

Requiring the conservation of primary constraints,

$$\dot{\mathcal{H}}_{(1)} \approx -\frac{2}{3} \partial^i \partial_i \delta \pi \approx 2 \partial_i \partial_j \Psi_{(1)}^{ij}, \quad \dot{\mathcal{H}}_{(1)}^i \approx 0, \quad \dot{\Phi}_{(1)}^{ij} \approx 2 \Psi_{(1)}^{ij}, \quad (3.5)$$

now generates the linearised secondary traceless constraint,

$$\Psi_{(1)}^{ij} = \delta \pi^{ij} - \frac{1}{3} \delta^{ij} \delta \pi. \quad (3.6)$$

As anticipated from the fact that $\bar{\rho}=0$ for Minkowski space, and from the general results for the Poisson brackets between traceless constraints in (2.32), the two linearised traceless constraints are found to commute,

$$\{\Phi_{(1)}^{ij}[f_{ij}], \Psi_{(1)}^{kl}[s_{kl}]\} \approx 0. \quad (3.7)$$

This makes all the constraints first-class, and the conservation of the secondary constraint generates no further constraints.

Naively speaking, we would now count $N_{1st}=14$ first-class constraints and $N_{2nd}=0$ second-class ones. Given that the number of canonical variables in our Hamiltonian formulation

is $N_{\text{can}} = 24$, this would produce a paradoxical answer of $N_{\text{phy}} = -2$ propagating degrees of freedom. Of course, this counting is not correct, and the correct counting is a little more subtle. We should notice that the linearised momentum constraint in (3.3) also changes character compared to its counterpart in the full theory. Because of the secondary traceless constraint (3.6), it is expressible as a gradient of a scalar function,

$$\mathcal{H}_{(1)}^i \longrightarrow -\frac{2}{3}\partial^i\delta\pi, \quad (3.8)$$

which means it loses its transverse part. This implies that the momentum constraint counts as a single first-class constraint instead of three.⁹ This brings the total number of first-class constraints to $N_{\text{1st}} = 12$. Therefore, for the action truncated at quadratic order, according to formula in (2.26), one counts no propagating degrees of freedom,

$$N_{\text{phy}} = \frac{1}{2}(24 - 2 \times 12 - 0) = 0. \quad (3.9)$$

Thus, we have uncovered the mechanism behind emptying the spectrum of linearised perturbations around Minkowski space in the theory that propagates three degrees of freedom otherwise. While truncating the action at higher orders than quadratic will remove this artifact, insisting on perturbation theory around Minkowski space will not uncover any degrees of freedom at any order, as we show in the following subsection.

3.2 Nonlinear perturbations

There is a difference between (i) truncating the action at some nonlinear order and then solving the equations of motion exactly, and (ii) solving the equations of motion perturbatively as a power series in small fluctuations. Provided that the truncation does not interfere with constraints, the truncated theory should remain faithful to the full one when it comes to counting the degrees of freedom. However, this approach is not directly applicable here because the truncation in the powers of fluctuations preserves diffeomorphisms only perturbatively. The latter strategy, that we consider here, is not guaranteed to remain faithful, and in fact, as we shall see, consistently yields no degrees of freedom at an arbitrary order in perturbations. This result contradicts the results of the full Hamiltonian analysis given in Sec. 2, that applies arbitrarily close to Minkowski space. Rather than being a physical property of the theory in the vicinity of Minkowski space, we should conclude that perturbation theory centered around Minkowski space is not the appropriate scheme to describe this regime of the theory. The behaviour of perturbations in this regime is essentially nonperturbative, as already pointed out in [4, 30].

The natural assumption when considering small perturbations around a particular background is to quantify their evolution as a power series organized in the powers of perturbation fields. If we append a bookkeeping parameter ε to each perturbation field in (2.39), then this means we are looking for solutions in the following power series form:

$$h_{ij} \approx \delta_{ij} + \varepsilon\delta\bar{h}_{ij}^1 + \varepsilon^2\delta\bar{h}_{ij}^2 + \varepsilon^3\delta\bar{h}_{ij}^3 + \dots, \quad (3.10a)$$

$$\pi^{ij} \approx 0 + \varepsilon\delta\pi^{ij} + \varepsilon\delta\bar{\pi}_1^{ij} + \varepsilon^2\delta\bar{\pi}_2^{ij} + \varepsilon^3\delta\bar{\pi}_3^{ij} + \dots, \quad (3.10b)$$

$$\mathcal{K}_{ij} \approx 0 + \varepsilon\delta\mathcal{K}_{ij} + \varepsilon\delta\bar{\mathcal{K}}_1^{ij} + \varepsilon^2\delta\bar{\mathcal{K}}_2^{ij} + \varepsilon^3\delta\bar{\mathcal{K}}_3^{ij} + \dots, \quad (3.10c)$$

$$\rho^{ij} \approx 0 + \varepsilon\delta\rho^{ij} + \varepsilon\delta\bar{\rho}_1^{ij} + \varepsilon^2\delta\bar{\rho}_2^{ij} + \varepsilon^3\delta\bar{\rho}_3^{ij} + \dots, \quad (3.10d)$$

$$N \approx 1 + \varepsilon\delta\bar{N}_1 + \varepsilon^2\delta\bar{N}_2 + \varepsilon^3\delta\bar{N}_3 + \dots, \quad (3.10e)$$

$$N_i \approx 0 + \varepsilon\delta\bar{N}_1^i + \varepsilon^2\delta\bar{N}_2^i + \varepsilon^3\delta\bar{N}_3^i + \dots, \quad (3.10f)$$

$$\lambda_{ij} \approx 0 + \varepsilon\delta\bar{\lambda}_{ij}^1 + \varepsilon^2\delta\bar{\lambda}_{ij}^2 + \varepsilon^3\delta\bar{\lambda}_{ij}^3 + \dots, \quad (3.10g)$$

⁹Strictly speaking the zero mode remains undetermined by the longitudinal momentum constraint, but here we consider only normalizable modes.

where there is no ε dependence except for the explicitly indicated. Organizing the theory in powers of ε we can derive the action, and consequently dynamical equations, for each order of the perturbative correction.

In Sec. 3.1 we found the solutions for the first order: there are no propagating degrees of freedom, two canonical momenta are found to vanish,

$$\delta\bar{\pi}_1^{ij} = 0, \quad \delta\bar{\rho}_1^{ij} = 0, \quad (3.11)$$

and all the other first order perturbations are undetermined, and depend on the gauge choice. This, together with the background fields, serves as a starting point for determining the properties of higher order perturbations.

Second order perturbation. We proceed to find the second order perturbation by shifting the fields in the original action by the background plus first order solution,

$$\begin{aligned} h_{ij} &\longrightarrow \delta_{ij} + \varepsilon\delta\bar{h}_{ij}^1 + \varepsilon^2\delta h_{ij}, & \pi^{ij} &\longrightarrow \varepsilon^2\delta\pi^{ij}, & \mathcal{K}_{ij} &\longrightarrow \varepsilon\delta\bar{\mathcal{K}}_{ij}^1 + \varepsilon^2\delta\mathcal{K}_{ij}, & \rho^{ij} &\longrightarrow \varepsilon^2\delta\rho^{ij}, \\ N &\longrightarrow 1 + \varepsilon\delta\bar{N}_1 + \varepsilon^2\delta N, & N_i &\longrightarrow \varepsilon\delta\bar{N}_i^1 + \varepsilon^2\delta N_i, & \lambda_{ij} &\longrightarrow \varepsilon\delta\bar{\lambda}_{ij}^1 + \varepsilon^2\delta\lambda_{ij}. \end{aligned} \quad (3.12)$$

Solving for the ε -independent part of the new dynamical variables will then determine the second order perturbation. The equations of motion at this order are encoded in the lowest order shifted action, that is obtained by plugging in (3.12) into (2.12) and keeping only relevant terms,

$$\begin{aligned} \mathcal{S}_2[\delta N, \delta N_i, \delta\lambda_{ij}, \delta h_{ij}, \delta\pi^{ij}, \delta\mathcal{K}_{ij}, \delta\rho^{ij}] \\ = \varepsilon^4 \int d^4x \left[\delta\pi^{ij}\delta\dot{h}_{ij} + \delta\rho^{ij}\delta\dot{\mathcal{K}}_{ij} - \mathcal{H}_2 - \delta N\mathcal{H}_{(1)} - \delta N_i\mathcal{H}_{(1)}^i - \delta\lambda_{ij}\Phi_{(1)}^{ij} \right] + \mathcal{O}(\varepsilon^5). \end{aligned} \quad (3.13)$$

We see that the dynamics of second order perturbations is given by the quadratic action. The primary constraints are linear, the same ones as for linear perturbations in (3.3). The quadratic part of the Hamiltonian density, $\mathcal{H}_2 = \mathcal{H}_{(2)} + \Delta\mathcal{H}_2$, is also the same as for first order perturbations in (3.4), but in addition a linear part appears,

$$\begin{aligned} \Delta\mathcal{H}_2 = \frac{1}{6} \Big[& (\delta\bar{\mathcal{K}}_1)^2 - 3\delta\bar{\mathcal{K}}_1^{ij}\delta\bar{\mathcal{K}}_{ij}^1 - 2\partial_i(\delta\bar{h}_1^{ij}\partial^k\delta\bar{h}_{jk}^1) + \partial_i(\delta\bar{h}_1^{ij}\partial_j\delta\bar{h}_1) + \partial^k(\delta\bar{h}_1^{ij}\partial_k\delta\bar{h}_{ij}^1) \\ & + (\partial_i\delta\bar{h}_1^{ik})(\partial^j\delta\bar{h}_{jk}^1) - \frac{1}{2}(\partial^i\delta\bar{h}_1^{jk})(\partial_j\delta\bar{h}_{ik}^1) - \frac{1}{4}(\partial^i\delta\bar{h}_1)(\partial_i\delta\bar{h}_1) - \frac{1}{4}(\partial^k\delta\bar{h}_1^{ij})(\partial_k\delta\bar{h}_{ij}^1) \\ & + \frac{1}{3}(3\delta\bar{N}_1 + \delta\bar{h}_1)(\partial^i\partial^j\delta\bar{h}_{ij}^1 - \partial^k\partial_k\delta\bar{h}_1) + (2\partial^i\delta\bar{h}_{ik}^1 - \partial_k\delta\bar{h}_1)\partial^k\delta\bar{N}_1 \\ & + 2(\delta\bar{h}_{ij}^1\partial^i\partial^j\delta\bar{N}_1 - \frac{1}{3}\delta\bar{h}_1\partial^i\partial_i\delta\bar{N}_1) + 2\delta\bar{N}_i^1\partial^i\delta\bar{\mathcal{K}}_1 + 4\delta\bar{\mathcal{K}}_1^{ij}\partial_i\delta\bar{N}_j^1 \Big] \delta\rho \\ & - \delta\bar{N}_1^k(2\partial_i\delta\bar{h}_{jk}^1 - \partial_k\delta\bar{h}_{ij}^1)\delta\pi^{ij}. \end{aligned} \quad (3.14)$$

Apart from this linear part in the Hamiltonian density, the canonical action second order perturbations is exactly the same as for the first order perturbations (3.2).

