

NOWHERE-ZERO FLOWS ON SIGNED SUPEREULERIAN GRAPHS

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ABSTRACT. In 1983, Bouchet conjectured that every flow-admissible signed graph admits a nowhere-zero 6-flow. We verify this conjecture for the class of flow-admissible signed graphs possessing a spanning even Eulerian subgraph, which includes as a special case all signed graphs with a balanced Hamiltonian circuit. Furthermore, we show that this result is sharp by citing a known infinite family of signed graphs with a balanced Hamiltonian circuit that do not admit a nowhere-zero 5-flow. Our proof relies on a construction that transforms signed graphs whose underlying graph admits a nowhere-zero 4-flow into a signed 3-edge-colorable cubic graph. This transformation has the crucial property of establishing a sign-preserving bijection between the bichromatic cycles of the resulting signed cubic graph and certain Eulerian subgraphs of the original signed graph. As an application of our main result, we also show that Bouchet's conjecture holds for all signed abelian Cayley graphs.

1. INTRODUCTION

All graphs in this paper are finite, loopless and may have multiple edges. Set $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$. For basic notation and terminology which are not defined here, we refer to [2, 19]. A nowhere-zero flow is a way of assigning an orientation and a nonzero value from an abelian group A to each edge of a graph, such that the Kirchhoff current law is satisfied at every vertex. This law requires that the sum of values flowing into a vertex equals the sum of values flowing out of it. The concept of integer flow was introduced by Tutte [15, 16] when he observed that each nowhere-zero k -flow on a plane graph corresponds to a k -face-coloring of it, and vice versa. Jaeger [6] further demonstrated that if a graph G has a k -face-colorable 2-cell embedding in an orientable surface, then it admits a nowhere-zero k -flow. Therefore, nowhere-zero flow and face coloring can be seen as dual concepts. Due to the duality between local tensions and flows on graphs embedded in nonorientable surfaces, Bouchet [3] systematically developed an analogous concept of a nowhere-zero flow using bidirected edges instead of directed ones in 1983. Since signed graphs provide a convenient language for describing such embeddings, the nowhere-zero flow on a signed graph is generally used to represent the nowhere-zero flow introduced by Bouchet.

Bouchet [3] conjectured in 1983 that every flow-admissible signed graph admits a nowhere-zero 6-flow, wherein he proved that such signed graphs admit a nowhere-zero 216-flow. This question has attracted a lot of attention since then. In 1987, Zýka [20] improved Bouchet's results to nowhere-zero 30-flow. Recently, Zýka's results were improved by DeVos et al. [5] to nowhere-zero 11-flow, which is the best current general approach to Bouchet's conjecture.

Our work focuses on a specific class of such graphs. Recall that a graph G is *supereulerian* if it contains a spanning Eulerian subgraph. We introduce the concept of an *even Eulerian signed graph*, defined as a signed Eulerian graph containing an even number of negative edges. It is a known result

2010 *Mathematics Subject Classification.* 05C21; 05C22.

Key words and phrases. Nowhere-zero flows, Signed graph, Supereulerian graph, Hamiltonian graph, Abelian Cayley graph.

that every supereulerian graph admits a nowhere-zero 4-flow. This fact, combined with a recent theorem by Luo et al. [8] which builds upon the results of Li et al. [7], provides a baseline for our investigation.

Theorem 1.1 ([8]). *Let (G, σ) be a flow-admissible signed graph. If G admits a nowhere-zero 4-flow, then (G, σ) admits a nowhere-zero 8-flow.*

An immediate consequence of Theorem 1.1 is that every flow-admissible signed supereulerian graph admits a nowhere-zero 8-flow. The main contribution of this paper is to improve this bound for signed supereulerian graphs that contain a spanning even Eulerian subgraph. We prove that this class of signed graphs admits a nowhere-zero 6-flow, thereby verifying Bouchet's conjecture in this special case.

Theorem 4.1. *Let (G, σ) be flow-admissible. If (G, σ) has a spanning even Eulerian subgraph, then (G, σ) admits a nowhere-zero 6-flow.*

If the spanning even Eulerian subgraph is a balanced Hamiltonian circuit, then the following theorem holds.

Theorem 4.3. *Let (G, σ) be flow-admissible. If (G, σ) has a balanced Hamiltonian circuit, then (G, σ) admits a nowhere-zero 6-flow.*

Consider the signed cubic graph (G_n, σ_n) derived from an even circuit C_{2n} , where n is odd positive integer. This signed graph is constructed by replacing every second edge with a pair of parallel edges and assigning a signature such that all single edges are positive, and exactly one edge in each pair of parallel edges is negative. Fig. 1.1 illustrates the signed graph (G_3, σ_3) . In our figures, negative edges are depicted by dashed lines. Note that (G_n, σ_n) contains a balanced Hamiltonian circuit which is a spanning even Eulerian subgraph. Máčajová et al. [9] and Schubert et al. [14] independently proved that (G_n, σ_n) admits a nowhere-zero 6-flow but does not admit any nowhere-zero 5-flow. Therefore, the value 6 in Theorems 4.1 and 4.3 is optimal.

In order to prove Theorems 4.1 and 4.3, we introduce a method reduces the general case to the cubic case. More precisely, we construct a signed 3-edge-colorable cubic graph from a signed 4-NZF-admissible graph, where a graph is 4-NZF-admissible if it admits a nowhere-zero 4-flow. Note that every 3-edge-colorable cubic graph is 4-NZF-admissible. Utilizing this method, we prove the following two theorems.

Theorem 3.4. *Let k be a positive integer. Then the following statements are equivalent:*

- (1) *Every flow-admissible signed 4-NZF-admissible graph admits a nowhere-zero k -flow;*
- (2) *Every flow-admissible signed 3-edge-colorable cubic graph admits a nowhere-zero k -flow.*

For a specific class of signed 4-NZF-admissible graphs, known as signed supereulerian graphs, and a specific class of signed 3-edge-colorable cubic graphs, referred to as signed Hamiltonian cubic graphs, we present the following theorem.

Theorem 3.5. *Let k be a positive integer. Then the following statements are equivalent:*

- (1) *Every flow-admissible signed supereulerian graph admits a nowhere-zero k -flow;*
- (2) *Every flow-admissible signed Hamiltonian cubic graph admits a nowhere-zero k -flow;*

(3) *Every flow-admissible signed Hamiltonian cubic graph admits a nowhere-zero k -flow.*

Moreover, we apply Theorem 4.3 to prove the following theorem for signed abelian Cayley graphs, which are a class of signed Hamiltonian graphs.

Theorem 5.1. *Every flow-admissible signed abelian Cayley graph admits a nowhere-zero 6-flow.*

The value 6 is optimal, as there exists a signed abelian Cayley graph without any nowhere-zero 5-flow, as shown in Fig. 1.2.

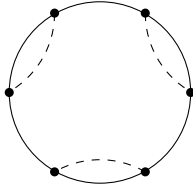


FIGURE 1.1. (G_3, σ_3) .

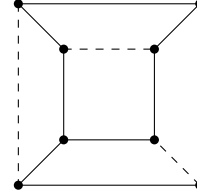


FIGURE 1.2. A flow-admissible signed abelian Cayley graph without any nowhere-zero 5-flow.

Inspired by the characterization of the flow number of signed Eulerian graphs [10], we characterize the flow number of a class of signed abelian Cayley graphs. The *flow number* of (G, σ) , denoted by $\Phi(G, \sigma)$, is the minimum k such that (G, σ) admits a nowhere-zero k -flow. Let $E_N(G, \sigma)$ denote the set of negative edges in (G, σ) .

Theorem 5.7. *Let A be a finite abelian group of odd order and $\Gamma = \text{Cay}(A, S)$ is connected. If (Γ, σ) is flow-admissible, then*

- (1) $\Phi(\Gamma, \sigma) = 2$ if and only if $|E_N(\Gamma, \sigma)|$ is even;
- (2) $\Phi(\Gamma, \sigma) = 3$ if and only if $|E_N(\Gamma, \sigma)|$ is odd and $\frac{|S|}{2} \geq 3$;
- (3) $\Phi(\Gamma, \sigma) = 4$ if and only if $|E_N(\Gamma, \sigma)|$ is odd and $\frac{|S|}{2} = 2$.

The organization of the rest of the paper is as follows. Basic notation and terminology are introduced in Section 2. In Section 3, we present the method that derives a signed 3-edge-colorable cubic graph from a signed 4-NZF-admissible graph. This section also includes the proofs of Theorems 3.4, and 3.5. Section 4 presents the proofs of Theorems 4.1 and 4.3, which establish sufficient conditions for a signed supereulerian graph to admit a nowhere-zero 6-flow. As an application of Theorem 4.3, Theorem 5.1 is proved in Section 5, which discusses signed abelian Cayley graphs. Additionally, Section 5 provides the characterization of the flow number of abelian Cayley graphs with an odd number of vertices, as stated in Theorem 5.7.

2. NOTATION AND TERMINOLOGY

We write G for a *graph*, with its vertex set and edge set denoted by $V(G)$ and $E(G)$, respectively. A *circuit* is a connected 2-regular graph. A graph G is said to be *even* if every vertex of G has an even degree. A graph G is called an *Eulerian graph* if it is both connected and even. A graph G is

called *supereulerian* if it contains a spanning Eulerian subgraph. Specifically, a *Hamiltonian* graph is a supereulerian graph that contains a spanning circuit.

A *signed graph* is defined as (G, σ) , where G is the underlying graph and $\sigma : E(G) \rightarrow \{\pm 1\}$ is a signature assigning a sign to each edge. An edge e of (G, σ) is *positive* if $\sigma(e) = +1$; otherwise, it is *negative*. Recall that $E_N(G, \sigma)$ denote the set of negative edges in (G, σ) . A signed graph (G, σ) is *all-positive* if $E_N(G, \sigma) = \emptyset$. In this paper, ordinary graphs are considered as all-positive signed graphs. Let F be a subgraph of G . The *sign of F* , denoted by $\sigma(F)$, is the product of the signs of its edges. Specifically, let $\sigma(F) = +1$ if $E(F) = \emptyset$. A circuit C is *balanced* if $\sigma(C) = +1$, and *unbalanced* otherwise. A signed graph (G, σ) is called *balanced* if there is no unbalanced circuit in (G, σ) , and *unbalanced* otherwise.

Switching is an operation on a signed graph. For a vertex $v \in V(G)$, *switching at v* negates the sign of each edge incident with v . For a vertex set U , *switching at U* means switching all vertices in U . It is worth noting that switching does not change the sign of any circuit. If the signed graph (G, σ') is obtained from (G, σ) by a sequence of switchings, then we say that (G, σ') is *switching equivalent* to (G, σ) . Switching equivalence is an equivalence relation.

Two signed graphs (G, σ) and (H, π) are *isomorphic*, denoted by $(G, \sigma) \cong (H, \pi)$ if there is an isomorphism f from G to H such that for any circuit C in G , $\sigma(C) = \pi(f(C))$. It is easy to see that if f is an isomorphism from G to H such that $\sigma(e) = \pi(f(e))$ for any $e \in E(G)$, then f is an isomorphism from (G, σ) to (H, π) .

Let G_1 be a subgraph of G . It is convenient to denote the signed graph $(G_1, \sigma|_{E(G_1)})$ by (G_1, σ) , where $\sigma|_{E(G_1)}$ is a restriction of σ to $E(G_1)$.

A *signed circuit* is a signed graph that belongs to one of the following three types:

- (1) A balanced circuit;
- (2) A short barbell, which is the union of two unbalanced circuits that meet at a single vertex;
- (3) A long barbell, which is the union of two disjoint unbalanced circuits with a path that meets the circuits only at its ends.

For an edge e with two ends u and v , it can be regarded as two half edges h_e^u and h_e^v , where h_e^u is incident with u and h_e^v is incident with v . Let $H(G)$ be the set of all half edges of G , and $H_G(u)$ be the set of all half edges incident with u . An *orientation* of (G, σ) is a mapping $\tau : H(G) \rightarrow \{\pm 1\}$ such that $\tau(h_e^u)\tau(h_e^v) = -\sigma(e)$ for each edge $e \in E(G)$. For a half edge $h_e^u \in H(G)$, we say h_e^u is *oriented away from u* if $\tau(h_e^u) = +1$; otherwise h_e^u is *oriented toward u* .

