EXTREMAL CONSTRUCTIONS FOR APEX PARTITE HYPERGRAPHS

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Abstract. We establish new lower bounds for the Turán and Zarankiewicz numbers of certain apex partite hypergraphs. Given a $(d-1)$-partite $(d-1)$-uniform hypergraph $\mathcal{H}$, let $\mathcal{H}(k)$ be the d-partite d-uniform hypergraph whose dth part has k vertices that share $\mathcal{H}$ as a common link. We show that $\mathrm{ex}(n,\mathcal{H}(k))=\Omega_{\mathcal{H}}(n^{d-\frac{1}{e(\mathcal{H})}})$ if k is at least exponentially large in $e(\mathcal{H})$. Our bound is optimal for all Sidorenko hypergraphs $\mathcal{H}$ and verifies a conjecture of Lee for such hypergraphs.

In particular, for the complete d-partite d-uniform hypergraphs $\mathcal{K}^{(d)}_{s_1,\ldots,s_d}$, our result implies that $\mathrm{ex}(n,\mathcal{K}^{(d)}_{s_1,\cdots,s_d})=\Theta(n^{d-\frac{1}{s_1\cdots s_{d-1}}})$ if s_d is at least exponentially large in terms of $s_1\cdots s_{d-1}$, improving the factorial condition of Pohoata and Zakharov and answering a question of Mubayi. Our method is a generalization of Bukh's random algebraic method [Duke Math. J. 2024] to hypergraphs, and extends to the sided Zarankiewicz problem.

1. Introduction

Given a d-uniform hypergraph $\mathcal{H}$, the Turán problem in Extremal Combinatorics studies the *Turán number* $\mathrm{ex}(n,\mathcal{H})$, which is the maximum number of edges in an n-vertex d-uniform hypergraph without containing $\mathcal{H}$ as a subgraph. A classical line of work going back to Erdős already highlights the role of complete multipartite configurations as $\mathcal{H}$ has a degenerate Turán number $O(n^{d-\Omega_H(1)})$ if and only if it is d-partite. Write $\mathcal{K}^{(d)}_{s_1,\ldots,s_d}$ for the complete d-partite d-uniform hypergraph whose parts have sizes $s_1,\ldots,s_d$. Erdős showed (see [11]) that $\mathrm{ex}(n,\mathcal{K}^{(d)}_{s_1,\ldots,s_d})=O_{d,s_1,\ldots,s_{d-1}}(n^{d-\frac{1}{s_1\cdots s_{d-1}}})$. This was conjectured to be sharp in the exponent.

Conjecture 1.1 ([22])**.** For any positive integers $s_1\le\cdots\le s_d$,

$$\mathrm{ex}(n,\mathcal{K}^{(d)}_{s_1,\ldots,s_d})=\Theta_{d,s_1,\ldots,s_{d-1}}\left(n^{d-\frac{1}{s_1\cdots s_{d-1}}}\right). \tag{1}$$

There has been some progress toward Conjecture 1.1 in regimes where the last part is very large. The best result to date is by Pohoata and Zakharov [26]; improving on [20] they showed that (1) holds for factorially large s_d, namely $s_d>((d-1)(s_1\cdots s_{d-1}-1))!$. When $d=2$, a recent breakthrough by Bukh [5] established an exponential bound for s_2 in terms of s_1.

Zarankiewicz variant. It is often useful in some applications (see e.g.[1, 29]) to distinguish the parts of a d-partite d-uniform hypergraph and forbid copies of $\mathcal{K}^{(d)}_{s_1,\ldots,s_d}$ in a *sided* sense. The *Zarankiewicz number* $\mathrm{z}(n_1,\ldots,n_d,\mathcal{K}^{(d)}_{s_1,\ldots,s_d})$ is the maximum number of edges in a d-partite d-uniform hypergraph $\mathcal{G}$ with parts of sizes $n_1,\ldots,n_d$ containing no copy of $\mathcal{K}^{(d)}_{s_1,\ldots,s_d}$ such that the set of size s_i in $\mathcal{K}^{(d)}_{s_1,\ldots,s_d}$ is embedded in the part of $\mathcal{G}$ of size n_i for each $i\in[d]$. When $n_1=\cdots=n_d=m$, we abbreviate $\mathrm{z}(m,\ldots,m,\mathcal{K}^{(d)}_{s_1,\ldots,s_d})$ to $\mathrm{z}(m,\mathcal{K}^{(d)}_{s_1,\ldots,s_d})$. Since the sided problem is more permissive, we trivially have $\mathrm{ex}(dm,\mathcal{K}^{(d)}_{s_1,\ldots,s_d})\le\mathrm{z}(m,\mathcal{K}^{(d)}_{s_1,\ldots,s_d})$. Recently, Mubayi [23] improved the factorial bound on s_d to exponential at the expense of a $o(1)$ error in the exponent; he showed that