While the linear part (3.14) influences the equations of motion for δh_{ij} and $\delta\mathcal{K}_{ij}$, it does not change the structure of constraints. Namely, the conservation of primary constraints generates the same secondary traceless constraint (3.6) as found at linear order. The additional linear piece (3.14) does not influence the dynamics of the secondary constraint either, and its conservation generates no further constraints. Thus, we end up with the same first-class constraints as for linear order, where the momentum constraint loses its transverse part in the same manner. Therefore, at second order in perturbations we again get that the two canonical momenta vanish,

$$\delta\bar{\rho}_2^{ij} = 0, \quad \delta\bar{\pi}_2^{ij} = 0, \quad (3.15)$$

and all the other dynamical variables remain undetermined and dependent on the gauge.

Third order perturbation. The third order perturbation is found by first shifting the fields by the background solution and solutions for the first two orders of perturbations,

$$\begin{aligned} h_{ij} &\longrightarrow \delta_{ij} + \varepsilon \delta \bar{h}_{ij}^1 + \varepsilon^2 \delta \bar{h}_{ij}^2 + \varepsilon^3 \delta h_{ij}, & \pi^{ij} &\longrightarrow \varepsilon^3 \delta \pi^{ij}, & \mathcal{K}_{ij} &\longrightarrow \varepsilon \delta \bar{\mathcal{K}}_{ij}^1 + \varepsilon^2 \delta \bar{\mathcal{K}}_{ij}^2 + \varepsilon^3 \delta \mathcal{K}_{ij}, \\ \rho^{ij} &\longrightarrow \varepsilon^3 \delta \rho^{ij}, & N &\longrightarrow 1 + \varepsilon \delta \bar{N}_1 + \varepsilon^2 \delta \bar{N}_2 + \varepsilon^3 \delta N, & N_i &\longrightarrow \varepsilon \delta \bar{N}_i^1 + \varepsilon^2 \delta \bar{N}_i^2 + \varepsilon^3 \delta N_i, \\ \lambda_{ij} &\longrightarrow \varepsilon \delta \bar{\lambda}_{ij}^1 + \varepsilon^2 \delta \bar{\lambda}_{ij}^2 + \varepsilon^3 \delta \lambda_{ij}. \end{aligned} \quad (3.16)$$

Plugging these shifted fields into the action produces the action for cubic perturbations,

$$\begin{aligned} \mathcal{S}_3[\delta N, \delta N_i, \delta \lambda_{ij}, \delta h_{ij}, \delta \pi^{ij}, \delta \mathcal{K}_{ij}, \delta \rho^{ij}] \\ = \varepsilon^6 \int d^4x \left[\delta \pi^{ij} \delta \dot{h}_{ij} + \delta \rho^{ij} \delta \dot{\mathcal{K}}_{ij} - \mathcal{H}_3 - \delta N \mathcal{H}_{(1)} - \delta N_i \mathcal{H}_{(1)}^i - \delta \lambda_{ij} \Phi_{(1)}^{ij} \right] + \mathcal{O}(\varepsilon^7). \end{aligned} \quad (3.17)$$

The primary constraints are once more unchanged, and so is the quadratic part of the Hamiltonian, $\mathcal{H}_3 = \mathcal{H}_{(2)} + \Delta \mathcal{H}_3$. The only updated part compared to the previous lower order is the linear part of the Hamiltonian density, that takes the same form as at the quadratic level,

$$\Delta \mathcal{H}_3 = A_3 \delta \rho + B_{ij}^3 \delta \pi^{ij}, \quad (3.18)$$

only with updated coefficients A_3 and B_{ij}^3 that depend on the solutions of lower orders. The concrete expressions for these coefficients are immaterial for the dynamics of constraints. We obtain the same constraint structure as for the linear order, implying again that momenta vanish,

$$\delta \bar{\rho}_2^{ij} = 0, \quad \delta \bar{\pi}_2^{ij} = 0, \quad (3.19)$$

and that other dynamical variables are undetermined.

n -th order perturbation. The pattern of behaviour observed for lowe perturbative orders continues to higher orders. It is proven in Appendix B that the quadrati part of the action for the n -th order perturbation has the same form at every order, just as at first order. The part of the action that is updated at each successive order is only the linear part, which here contributes to the Hamiltonian density only.

Here in addition we observe another pattern, which is the vanishing of the perturbation of the canonical momenta ρ^{ij} and π^{ij} at each order. It then follows that, after plugging in the variables shifted by the solutions for first $(n-1)$ orders, the action for the n -th order perturbation reads

$$\begin{aligned} \mathcal{S}^n[\delta N, \delta N_i, \delta \lambda_{ij}, \delta h_{ij}, \delta \pi^{ij}, \delta \mathcal{K}_{ij}, \delta \rho^{ij}] \\ = \varepsilon^{2n} \int d^4x \left[\delta \pi^{ij} \delta \dot{h}_{ij} + \delta \rho^{ij} \delta \dot{\mathcal{K}}_{ij} - \mathcal{H}_n - \delta N \mathcal{H}_{(1)} - \delta N_i \mathcal{H}_{(1)}^i - \delta \lambda_{ij} \Phi_{(1)}^{ij} \right] + \mathcal{O}(\varepsilon^{2n+1}). \end{aligned} \quad (3.20)$$

The Hamiltonian density always has the same quadratic part, $\mathcal{H}_n = \mathcal{H}_{(2)} + \Delta \mathcal{H}_n$, but the linear part changes from order to order,

$$\mathcal{H}_n = A_n \delta \rho + B_{ij}^n \delta \pi^{ij}, \quad (3.21)$$

where its form remains the same, and just the coefficients A_n and B_{ij}^n get updated. It is straightforward to prove by induction that this form of the Hamiltonian density is true at every order. It is then easy to see that the linear part of the Hamiltonian density does not participate in determining the conservation of constraints. That is why the conservation of primary constraints at n -th order generate a secondary constraint $\Psi_{(1)}^{ij}$, such that all constraints are first-class, and that the momentum constraint becomes longitudinal.

Implications and interpretation. The perturbative expansion of small perturbations around Minkowski space consistently produces the same constraint structure at each order, and yields an empty spectrum of propagating degrees of freedom. It might seem that this implies a discontinuous jump in the number of degrees of freedom when we approach close to Minkowski space. However, this conclusion would contradict the general result of the constraint analysis in Sec. 2 that is valid arbitrarily close to Minkowski space. The resolution of this seeming paradox is found upon closer examination of the perturbative analysis. The relevant fact is the vanishing of ρ to all orders, which is equivalent to the vanishing Ricci scalar at all orders. What this means is that perturbation theory does not admit small variations of the Ricci scalar, and forces us to exact Minkowski background at each order. This means that expansion in small perturbations is not appropriate to probe the vicinity of Minkowski space, where the dynamics of propagating degrees of freedom becomes nonperturbative (i.e. perturbations become strongly coupled), and is missed by the perturbative expansion centered on Minkowski background. Rather, more sophisticated analytic methods are called for to analyse this regime.

4 Perturbations around other singular backgrounds

The results of the Hamiltonian analysis of Sec. 2 suggests that Minkowski spacetime is not an isolated background exhibiting a strong coupling feature for linearised perturbations. Rather, we expect those to be all backgrounds for which the bracket (2.32) vanishes, signaling the change of second-class constraints to first-class. In this section we first provide a couple of examples of such spacetimes, and then show that all traceless-Ricci backgrounds, i.e. backgrounds with a vanishing Ricci scalar, exhibit the same strong coupling feature.

4.1 Schwarzschild spacetime

Spherically symmetric and static black hole spacetime is a vacuum solution of Einstein's general relativity, and is thus a Ricci-flat spacetime. It is consequently also a solution of the pure R^2 theory, where, as we show, it possesses an empty spectrum of linearised perturbations. Schwarzschild spacetime is described by the line element

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2), \quad f(r) = 1 - \frac{r_s}{r}, \quad (4.1)$$

where $r_s = 2G_N M$ is the Schwarzschild radius, M is the black hole mass, and G_N is the Newton constant. The lapse and shift variables of the ADM decomposition for the Schwarzschild metric are read off from this diagonal line element as $\bar{N} = \sqrt{f(r)}$ and $\bar{N}_i = 0$, and the induced spatial metric is diagonal and time-independent, as inferred from the line element on equal time hypersurfaces,

$$d\ell^2 = \bar{h}_{ij} dx^i dx^j = \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2). \quad (4.2)$$

It follows that the extrinsic curvature vanishes, $\bar{\mathcal{K}}_{ij} = 0$, and consequently from Eq. (2.20) that $\bar{\lambda}_{ij} = 0$. Then it follows from vanishing of the Ricci scalar $\bar{R} = 0$ from Eqs. (2.6) and (2.7) that

$$\bar{\mathcal{R}}_{ij} = \frac{2}{\bar{N}} \bar{\nabla}_i \bar{\nabla}_j \bar{N}. \quad (4.3)$$

The trace of Eq. (2.19), together with the primary traceless constraint, then tells us that $\bar{\rho}^{ij} = 0$, and subsequently Eq. (2.18) implies $\bar{\pi}^{ij} = 0$.