Definition 2.1. Let (G, σ) be a signed graph, A be an abelian group, and τ be an orientation of (G, σ) . Let $f : E(G) \rightarrow A$ be a function, and $k \geq 2$ be an integer.

- (1) For each vertex $v \in V(G)$, the boundary of f at v is

$$\partial f(v) = \sum_{h \in H_G(v)} \tau(h)f(e_h),$$

where e_h is the edge of G containing the half edge h .

- (2) The support of f , denoted by $\text{supp}(f)$, is the set of edges e for which $f(e)$ is not equivalent to the identity element of A .

(3) Let $A = \mathbb{Z}$. Then the ordered pair (τ, f) is a k -flow of (G, σ) if $\partial f(v) = 0$ for each $v \in V(G)$ and $|f(e)| < k$ for each $e \in E(G)$. A k -flow (τ, f) is a nowhere-zero k -flow if $\text{supp}(f) = E(G)$.

(4) Let $A = \mathbb{Z}_k$. Then the ordered pair (τ, f) is called a $\mathbb{Z}_k$ -flow of (G, σ) if $\partial f(v) = 0$ for each vertex v . A $\mathbb{Z}_k$ -flow (τ, f) is a nowhere-zero $\mathbb{Z}_k$ -flow if $\text{supp}(f) = E(G)$.

For convenience, we abbreviate “nowhere-zero k -flow” as k -NZF and “nowhere-zero $\mathbb{Z}_k$ -flow” as $\mathbb{Z}_k$ -NZF. If the orientation is understood from the context, we use f instead of (τ, f) to denote a flow.

Switching at a vertex v only reverses the directions of the half edges incident with v , while the directions of other half edges and the flow values of all edges remain unchanged. Thus, if (G, σ) is switching equivalent to (G, σ') and (G, σ) admits a k -NZF, then (G, σ') also admits a k -NZF.

A signed graph is considered *flow-admissible* if it admits a k -NZF for some integer k . The following characterization of flow-admissible signed graphs can be found in [3, 10].

Proposition 2.2. [10] *The following statements are equivalent for every connected unbalanced signed graph (G, σ) :*

- (a) (G, σ) is flow-admissible.
- (b) The edges of G can be covered with signed circuits.
- (c) (G, σ) has no edge e such that $(G \setminus e, \sigma)$ has a balanced component.

Based on statements (a) and (b) in Proposition 2.2, we have the following lemma.

Lemma 2.3. *Let $\mathcal{S}(G, \sigma)$ be the set of all signed circuits in the signed graph (G, σ) . Then (G, σ) is flow-admissible if and only if*

$$\bigcup_{(C, \sigma) \in \mathcal{S}(G, \sigma)} E(C) = E(G),$$

i.e., every edge of G is covered by signed circuits.

There is a direct corollary as follows.

Corollary 2.4. *The signed graph (G, σ) is flow-admissible if and only if every edge of G is contained in a flow-admissible signed subgraph of (G, σ) .*

For any ordinary graph G , we define $E_G(v) = \{e \in E(G) : e \text{ is incident with } v\}$, where v is a vertex in $V(G)$. For $F \subseteq E_G(v)$, we denote by $G_{[v; F]}$ the graph obtained from G by splitting all edges of F away from v and adding a new vertex v' as the end of these edges.

Note that, for a signed graph (G, σ) , the signature σ is a function defined on $E(G)$, and splittings do not change the edge set. Thus, $(G_{[v; F]}, \sigma)$ is a signed graph obtained from (G, σ) by performing a splitting at v with respect to F . Furthermore, if $(G_{[v; F]}, \sigma)$ admits a k -NZF, then so does (G, σ) .

3. SIGNED 4-NZF-ADMISSIBLE GRAPHS AND 3-EDGE-COLORABLE CUBIC GRAPHS

In this section, a method is developed for deriving a signed 3-edge-colorable cubic graph from a signed 4-NZF-admissible graph. Using this method, we establish an equivalence in the admission of k -NZF between signed 4-NZF-admissible graphs and signed 3-edge-colorable cubic graphs, as well as between signed supereulerian graphs and signed Hamiltonian cubic graphs. By applying these relationships, we show that every flow-admissible signed 4-NZF-admissible graph admits a 10-NZF, and every flow-admissible signed supereulerian graph admits an 8-NZF. Furthermore, we apply these relationships to prove that signed supereuler graphs with a spanning even Eulerian graph admit a $\mathbb{Z}_4$ -NZF.

To achieve these results, we first introduce and explore several properties of 4-NZF-admissible graphs. The following theorem illustrates how two 2-flows contribute to our understanding of the structure of graphs that are 4-NZF-admissible.

Theorem 3.1. [19] *Let G be a graph and k_1, k_2 be two integers. Then G admits a nowhere-zero $k_1 k_2$ -flow if and only if G admits a k_1 -flow f_1 and a k_2 -flow f_2 such that $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$.*

Hence, the graph G is 4-NZF-admissible if and only if G admits a 2-flow f_1 and a 2-flow f_2 such that $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$. Note that if f is a 2-flow in G , then $\text{supp}(f)$ induces an even subgraph in G . Consequently, a 4-NZF-admissible graph can be covered by two even subgraphs.

We need more notation and terminology.

To *contract* an edge e of a graph G means to delete the edge e and then identify its ends. The resulting graph is denoted by G/e . For $S \subseteq E(G)$, let G/S denote the graph obtained from G by contracting all edges of S .

Let $C(G)$ be the set of components of graph G . The degree of a vertex v in the graph G , denoted by $d_G(v)$.

Let X and Y be two disjoint vertex sets of G . We denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other end in Y . For a subgraph H of G , denote the boundary of H , $E_G[V(H), V(G) \setminus V(H)]$, by $\partial(H)$.

The following lemma presents the method for deriving a signed 3-edge-colorable graph from a signed 4-NZF-admissible graph, while preserving certain properties.

Lemma 3.2. *Let G be a 4-NZF-admissible graph, with f_1 and f_2 being two 2-flows on G such that $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$.*

For a signed 4-NZF-admissible graph (G, σ) , there exists a signed 3-edge-colorable graph (G', σ') such that the following statements hold:

- (1) *If (G, σ) is flow-admissible, then (G', σ') is flow-admissible;*
- (2) *Let H_1 be a spanning subgraph of G with edge set $\text{supp}(f_1)$. There exists a 2-factor J of G' such that there is a bijection $f : C(H_1) \rightarrow C(J)$, and for any $I \in C(H_1)$, we have $\sigma(I) = \sigma'(f(I))$;*
- (3) *Let $S = E(G') \setminus E(G)$. For any edge $e \in S$, we have $\sigma'(e) = +1$. Furthermore, $(G'/S, \sigma' |_{E(G'/S)}) \cong (G, \sigma)$.*

Proof. Since a cubic graph is 3-edge-colorable if and only if it has a 2-factor and each component of the 2-factor forms an even circuit, our objective is to derive an even circuit from every component of H_1 . Meanwhile, we must ensure that the flow-admissible property is maintained if (G, σ) is flow-admissible. Thus, we may always assume that (G, σ) is flow-admissible.

Since a component I of H_1 is Eulerian, I has an Euler tour, denoted by T . Let $T = v_0 e_1 v_1 e_2 \cdots v_k e_0 v_0$. We will use a series of splittings such that I converts into a circuit. Let $G^1 = G$. For $x \in [2, k]$, $G^x = G^{x-1}_{[v_x; \{e_x, e_{x+1}\}]}$ if $|E_{G^{x-1}}(v_x) \cap E(I)| > 2$, otherwise $G^x = G^{x-1}$, where $e_{k+1} = e_0$. This iteration will be carried out $k-1$ times, and the resulting graph is G^k . Meanwhile, the resulting signed graph is (G^k, σ) . Let I^k be the subgraph of G^k induced by edge set $E(I)$. Since I is Eulerian and T is the Euler tour of I , we have I^k is a circuit. Note that (G^k, σ) may not be flow-admissible. We will add some positive edges to ensure the property of flow-admissibility.

For any $v \in V(G)$, let $\{v'_0, v'_1, v'_2, \dots, v'_l\}$, where $l \geq 0$ and $v'_0 = v$, be a vertex set whose elements are obtained by splitting v . If $l = 1$, we add two positive edges (multiple edges) to connect v'_0 and v'_1 . If $l > 1$, we add $\frac{l(l+1)}{2}$ positive edges to (G^k, σ) such that $\{v'_0, v'_1, v'_2, \dots, v'_l\}$ induce an all-positive complete graph. Denote the new signed graph by (G_1, σ_1) . Since all-positive digons and all-positive complete graphs are flow-admissible, the added positive edges are covered by flow-admissible subgraphs. Next, we need to verify that every edge in $E(G)$ is also covered by a flow-admissible subgraph.

More precisely, since (G, σ) is flow-admissible, every edge in $E(G)$ is covered by a signed circuit in (G, σ) . Thus, our goal is to show that each signed circuit in (G, σ) can be extended to a signed

circuit in (G_1, σ_1) . Let (C, σ) be a signed circuit of (G, σ) and (C', σ_1) be subgraph of (G_1, σ_1) induced by $E(C)$. If $(C, \sigma) \cong (C', \sigma_1)$, then we are done. Hence, we may always assume that (C, σ) is not isomorphic to (C', σ_1) , where there are three cases as follows.

Case 1. (C, σ) is a balanced circuit in (G, σ) .

Since C' is obtained from C by a sequence of splittings and C is a circuit, the vertices of C' have degree 2 or 1. Let v_1 be a vertex of degree 1 in C' . Then v_1 is split from a vertex v in C . Since v has degree 2 in C , there is another vertex, say v_2 , in C' that is also split from v . It is easy to see that $d_{C'}(v_2) = 1$. Note that there is a positive edge v_1v_2 in (G_1, σ_1) . Then we add the positive edge v_1v_2 to C' , and we still denote the resulting graph by C' . We repeat this operation until there are no vertices of degree 1 in C' . Then (C', σ_1) is a balanced circuit in (G_1, σ_1) .

Case 2. (C, σ) is a short barbell in (G, σ) .

For two vertices of C' that are split from a vertex of degree 2 in C , we can add a positive edge to connect them, as in Case 1. Hence, for convenience, we may assume that only the vertex of degree 4 in C was split. Let v be the vertex of degree 4 in C , which split into several vertices in C' .

Subcase 2.1. The vertex v has been split into two vertices v_1 and v_2 in C' .

We only need to consider the following two cases. Namely, $d_{C'}(v_1) = d_{C'}(v_2) = 2$, or one of $d_{C'}(v_1)$ and $d_{C'}(v_2)$ is 3 and the other is 1.

If $d_{C'}(v_1) = d_{C'}(v_2) = 2$, then (C', σ_1) is either a balanced circuit or a union of two unbalanced circuits. For the first case, (C', σ_1) is already a signed circuit. For the second case, adding a positive edge v_1v_2 to (C', σ_1) , the resulting signed subgraph is a long barbell in (G_1, σ_1) .

If one of $d_{C'}(v_1)$ and $d_{C'}(v_2)$ is 3 and the other is 1, then we add a positive edge v_1v_2 of (G_1, σ_1) to (C', σ_1) . The resulting signed subgraph is a short barbell in (G_1, σ_1) .

Subcase 2.2. The vertex v has been split into three vertices v_1, v_2 and v_3 in C' .

Since $d_C(v) = 4$, there is a vertex v_i that has degree 2 in C' , $i \in [1, 3]$, say v_1 . Then $d_{C'}(v_2) = d_{C'}(v_3) = 1$. Then either (C', σ_1) is a path with positive sign or a union of a path with negative sign and an unbalanced circuit.

For the first case, adding a positive edge v_2v_3 to (C', σ') , the resulting signed subgraph is a balanced circuit in (G_1, σ_1) .

For the second case, adding two positive edges v_1v_2 and v_2v_3 to (C', σ_1) , the resulting signed subgraph is a long barbell in (G_1, σ_1) .

Subcase 2.3. The vertex v has been split into four vertices v_1, v_2, v_3 and v_4 in C' .