$$\mathrm{z}(m,\mathcal{K}^{(d)}_{s_1,\ldots,s_d})=m^{d-\frac{1}{s_1\cdots s_{d-1}}-o(1)},\quad \text{if } s_d>3^{(1+o(1))\,s_1\cdots s_{d-1}}. \tag{2}$$

However, his method applies only to the sided Zarankiewicz problem, and he asked whether a similar bound can be achieved in the Turán setting.

Connection to Sidorenko exponents. For d-uniform hypergraphs $\mathcal{H}, \mathcal{G}$, denote by $\mathrm{Hom}(\mathcal{H}, \mathcal{G})$ the set of homomorphisms from $\mathcal{H}$ to $\mathcal{G}$. The *homomorphism density* of $\mathcal{H}$ in $\mathcal{G}$ is defined as

$$t_{\mathcal{H}}(\mathcal{G}) := \frac{|\mathrm{Hom}(\mathcal{H}, \mathcal{G})|}{|V(\mathcal{G})|^{|V(\mathcal{H})|}},$$

Let $\mathcal{K}_d^{(d)}$ denote the complete d-uniform hypergraph on d vertices. The *Sidorenko exponent* of a d-partite d-uniform hypergraph $\mathcal{H}$ is

$$s(\mathcal{H}) := \sup\Big\{ s \geq 0 : t_{\mathcal{H}}(\mathcal{G}) = t_{\mathcal{K}_d^{(d)}}(\mathcal{G})^s > 0 \text{ for some } \mathcal{G} \Big\}.$$

Sidorenko's conjecture, a central conjecture in Extremal Combinatorics, states that for every bipartite graph H, its Sidorenko exponent satisfies $s(H) = e(H)$. This conjecture remains open. It is known that (see e.g. [9, 24]) Sidorenko's conjecture is not true for hypergraphs. We call a hypergraph $\mathcal{H}$ *Sidorenko* if $s(\mathcal{H}) = e(\mathcal{H})$.

Very recently, Lee [19] discovered a connection between Sidorenko exponent and Turán problem. In particular, he used $s(\mathcal{H})$ to give an upper bound on the Turán number for certain 'apex' partite hypergraphs. Given a $(d-1)$-partite $(d-1)$-uniform hypergraph $\mathcal{H}$ and $k \in \mathbb{N}$, let $\mathcal{H}(k)$ be the d-partite d-uniform hypergraph whose dth part has k vertices that have $\mathcal{H}$ as a common link graph. Lee [19] proved that $\mathrm{ex}(n, \mathcal{H}(k)) = O_{\mathcal{H},k}\Big(n^{d-\frac{1}{s(\mathcal{H})}}\Big)$. He further conjectured that this bound is best possible.

Conjecture 1.2 ([19])**.** Let $d \geq 2$ and $\mathcal{H}$ be a $(d-1)$-partite $(d-1)$-uniform hypergraph. There exists a constant $C = C(\mathcal{H})$ such that for all $k \geq C$,

$$\mathrm{ex}(n, \mathcal{H}(k)) = \Omega_{\mathcal{H}}\Big(n^{d-\frac{1}{s(\mathcal{H})}}\Big).$$

As an interesting test case, Lee asked whether $\mathrm{ex}(n, C_6(k)) = \Omega\Big(n^{\frac{17}{6}}\Big)$ for large k.