The canonical action for linear perturbations around Schwarzschild spacetime reads

$$\mathcal{S}_{(2)}[\delta N, \delta N_i, \delta \lambda_{ij}, \delta h_{ij}, \delta \pi^{ij}, \delta \mathcal{K}_{ij}, \delta \rho^{ij}]$$

$$= \int d^4x \left[\delta\pi^{ij} \delta\dot{g}_{ij} + \delta\rho^{ij} \delta\dot{\mathcal{K}}_{ij} - \mathcal{H}_{(2)} - \delta N \mathcal{H}_{(1)} - \delta N_i \mathcal{H}_{(1)}^i - \delta\lambda_{ij} \Phi_{(1)}^{ij} \right], \quad (4.4)$$

where only the second perturbation of the Hamiltonian constraint contributes to the quadratic canonical Hamiltonian density,

$$\mathcal{H}_{(2)} = \bar{N} \sqrt{\bar{h}} \left[\frac{1}{144} \left(\frac{\delta\rho}{\sqrt{\bar{h}}} \right)^2 - 2\delta\mathcal{K}_{ij} \frac{\delta\pi^{ij}}{\sqrt{\bar{h}}} + \frac{1}{6} (\bar{\nabla}_i \delta h) \bar{\nabla}^i \left(\frac{\delta\rho}{\sqrt{\bar{h}}} \right) + \frac{1}{6} \delta h_{ij} \left(\bar{\nabla}^i \bar{\nabla}^j - \frac{1}{2} \bar{\mathcal{R}}^{ij} \right) \frac{\delta\rho}{\sqrt{\bar{h}}} \right], \quad (4.5)$$

where $\delta h = \bar{h}^{ij} \delta h_{ij}$, and where $\bar{N} \sqrt{\bar{h}} = r^2 \sin(\theta)$ is the volume Jacobian of the spherical coordinate system. The linearised primary constraints are given by

$$\mathcal{H}_{(1)} = -\frac{1}{3} \sqrt{\bar{h}} \left(\bar{\nabla}^i \bar{\nabla}_i - \frac{1}{2} \bar{\mathcal{R}} \right) \frac{\delta\rho}{\sqrt{\bar{h}}}, \quad \mathcal{H}_{(1)}^i = -2\sqrt{\bar{h}} \bar{\nabla}_j \left(\frac{\delta\pi^{ij}}{\sqrt{\bar{h}}} \right), \quad \Phi_{(1)}^{ij} = \delta\rho^{ij} - \frac{1}{3} \bar{h}^{ij} \delta\rho. \quad (4.6)$$

All primary constraints are mutually first-class, and their conservation,

$$\dot{\mathcal{H}}_{(1)} = -\frac{2}{3} \sqrt{\bar{h}} \left[\bar{N} \bar{\nabla}^i \bar{\nabla}_i + 2(\bar{\nabla}^i \bar{N}) \bar{\nabla}_i \right] \frac{\delta\pi}{\sqrt{\bar{h}}} \approx 2\sqrt{\bar{h}} \left[\bar{N} \bar{\nabla}_i \bar{\nabla}_j + 2(\bar{\nabla}_i \bar{N}) \bar{\nabla}_j \right] \frac{\Psi_{(1)}^{ij}}{\sqrt{\bar{h}}}, \quad (4.7a)$$

$$\dot{\mathcal{H}}_{(1)}^i \approx 0, \quad \dot{\Phi}_{(1)}^{ij} \approx 2\bar{N} \Psi_{(1)}^{ij} \quad (4.7b)$$

generates a secondary traceless constraint,

$$\Psi_{(1)}^{ij} = \delta\pi^{ij} - \frac{1}{3} \bar{h}^{ij} \delta\pi. \quad (4.8)$$

The conservation of this secondary constraint generates no further constraints. Moreover, it is first-class with all the primary constraints. including the traceless one,

$$\{ \Phi_{(1)}^{ij} [f_{ij}], \Psi_{(1)}^{kl} [s_{kl}] \} \approx 0. \quad (4.9)$$

This makes all constraints first-class; the naive count gives $N_{1st} = 14$ of them. However, the secondary constraint removes the transverse part of the momentum constraint in (4.6),

$$\mathcal{H}_{(1)}^i \approx -\frac{2}{3} \sqrt{\bar{h}} \bar{\nabla}^i \left(\frac{\delta\pi}{\sqrt{\bar{h}}} \right), \quad (4.10)$$

reducing the number of first-class constraints to $N_{1st} = 12$. This brings us to the conclude there are no linear degrees of freedom around Schwarzschild black hole spacetime, owing to the same mechanism observed for linear perturbations around Minkowski space. This result contradicts previous work [31] that reported the stability analysis of a linearised scalar degree of freedom in pure R^2 theory around the Schwarzschild black hole.

4.2 Radiation-dominated cosmological spacetime

Homogeneous, isotropic, and spatially flat expanding cosmological spacetime is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) line element,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\vec{x}^2, \quad (4.11)$$

where the scale factor $a(t)$ encodes the dynamics of the expansion. It is not a vacuum solution of Einstein's general relativity, and in general its Ricci tensor does not vanish. In the special case when the expansion is sourced by conformal matter, the Ricci tensor is traceless. This is the case of radiation-dominated universe, where the Hubble rate, $H = \dot{a}/a$, satisfies $\dot{H} = -2H^2$. Such a spacetime is indeed a vacuum solution of the pure R^2 theory, on account of its vanishing Ricci scalar.

The background ADM variables (2.39) for this spacetime are inferred from the line element (4.11), the canonical equations of motion (2.18)–(2.21), and constraints (2.25) and (2.30),

$$\bar{N} = 1, \quad \bar{N}_i = 0, \quad \bar{h}_{ij} = a^2 \delta_{ij}, \quad \bar{\pi}^{ij} = 0, \quad \bar{\mathcal{K}}_{ij} = -H \bar{h}_{ij}, \quad \bar{\rho}^{ij} = 0, \quad \bar{\lambda}_{ij} = 0. \quad (4.12)$$

The canonical action for linearised perturbations around this background is then given by

$$\begin{aligned} \mathcal{S}_{(2)}[\delta N, \delta N_i, \delta \lambda_{ij}, \delta h_{ij}, \delta \pi^{ij}, \delta \mathcal{K}_{ij}, \delta \rho^{ij}] \\ = \int d^4x \left[\delta \pi^{ij} \delta \dot{h}_{ij} + \delta \rho^{ij} \delta \dot{\mathcal{K}}_{ij} - \mathcal{H}_{(2)} - \delta N \mathcal{H}_{(1)} - \delta N_i \mathcal{H}_{(1)}^i - \delta \lambda_{ij} \Phi_{(1)}^{ij} \right]. \end{aligned} \quad (4.13)$$

where the linearised constraints are given by

$$\mathcal{H}_{(1)} = -\frac{1}{3} \partial^i \partial_i \delta \rho + 2H a^2 \delta \pi, \quad \mathcal{H}_{(1)}^i = -2 \partial_j \delta \pi^{ij} + \frac{2}{3} H \partial^i \delta \rho, \quad \Phi_{(1)}^{ij} = \delta \rho^{ij} - \frac{1}{3} \delta^{ij} \delta \rho. \quad (4.14)$$

Note that in this subsection the indices are raised and lowered by the Kronecker delta symbol, and traces are defined accordingly, $\delta \pi = \delta_{ij} \delta \pi^{ij}$, $\delta \rho = \delta_{ij} \delta \rho^{ij}$. The quadratic Hamiltonian density receives contribution from the second perturbation of the Hamiltonian constraint only,

$$\mathcal{H}_{(2)} = \frac{a}{144} \delta \rho^2 - 2 \delta \mathcal{K}_{ij} \delta \pi^{ij} + \frac{1}{6a^2} \delta \rho (\partial^i \partial^j - \delta^{ij} \partial^k \partial_k) \delta h_{ij}, \quad (4.15)$$

where we have discarded total derivatives.

All primary constraints are mutually first-class. Their conservation,

$$\dot{\mathcal{H}}_{(1)} \approx -\frac{2}{3} \partial^i \partial_i (\delta \pi - H \delta \rho) \approx 2 \partial_i \partial_j \Psi_{(1)}^{ij}, \quad \dot{\mathcal{H}}_{(1)}^i \approx -4H \partial_j \Psi_{(1)}^{ij}, \quad \dot{\Phi}_{(1)}^{ij} \approx 2 \Psi_{(1)}^{ij}, \quad (4.16)$$

generates a secondary traceless constraint,

$$\Psi_{(1)}^{ij} = \delta \pi^{ij} - \frac{1}{3} \delta^{ij} \delta \pi. \quad (4.17)$$

This constraint is also first-class with all others, and its conservation generates no further constraints. Furthermore, this secondary constraint makes the momentum constraint longitudinal,

$$\mathcal{H}_{(1)}^i \approx -\frac{2}{3} \partial^i (\delta \pi - H \delta \rho), \quad (4.18)$$

meaning it should really be interpreted as a scalar constraint $\delta \pi \approx H \delta \rho$, which then turns the Hamiltonian constraint into

$$\mathcal{H}_{(1)} \approx -\frac{1}{3} (\partial^i \partial_i - a^2 H^2) \delta \rho \approx 0. \quad (4.19)$$

This constraint takes the form of the modified Helmholtz equation with a time-dependent mass, and is equivalent to $\delta \rho \approx 0$, which then implies $\delta \pi \approx 0$. The feature of losing two constraints from the momentum constraint brings the total number of first-class constraints to $N_{1st} = 12$, which implies no propagating degrees of freedom in the linearised spectrum.

4.3 General traceless-Ricci spacetimes

The two examples of preceding subsections support what is already clear from the bracket (2.32) between primary and secondary traceless constraints: singular behaviour is not innate to perturbations around Minkowski space background, but is expected whenever $\bar{\rho}$ vanishes. Together with the constraint in (2.15) that is always valid, which implies singular points will have a vanishing canonical momentum associated to extrinsic curvature,

$$\bar{\rho}^{ij} = 0. \quad (4.20)$$

By taking the trace of Eq. (2.20) and inspecting Eqs. (2.6) and (2.7),

$$\frac{\rho}{\sqrt{h}} \approx -12 \left(2F + K^2 - K_{ij} K^{ij} + \mathcal{R} \right) = -12R, \quad (4.21)$$

it is possible to interpret such spacetime points as those for which the covariant Ricci scalar vanishes, $\bar{R}=0$.

While we expect the singular features for perturbations to appear locally where this happens, the local characterization of the constraint structure would require a more detailed analysis. Here we consider general traceless-Ricci spacetimes, for which (4.20) is true everywhere and for all times. For such spacetimes combining the condition in (4.20) with equation of motion (2.21) implies vanishing of the canonical momentum associated to the spatial metric,

$$\bar{\pi}^{ij} = 0. \quad (4.22)$$

The rest of the variables are not constrained, apart from having to satisfy the equations of motion that remain from (2.18)–(2.19),

$$\dot{\bar{h}}_{ij} \approx -2\bar{N}\bar{\mathcal{K}}_{ij} + 2\bar{\nabla}_{(i}\bar{N}_{j)}, \quad (4.23)$$

$$\dot{\bar{\mathcal{K}}}_{ij} \approx \frac{\bar{h}_{ij}}{3} \left[\frac{\bar{N}}{2} \left(\bar{\mathcal{K}}^2 - 3\bar{\mathcal{K}}^{kl}\bar{\mathcal{K}}_{kl} + \bar{\mathcal{R}} \right) - \bar{\nabla}^k \bar{\nabla}_k \bar{N} + \bar{N}^k \bar{\nabla}_k \bar{\mathcal{K}} + 2\bar{\mathcal{K}}^{kl} \bar{\nabla}_k \bar{N}_l \right] + \bar{N} \bar{\lambda}_{ij}. \quad (4.24)$$

The three spacetimes considered in sections 3.1, 4.1, and 4.2 are just special instances of traceless-Ricci spacetimes, with the first one being Ricci-flat. Some further examples [32–34] of exact Ricci-flat spacetimes are Kerr spacetime (and Taub-NUT spacetime that generalizes it), Kasner spacetime, and pp-wave spacetime; all of them are exact vacuum solutions of pure R^2 theory and exhibit an empty spectrum of linearised perturbations.