Since $d_C(v) = 4$, we have $d_{C'}(v_1) = d_{C'}(v_2) = d_{C'}(v_3) = d_{C'}(v_4) = 1$. Hence, (C', σ_1) is a union of two paths with negative sign, say (P_1, σ_1) and (P_2, σ_1) . Without loss of generality, let $\{v_1, v_2\} \subseteq V(P_1)$, and $\{v_3, v_4\} \subseteq V(P_2)$. Then v_1, v_2, v_3 and v_4 are ends of P_1 and P_2 , respectively. By adding two positive edges v_1v_3 and v_2v_4 to (C', σ_1) , the resulting signed subgraph forms a balanced circuit in (G_1, σ_1) .

Case 3. (C, σ) is a long barbell in (G, σ) .

For convenience, we assume that only the vertices of degree 3 in C have been split. By symmetry, we can further assume that only one vertex of degree 3, say v , has been split.

Let (C^*, σ) be the unbalanced circuit in (C, σ) that contained v as a vertex, and (P^*, σ) be the path in (C, σ) that meets the unbalanced circuits only at its ends. Let (L, σ_1) be a subgraph of (C', σ_1) , where L is induced by $E(C^*) \cup E(P^*)$.

Subcase 3.1. The vertex v has been split into two vertices v_1 and v_2 in C' .

Since $d_C(v) = 3$, there exists a vertex v_i of degree 2 in C' , where $i \in [1, 2]$, say v_1 . Consequently, the degree of v_2 is 1. The signed graph (L, σ_1) is either a path or a union of an unbalanced circuit and a path. Therefore, adding a positive edge v_1v_2 to (C', σ_1) , the resulting signed subgraph forms a long barbell in (G_1, σ_1) .

Subcase 3.2. The vertex v has been split into three vertices v_1, v_2 and v_3 in C' .

Since $d_C(v) = 3$, we have $d_C(v_1) = d_C(v_2) = d_C(v_3) = 1$. Then (L, σ_1) is a union of a path and a path with negative sign, say (P_1, σ_1) and (P_2, σ_1) . Without loss of generality, let v_1 be an end in P_1 , and let v_2 and v_3 be ends in P_2 . Then, when we add two positive edges v_1v_2 and v_1v_3 to (C', σ_1) , the resulting signed subgraph is a long barbell in (G_1, σ_1) .

Since every edge of (G_1, σ_1) is contained in a flow-admissible signed subgraph, (G_1, σ_1) is also flow-admissible.

Next, we aim to transform all the vertices in $V(I^k)$ into vertices of degree 3 through a series of blow up. For technical reasons, a digon is considered a circuit of length 2, denoted by C_2 . For a vertex $v \in V(I^k)$, we replace v by an all-positive circuit $(C_v, +)$ of length $d(v)$ and define the incidence relation between the edges of $E_{G_1}(v)$ and the vertices of $(C_v, +)$ as follows. Let the two edges in $E(I^k) \cap E_{G_1}(v)$ be incident with v_1 and v_2 , respectively, where v_1 and v_2 are adjacent in C_v . Then, $E(I^k)$ combined with $E(C_v) \setminus \{v_1v_2\}$ can be extended to form a new circuit that contains all vertices of I^k and C_v . Note that $|E_{G_1}(v) \setminus E(I^k)| = |V(C_v) \setminus \{v_1, v_2\}| = d_{G_1}(v) - 2$. Let φ be an arbitrary bijection from $E_{G_1}(v) \setminus E(I^k)$ to $V(C_v) \setminus \{v_1, v_2\}$. Then an edge $e \in E_{G_1}(v) \setminus E(I^k)$ is incident with a vertex $v \in V(C_v) \setminus \{v_1, v_2\}$ if and only if $\varphi(e) = v$. Since $(C_v, +)$ is a balanced circuit, every edge of $E(C_v)$ is covered by a signed circuit. For any signed circuit in (G_1, σ_1) containing v , it is easy to verify that replacing vertex v by an all-positive circuit still maintains flow-admissibility. Therefore, every edge of the resulting signed graph is contained in a flow-admissible subgraph, i.e., the resulting signed graph is flow-admissible. We repeat this operation for all vertices of I^k . Denote the new circuit obtained from I^k by I_2 , and the resulting signed graph by (G_2, σ_2) . It is easy to see that every vertex in $V(I_2)$ has degree 3 in G_2 and (G_2, σ_2) is flow-admissible.

It remains to prove that I_2 is an even circuit. Note that any edge which is incident with two vertices of I_2 contributes an even number of vertices to I_2 . Thus, in graph G_2 , we only need to consider the number of edges that have only one end in I_2 . We need to note that the series of operations we performed to transform I into I_2 do not change the number of edges connecting this component to the outside. Hence, we only need to consider the number of edges that have only one end in I in graph G , i.e., the number of elements in the boundary $\partial_G(I)$. Recall that f_1 and f_2 are two 2-flows in G such that $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$, and H_i is a spanning subgraph in G with edge set $\text{supp}(f_i)$, $i \in \{1, 2\}$, and I is a component of H_1 . Thus, $\partial_G(I) \subseteq E(H_2)$. Since $\partial_G(I)$ is an edge-cut of G , we have that $\partial_G(I)$ is also an edge-cut of H_2 . Note that H_2 is an even graph. Therefore, $|\partial_G(I)|$ is even. Thus, all edges of $\partial_G(I)$ contribute an even number of vertices to I_2 . Hence, I_2 is an even circuit.

The process of converting (I, σ) into (I_2, σ_2) is called 3-regularizing of I , and we call I_2 the 2-normal graph of I . Let (G', σ') be the signed graph obtained from (G, σ) by 3-regularizing all

components of H_1 , and let (J, σ') be the union of the 2-normal graphs of all components of H_1 . It is easy to see that (G', σ') is a flow-admissible signed cubic graph and J is a 2-factor of G' . Since every component of J is an even circuit, (G', σ') is a flow-admissible signed 3-edge-colorable cubic graph. Thus, Statement (1) holds.

Each component in H_1 has a unique 2-normal graph in J , which is a component in J . Conversely, every component in J is obtained from a component in H_1 by 3-regularizing. Therefore, there exists a natural bijection $f : C(H_1) \rightarrow C(J)$ such that for any $I \in C(H_1)$, $f(I)$ is a component of J obtained from I by 3-regularizing. Since every edge we added is positive, we have $\sigma(I) = \sigma'(f(I))$. Consequently, Statement (2) holds.

It is easy to see that $S = E(G') \setminus E(G)$ is the set of all the edges we added. Thus, for any edge $e \in S$, we have $\sigma'(e) = +1$. In order to show the structure of $(G'/S, \sigma'|_{E(G'/S)})$, we will show that there is a decomposition $\{E(S_u) : u \in V(G)\}$ of S , where (S_u, σ') is an induced all-positive subgraph in (G', σ') . Next, we introduce the vertex set of the graph S_u .

Let u be a vertex of G that is in a component I' of H_1 . In the process of 3-regularizing I' , the vertex u is initially split into $\frac{d_{I'}(u)}{2}$ vertices, denoted by $u_0, u_1, u_2, \dots, u_{\frac{d_{I'}(u)}{2}-1}$, where $u_0 = u$.

Subsequently, for any $i \neq 0$, each vertex u_i is blown up into $\frac{d_{I'}(u)}{2} + 1$ vertices $u_i^1, u_i^2, \dots, u_i^{\frac{d_{I'}(u)}{2}+1}$ if $d_{I'}(u) \neq 4$, and each vertex u_i is blown up into $\frac{d_{I'}(u)}{2} + 2 = 4$ vertices $u_i^1, u_i^2, u_i^3, u_i^4$ if $d_{I'}(u) = 4$. For $i = 0$, vertex u_0 is blown up into $d_G(u) - \frac{d_{I'}(u)}{2} + 1$ vertices $u_0^1, u_0^2, \dots, u_0^{d_G(u) - \frac{d_{I'}(u)}{2} + 1}$ if $d_{I'}(u) \neq 4$, and vertex u_0 is blown up into $d_G(u) - \frac{d_{I'}(u)}{2} + 2 = d_G(u)$ vertices $u_0^1, u_0^2, \dots, u_0^{d_G(u)}$ if $d_{I'}(u) = 4$. Let $B(u_i)$ be the set of vertices blown up from u_i , $i \in [0, \frac{d_{I'}(u)}{2} - 1]$. Therefore, the vertex set $V(S_u) = \bigcup_{i \in [0, \frac{d_{I'}(u)}{2} - 1]} B(u_i)$.

Note that every edge of S_u is an element in S . Thus, (S_u, σ') is all-positive. Conversely, for any edge $e \in S$, the edge e is an element in some $E(S_u)$, where $u \in V(G)$. Thus, $\bigcup_{u \in V(G)} E(S_u) = S$. It is evident that $V(S_y) \cap V(S_z) = \emptyset$ if y and z are distinct vertices of G . Therefore, $\{E(S_u) : u \in V(G)\}$ is a decomposition of S . Furthermore, $\partial_{G'}(S_u) = \partial_G(u)$, for any vertex $u \in V(G)$.

Let u' be the vertex in the graph G'/S obtained by contracting the edges in the set $E(S_u)$. Define a mapping $g : G'/S \rightarrow G$ such that $g(u') = u$. Consider a and b as two distinct vertices of G . Then $E_{G'}[V(S_a), V(S_b)] = E_G[\{a\}, \{b\}]$. Thus, g establishes an isomorphism between G'/S and G .

If G has no multiple edges, then g also acts as an isomorphism between $(G'/S, \sigma'|_{E(G'/S)})$ and (G, σ) , where $\sigma'|_{E(G'/S)}$ denotes the restriction of σ' to $E(G'/S)$. Note that $E(G'/S) = E(G)$. In cases where G contains multiple edges, let $g(e) = e$. Then g remains an isomorphism between $(G'/S, \sigma'|_{E(G'/S)})$ and (G, σ) . Therefore, $(G'/S, \sigma'|_{E(G'/S)}) \cong (G, \sigma)$, validating Statement (3). $\square$

Remark 3.3. We note that a similar reduction method was introduced in [8]. However, our reduction distinguishes itself by explicitly transforming Eulerian subgraphs in 4-NZF-admissible graphs into bichromatic circuits in the resulting 3-regular graph, while crucially preserving the sign of these Eulerian subgraphs throughout the transformation.

Let f be a k -NZF of the signed graph (G, σ) , and let S be a set of positive edges in (G, σ) . Consider the signed graph $(G/S, \sigma|_{G/S})$, which is obtained by contracting all edges in S . For simplicity, we denote it by $(G/S, \sigma)$. After contracting the edges in S , there exists a k -NZF, denoted by $f|_{G/S}$, in $(G/S, \sigma)$. Here, $f|_{G/S}$ represents the restriction of f to the edge set $E(G/S)$. Let us recall Theorem 3.4.

Theorem 3.4. Let k be a positive integer. Then the following statements are equivalent:

- (1) *Every flow-admissible signed 4-NZF-admissible graph admits a nowhere-zero k -flow;*
- (2) *Every flow-admissible signed 3-edge-colorable cubic graph admits a nowhere-zero k -flow.*

Proof. It is straightforward that (1) implies (2) since every 3-edge-colorable cubic graph is 4-NZF-admissible. Therefore, we only need to prove that (2) implies (1).

Let (G, σ) be a flow-admissible signed 4-NZF-admissible graph. According to Lemma 3.2, there exists a flow-admissible signed 3-edge-colorable cubic graph (G', σ') such that $(G'/S, \sigma') \cong (G, \sigma)$, where S is a set of positive edges. Since every flow-admissible signed 3-edge-colorable cubic graph admits a k -NZF, (G', σ') admits a k -NZF. Consequently, $(G'/S, \sigma')$ admits a k -NZF, and therefore, so does (G, σ) . $\square$

Let $G_1, G_2, \dots, G_t$ be subgraphs of G . The notation $G_1 \triangle G_2 \triangle \dots \triangle G_t$ represents the symmetric difference of these subgraphs. The following theorem shows the equivalence in the admission of k -NZF among signed supereulerian graphs, signed Hamiltonian graphs and signed Hamiltonian cubic graphs.

Theorem 3.5. *Let k be a positive integer. Then the following statements are equivalent:*

- (1) *Every flow-admissible signed supereulerian graph admits a nowhere-zero k -flow;*
- (2) *Every flow-admissible signed Hamiltonian graph admits a nowhere-zero k -flow;*
- (3) *Every flow-admissible signed Hamiltonian cubic graph admits a nowhere-zero k -flow.*

Proof. It is trivial that (1) implies (2) and (2) implies (3). Thus, we only need to prove that (3) implies (1).

Since G is a supereulerian graph, it contains a spanning Eulerian subgraph H_1 . For any edge $e \in E(G) \setminus E(H_1)$, there exists a circuit in $H_1 \cup e$ that contains e , we denote it by C_e . Let $H_2 = \triangle_{e \in E(G) \setminus E(H_1)} C_e$. Then H_2 is an even graph. Let f_1 be a 2-flow with $\text{supp}(f_1) = E(H_1)$ and f_2 be a 2-flow with $\text{supp}(f_2) = E(H_2)$. Therefore, $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$.

Let (G, σ) be a flow-admissible signed supereulerian graph. By Lemma 3.2, there exists a flow-admissible signed 3-edge-colorable cubic graph (G', σ') such that $(G'/S, \sigma') \cong (G, \sigma)$, where S is a set of positive edges. And there exists a 2-factor J of G' such that there is a bijection $f : C(H_1) \rightarrow C(J)$. Since $|C(H_1)| = 1$, we have $|C(J)| = 1$. Thus, J is a Hamiltonian circuit of G' . Therefore, (G', σ') is a signed Hamiltonian cubic graph. Since every flow-admissible signed Hamiltonian cubic graph admits a k -NZF, (G', σ') admits a k -NZF. Thus, $(G'/S, \sigma')$ admits a k -NZF, and so does (G, σ) . $\square$

For $\mathbb{Z}_k$ -flow, we can prove that a class of flow-admissible signed supereulerian graphs admits a $\mathbb{Z}_4$ -NZF. To conclude this section with an application of Lemma 3.2, we prove that every signed supereulerian graph with a spanning even Eulerian subgraph admits a $\mathbb{Z}_4$ -NZF. Before proceeding, it is necessary to define some terms and introduce relevant lemmas. A signed graph (G, σ) is called *antibalanced* if all even circuits in G are balanced and all odd circuits are unbalanced. Máčajová et al. [9] provide the following characterization of signed cubic graphs that admit a $\mathbb{Z}_4$ -NZF.

Theorem 3.6. [9] *A signed cubic graph admits a $\mathbb{Z}_4$ -NZF if and only if it has an antibalanced 2-factor.*

Recall that an even Eulerian graph is a signed Eulerian graph with an even number of negative edges. For $\mathbb{Z}_4$ -NZF, we have the following corollary.

Corollary 3.7. *Every signed supereulerian graph with a spanning even Eulerian subgraph admits a $\mathbb{Z}_4$ -NZF.*

Proof. Let (G, σ) be a signed supereulerian graph with a spanning even Eulerian subgraph (H, σ) . As mentioned in the proof of Theorem 3.5, there are two 2-flows, f_1 and f_2 , on G such that $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$ and $\text{supp}(f_1) = E(H)$. According to Lemma 3.2, there exists a signed 3-edge-colorable cubic graph (G', σ') with a balanced Hamiltonian circuit such that $(G'/S, \sigma') \cong (G, \sigma)$, where S is a set of positive edges. Note that an even circuit with positive sign is both balanced and antibalanced. Consequently, if a Hamiltonian circuit in a cubic graph is antibalanced, it is also balanced due to its even length. Thus, according to Theorem 3.6, every signed Hamiltonian cubic graph with a balanced Hamiltonian circuit admits a $\mathbb{Z}_4$ -NZF. Therefore, (G', σ') admits a $\mathbb{Z}_4$ -NZF, and so does (G, σ) . $\square$

4. NOWHERE-ZERO 6-FLOWS ON SIGNED SUPEREULERIAN GRAPHS WITH A SPANNING EVEN EULERIAN SUBGRAPH

In this section, we discuss the existence of a 6-NZF on a signed supereulerian graph with a spanning even Eulerian subgraph, as follows.

Theorem 4.1. *Let (G, σ) be flow-admissible. If (G, σ) has a spanning even Eulerian subgraph, then (G, σ) admits a nowhere-zero 6-flow.*

The following lemma shows that Theorem 4.1 can be reduced to the problem of deciding whether a flow-admissible signed Hamiltonian graph with a balanced Hamiltonian circuit admits a 6-NZF.

Lemma 4.2. *Let k be a positive integer. Then the following statements are equivalent:*

- (1) *Every flow-admissible signed supereulerian graph with a spanning even Eulerian subgraph admits a k -NZF;*
- (2) *Every flow-admissible signed Hamiltonian graph with a balanced Hamiltonian circuit admits a k -NZF;*
- (3) *Every flow-admissible signed Hamiltonian cubic graph with a balanced Hamiltonian circuit admits a k -NZF.*

Proof. We only need to show that (3) implies (1). Let (G, σ) be a flow-admissible signed supereulerian graph with a spanning even Eulerian subgraph (H, σ) . As mentioned in the proof of Theorem 3.5, there are two 2-flows, f_1 and f_2 , on G such that $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$ and $\text{supp}(f_1) = E(H)$. By Lemma 3.2, there exists a flow-admissible signed Hamiltonian cubic graph (G', σ') with a balanced Hamiltonian circuit such that $(G'/S, \sigma') \cong (G, \sigma)$, where S is a set of positive edges. By Statement (3), (G', σ') admits a k -NZF, so does $(G'/S, \sigma')$. Therefore, (G, σ) admits a k -NZF. $\square$

By Lemma 4.2, in order to prove Theorem 4.1, it therefore suffices to prove Theorem 4.3.

Theorem 4.3. *Let (G, σ) be flow-admissible. If (G, σ) has a balanced Hamiltonian circuit, then (G, σ) admits a nowhere-zero 6-flow.*

Before we proceed, we require the following lemma.

Lemma 4.4. [4] *If a signed graph (G, σ) is connected and admits a $\mathbb{Z}_2$ -flow f_1 such that $\text{supp}(f_1)$ has an even number of negative edges, then it also admits a 3-flow f_2 with $\text{supp}(f_1) = \{e \in E(G) : f_2(e) = \pm 1\}$.*

By Lemma 4.4, we have the following lemma that shows the existence of a 3-flow in a signed graph with an all-positive Hamiltonian circuit.

Lemma 4.5. *Let (G, σ) be a signed graph with an all-positive Hamiltonian circuit (H, σ) . If $|E_N(G, \sigma)|$ is even, then (G, σ) admits a 3-flow f such that $E(G) \setminus E(H) \subseteq \{e \in E(G) : f(e) = \pm 1\}$.*

Proof. For an edge $e \in H$, there exists a Hamiltonian path $H \setminus e$ of G , denoted by P . Then, for any edge $e_1 \in E(G) \setminus E(H)$, we have $P \cup \{e_1\}$ forms a unique circuit, denoted by C_{e_1} .

The symmetric difference $\bigtriangleup_{e \in E(G) \setminus E(H)} C_e$, denoted by H' , contains all edges of $E(G) \setminus E(H)$. Therefore, (H', σ) is an even graph with an even number of negative edges. It is evident that every signed even graph admits a $\mathbb{Z}_2$ -NZF. Consequently, (H', σ) also admits a $\mathbb{Z}_2$ -NZF f' .

Since G is connected, by Lemma 4.4, (G, σ) admits a 3-flow f such that $E(G) \setminus E(H) \subseteq \{e \in E(G) : f(e) = \pm 1\}$. $\square$

The following theorem shows that if a signed graph with an all-positive Hamiltonian circuit has an even number of negative edges, then it admits a 6-NZF.

Lemma 4.6. *Let (G, σ) be a signed graph with an all-positive Hamiltonian circuit (H, σ) . If $|E_N(G, \sigma)|$ is even, then (G, σ) admits a 6-NZF.*

Proof. According to Lemma 4.5, (G, σ) admits a 3-flow f_1 such that $\text{supp}(f_1) \supseteq E(G) \setminus E(H)$. It is important to note that (G, σ) admits a 2-flow f_2 with $\text{supp}(f_2) = E(H)$, since (H, σ) is all-positive. Therefore, $f_1 + 3f_2$ forms a 6-NZF on (G, σ) . $\square$

Let H be a Hamiltonian circuit in the graph G , with the vertex sequence $v_0 v_1 \cdots v_{n-1} v_0$. Let e_1 and e_2 be two edges in $E(G) \setminus E(H)$. Suppose the ends of e_1 are v_i and v_j , and the ends of e_2 are v_k and v_l . We say that e_1 and e_2 are *intersect* along H if $i < k < j < l$. Meanwhile, e_1 and e_2 are said to be *parallel* along H if $k < i < j < l$ or $i < j < k < l$. These terms originate from plane geometry. When we draw the Hamiltonian circuit H as a circle on a plane, and connect four distinct points with two line segments, these segments either intersect or do not intersect.

The following lemma discusses the existence of a 6-NZF in a signed graph that contains an all-positive Hamiltonian circuit and has two negative edges that intersecting along this circuit.

Lemma 4.7. *Let (G, σ) be a flow-admissible signed graph with an all-positive Hamiltonian circuit (H, σ) . If two negative edges intersect along H , then (G, σ) admits a 6-NZF.*

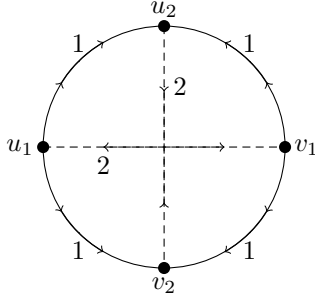
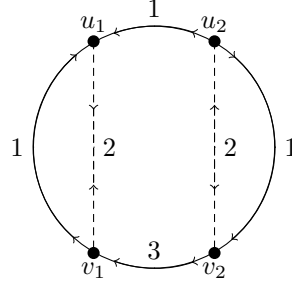
Proof. If $|E_N(G, \sigma)|$ is even, then it is a direct corollary of Lemma 4.6. Therefore, for the remainder of the proof, we assume that $|E_N(G, \sigma)|$ is odd. Let e_1 and e_2 be two negative edges that intersect along H . Then $(G \setminus e_1, \sigma)$ is a signed graph with an all-positive Hamiltonian circuit, and $|E_N(G \setminus e_1, \sigma)|$ is even. By Lemma 4.5, $(G \setminus e_1, \sigma)$ admits a 3-flow f_1 and $E(G) \setminus (E(H) \cup \{e_1\}) \subseteq \text{supp}(f_1)$. Let $e_1 = u_1 v_1$ and $e_2 = u_2 v_2$. There exists a 3-flow of $(H \cup \{e_1, e_2\}, \sigma)$, denoted by f_2 , as illustrated in Fig. 4.1 (omitting edges with a weight of 0 and vertices of degree 2). This 3-flow is constructed as follows:

$$f_2(e) = \begin{cases} 2, & \text{if } e \in \{e_1, e_2\}; \\ 1, & \text{if } e \in E(H). \end{cases}$$

Since $f_1(e_2) = \pm 1$, $f_2(e_2) = 2$, and $f_2(E(H)) = \{1\}$, we conclude that either $2f_1 + f_2$ or $2f_1 - f_2$ is a 6-NZF on (G, σ) . $\square$

We present the proof of Theorem 4.3 below.

The proof of Theorem 4.3. We assume that the balanced Hamiltonian circuit (H, σ) is all-positive; otherwise, we switch at some vertices of H to ensure that every edge in H is positive. By Lemma 4.2, we assume that (G, σ) is a signed cubic graph. Consequently, any two edges in $E(G) \setminus E(H)$

FIGURE 4.1. A 3-NZF f_2 on $(H \cup \{e_1, e_2\}, \sigma)$.FIGURE 4.2. A 4-NZF f_1 on (G_1, σ) .

are either intersecting or parallel along H . By Lemma 4.6 and Lemma 4.7, we can further assume that $|E_N(G, \sigma)|$ is odd and that any two negative edges are parallel along H .

Let e^1 and e^2 be two negative edges with ends u_1, v_1 and u_2, v_2 , respectively. Consider the path $P = v_1 e_0 w_1 e_1 w_2 e_2 \cdots w_k e_k v_2$, where e_i is an edge and w_i is a vertex, $i \in [0, k]$. This path in H that connects v_1 and v_2 and does not contain u_1 and u_2 as vertices. We may assume that every vertex in $V(P) \setminus \{v_1, v_2\}$ is incident only with positive edges. Otherwise, we replace e_1 by the negative edge incident to some vertex w_i , where $i \in [1, k]$. Thus, these two negative edges e_1, e_2 and path P can definitely be found in the signed graph (G, σ) .