Canonical structure of linearised perturbations. Shifting the variables as in (2.39), and retaining only quadratic terms in the canonical action gives

$$\begin{aligned} \mathcal{S}_{(2)} [\delta N, \delta N_i, \delta \lambda_{ij}, \delta h_{ij}, \delta \pi^{ij}, \delta \mathcal{K}_{ij}, \delta \rho^{ij}] \\ = \int d^4x \left[\delta \pi^{ij} \delta \dot{h}_{ij} + \delta \rho^{ij} \delta \dot{\mathcal{K}}_{ij} - \mathcal{H}_{(2)} - \delta N \mathcal{H}_{(1)} - \delta N_i \mathcal{H}_{(1)}^i - \delta \lambda_{ij} \Phi_{(1)}^{ij} \right], \end{aligned} \quad (4.25)$$

where throughout we tacitly shifted and rescaled the perturbation of the Lagrange multiplier $\delta \lambda_{ij}$ to absorb all the traceless parts of $\delta \rho^{ij}$. The linearised Hamiltonian and momentum constraints are now given by

$$\mathcal{H}_{(1)} = \sqrt{\bar{h}} \left[-2\bar{\mathcal{K}}_{ij} \frac{\delta \pi^{ij}}{\sqrt{\bar{h}}} + \frac{1}{6} \left(\bar{\mathcal{K}}^2 - 3\bar{\mathcal{K}}^{ij} \bar{\mathcal{K}}_{ij} + \bar{\mathcal{R}} \right) \frac{\delta \rho}{\sqrt{\bar{h}}} - \frac{1}{3} \bar{\nabla}^i \bar{\nabla}_i \left(\frac{\delta \rho}{\sqrt{\bar{h}}} \right) \right], \quad (4.26)$$

$$\mathcal{H}_{(1)}^i = \sqrt{\bar{h}} \left[-2\bar{\nabla}_j \left(\frac{\delta \pi^{ij}}{\sqrt{\bar{h}}} \right) + \frac{1}{3} \frac{\delta \rho}{\sqrt{\bar{h}}} \bar{\nabla}^i \bar{\mathcal{K}} - \frac{2}{3} \bar{\nabla}_j \left(\bar{\mathcal{K}}^{ij} \frac{\delta \rho}{\sqrt{\bar{h}}} \right) \right]. \quad (4.27)$$

and the linearised primary traceless constraint reads

$$\Phi_{(1)}^{ij} = \delta \rho^{ij} - \frac{1}{3} \bar{h}^{ij} \delta \rho. \quad (4.28)$$

The quadratic canonical Hamiltonian density that governs the dynamics of linearized perturbations,

$$\mathcal{H}_{(2)} = \bar{N} (\mathcal{H}^{(2)} + \bar{\lambda}_{ij} \Phi_{(2)}^{ij}) + \bar{N}_i \mathcal{H}_{(2)}^i, \quad (4.29)$$

is composed of second variations of the primary constraints,

$$\mathcal{H}_{(2)} = \sqrt{\bar{h}} \left[\frac{1}{144} \left(\frac{\delta \rho}{\sqrt{\bar{h}}} \right)^2 - 2\delta \mathcal{K}_{ij} \frac{\delta \pi^{ij}}{\sqrt{\bar{h}}} + \left(\bar{\mathcal{K}}^{ij} - \frac{1}{3} \bar{h}^{ij} \bar{\mathcal{K}} \right) \left(\bar{\mathcal{K}}_i^k \delta h_{kj} - \frac{1}{6} \bar{\mathcal{K}}_{ij} \delta h - \delta \mathcal{K}_{ij} \right) \frac{\delta \rho}{\sqrt{\bar{h}}} \right]$$

$$\begin{aligned}
& -\frac{1}{6}(\bar{\mathcal{R}}^{ij} - \frac{1}{3}\bar{h}^{ij}\bar{R})\delta h_{ij}\frac{\delta\rho}{\sqrt{h}} + \frac{1}{3}(\delta_k^{(i}\delta_l^{j)} - \frac{1}{3}\bar{h}^{ij}\bar{h}_{kl})\bar{\nabla}^k(\delta h_{ij}\bar{\nabla}^l\frac{\delta\rho}{\sqrt{h}}) \\
& + \frac{1}{6}\left(\frac{\delta\rho}{\sqrt{h}}\bar{\nabla}^i\bar{\nabla}^j\delta h_{ij} + (\bar{\nabla}^k\delta h\bar{\nabla}_k)\frac{\delta\rho}{\sqrt{h}}\right)\Big], \tag{4.30}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_{(2)}^i = \sqrt{h}\Big[& -2\frac{\delta\pi^{jk}}{\sqrt{h}}\left(\bar{\nabla}_k\delta h_j^i - \frac{1}{2}\bar{\nabla}^i\delta h_{jk}\right) + \frac{2}{3}\left(\delta_j^k\bar{\nabla}^l - \frac{1}{3}\bar{h}^{kl}\bar{\nabla}_j\right)\left(\bar{\mathcal{K}}^{ij}\delta h_{kl}\frac{\delta\rho}{\sqrt{h}}\right) \\
& - \frac{1}{3}\left(\delta h^{ij} - \frac{1}{3}\bar{h}^{ij}\delta h\right)(\bar{\nabla}_j\bar{\mathcal{K}})\frac{\delta\rho}{\sqrt{h}} + \frac{1}{3}\frac{\delta\rho}{\sqrt{h}}\left(\bar{\nabla}^i\delta\mathcal{K} - \delta h_{jk}\bar{\nabla}^i\bar{\mathcal{K}}^{jk}\right) \\
& + \frac{2}{3}\delta h^{ij}\bar{\nabla}^k\left(\bar{\mathcal{K}}_{jk}\frac{\delta\rho}{\sqrt{h}}\right) - \frac{2}{3}\bar{\nabla}_j\left(\delta\mathcal{K}^{ij}\frac{\delta\rho}{\sqrt{h}}\right)\Big], \tag{4.31}
\end{aligned}$$

$$\Phi_{(2)}^{ij} = \frac{1}{3}\left(\delta h^{ij} - \frac{1}{3}\bar{h}^{ij}\delta h\right)\delta\rho. \tag{4.32}$$

where $\delta h = \bar{h}^{ij}\delta h_{ij}$.

We notice there are no terms quadratic in coordinates in the parts (4.30)–(4.32) making up the canonical Hamiltonian. This means that the conservation of constraints (4.26)–(4.28) which are linear in the perturbed momenta can only give rise to constraints which are also linear in the perturbed momenta. Indeed, the conservation of linearised primary constraints

$$\dot{\mathcal{H}}_{(1)} \approx 2\sqrt{h}\left(\bar{N}\bar{\nabla}_i + 2\bar{\nabla}_i\bar{N}\right)\bar{\nabla}_j\left(\frac{\Psi_{(1)}^{ij}}{\sqrt{h}}\right) + 2\left(2\bar{\mathcal{K}}_{ik}\bar{\nabla}_j\bar{N}^k + \bar{N}^k\bar{\nabla}_k\bar{\mathcal{K}}_{ij}\right)\Psi_{(1)}^{ij} - 2\bar{N}\bar{\lambda}_{ij}\Psi_{(1)}^{ij}, \tag{4.33a}$$

$$\dot{\mathcal{H}}_{(1)}^i \approx 4\sqrt{h}\bar{\nabla}_j\left(\bar{N}\bar{\mathcal{K}}_k^i\frac{\Psi_{(1)}^{jk}}{\sqrt{h}}\right) - 2\bar{N}\Psi_{(1)}^{jk}\bar{\nabla}^i\bar{\mathcal{K}}_{jk}, \quad \dot{\Phi}_{(1)}^{ij} \approx 2\bar{N}\Psi_{(1)}^{ij}, \tag{4.33b}$$

implies that

$$\Psi_{(1)}^{ij} = \delta\pi^{ij} - \frac{1}{3}\bar{h}^{ij}\delta\pi + \frac{\delta\rho}{6}\left(\bar{\mathcal{K}}^{ij} - \frac{1}{3}\bar{h}^{ij}\bar{\mathcal{K}}\right), \tag{4.34}$$

appears as a secondary traceless constraint. Note that in deriving Eqs. (4.33) it is necessary to use equations of motion (2.18)–(2.21) given that primary constraints are now explicitly time-dependent via the background quantities.

All the linearised constraints (being linear in perturbed momenta) trivially commute, and are therefore all first-class. Consequently, no further constraints are generated by the conservation of the secondary constraint. Naively we would count 14 first-class constraints — 4 for Hamiltonian and momentum constraints, and 10 for primary and secondary traceless constraints. However, that would not be consistent, given that the number of canonical variables is 24. It should not be overlooked that Hamiltonian and momentum constraints are impacted by the secondary traceless constraint, that eliminates the traceless part of $\delta\pi^{ij}$,

$$\mathcal{H}_{(1)} \approx \sqrt{h}\left[-\frac{2\bar{\mathcal{K}}}{3}\frac{\delta\pi}{\sqrt{h}} + \frac{1}{18}\left(\bar{\mathcal{K}}^2 - 3\bar{\mathcal{K}}^{ij}\bar{\mathcal{K}}_{ij} + 3\bar{\mathcal{R}}\right)\frac{\delta\rho}{\sqrt{h}} - \frac{1}{3}\bar{\nabla}^i\bar{\nabla}_i\left(\frac{\delta\rho}{\sqrt{h}}\right)\right], \tag{4.35}$$

$$\mathcal{H}_{(1)}^i \approx \sqrt{h}\left[-\frac{2}{3}\bar{\nabla}^i\left(\frac{\delta\pi}{\sqrt{h}}\right) - \frac{\bar{\mathcal{K}}}{9}\bar{\nabla}^i\left(\frac{\delta\rho}{\sqrt{h}}\right) + \frac{2}{9}\frac{\delta\rho}{\sqrt{h}}\bar{\nabla}^i\bar{\mathcal{K}} - \frac{1}{3}\bar{\nabla}_j\left(\bar{\mathcal{K}}^{ij}\frac{\delta\rho}{\sqrt{h}}\right)\right]. \tag{4.36}$$

This way the formally four first-class constraints depend on the two scalar perturbations only. Because of this they have to be counted as only two first-class constraints, and essentially reduce to $\delta\rho \approx 0$ and $\delta\pi \approx 0$. This brings the count of the total number of first-class constraints to $N_{1st} = 12$, and consequently the number of physical propagating degrees of freedom to zero,

$$N_{\text{phy}} = \frac{1}{2}(24 - 2 \times 12 - 0) = 0. \tag{4.37}$$

Accidental gauge symmetry. The change of character of traceless constraints from second-class to first-class, and the loss of transverse components of the momentum constraint suggest that the structure of local symmetries of the theory have been modified by the linearisation process. Indeed, the singularity of traceless-Ricci backgrounds in the Lagrangian formulation for linearised perturbations,

$$S[\delta g_{\mu\nu}] = \int d^4x \sqrt{-\bar{g}} \left[\left(\bar{D}^\mu \bar{D}^\nu - \bar{g}^{\mu\nu} \bar{D}^\rho \bar{D}_\rho - \bar{R}^{\mu\nu} \right) \delta g_{\mu\nu} \right]^2, \quad (4.38)$$

manifests itself through an accidental gauge symmetry for linear perturbations,

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} + \mathcal{P}_{\mu\nu}{}^{\rho\sigma} \xi_{\rho\sigma}, \quad (4.39)$$

where $\xi_{\mu\nu}$ is an arbitrary symmetric 2-tensor field, and $\mathcal{P}_{\mu\nu}{}^{\rho\sigma}$ is the appropriate projector.

While finding the projector in (4.39) explicitly is generally not a straightforward task, it is not difficult to construct it for more special Ricci-flat backgrounds, where $\bar{R}_{\mu\nu} = 0$. This is accomplished in Appendix A with the help of the Bach tensor, that is both transverse and traceless on arbitrary backgrounds, and vanishes for Ricci-flat backgrounds. The projector can be read off from the linear perturbation of the Bach tensor around Ricci-flat backgrounds, and written in a more compact form upon commuting some covariant derivatives,

$$\mathcal{P}_{\mu\nu}{}^{\rho\sigma} = \Pi^{(\rho}{}_{(\mu} \Pi^{\sigma)}{}_{\nu)} - \frac{1}{3} \Pi_{(\mu\nu)} \Pi^{(\rho\sigma)} + \bar{R}_{(\mu}{}^{\alpha}{}_{\nu)}{}^{(\rho} \Pi^{\sigma)}{}_{\alpha} + \frac{1}{2} \bar{R}_{(\mu}{}^{\alpha}{}_{\nu)}{}^{\beta} \bar{R}_{\alpha}{}^{(\rho}{}_{\beta}{}^{\sigma)} - \bar{D}^\alpha \bar{D}_{(\mu} \bar{R}_{\nu)}{}^{(\rho}{}_{\alpha}{}^{\sigma)}, \quad (4.40)$$

where

$$\Pi_{\mu\nu} = \bar{g}_{\mu\nu} \bar{D}^\alpha \bar{D}_\alpha - \bar{D}_\mu \bar{D}_\nu \quad (4.41)$$

is a transverse projector when acting on a vector, and where derivatives act on everything to the right of them. The projector in (4.40) is transverse and traceless when contracted onto a 2-tensor, so that the transformation in (4.39) is guaranteed to be a symmetry for an arbitrary symmetric tensor $\xi_{\mu\nu}$. Furthermore, it reduces to the transverse-traceless projector in flat space, and reproduces the accidental gauge symmetry found in [4].

Nonlinear perturbations. While the spectrum of linear perturbations around traceless-Ricci backgrounds is empty, it is not immediately clear what the conclusion would be at higher order in perturbation theory. This question can be addressed in the same manner as for the special case of Minkowski background in Sec. 3.2, by assuming a power series expansion for all variables around the background,

$$h_{ij} \approx \bar{h}_{ij} + \varepsilon \delta \bar{h}_{ij}^1 + \varepsilon^2 \delta \bar{h}_{ij}^2 + \varepsilon^3 \delta \bar{h}_{ij}^3 + \dots, \quad (4.42a)$$

$$\pi^{ij} \approx 0 + \varepsilon \delta \bar{\pi}_1^{ij} + \varepsilon^2 \delta \bar{\pi}_2^{ij} + \varepsilon^3 \delta \bar{\pi}_3^{ij} + \dots, \quad (4.42b)$$

$$\mathcal{K}_{ij} \approx \bar{\mathcal{K}}_{ij} + \varepsilon \delta \bar{\mathcal{K}}_{ij}^1 + \varepsilon^2 \delta \bar{\mathcal{K}}_{ij}^2 + \varepsilon^3 \delta \bar{\mathcal{K}}_{ij}^3 + \dots, \quad (4.42c)$$

$$\rho^{ij} \approx 0 + \varepsilon \delta \bar{\rho}_1^{ij} + \varepsilon^2 \delta \bar{\rho}_2^{ij} + \varepsilon^3 \delta \bar{\rho}_3^{ij} + \dots, \quad (4.42d)$$

$$N \approx \bar{N} + \varepsilon \delta \bar{N}_1 + \varepsilon^2 \delta \bar{N}_2 + \varepsilon^3 \delta \bar{N}_3 + \dots, \quad (4.42e)$$

$$N_i \approx \bar{N}_i + \varepsilon \delta \bar{N}_1^i + \varepsilon^2 \delta \bar{N}_2^i + \varepsilon^3 \delta \bar{N}_3^i + \dots, \quad (4.42f)$$

$$\lambda_{ij} \approx \bar{\lambda}_{ij} + \varepsilon \delta \bar{\lambda}_{ij}^1 + \varepsilon^2 \delta \bar{\lambda}_{ij}^2 + \varepsilon^3 \delta \bar{\lambda}_{ij}^3 + \dots. \quad (4.42g)$$

where the bookkeeping parameter ε keeps track of the powers of perturbation fields.

We will see that perturbation theory fails to capture any propagating degrees of freedom around all traceless-Ricci backgrounds. Proving this is analogous, though much more tedious, to the procedure in Sec. 3.2. It relies on the result from Appendix B which tell us that the action at each successive perturbative order remains quadratic,

$$\mathcal{S}_n[\delta N, \delta N_i, \delta \lambda_{ij}, \delta h_{ij}, \delta \pi^{ij}, \delta \mathcal{K}_{ij}, \delta \rho^{ij}]$$

$$= \varepsilon^{2n} \int d^4x \left[\delta\pi^{ij} \delta h_{ij} + \delta\rho^{ij} \delta\dot{K}_{ij} - \mathcal{H}_n - \delta N \mathcal{H}_{(1)} - \delta N_i \mathcal{H}_{(1)}^i - \delta\lambda_{ij} \Phi_{(1)}^{ij} \right] + \mathcal{O}(\varepsilon^{2n+1}), \quad (4.43)$$

such that the constraints, given in (4.26)–(4.28), remain the same, and that the Hamiltonian density,

$$\mathcal{H}_n = \mathcal{H}_{(2)} + \Delta\mathcal{H}_n, \quad (4.44)$$

contains the quadratic part $\mathcal{H}_{(2)}$, given in (4.29), that does not change, and a linear part $\Delta\mathcal{H}_n$ that changes from order to order. Furthermore, the perturbations of the momenta $\delta\pi^{ij}$ and $\delta\rho^{ij}$ vanish on-shell at every order, so the linear part of the Hamiltonian has to take the form

$$\Delta\mathcal{H}_n = A_n \delta\rho + B_{ij}^n \delta\pi^{ij}, \quad (4.45)$$

where it is only the coefficient functions A_n and B_{ij}^n that are updated at each order. This is why the linear part of the Hamiltonian density does not influence the constraint structure, that remains the same at each order. Consequently, we find no propagating degrees of freedom at each perturbative order.

Attention should be paid to the detail that primary constraints are now explicitly time-dependent, but only through the background field dependence. This time dependence, that is the same at each order, has to be accounted for when demanding conservation of constraints.

Therefore, the degenerate constraint structure for perturbations around traceless-Ricci backgrounds does not change at any order, and consistently yields zero propagating degrees of freedom. However, absence of degrees of freedom is not a physical feature of the vicinity of traceless-Ricci spacetimes. The interpretation of this feature is the same as in the Minkowski case from Sec. (3.2): perturbation theory is not appropriate for examining the vicinity of traceless-Ricci spacetimes. The dynamics of the variable $\delta\rho$, that measures departure from the vanishing Ricci scalar, becomes nonperturbative close to traceless-Ricci spacetimes, and different methods have to be used to quantify this region in field space.

5 Cosmological phase space

Examples of singular spacetimes that exhibit the strong coupling feature given in Sec. 4 are all spacetimes with globally and eternally vanishing Ricci scalar. Slight deviations from such spacetimes cannot be described by perturbation theory. This begs the question of whether such backgrounds can be reached dynamically, or whether they are in some sense isolated from the rest of the phase space. In order to address this question, at least in part, in this section we give the dynamical system analysis for spatially flat cosmology of pure R^2 theory. We find that $R = 0$ point is indeed crossed by some trajectories of the evolution. We focus on demonstrating this by providing various phase space plots, and consequently we do not delve into analysing various attractors and singular points of the phase space flow.

It might be tempting to perform this analysis in the Einstein frame. However, we refrain from this on account of the singularities introduced by the conformal transformation itself [18]. Such singularities are located precisely at points $R=0$ that we seek to explore. For this reason we perform the dynamical systems analysis directly in the Jordan frame. Even then, there remain many different ways to formulate a dynamical system [35–40]. The case at hand is most similar to the $f(R)=R^n$ theory studied in [41]. However, the variables used there were developed for a different purpose, and cannot be used in practice to probe the strongly coupled region that we are interested in. Accordingly, here we take a different approach.

The line element for spatially flat FLRW spacetime was already given in (4.11). For this spacetime the Ricci tensor is diagonal,

$$R_{00} = -3(\dot{H} + H^2), \quad R_{ij} = a^2 \delta_{ij} (\dot{H} + 3H^2). \quad (5.1)$$

while the the Ricci scalar reads

$$R = 6(\dot{H} + 2H^2). \quad (5.2)$$

Equations of motion (2.2) are also diagonal, and reduce to the two Friedmann equations,

$$H\dot{R} + H^2R - \frac{R^2}{12} = 0, \quad \ddot{R} + 2H\dot{R} - H^2R + \frac{R^2}{12} = 0. \quad (5.3)$$

The three equations in (5.2) and (5.3) form the basis of the dynamical system formulation. However, not all three equations are independent. For instance, the second Friedmann equation in (5.3) can be derived from the first Friedmann equation, and the definition of the Ricci scalar (5.2). In the remainder of this section we first give three different two-dimensional dynamical system formulations, analogous to the ones considered in [42], with the goal of illustrating the dynamics from multiple perspectives. To this end, it is convenient to define dimensionless quantities,

$$X = H/\kappa, \quad Y = R/\kappa^2, \quad Z = \dot{R}/\kappa^3, \quad (5.4)$$

where κ is some arbitrary scale. It is also convenient to rescale time with the same scale, $T = \kappa t$, which makes the problem dimensionless. The phase space is broken up into four distinct partitions in two-dimensional formulations of the dynamical system. These are uniquely characterised by the sign of the Hubble rate, and the sign of the velocity of the Ricci scalar, that never change during the evolution. Color coding of the partitions, that we use in figures throughout the section, is given in Table 1 below. We conclude the section with a one-dimensional formulation that clearly reveals that evolution can cross the point $R = 0$.