Define $G_1 = H \cup \{e^1, e^2\}$. It is easy to see that (G_1, σ) admits a 4-NZF f_1 (see Fig. 4.2, we omit the edge which weighted by 0, and the vertices of degree 2). Next, we will construct a 3-flow f_2 on (G, σ) such that $f_1 + 2f_2$ or $f_1 - 2f_2$ is a 6-NZF on (G, σ) .

Let $M_P = \{e \in E(G) : e \text{ has at least one end in } P\} \setminus \{e^1, e^2\}$. Note that, $\sigma(e) = +1$ for all $e \in M_P$. Let $M = E(G) \setminus (E(H) \cup \{e^2\})$. We will remove certain edges from (G, σ) to obtain a signed subgraph (G_2, σ) of (G, σ) . Depending on the parity of $|E(P)|$, we will discuss the structure of (G_2, σ) . There are two distinct cases to consider.

Case 1. $|E(P)|$ is odd.

After removing all edges of $\{e^2, e_0, e_2, \dots, e_k\}$ from (G, σ) , denote the resulting signed graph by (G_2, σ) . Since $\{e_0, e_2, \dots, e_k\}$ is a matching in G , each vertex in $V(P) \setminus \{v_1, v_2\}$ has degree 2 in G_2 . Given that M_P and $\{e_1, e_3, \dots, e_{k-1}\}$ are disjoint matchings, $M_P \cup \{e_1, e_3, \dots, e_{k-1}\}$ induces a disjoint union of paths and circuits, denoted by $P_1, P_2, \dots, P_x$ and $C_1, C_2, \dots, C_y$, respectively. Note that each circuit C_i is a component of G_2 , and (C_i, σ) is all-positive for each $i \in [1, y]$. Let P' be a path induced by the edge set $E(H) \setminus E(P)$, and let $M' = M \setminus M_P$. Note that in G_2 , there is a single vertex v_2 of degree 1, and no edge in M' has v_2 as an end. Additionally, there is no path P_j containing v_2 as an end, for $j \in [1, x]$. Therefore, the ends of every P_j and each edge in M' are vertices of degree 3 in G_2 . As a result, all edges in M' have their ends in $V(P') \setminus \{v_2\}$, and likewise, each P_j has its ends in $V(P') \setminus \{v_2\}$, for $j \in [1, x]$. Thus, G_2 has $y + 1$ components.

Case 2. $|E(P)|$ is even.

Let e^* be an edge incident with v_1 that is different from e^1 and e_0 . Note that $e^* \in E(H)$. By removing the edges in the set $\{e^*, e^2, e_1, e_3, \dots, e_k\}$ from (G, σ) , the resulting signed graph is denoted by (G_2, σ) . Since the set $\{e^*, e_1, e_3, \dots, e_k\}$ forms a matching in G , every vertex of $V(P) \setminus \{v_2\}$ has degree 2 in G_2 . Since $M_P \cup \{e^1\}$ and $\{e_0, e_2, \dots, e_{k-1}\}$ are disjoint matchings, the union

$M_P \cup \{e^1, e_0, e_2, \dots, e_{k-1}\}$ induces a disjoint union of paths and circuits. These are denoted by $P_1, P_2, \dots, P_x$ and $C_1, C_2, \dots, C_y$, respectively. It is easy to see that each circuit C_i is a component of G_2 . Since e^1 has only one end in P , it cannot be present in any circuit C_i , for $i \in [1, y]$. Thus, each (C_i, σ) is all-positive and $e^1 \in E(P_j)$ for a unique $j \in [1, y]$. Let P' be a path induced by the edge set $E(H) \setminus (E(P) \cup \{e^*\})$ and let $M' = M \setminus (M_P \cup \{e^1\})$. Note that in G_2 , there is only one vertex v_2 has degree 1, and no edge in M' containing v_2 as an end. Additionally, no path P_j terminates at v_2 for $j \in [1, x]$. Therefore, the ends of every P_j and every edge in M' must be vertices of degree 3 in G_2 . Consequently, the ends of all edges in M' are in $V(P') \setminus \{v_2\}$, and the ends of each P_j are also within $V(P') \setminus \{v_2\}$ for $j \in [1, x]$. Thus, G_2 has $y + 1$ components.

Let G_3 be the component of G_2 that contains the P' . Define the set

$$\mathcal{C} = \{C : C \text{ is the unique circuit in } P' \cup e \text{ or } P' \cup P_j, j \in [1, x]\}.$$

The symmetric difference $\Delta_{C \in \mathcal{C}} C$, denoted by G_4 , contains all edges in $E(G_3) \setminus E(P') = M' \cup (\bigcup_{j \in [1, y]} E(P_j)) = M \cup \{e_1, e_3, \dots, e_{k-1}\}$. Let $M^* = M \cup \{e_1, e_3, \dots, e_{k-1}\}$. Since G_4 is an even graph, (G_4, σ) is a signed even graph. Every signed even graph admits a $\mathbb{Z}_2$ -NZF, even if it is not necessarily flow-admissible. Thus, (G_4, σ) admits a $\mathbb{Z}_2$ -NZF f_4 , and $M^* \subseteq \text{supp}(f_4)$. Since $e^2 \notin E(G_4)$ and $M \subseteq E(G_4)$, the signed graph (G_4, σ) has an even number of negative edges. This means that $\text{supp}(f_4)$ has an even number of negative edges. Given that G_3 is connected and f_4 is a $\mathbb{Z}_2$ -flow on (G_3, σ) , the signed graph (G_3, σ) admits a 3-flow f_3 such that $M^* \subseteq \{e \in E(G_4) : f_3(e) = \pm 1\}$. Since each (C_i, σ) is all-positive, (C_i, σ) admits a 2-NZF g_i for each i . By combining these, $f_2 = f_3 + \sum_{i \in [1, y]} g_i$ forms a 3-flow on (G_2, σ) . We can verify that $f_1 + 2f_2$ or $f_1 - 2f_2$ is a 6-NZF on (G, σ) .

For $E(P)$, we have $f_1(E(P)) = \{+3\}$ and $f_2(E(P)) \subseteq \{0, +1, -1\}$. Therefore, $(f_1 \pm 2f_2)(E(P)) \subseteq \{3, +5, -5\}$.

For $E(H) \setminus E(P)$, we have $f_1(E(H) \setminus E(P)) = \{+1\}$ and $f_2(E(H) \setminus E(P)) \subseteq \{0, +1, -1, +2, -2\}$. Therefore, $(f_1 \pm 2f_2)(E(H) \setminus E(P)) \subseteq \{+1, -1, +3, -3, +5\}$.

For $M \setminus \{e_1\}$, we have $f_1(M \setminus \{e_1\}) = \{0\}$ and $f_2(M \setminus \{e_1\}) \subseteq \{+1, -1\}$. Therefore, $(f_1 \pm 2f_2)(M \setminus \{e_1\}) \subseteq \{+2, -2\}$.

For e_1 , we have $f_1(e_1) = +1$ and $f_2(e_1) \in \{+1, -1\}$. Therefore, $(f_1 \pm 2f_2)(e_1) \in \{-1, 3\}$.

For e_2 , we have $f_1(e_2) = +1$ and $f_2(e_2) = 0$. Therefore, $(f_1 \pm 2f_2)(e_2) = +1$.

Thus, $f_1 + 2f_2$ or $f_1 - 2f_2$ is a 6-NZF on (G, σ) . $\square$

A *Kotzig graph* is a cubic graph that has three 1-factors such that the union of any two of them induces a Hamiltonian circuit. Schubert et al. [14] prove that every flow-admissible signed Kotzig graph admits a 6-NZF, i.e., Theorem 4.8. According to Theorem 4.3, we provide an alternative proof of Theorem 4.8 as follows.

Theorem 4.8. [14] *Let (G, σ) be a flow-admissible signed cubic graph. If G is a Kotzig graph, then (G, σ) admits a 6-NZF.*

Proof. Let F_1, F_2 and F_3 be the three 1-factors of G such that the union of any two of them induces a Hamiltonian circuit. By the Pigeonhole Principle, there exist distinct indices $i, j \in \{1, 2, 3\}$ such that $|E_N(F_i, \sigma)| \equiv |E_N(F_j, \sigma)| \pmod{2}$. Thus, $(F_i \cup F_j, \sigma)$ is a balanced Hamiltonian circuit of (G, σ) . By Theorem 4.3, (G, σ) admits a 6-NZF. $\square$

5. NOWHERE-ZERO FLOWS ON SIGNED ABELIAN CAYLEY GRAPHS

In this section, the nowhere-zero flows on signed abelian Cayley graphs are studied. All groups considered in this paper are finite.

5.1. Nowhere-zero 6-flows on signed abelian Cayley graphs.

In this subsection, it is shown that every flow-admissible signed abelian Cayley graph admits a 6-NZF.

Let X be a group and let S be a subset of X that is closed under taking inverses and does not contain the identity. The Cayley graph $\text{Cay}(X, S)$ is defined with vertex set X , where two vertices g and h are adjacent if and only if $hg^{-1} \in S$. A Cayley graph $\text{Cay}(X, S)$ is said to be abelian if X is abelian.

A graph in which every vertex has equal degree k is called regular of valency k . Because connected abelian Cayley graphs possess Hamiltonian circuits, they are supereulerian, and thus admit a 4-NZF. Moreover, Potočník et al. [12] and Nánásiová et al. [11] showed that every abelian Cayley graph of valency at least 5 admits a 3-NZF.

The main result of this subsection shows that such a class of flow-admissible signed 4-NZF-admissible graphs admit a 6-NZF, as follows.

Theorem 5.1. *Every flow-admissible signed abelian Cayley graph admits a nowhere-zero 6-flow.*

To prove Theorem 5.1, we introduce two fundamental structures in abelian Cayley graphs: the circular ladder and the Möbius ladder. Let $n \geq 1$ be an integer. A cubic graph is called a *circular ladder* if it is isomorphic to $C_n \square K_2$, denoted by CL_n . (For the definition of the Cartesian product of graphs, see [2] p. 30.) Let $V(CL_n) = \{x_0, x_1, \dots, x_{n-1}, y_0, y_1, \dots, y_{n-1}\}$, and $E(CL_n) = \{x_i y_i : i \in \mathbb{Z}_n\} \cup \{x_i x_{i+1} : i \in \mathbb{Z}_n\} \cup \{y_i y_{i+1} : i \in \mathbb{Z}_n\}$. A cubic graph is defined as a *Möbius ladder* if it can be obtained from CL_n by removing edges $x_{n-1}x_0$ and $y_{n-1}y_0$, and adding edges $x_{n-1}y_0$ and x_0y_{n-1} . This graph is denoted by ML_n .

The following lemma shows that every connected cubic abelian Cayley graph is isomorphic to either CL_n or ML_n . An element a of a group X is an involution if $a^2 = 1_e$, where 1_e is the identity element of X . Specifically, a is a central involution if it is an involution and commutes with every element $b \in X$, i.e., $ab = ba$ for all $b \in X$. If $\text{Cay}(X, S)$ is cubic, then S includes a involution of X , because $|S| = 3$ and it is closed under taking inverses. Furthermore, if X is abelian, then S includes a central involution of X .

Lemma 5.2. [11] *Let $\text{Cay}(X, S)$ be a connected cubic Cayley graph such that S contains a central involution of X . Then $\text{Cay}(X, S)$ is isomorphic to CL_n or ML_n .*

Thus, a connected cubic abelian Cayley graph is isomorphic to either a circular ladder or a Möbius ladder. The following theorem shows that, to prove Theorem 5.1, it suffices to show that every flow-admissible (CL_n, σ) and (ML_n, τ) admits a 6-NZF. Note that every component of $\text{Cay}(X, S)$ is $|S|$ -edge-connected because $\text{Cay}(X, S)$ is vertex-transitive. Raspaud et al. [13] showed that every flow-admissible signed 4-edge-connected graph admits a 4-NZF, as follows.

Theorem 5.3. [13] *Let G be a 4-edge-connected graph. If (G, σ) is flow-admissible, then (G, σ) admits 4-NZF.*

Although we have not yet proven that every flow-admissible (CL_n, σ) and (ML_n, τ) admits a 6-NZF, we present a proof of Theorem 5.1 here.