$\text{sgn}(H)$	+	+	−	−
$\text{sgn}(\dot{R})$	+	−	+	−
colour				

Table 1: Four partitions of the cosmological evolution in the pure R^2 theory, classified by the sign of the Hubble rate, and the sign of the velocity of the Ricci scalar.

First two-dimensional formulation. In the first representation of the system, we simply take the definition of the Ricci scalar and the first Friedmann equation as independent equations, which in dimensionless variables read

$$\frac{dX}{dT} = \frac{Y}{6} - 2X^2, \quad \frac{dY}{dT} = \frac{Y^2}{12X} - XY. \quad (5.5)$$

The compactified phase space plot following from these equations is given in Fig. 1. The autonomous system in (5.5) has the advantage of being described by rational functions. Accordingly, almost the whole volume of the infinite two-dimensional plane corresponds to a well-defined flow, with the exceptions arising along $X = 0$, which is a set of measure zero. Four distinct cosmological evolutions are visible, and in line with expectation that the strongly coupled surface $Y = 0$ mostly manifests as a separatrix to partition these. This partitioning appears to break down, however, at precisely the point $X = Y = 0$ where the system is no longer faithful to the physics. As illustrated in Fig. 1, we actually claim that two of the four evolutions penetrate through the strongly coupled surface at this point: since the point in question involves the convergence of infinitely many flow lines, and a corresponding loss of information, we must transition to improved coordinates to defend this claim. It will be clear that this feature is a consequence of the two-dimensional projection of what is in essence a two-dimensional curved surface embedded in a three-dimensional phase space.

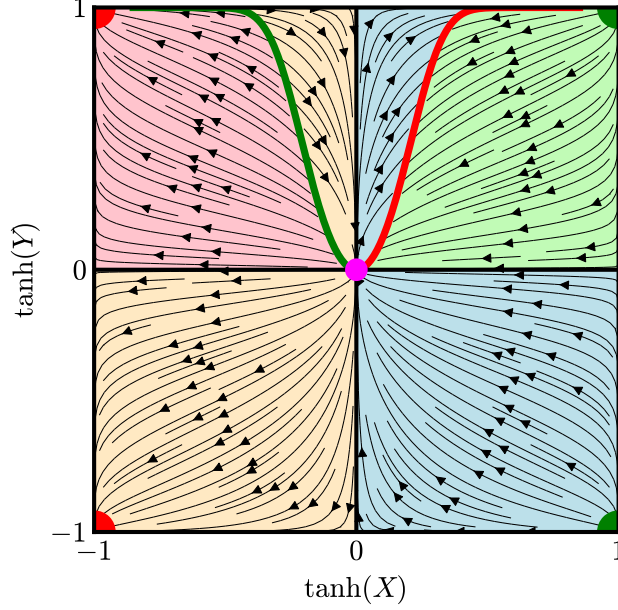


Figure 1: Compactified phase space flow for the system of autonomous equations in (5.5). The four sectors of evolution are indicated in different pastel colors defined in Table 1. The stream lines in green and blue partitions originate in respective sources indicated by green dots in the corners of the plot, and terminate at the red curve representing an expanding de Sitter attractor; the stream lines in red and yellow partitions originate at the green curve representing a contracting de Sitter repulsor, and terminate at respective sinks indicated by red dots in corners. Sources and sinks appear confined at the corners, as a result of our hyperbolic compactification. The lines at $X=0$ and $Y=0$ partition the flow at all points except for the magenta point at the origin— at this point we claim that the blue and green evolutions penetrate the strongly coupled surface. The upper-right quadrant matches that given in [40].

Second two-dimensional formulation. To more thoroughly explore what happens at the origin of Fig. 1, we choose the Ricci scalar and its velocity to span the phase space. In order to do this we need to solve for the Hubble rate from the first Friedmann equation in terms of the other two variables, so as to eliminate it algebraically from the system. In terms of the variable X , the first Friedmann equation is a quadratic that produces two branches

$$X = \bar{X}_{\pm}(Y, Z) \equiv \frac{1}{2Y} \left[-Z \pm \sqrt{\frac{3Z^2 + Y^3}{3}} \right]. \quad (5.6)$$

Note that the discriminant excludes a part of the phase space for which $3Z^2 + Y^3 < 0$. We plug this solution into the remaining equations to form the dynamical system,

$$\frac{dY}{dT} = Z, \quad \frac{dZ}{dT} = -3\bar{X}_{\pm}Z. \quad (5.7)$$

Note that the first equation is the trivial definition of the Ricci scalar derivative, so that only the second equation is sensitive to the branching. The corresponding plot of the phase space flow is given in Fig. 2. Despite the mild complication of having to glue the branches along black curved dashed cuts where the discriminant vanishes, the essential point is clearly visible, that the blue and yellow evolutions penetrate the strongly coupled surface at $Y=0$, indicated by the dashed magenta line.

Third two-dimensional formulation. The last combination of quantities we can use to form a two-dimensional phase space flow are the Hubble rate and the Ricci scalar velocity. In

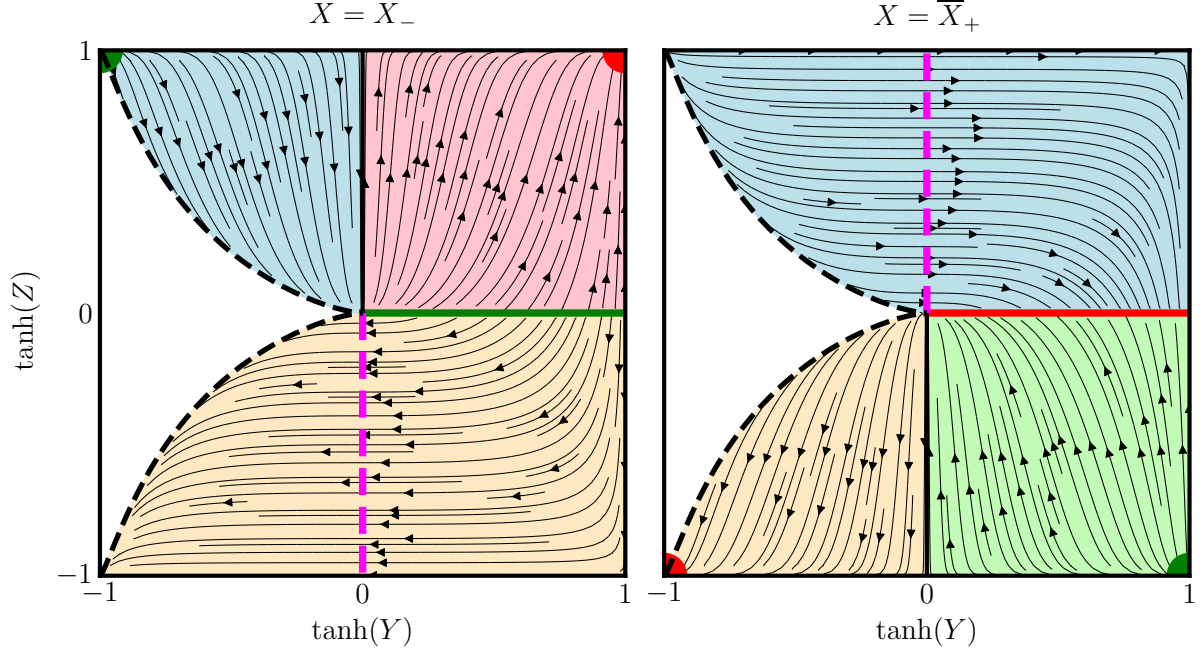


Figure 2: Compactified phase space flow for the system of autonomous equations in (5.7), producing an alternative perspective to that shown in Fig. 1. In these coordinates, the blue and yellow evolutions are split over the two branches, which must be glued along the matching cuts indicated by black dashed lines. The flow lines bounce tangentially from these dashed lines when transitioning from one branch to another. Both blue and yellow evolutions clearly pass through the strongly coupled surface at $Y = 0$, indicated by the dashed magenta lines, which degenerate into the single magenta point at the origin in Fig. 1. The de Sitter repulsor (green) and attractor (red) lie along the line $Z = 0$ for $Y > 0$. The sources and sinks in the corners are indicated by green and red dots, respectively.

this case we again solve the first Friedmann equation algebraically, but this time for the Ricci scalar,

$$Y = \bar{Y}_{\pm}(X, Z) \equiv 6 \left[X^2 \pm \sqrt{\frac{X(3X^3 + Z)}{3}} \right]. \quad (5.8)$$

The discriminant again cuts out part of the phase space for which $X(3X^3 + Z) < 0$. Since this discriminant factorises, the result will be an awkward pair of cuts. Plugging the solution into the definition of the Ricci scalar and the second Friedmann equation then gives the two equations of the dynamical system,

$$\frac{dX}{dT} = \frac{\bar{Y}_{\pm}}{6} - 2X^2, \quad \frac{dZ}{dT} = -3XZ. \quad (5.9)$$

The plot of the phase space flow is given in 3. This is provided largely out of completeness, since the essential feature of penetration through the strongly coupled surface is already clear from Figs. 1 and 2.

One-dimensional formulation. Finally, we present an alternative to the three two-dimensional dynamical system formulations presented above. Given that none of the four partitions cross the $\dot{R} = 0$ point, we may adopt $\dot{R}$ as the natural scale of the problem.¹⁰ Therefore, we can adopt

¹⁰One should be careful not to introduce singular points in the dynamical system that are just the property of the variables used, as opposed to the genuine singular points of the evolution [43].