The proof of Theorem 5.1. Let $\Gamma = \text{Cay}(X, S)$ be an abelian Cayley graph. Note that if $\Gamma = \text{Cay}(X, S)$ is not connected, then each component of $\Gamma = \text{Cay}(X, S)$ is isomorphic to an abelian Cayley graph $\Gamma_1 = \text{Cay}(X_1, S)$, where X_1 is a proper subgroup of X and generated by S .

Let (Γ, σ) be flow-admissible. Since (Γ, σ) is flow-admissible if and only if each component of (Γ, σ) is flow-admissible, we can assume that each component of (Γ, σ) is flow-admissible. Furthermore, if Γ is not connected, then each component of (Γ, σ) is isomorphic to a flow-admissible signed abelian Cayley graph. Thus, without loss of generality, we assume that Γ is connected.

If $|S| \geq 4$, then Γ is 4-edge-connected. By Theorem 5.3, (Γ, σ) admits a 4-NZF.

When $|S| = 3$, Γ is isomorphic to either CL_n or ML_n . By Theorems 5.4 and 5.6, (Γ, σ) admits a 6-NZF.

When $|S| = 2$, the signed graph (Γ, σ) is a balanced circuit since (Γ, σ) is flow-admissible. Consequently, there is a 2-NZF in (Γ, σ) .

For $|S| = 1$, (Γ, σ) is not flow-admissible, leading to a contradiction. $\square$

Let G be a circular ladder or Möbius ladder. In the remainder of this subsection, we will prove that every flow-admissible (G, σ) admits a 6-NZF, as stated in Theorem 5.4 and Theorem 5.6. In most cases, we can find a balanced Hamiltonian circuit in (G, σ) , and we usually assume that the balanced Hamiltonian circuit is all-positive due to the switching operation. Note that (G, σ) is not flow-admissible if $|E_N(G, \sigma)| = 1$.

The following theorem shows that every flow-admissible signed Möbius ladder admits a 6-NZF.

Theorem 5.4. *Every flow-admissible (ML_n, σ) admits a 6-NZF.*

Proof. By Theorem 4.3, it suffices to prove that there is a balanced Hamiltonian circuit in (ML_n, σ) . The edge set $E(ML_n) \setminus \{e \in E(ML_n) : e = x_i y_i, i \in \mathbb{Z}_n\}$ induces a Hamiltonian circuit in ML_n , denoted by C . If (C, σ) is balanced, then we are done. Therefore, for the remainder of the proof, we assume that (C, σ) is unbalanced. We may assume that (C, σ) has precisely one negative edge, $x_0 y_{n-1}$. Otherwise we switch at some vertices of C such that $x_0 y_{n-1}$ is negative and other edges are positive.

If $\sigma(x_0 y_0) = \sigma(x_{n-1} y_{n-1})$, then the sequence $x_0 x_1 x_2 \cdots x_{n-1} y_{n-1} y_{n-2} \cdots y_0 x_0$ forms a balanced Hamiltonian circuit.

Suppose now that $\sigma(x_0 y_0) \neq \sigma(x_{n-1} y_{n-1})$. Without loss of generality, let $x_{n-1} y_{n-1}$ be negative. We claim that there exists another negative edge in $\{x_i y_i : i \in [1, n-2]\}$. Suppose, to the contrary, that $\{x_i y_i : i \in [1, n-2]\}$ contains no negative edges. Then we switch at y_{n-1} . The resulting signed graph has only one negative edge, $y_{n-2} y_{n-1}$, which contradicts the fact that (ML_n, σ) is flow-admissible.

Meanwhile, we claim that there is another positive edge in $\{x_i y_i : i \in [1, n-2]\}$. Suppose, to the contrary, that $\{x_i y_i : i \in [1, n-2]\}$ contains no positive edges. Then we switch at $\{y_{n-1}, y_{n-2}, \dots, y_2, y_1\}$. The resulting signed graph has only one negative edge, $y_0 y_1$, which leads to a contradiction.

Hence, there exists a pair $(j, j+1)$, where $j \in [1, n-2]$, such that $\sigma(x_j y_j) \neq \sigma(x_{j+1} y_{j+1})$. Without loss of generality, we assume that $\sigma(x_j y_j) = -1$. Then we switch at $\{x_0, x_1, x_2, \dots, x_j\}$. The resulting signed graph is denoted by (ML_n, σ') . In this resulting signed graph, the subgraph (C, σ') has only one negative edge, $x_j x_{j+1}$. Furthermore, both $x_j y_j$ and $x_{j+1} y_{j+1}$ are positive in (ML_n, σ') . Thus, $(C \setminus \{x_j x_{j+1}, y_j y_{j+1}\}) \cup \{x_j y_j, x_{j+1} y_{j+1}\}$ forms an all-positive Hamiltonian circuit

$$x_j y_j y_{j-1} \cdots y_0 x_{n-1} x_{n-2} \cdots x_{j+1} y_{j+1} y_{j+2} \cdots y_{n-1} x_0 x_1 \cdots x_j.$$

Therefore, there is a balanced Hamiltonian circuit in (ML_n, σ) . $\square$

In the remainder of this subsection, we will prove that every flow-admissible (CL_n, σ) admits a 6-NZF. Before we proceed, we need to introduce some notation and terminology. Let $C_x = x_0x_1 \cdots x_{n-1}x_0$ and $C_y = y_0y_1 \cdots y_{n-1}y_0$ be two circuits of CL_n , and let M be the 1-factor of CL_n with edge set $\{x_iy_i : i \in [0, n-1]\}$. These three subgraphs are edge disjoint, and $CL_n = C_x \cup C_y \cup M$.

For (CL_n, σ) with $\sigma(x_ix_{i+1}) = \sigma(y_iy_{i+1}) = +1$, where the indices i and $i+1$ are taken modulo n , and $n \geq 3$. An (m, i) -extender of (CL_n, σ) is a signed graph obtained from (CL_n, σ) by replacing x_ix_{i+1} and y_iy_{i+1} by two all-positive paths of length $m+1$, denoted by $P_{x_i} = x_ix_1^ix_2^i \cdots x_m^ix_{i+1}$ and $P_{y_i} = y_iy_1^iy_2^i \cdots y_m^iy_{i+1}$, respectively, and adding edges $x_j^iy_j^i$ for $j \in [1, m]$, where $x_j^iy_j^i$ is negative if j is odd, and positive if j is even. There is an example, as shown in Fig. 5.1. It is easy to see

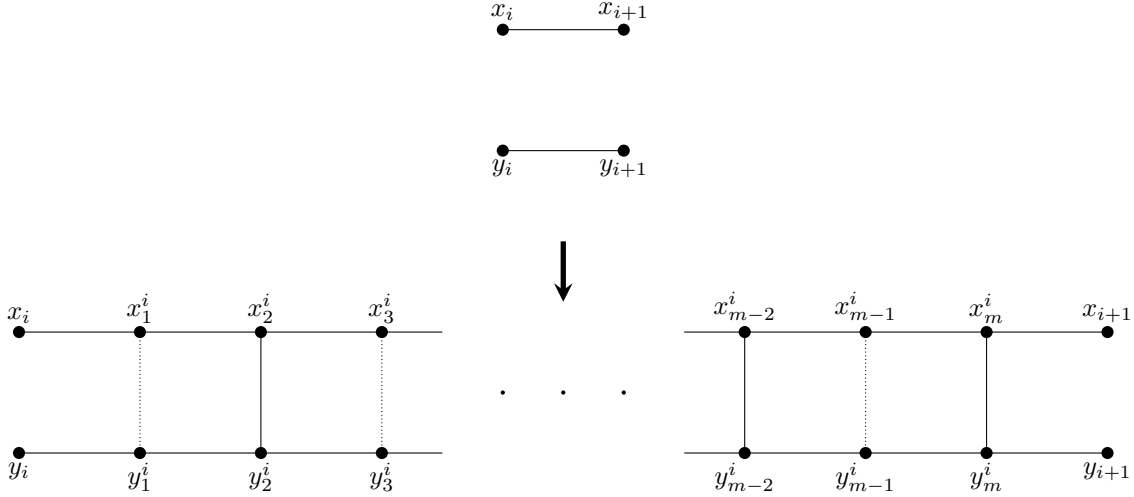


FIGURE 5.1. A extending of x_ix_{i+1} and y_iy_{i+1} , where m is even.

that the (m, i) -extender of (CL_n, σ) is isomorphic to a signed circular ladder with underlying graph CL_{n+m} . Additionally, the $(0, i)$ -extender of (CL_n, σ) is simply (CL_n, σ) .

The following lemma shows that a k -NZF of (CL_n, σ) can be extended to a k -NZF of the $(4q, i)$ -extender of (CL_n, σ) in certain cases, where $k \geq 4$ and q are integers.

Lemma 5.5. *Let $k \geq 4$, $n \geq 3$ and $q \geq 0$ be integers, and let $\sigma(x_ix_{i+1}) = \sigma(y_iy_{i+1}) = +1$ in (CL_n, σ) , where the indices are considered modulo n .*

- (1) *If there exists a k -NZF f on (CL_n, σ) such that $f(x_ix_{i+1}) = \pm 1$ and $f(y_iy_{i+1}) = \pm 2$, then the $(4q, i)$ -extender of (CL_n, σ) admits a k -NZF.*
- (2) *If there exists a k -NZF f on (CL_n, σ) such that $f(x_ix_{i+1}) = \pm 1$ and $f(y_iy_{i+1}) = \pm 1$, then the $(4q, i)$ -extender of (CL_n, σ) admits a k -NZF.*

Proof. We may assume that $q \geq 1$ since the statements hold trivially when $q = 0$. Denote the $(4q, i)$ -extender of (CL_n, σ) by (G, τ) . Let $P_{x_i} = x_ix_1^ix_2^i \cdots x_{4q}^ix_{i+1}$ and $P_{y_i} = y_iy_1^iy_2^i \cdots y_{4q}^iy_{i+1}$. Set $M^* = \{x_j^iy_j^i \in E(G) : j \in [1, 4q]\}$. There exists a k -flow of (G, τ) , denoted by f_1 , obtained from

f , as follows.

$$f_1(e) = \begin{cases} f(e), & e \in E(CL_n) \setminus \{x_i x_{i+1}, y_i y_{i+1}\}; \\ f(x_i x_{i+1}), & e \in E(P_{x_i}); \\ f(y_i y_{i+1}), & e \in E(P_{y_i}); \\ 0, & e \in M^*. \end{cases}$$

Namely, $\text{supp}(f_1) = E(G) \setminus M^*$. Next, we construct another 3-flow f_2 on (G, σ) such that $M^* \subseteq \text{supp}(f_2)$. The expression for f_2 is detailed below, and we suggest readers refer to Fig. 5.2 for a visual representation to aid understanding.

$$f_2(e) = \begin{cases} 1, & e \in M^*; \\ 2, & e \in \{x_{4l+2}^i x_{4l+3}^i : l \in [0, q-1]\}; \\ 1, & e \in \{x_{2l+1}^i x_{2l+2}^i : l \in [0, 2q-1]\}; \\ 1, & e \in \{y_{4l+1}^i y_{4l+2}^i : l \in [0, q-1]\}; \\ -1, & e \in \{y_{4l+3}^i y_{4l+4}^i : l \in [0, q-1]\}; \\ 0, & \text{otherwise.} \end{cases}$$

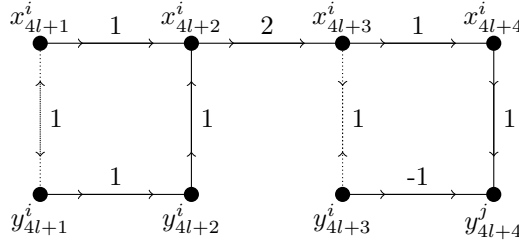


FIGURE 5.2. A fragment of f_2 .

(1) Given $f(x_i x_{i+1}) = \pm 1$ and $f(y_i y_{i+1}) = \pm 2$, there are four cases that need to be considered. As illustrated in Fig. 5.3, for any $e \in E(P_x) \cup E(P_y) \cup M$, it holds that $|(f_1 + f_2)(e)| \leq 3$ or $|(f_1 - f_2)(e)| \leq 3$. Given that f_1 is a k -flow with $k \geq 4$, it follows that $f_1 + f_2$ or $f_1 - f_2$ is a k -NZF on (G, τ) .