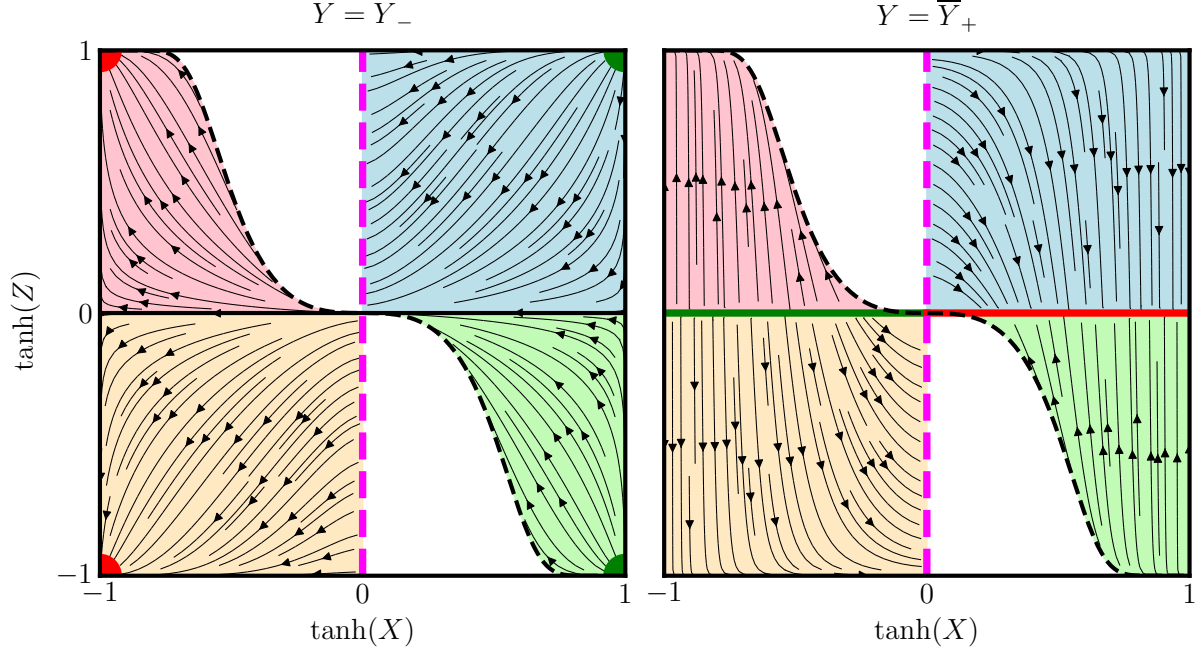


Figure 3: Compactified phase space flow for the system of autonomous equations in (5.9), producing an alternative perspective to that shown in Figs. 1 and 2. In these coordinates, all evolutions are split over the two branches, which must be glued along the matching cuts indicated by dashed lines. The flow lines bounce tangentially from the black dashed lines, and continue orthogonally through the magenta dashed lines, that represent the singular surface of the vanishing Ricci scalar. Sources and sinks in corners on the left plot are indicated by green and red dots, respectively, while de Sitter repulsor and attractor on the right plot are indicated by the green and red lines, respectively.

the Hubble rate and the Ricci scalar as variables in these natural units,

$$x = H/\dot{R}^{1/3} = X/Z^{1/3}, \quad y = R/\dot{R}^{2/3} = Y/Z^{2/3}. \quad (5.10)$$

Together with the definition of dimensionless time,

$$\tau = t\dot{R}^{1/3}, \quad (5.11)$$

this reduces the three equations (5.2) and (5.3) into a single first order equation, supplemented by an algebraic equation,

$$\frac{dx}{d\tau} = -x^2 + \frac{y}{6}, \quad x + x^2y - \frac{y^2}{12} = 0, \quad (5.12)$$

with the third equation becoming redundant. The effectively one-dimensional dynamical system formulation in (5.12) is possible for monomial $f(R) = R^n$, but not for general $f(R)$ theories. Fig. 4 gives the solutions of this system, represented by the two disjoint curves. The direction of flow along these two curves is either clockwise or anti-clockwise, and is determined only when the sign of the Ricci scalar velocity is chosen. In this way the curve that passes through $R=0$ point corresponds to blue and yellow partitions from two-dimensional formulations, and the curved that does not pass through that point corresponds to red and green partitions. Once again, we clearly see that the points at which $R=0$ can be traversed during the cosmological evolution.

6 Discussion

In this work, we have performed a full Hamiltonian constraint analysis of pure R^2 theory of gravity in order to address recent controversies regarding its particle spectrum. Our analysis

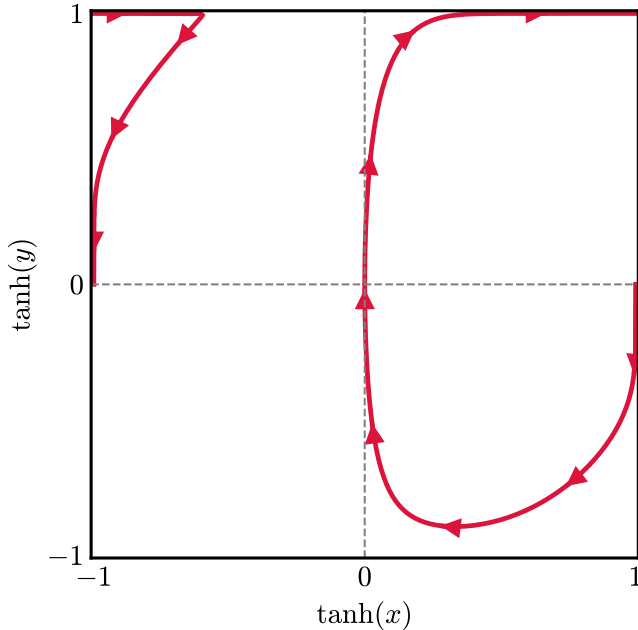


Figure 4: Compactified flow trajectory for the system in (5.12). The direction of the flow is with respect to the dimensionless time defined in (5.11). However, note that the dimensionful time follows this flow if the Ricci scalar velocity is positive, $\dot{R} > 0$, and actually flows in reverse direction when $\dot{R} < 0$. This is the reason why the right curve captures both the blue and the yellow sectors defined in Table 1, and the left curve captures the green and red sectors. For sectors with positive Ricci velocity the evolution in dimensionful time follows the clockwise direction indicated, while for sectors with negative Ricci velocity this flow is anti-clockwise. The penetration through the strongly coupled surface at $y = 0$ is evident for the right curve, that captures both blue and yellow partitions from Figs. 1–3.

confirms that in general the full theory propagates three degrees of freedom, corresponding to a massive spin-two graviton and a scalar, in agreement with established results [6]. However, the constraint structure reveals singular points in field space where constraints change character. This feature is behind the reported absence of propagating linearised perturbations around Minkowski space [1, 3, 4]¹¹.

Furthermore, we have shown that the absence of propagating modes in the linearised theory is not a phenomenon specific to Minkowski space, but rather a generic feature of any background with a vanishing Ricci scalar, $R = 0$, i.e. traceless-Ricci background. The perturbative analysis fails on these singular surfaces because the constraint algebra is fundamentally altered: ten second class constraints of the full theory become first class, while the momentum constraint becomes degenerate as it loses its transverse part. This feature is responsible for removing all degrees of freedom from the linearised spectrum. Moreover, considering higher order perturbation theory around such surfaces, organized in powers of the perturbations, yields no propagating degrees of freedom at every order. While such a perturbation theory ought to probe the vicinity of traceless-Ricci spacetimes, it is in conflict with the general result of three propagating degrees of freedom that is valid arbitrarily close to traceless-Ricci spaces. Thus, this result should be interpreted as a limitation of the perturbation theory, and its inability to describe this regime of the theory. Rather, the dynamics of perturbations should become nonperturbative in this regime.

It is worth pointing out that a similar issue has also been reported for the Palatini (metric-

¹¹The absence of linearised degrees of freedom around Minkowski space has also been shown in D spacetime dimensions [46].

affine) formulation of pure R^2 theory, where no degrees of freedom are found in the linearised spectrum around Minkowski spacetime [44]. In that context, the full theory is known to propagate just the two degrees of freedom of the graviton, a result which has also been confirmed by Hamiltonian analysis [45]. The recurrence of this pathology underscores that the surfaces where the theory becomes strongly coupled are a key, non-trivial feature of higher-derivative gravity models. Similar features where a subset of degrees of freedom disappears from the linearised spectrum have been observed for the cusciton model [47], and also for the Einsteinian Cubic Gravity and its generalizations [48]. In the latter instance it was found that these singular surfaces in field space are shielded from the rest of the phase space by the nonperturbative behaviour of degrees of freedom in their vicinity. This begs the question whether something like this happens in the pure R^2 model we consider here.

The cosmological phase-space analysis we give in Sec. 5 demonstrates that the singular $R=0$ surface is dynamically accessible. This implies that the strong-coupling feature is not merely a mathematical curiosity confined to static, eternal spacetimes, but is of direct physical relevance for evolving cosmologies. This opens several avenues for future investigation. First, it is unclear what happens to perturbations as the cosmological background evolves smoothly through an $R=0$ phase. We believe this question requires a dedicated study using full numerical evolution to properly capture the non-linear dynamics.

Second, we conjecture that this phenomenon is a general feature of all $f(R)$ theories, manifesting on backgrounds where $f'(R)=0$. There is already some indication of this in the literature [49], and a full Hamiltonian analysis would be the appropriate tool to verify it. Confirming this conjecture would provide a unified understanding of the connection between theories like $R+R^2$ gravity and pure R^2 gravity. Rather than being disconnected theories, as has been suggested in [1] and [4], their features would be smoothly related, with the primary difference being the location of the singular, strong-coupling surface in the phase space.

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A Perturbing Bach tensor

The Bach tensor for 4-dimensional spacetimes,

$$B_{\mu\nu} = W_{\mu\rho\nu\sigma}S^{\rho\sigma} + D^\rho D_\rho S_{\mu\nu} - D^\rho D_{(\mu}S_{\nu)\rho}, \quad (\text{A.1})$$

is defined in terms of the Weyl tensor,

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - 2g_{\mu[\rho}R_{\sigma]\nu} + \frac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu}, \quad (\text{A.2})$$

and the Schouten tensor,

$$S_{\mu\nu} = \frac{1}{2}R_{\mu\nu} - \frac{1}{12}Rg_{\mu\nu}. \quad (\text{A.3})$$

It is both transverse, $D^\mu B_{\mu\nu} = 0$, and traceless, $g^{\mu\nu} B_{\mu\nu} = 0$, for arbitrary spacetimes. For Ricci-flat spacetimes it vanishes, $B_{\mu\nu} = 0$, given that the Schouten tensor (A.3) vanishes there. Therefore, we have that the linearised perturbation of the Bach tensor around Ricci-flat background, $R_{ij} = 0$, is also traceless and transverse,

$$\delta(D^\mu B_{\mu\nu}) = \overline{D}^\mu \delta B_{\mu\nu} = 0, \quad \delta(g^{\mu\nu} B_{\mu\nu}) = \overline{g}^{\mu\nu} \delta B_{\mu\nu} = 0, \quad (\text{A.4})$$

where $\overline{g}_{\mu\nu}$ is the background metric, and $\overline{D}_\mu$ the covariant derivative with respect to the background metric. This perturbation of the Bach tensor receives contributions from the perturbation of the Schouten tensor only,

$$\delta S_{\mu\nu} = -\frac{1}{4} \mathbb{P}_{\mu\nu}{}^{\rho\sigma} \delta g_{\rho\sigma}, \quad (\text{A.5})$$

defined in terms of the projector

$$\mathbb{P}_{\mu\nu}{}^{\rho\sigma} = -2\overline{D}^{(\rho} \delta_{(\mu}^{\sigma)} \overline{D}_{\nu)} + \delta_{(\mu}^\rho \delta_{\nu)}^\sigma \overline{D}^\alpha \overline{D}_\alpha + \overline{g}^{\rho\sigma} \overline{D}_{(\mu} \overline{D}_{\nu)} + \frac{\overline{g}_{\mu\nu}}{3} (\overline{D}^{(\rho} \overline{D}^{\sigma)} - \overline{g}^{\rho\sigma} \overline{D}^\alpha \overline{D}_\alpha). \quad (\text{A.6})$$