(2) In a similar manner, either $f_1 + 2f_2$ or $f_1 - 2f_2$ forms a k -NZF on (G, τ) , as depicted in Fig. 5.4. □

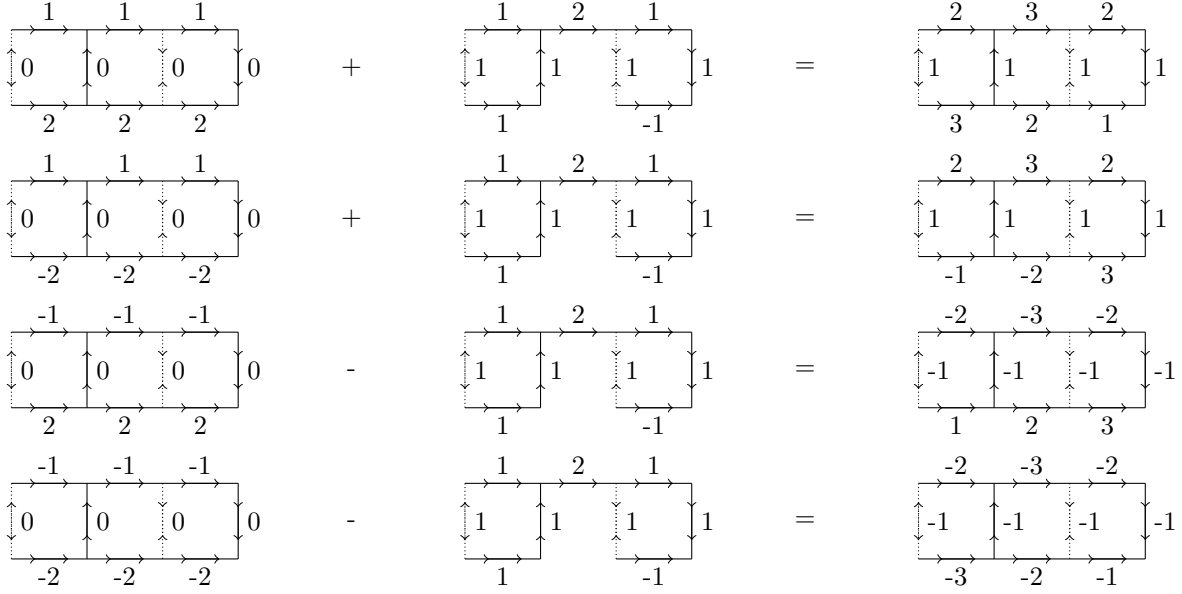
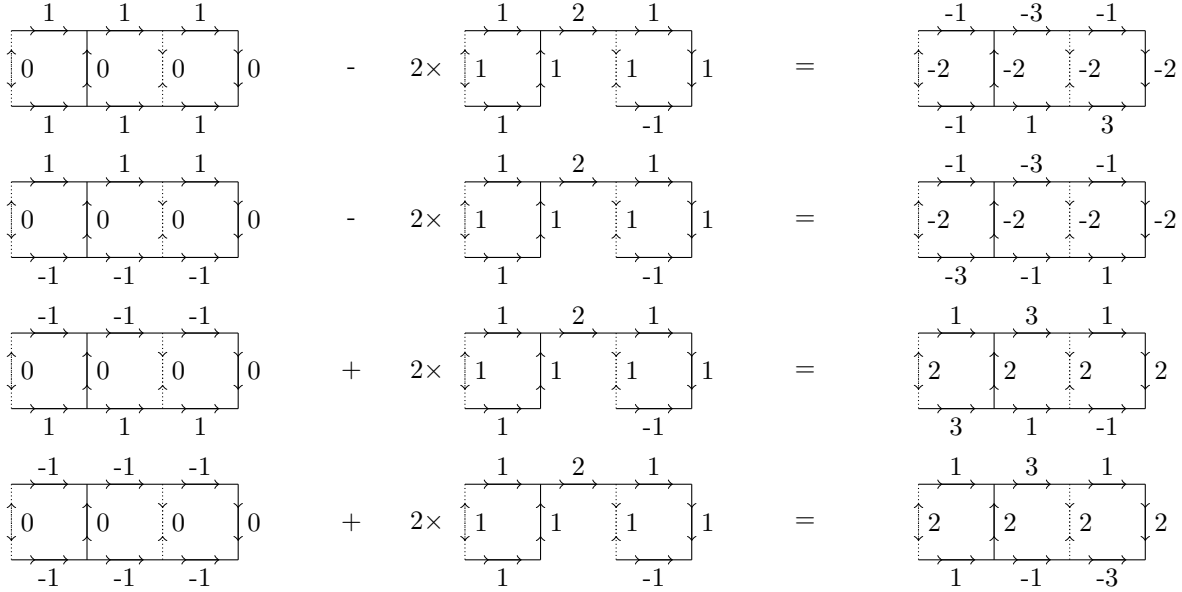
The following theorem shows that every flow-admissible signed circular ladder admits a 6-NZF.

Theorem 5.6. *Every flow-admissible (CL_n, σ) admits a 6-NZF.*

Proof. We consider three cases based on the sign of $\sigma(C_x)$ and $\sigma(C_y)$.

Case 1. $\sigma(C_x) = \sigma(C_y) = +1$.

Without loss of generality, we assume that (C_x, σ) and (C_y, σ) are all-positive; otherwise we switch at some vertex in $V(C_x) \cup V(C_y)$ to ensure that every edge in C_x and C_y is positive. Since (CL_n, σ) is flow-admissible, it follows that $n \geq 2$.

FIGURE 5.3. A fragment of $f_1 + f_2$ and $f_1 - f_2$.FIGURE 5.4. A fragment of $f_1 + 2f_2$ and $f_1 - 2f_2$.

Subcase 1.1. There exists an $i \in [0, n - 1]$ such that $\sigma(x_i y_i) = \sigma(x_{i+1} y_{i+1})$ modulo n .

Assume that $\sigma(x_i y_i) = \sigma(x_{i+1} y_{i+1}) = +1$; otherwise, we switch at $V(C_x)$. Then, $(C_x \cup C_y \cup \{x_i y_i, x_{i+1} y_{i+1}\}) \setminus \{x_i x_{i+1}, y_i y_{i+1}\}$ forms an all-positive Hamiltonian circuit. By Theorem 4.3, we conclude that (CL_n, σ) admits a 6-NZF.

Subcase 1.2. There is no $i \in [0, n-1]$ such that $\sigma(x_i y_i) = \sigma(x_{i+1} y_{i+1})$ modulo n .

For any $i \in [0, n-1]$, we have $\sigma(x_i y_i) \neq \sigma(x_{i+1} y_{i+1})$. Thus, n is even; otherwise, there exists a $j \in [0, n-1]$ such that $\sigma(x_j y_j) = \sigma(x_{j+1} y_{j+1})$ modulo n . Because (CL_n, σ) is flow-admissible, it follows that $n \geq 4$. Otherwise, there would be only one negative edge, a contradiction. Assume that $\sigma(x_0 y_0) = +1$. Otherwise, perform a switching at $V(C_1)$.

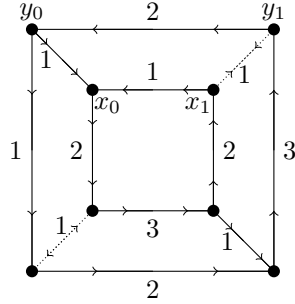


FIGURE 5.5. A 4-NZF f_1 on (CL_4, σ_1) .

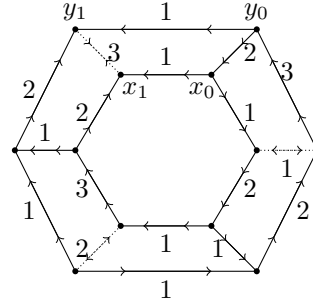


FIGURE 5.6. A 4-NZF f_2 on (CL_6, σ_2) .

We claim that (CL_{4k}, σ) admits a 4-NZF, where $k \geq 1$ is an integer. We consider a signed circular ladder (CL_4, σ_1) which is isomorphic to (CL_{4k}, σ) if $k = 1$. Fig. 5.5 shows that (CL_4, σ_1) admits a 4-NZF f_1 . Note that $\sigma_1(x_0 x_1) = \sigma_1(y_0 y_1) = +1$, $f_1(x_0 x_1) = \pm 1$ and $f_1(y_0 y_1) = \pm 2$. Therefore, the $(4(k-1), 0)$ -extender of (CL_4, σ_1) admits a 4-NZF, by Lemma 5.5. Note that, (CL_{4k}, σ) is isomorphic to the $(4(k-1), 0)$ -extender of (CL_4, σ_1) . Thus, (CL_{4k}, σ) admits a 4-NZF.

We claim that (CL_{4k+2}, σ) admits a 4-NZF, where $k \geq 1$ is an integer. Consider a signed circular ladder (CL_6, σ_2) , as shown in Fig. 5.6. Additively, (CL_{4k+2}, σ) is isomorphic to the $(4(k-1), 0)$ -extender of (CL_6, σ_2) . Fig. 5.6 shows that (CL_6, σ_2) admits a 4-NZF f_2 that satisfies the conditions of Lemma 5.5. Therefore, the $(4(k-1), 0)$ -extender of (CL_6, σ_2) admits a 4-NZF, and so does (CL_{4k+2}, σ) .

Case 2. $\sigma(C_x) \neq \sigma(C_y)$.

Without loss of generality, assume that $\sigma(C_x) = -1$ and $\sigma(C_y) = +1$. Suppose that (C_x, σ) has only one negative edge, say $x_0 x_1$, and (C_y, σ) is all-positive. Since (CL_n, σ) is flow-admissible, it follows that $n \geq 2$. We shall consider two subcases with respect to the signs of $\sigma(x_0 y_0)$ and $\sigma(x_1 y_1)$.

Subcase 2.1. $\sigma(x_0 y_0) = \sigma(x_1 y_1)$.

Assume that $x_0 y_0$ and $x_1 y_1$ are positive. Otherwise, we switch at $V(C_x)$. Then, there exists an all-positive Hamiltonian circuit $(C_x \cup C_y \cup \{x_0 y_0, x_1 y_1\}) \setminus \{x_0 x_1, y_0 y_1\}$. By Theorem 4.3, (CL_n, σ) admits a 6-NZF.

Subcase 2.2. $\sigma(x_0 y_0) \neq \sigma(x_1 y_1)$.

Without loss of generality, assume that $\sigma(x_0 y_0) = -1$ and $\sigma(x_1 y_1) = +1$. We claim that there is another negative edge in $M \setminus \{x_0 y_0, x_1 y_1\}$. Otherwise, we switch at x_0 such that $E(CL_n)$ has only one

negative edge, leading to a contradiction. Hence, there exists an $i \in [2, n-1]$ such that $\sigma(x_i y_i) \neq \sigma(x_{i-1} y_{i-1})$. Without loss of generality, assume $\sigma(x_i y_i) = -1$ and $\sigma(x_{i-1} y_{i-1}) = +1$. Then we switch at $\{x_0, x_{n-1}, x_{n-2} \cdots x_i\}$, and denote the resulting signed graph by (CL_n, σ') . In (CL_n, σ') , C_x has only one negative edge $x_i x_{i-1}$, C_y remains all-positive, and $\sigma'(x_i y_i) = \sigma'(x_{i-1} y_{i-1}) = +1$. Then there exists an all-positive Hamiltonian circuit $(C_x \cup C_y \cup \{x_i y_i, x_{i-1} y_{i-1}\}) \setminus \{x_i x_{i-1}, y_i y_{i-1}\}$ in (CL_n, σ') . By Theorem 4.3, (CL_n, σ') admits a 6-NZF, and so does (CL_n, σ) .

Case 3. $\sigma(C_x) = \sigma(C_y) = -1$.

Without loss of generality, suppose that (C_x, σ) has only one negative edge $x_0 x_1$ and (C_y, σ) has only one negative edge $y_0 y_1$. Since (CL_n, σ) is flow-admissible, it follows that $n \geq 1$. If $n = 1$, then (CL_n, σ) is isomorphic to a long barbell. Thus, (CL_n, σ) admits a 3-NZF if $n = 1$. Now, consider $n \geq 2$.

Subcase 3.1. There exists an $i \in [0, n-1]$ such that $\sigma(x_i y_i) = \sigma(x_{i+1} y_{i+1})$ modulo n .