The perturbation of the Bach tensor is then given by

$$\delta B_{\mu\nu} = -\frac{1}{4} \mathcal{P}_{\mu\nu}{}^{\rho\sigma} \delta g_{\rho\sigma}, \quad (\text{A.7})$$

where the projector is defined as

$$\mathcal{P}_{\mu\nu}{}^{\rho\sigma} = \overline{D}^\alpha \overline{D}_\alpha \mathbb{P}_{\mu\nu}{}^{\rho\sigma} - \overline{D}^\alpha \overline{D}_{(\mu} \mathbb{P}_{\nu)\alpha}{}^{\rho\sigma} + \overline{R}_\mu{}^\alpha{}_\nu{}^\beta \mathbb{P}_{\alpha\beta}{}^{\rho\sigma}. \quad (\text{A.8})$$

The very convenient property of this projector is that any 2-tensor contracted into it will automatically be traceless and transverse on Ricci-flat backgrounds. It is given in a more compact form in (4.40) in the main text.

B Higher order perturbations

When studying small perturbations around certain field configurations the perturbative expansion organized in powers of the perturbation fields seems appropriate. This is often much easier implemented at the level of equations of motion. However, analyzing theories with constraints is more conveniently performed at the level of the action. That is why here we outline perturbation theory organized as an expansion in powers of perturbation fields adapted for the level of the action. In sections 3.2 and 4.3 we apply this method to study higher order perturbations around traceless-Ricci backgrounds in pure R^2 theory.

Consider some action $S[X]$ that is a functional of the fields $X^{\mathbf{a}}(x)$, where index $\mathbf{a}$ labels all the fields and their components. In this appendix we greatly benefit from employing DeWitt's shorthand notation, in which the capital Latin index labels both the type of the field and its coordinate dependence, $X^A = X^{\mathbf{a}}(x)$. For this appendix it is not relevant whether the action is Lagrangian or Hamiltonian, nor what precisely the dynamical fields are. What is important is that the first variation of this action generates equations of motion,

$$S_{,A}[\overline{X}] \equiv \left. \frac{\delta S[X]}{\delta X^{\mathbf{a}}(x)} \right|_{X=\overline{X}} = 0, \quad (\text{B.1})$$

where we have written the expression using DeWitt's shorthand notation for variational derivatives. The last beneficial element of this shorthand notation we need, before constructing the perturbative expansion, is the notation for contracted capital Latin indices,

$$X^A S_{,A}[X] = \int d^4x \sum_{\mathbf{a}} X^{\mathbf{a}}(x) \frac{\delta S[X]}{\delta X^{\mathbf{a}}(x)}, \quad (\text{B.2})$$

that denotes the sum over field types as well as an integral over coordinate dependence.

Equation (B.1) defines the background solution $X^A \approx \bar{X}^A$. We want to derive dynamical equations for small perturbations δX^A around that background. Given the assumption that the perturbations are small, the natural expansion is in powers of the perturbation fields. That is conveniently kept track of by introducing a bookkeeping parameter ε that we set to unity at the end of the computation, $X^A = \bar{X}^A + \varepsilon \delta X^A$. We look for the solution for perturbation fields as a power series in ε ,

$$\varepsilon \delta X^A \approx \varepsilon \delta \bar{X}_1^A + \varepsilon^2 \delta \bar{X}_2^A + \varepsilon^3 \delta \bar{X}_3^A + \dots, \quad (\text{B.3})$$

where the coefficient functions $\delta \bar{X}_n^A$ are independent of ε . As per usual in perturbation theory, equations of motion for each order are derived iteratively, where lower order solutions feed into equations for higher orders.

1st order. The action for the leading order perturbation is obtained from the starting action $S[X]$ by first shifting the fields by the background solution,

$$X^A \longrightarrow \bar{X}^A + \varepsilon \delta X^A, \quad (\text{B.4})$$

and plugging this shift into the action. We then expand the resulting action explicitly in powers of ε , keep only the lowest nonvanishing order, and drop the terms independent of δX^A that do not contribute to equations of motion,

$$S^1[\delta X] \equiv S[\bar{X} + \varepsilon \delta X] - S[\bar{X}] = \varepsilon \underbrace{S_{,A}[\bar{X}]}_{=0} \delta X^A + \frac{\varepsilon^2}{2} S_{,AB}[\bar{X}] \delta X^A \delta X^B + \mathcal{O}(\varepsilon^3). \quad (\text{B.5})$$

The term linear in ε vanishes owing to the background equation of motion (B.1), which makes the lowest nonvanishing order ε^2 . Varying this truncated action then generates equations of motion for the first perturbative correction $\delta \bar{X}_1^A$ in (B.3),

$$S_{,AB}[\bar{X}] \delta \bar{X}_1^B = 0. \quad (\text{B.6})$$

It is convenient to encode the zeroth order and the first order equations into a single expression,

$$\sum_{k=0}^1 \frac{\varepsilon^k}{k!} \left(\frac{\partial^k}{\partial \varepsilon^k} S_A[\bar{X} + \varepsilon \delta \bar{X}_1] \Big|_{\varepsilon=0} \right) = 0. \quad (\text{B.7})$$

2nd order. We proceed in the same manner at second order: by shifting the fields by the background solution and the first perturbative correction,

$$X^A \longrightarrow \bar{X}^A + \varepsilon \delta \bar{X}_1^A + \varepsilon^2 \delta X^A, \quad (\text{B.8})$$

and expanding the action accordingly, keeping the lowest nonvanishing order,

$$\begin{aligned} S^2[\delta X] \equiv S[\bar{X} + \varepsilon \delta \bar{X}_1 + \varepsilon^2 \delta X] - S[\bar{X} + \varepsilon \delta \bar{X}_1] &= \varepsilon^2 \underbrace{S_{,A}[\bar{X}]}_{=0} \delta X^A + \varepsilon^3 \underbrace{S_{,AB}[\bar{X}] \delta \bar{X}_1^A}_{=0} \delta X^B \\ &+ \frac{\varepsilon^4}{2} S_{,ABC}[\bar{X}] \delta \bar{X}_1^A \delta \bar{X}_1^B \delta X^C + \frac{\varepsilon^4}{2} S_{,AB}[\bar{X}] \delta X^A \delta X^B + \mathcal{O}(\varepsilon^5). \end{aligned} \quad (\text{B.9})$$

Orders lower than ε^4 drop out because of lower order equations of motion, where we again dropped the parts independent of the dynamical variables. Equation of motion for the second perturbative correction is then obtained by varying the truncated action,

$$S_{,AB}[\bar{X}] \delta \bar{X}_2^B + \frac{1}{2} S_{,ABC}[\bar{X}] \delta \bar{X}_1^B \delta \bar{X}_1^C = 0. \quad (\text{B.10})$$

There is again a useful compact form that captures the first three orders of equations of motion,

$$\sum_{k=0}^2 \frac{\varepsilon^k}{k!} \left(\frac{\partial^k}{\partial \varepsilon^k} S_A[\bar{X} + \varepsilon \delta \bar{X}_1 + \varepsilon^2 \delta \bar{X}_2] \Big|_{\varepsilon=0} \right) = 0. \quad (\text{B.11})$$

n -th order. The pattern that we see emerging for lower orders continues at each subsequent order. We can uncover it by following the same steps. First we shift the variable by solutions for all lower orders,

$$X^A \longrightarrow \overline{X}^A + \varepsilon \delta \overline{X}_1^A + \dots + \varepsilon^{n-1} \delta \overline{X}_{n-1}^A + \varepsilon^n \delta X^A, \quad (\text{B.12})$$

upon which we plug it into the action and expand it to order ε^{2n} ,

$$\begin{aligned} S^n[\delta X_n] &= S[\overline{X} + \dots + \varepsilon^{n-1} \delta \overline{X}_{n-1} + \varepsilon^n \delta X] - S[\overline{X} + \dots + \varepsilon^{n-1} \delta \overline{X}_{n-1}] \\ &= \frac{\varepsilon^{2n}}{n!} \left(\frac{\partial^n}{\partial \varepsilon^n} S_{,A}[\overline{X} + \dots + \varepsilon^{n-1} \delta \overline{X}_{n-1}] \Big|_{\varepsilon=0} \right) \delta X^A + \frac{\varepsilon^{2n}}{2} S_{,AB}[\overline{X}] \delta X^A \delta X^B + \mathcal{O}(\varepsilon^{2n+1}). \end{aligned} \quad (\text{B.13})$$

All the lower orders in this action vanish because of lower order equations of motion, that are compactly written as

$$\sum_{k=0}^{n-1} \frac{\varepsilon^k}{k!} \left(\frac{\partial^k}{\partial \varepsilon^k} S_A[\overline{X} + \varepsilon \delta \overline{X}_1 + \dots + \varepsilon^{n-1} \delta \overline{X}_{n-1}] \Big|_{\varepsilon=0} \right) = 0. \quad (\text{B.14})$$

The equation of motion for the n -th order perturbation is obtained by varying the action,

$$S_{,AB}[\overline{X}] \delta \overline{X}_n^B + \frac{1}{n!} \left(\frac{\partial^n}{\partial \varepsilon^n} S_{,A}[\overline{X} + \dots + \varepsilon^{n-1} \delta \overline{X}_{n-1}] \Big|_{\varepsilon=0} \right) = 0, \quad (\text{B.15})$$

that can be written condensely together with all other lower order equations as

$$\sum_{k=0}^n \frac{\varepsilon^k}{k!} \left(\frac{\partial^k}{\partial \varepsilon^k} S_A[\overline{X} + \varepsilon \delta \overline{X}_1 + \dots + \varepsilon^{n-1} \delta \overline{X}_{n-1} + \varepsilon^n \delta \overline{X}_n] \Big|_{\varepsilon=0} \right) = 0. \quad (\text{B.16})$$

This essentially completes the proof by induction of the action (B.13) at arbitrary order.

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