Assume that $\sigma(x_i y_i) = \sigma(x_{i+1} y_{i+1}) = +1$; otherwise, perform a switching at $V(C_x)$. Define $H = x_{i+1} x_{i+2} \cdots x_{n-1} x_0 x_1 \cdots x_i y_i y_{i-1} \cdots y_0 y_{n-1} y_{n-2} \cdots y_{i+1} x_{i+1}$. It is easy to verify that H forms a Hamiltonian circuit of CL_n . Additionally, there are only two negative edges $x_0 x_1$ and $y_0 y_1$, in (H, σ) . Thus, (H, σ) is a balanced Hamiltonian circuit in CL_n . By Theorem 4.3, (CL_n, σ) admits a 6-NZF.

Subcase 3.2. There is no $i \in [0, n-1]$ such that $\sigma(x_i y_i) \neq \sigma(x_{i+1} y_{i+1})$ modulo n .

It is evident that n is even. Assume that $\sigma(x_1 y_1) = +1$; otherwise, perform a switching at $V(C_1)$.

We claim that (CL_{4k}, σ) admits a 6-NZF, where $k \geq 1$ is an integer. Fig. 5.7 shows that the signed circular ladder (CL_4, σ_3) admits a 6-NZF f_3 that satisfies the conditions of Lemma 5.5. Therefore, the $(4(k-1), 3)$ -extender of (CL_4, σ_3) admits a 6-NZF. Additionally, (CL_{4k}, σ) is isomorphic to the $(4(k-1), 3)$ -extender of (CL_4, σ_3) . Thus, (CL_{4k}, σ) admits a 6-NZF.

We claim that (CL_{4k+2}, σ) admits a 4-NZF, where $k \geq 0$ is an integer. For $k = 0$, Fig. 5.8 shows that (CL_{4k+2}, σ) admits a 4-NZF. Now, suppose that $k \geq 1$. Fig. 5.9 illustrates that (CL_6, σ_4) admits a 4-NZF f_4 . By Lemma 5.5, the $(4(k-1), 1)$ -extender of (CL_6, σ_4) admits a 4-NZF. Since (CL_{4k+2}, σ) is isomorphic to the $(4(k-1), 1)$ -extender of (CL_6, σ_4) , it follows that (CL_{4k+2}, σ) admits a 4-NZF. $\square$

5.2. Flow number of signed Cayley graphs on abelian groups of odd order.

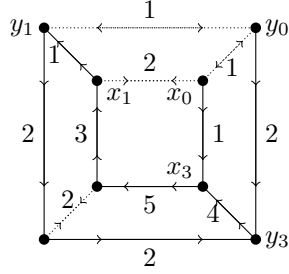
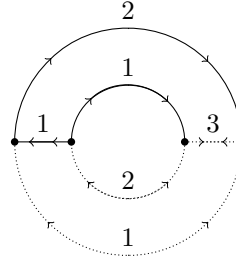
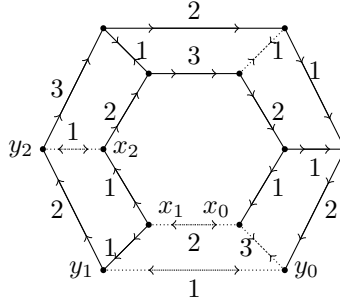
In this subsection, we characterize the flow number of flow-admissible signed Cayley graphs on abelian groups of odd order. In order to present this characterization, we also characterize the flow number of flow-admissible signed Hamilton-decomposable graphs. A graph is termed *Hamilton-decomposable* if it can be decomposed into several edge-disjoint Hamiltonian circuits.

Recall that the *flow number* of (G, σ) , denoted by $\Phi(G, \sigma)$, is the minimum k such that (G, σ) admits a k -NZF. The main result of this subsection is as follows.

Theorem 5.7. *Let A be a finite abelian group of odd order and $\Gamma = \text{Cay}(A, S)$ is connected. If (Γ, σ) is flow-admissible, then*

- (1) $\Phi(\Gamma, \sigma) = 2$ if and only if $|E_N(\Gamma, \sigma)|$ is even;
- (2) $\Phi(\Gamma, \sigma) = 3$ if and only if $|E_N(\Gamma, \sigma)|$ is odd and $\frac{|S|}{2} \geq 3$;
- (3) $\Phi(\Gamma, \sigma) = 4$ if and only if $|E_N(\Gamma, \sigma)|$ is odd and $\frac{|S|}{2} = 2$.

Let A be an abelian group of odd order. By Lagrange's Theorem, for any $x \in A$, the order of x is odd. Thus, there is no element $x \in A$ such that $x^2 = 1_e$, meaning there are no involutions in A .

FIGURE 5.7. A 6-NZF f_3 on (CL_4, σ_3) .FIGURE 5.8. A 4-NZF on (CL_2, σ) .FIGURE 5.9. A 4-NZF on (CL_6, σ_4) .

Consider the Cayley graph $\Gamma = \text{Cay}(A, S)$. Since $|A|$ is odd and S is closed under taking inverses, it follows that $|S|$ is even. Therefore, $\Gamma = \text{Cay}(A, S)$ is an even graph. If Γ is connected, then Γ is Eulerian. Thus, the Cayley graph Γ discussed in Theorem 5.7 is Eulerian. Consequently, the following result is necessary.

Theorem 5.8. [10] *Let (G, σ) be a signed Eulerian graph. Then*

- (1) (G, σ) has no nowhere-zero flow if and only if (G, σ) is unbalanced and $(G \setminus e, \sigma)$ is balanced for some edge e ;
- (2) $\Phi(G, \sigma) = 2$ if and only if (G, σ) has an even number of negative edges;
- (3) $\Phi(G, \sigma) = 3$ if and only if (G, σ) can be decomposed into three Eulerian subgraphs, with an odd number of negative edges each, that share a common vertex;
- (4) $\Phi(G, \sigma) = 4$ otherwise.

Alspach [1] conjectured that any $2k$ -regular connected Cayley graph on an abelian group has a Hamiltonian decomposition. Westlund et al. [18] validated Alspach's conjecture for the case $k = 3$, under the condition that the abelian group has an odd order.

Theorem 5.9. [18] *Every connected 6-regular Cayley graph on an abelian group of odd order is decomposable into three Hamiltonian circuits.*

Thus, to prove Theorem 5.7, we need to characterize the flow number of flow-admissible signed Hamilton-decomposable graphs. If G can be decomposed into l edge-disjoint Hamiltonian circuits, then G is $2l$ -edge-connected. Therefore, the following result is necessary.

Theorem 5.10. [17] *Every flow-admissible 8-edge-connected signed graph admits a nowhere-zero 3-flow.*

A path P is referred to as an xy -path if it connects the vertices x and y . The characterization of the flow number of flow-admissible signed Hamilton-decomposable graphs is as follows.

Theorem 5.11. *Let graph G be $2k$ -regular and Hamilton-decomposable. If (G, σ) is flow-admissible, then*

- (1) $\Phi(G, \sigma) = 2$ if and only if $|E_N(G, \sigma)|$ is even;
- (2) $\Phi(G, \sigma) = 3$ if and only if $|E_N(G, \sigma)|$ is odd and $k \geq 3$;
- (3) $\Phi(G, \sigma) = 4$ if and only if $|E_N(G, \sigma)|$ is odd and $k = 2$.

Proof. Statement (1) follows directly as a corollary of Theorem 5.8. If (G, σ) contains an odd number of negative edges, then according to Theorem 5.8, $3 \leq \Phi(G, \sigma) \leq 4$. Since a 4-regular graph cannot be decomposed into three Eulerian subgraphs, this confirms the validity of Statement (3). According to Theorem 5.10, if (G, σ) has an odd number of negative edges and $k > 3$, then (G, σ) admits a 3-NZF because G is 8-edge-connected. Therefore, it is sufficient to prove that $\Phi(G, \sigma) = 3$ when $|E_N(G, \sigma)|$ is odd and $k = 3$.

Consider three edge-disjoint Hamiltonian circuits C^1 , C^2 , and C^3 in G , such that their edge sets satisfy $E(C^1) \cup E(C^2) \cup E(C^3) = E(G)$. If all circuits in $\{C^1, C^2, C^3\}$ are unbalanced, then (G, σ) admits a 3-NZF by Statement (3) of Theorem 5.8. If only two circuits in $\{C^1, C^2, C^3\}$ are unbalanced, then (G, σ) has an even number of negative edges, leading to a contradiction. Therefore, it remains to prove that if there is only one unbalanced circuit, say C^1 , within $\{C^1, C^2, C^3\}$, then (G, σ) admits a 3-NZF. Without loss of generality, assume that (C^2, σ) is all-positive; if not, we switch at certain vertices of C^2 to make all its edges positive. We will consider two cases based on the signature of (C^3, σ) .

Case 1. (C^3, σ) is not all-positive.

There exists a negative edge e within $E(C^3)$, and $|E_N(C^3, \sigma)|$ is even. Let the ends of e be u and v . Since C^2 is a Hamiltonian circuit, it can be decomposed into two edge-disjoint uv -paths P_a^2 and P_b^2 . Consequently, $P_a^2 \cup \{e\}$ and $P_b^2 \cup (C^3 \setminus e)$ form two Eulerian subgraphs of G , each containing an odd number of negative edges. Thus, (G, σ) can be decomposed into three Eulerian subgraphs $P_a^2 \cup e$, $P_b^2 \cup (C^3 \setminus e)$, and C^1 , each having an odd number of negative edges and sharing the common vertices u and v . Therefore, $\Phi(G, \sigma) = 3$ by Statement (3) of Theorem 5.8.

Case 2. (C^3, σ) is all-positive.

Since (G, σ) is flow-admissible and $|E_N(G, \sigma)|$ is odd, there are at least three negative edges in $E_N(G, \sigma)$. Moreover, because both (C^2, σ) and (C^3, σ) are all-positive, there are at least three negative edges in (C^1, σ) . Let e_1 and e_2 denote two negative edges in (C^1, σ) . Let the ends of e_1 be u_1 and v_1 . Then, in C^2 , there exist two u_1v_1 -paths, denoted by P_α^2 and P_β^2 . Let the ends of e_2 be u_2 and v_2 . Thus, in C^3 , there exist two u_2v_2 -paths, denoted by P_γ^3 and P_δ^3 . Since C^3 is a Hamiltonian circuit of G , one of paths P_γ^3 or P_δ^3 contains u_1 as a vertex, say P_δ^3 . Consequently, (G, σ) can be decomposed into three Eulerian subgraphs: $(C^1 \setminus \{e_1, e_2\}) \cup P_\alpha^2 \cup P_\gamma^3$, $P_\beta^2 \cup \{e_1\}$, and $P_\delta^3 \cup \{e_2\}$. Each subgraph contains an odd number of negative edges and share a common vertex u_1 . Therefore, by Statement (3) of Theorem 5.8, $\Phi(G, \sigma) = 3$. $\square$

Now, we can complete the proof of Theorem 5.7.

Proof of Theorem 5.7. Given that Γ is Eulerian, $\Phi(G, \sigma) = 2$ if and only if $|E_N(\Gamma, \sigma)|$ is even, by Theorem 5.8. Thus, Statement (1) holds.

If (Γ, σ) contains an odd number of negative edges, then $3 \leq \Phi(G, \sigma) \leq 4$, according to Theorem 5.8. Given that Γ is $|S|$ -regular, it cannot be decomposed into three Eulerian subgraphs when $\frac{|S|}{2} = 2$. Thus, Statement (3) holds.

According to Theorem 5.10, if (Γ, σ) has an odd number of negative edges and $\frac{|S|}{2} > 3$, then (Γ, σ) admits a 3-NZF because Γ is 8-edge-connected. Therefore, it is sufficient to consider cases where $|E_N(\Gamma, \sigma)|$ is odd and $\frac{|S|}{2} = 3$. According to Theorem 5.9, (Γ, σ) is a flow-admissible signed Hamilton-decomposable graph. Thus, (Γ, σ) admits a 3-NZF because the Statement (2) of Theorem 5.11. $\square$

ACKNOWLEDGEMENTS

This work is supported by National Natural Science Foundation of China (Grant No 12461006), Guizhou Provincial Basic Research Program (Grant No. ZD[2025]085) and Scientific Research Foundation of Guizhou University (Grant No. [2023]41).

DECLARATIONS

Conflict of interest The authors declare that they have no conflict of interest.

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