1.1. Main results.

Our first result provides lower bounds for all 'apex' partite hypergraphs $\mathcal{H}(k)$.

Theorem 1.3. *Let $d \geq 2$ and $\mathcal{H}$ be a $(d-1)$-partite $(d-1)$-uniform hypergraph. There exists a constant c such that*

$$\mathrm{ex}(n, \mathcal{H}(k)) = \Omega_{\mathcal{H}}\Big(n^{d-\frac{1}{e(\mathcal{H})}}\Big) \qquad \text{if} \quad k > c^{e(\mathcal{H})}.$$

The bound in Theorem 1.3 is optimal for all Sidorenko hypergraphs, thereby confirming Theorem 1.2 for a wide class of hypergraphs. In particular, some well-known Sidorenko (hyper)graphs includes complete partite hypergraphs[1], even cycles, hypercubes and some symmetric graphs arising from finite reflection groups (see e.g. [8]).

Corollary 1.4. *Let $d, \ell, r \geq 2$ and $s_1, \dots, s_{d-1} \in \mathbb{N}$.*

- *For the complete partite hypergraphs, if $s_d > 9^{(1+o(1))s_1 \cdots s_{d-1}}$, then*
$$\mathrm{ex}(n, \mathcal{K}^{(d)}_{s_1,\dots,s_d}) = \Theta_{s_1,\dots,s_{d-1}}\left(n^{d-\frac{1}{s_1\cdots s_{d-1}}}\right).$$
- *For the even cycle $C_{2\ell}$, if $k > 9^{(1+o(1))2\ell}$, then*
$$\mathrm{ex}(n, C_{2\ell}(k)) = \Theta\Big(n^{3-\frac{1}{2\ell}}\Big).$$
- *For the hypercube Q_r, if $k > 9^{(1+o(1))r2^{r-1}}$, then*
$$\mathrm{ex}(n, Q_r(k)) = \Theta\Big(n^{3-\frac{1}{r2^{r-1}}}\Big).$$

[1]Note that $\mathcal{K}^{(d)}_{s_1,\dots,s_d} = \mathcal{K}^{(d-1)}_{s_1,\dots,s_{d-1}}(s_d)$.

For the complete partite hypergraphs $\mathcal{K}^{(d)}_{s_1,\dots,s_d}$, our result improves the previously best known factorial condition on s_d by Pohoata and Zakharov [26] to an exponential one, which answers positively the question posed by Mubayi [23] in a strong sense without error term in the exponent in (2). Moreover, for $d = 2$, our bound recovers the one by Bukh [5]: $s_2 > 9^{s_1} s_1^{4s_1^{2/3}}$.

Our second result generalizes Bukh's result [5] on Zarankiewicz problem to hypergraphs.

Theorem 1.5. *Let $s_1, \dots, s_d, n_1, \dots, n_d \in \mathbb{N}$ and let $\mathcal{H}$ be a $(d-1)$-partite $(d-1)$-uniform hypergraph whose $(d-1)$ parts have $s_1, \dots, s_{d-1}$ vertices, respectively. There exists $C = C(s_1, \cdots, s_{d-1})$ such that if $s_d > C\left(\log_{n_d}(n_1^{s_1} \cdots n_{d-1}^{s_{d-1}})\right)^{2\sqrt{e(\mathcal{H})}+1}$, then*

$$\mathrm{z}(n_1, \dots, n_d, \mathcal{H}(s_d)) = \Omega_{s_1,\dots,s_{d-1},s_d}\left(n_1 \cdots n_{d-1}\, n_d^{1-\frac{1}{e(\mathcal{H})}}\right).$$

In particular, when $\mathcal{H} = \mathcal{K}^{(d-1)}_{s_1,\dots,s_{d-1}}$ and $s_d > C\left(\log_{n_d}(n_1^{s_1} \cdots n_{d-1}^{s_{d-1}})\right)^{2\sqrt{s_1 \cdots s_{d-1}}+1}$, we have

$$\mathrm{z}(n_1, \dots, n_d, \mathcal{K}^{(d)}_{s_1,\dots,s_d}) = \Omega_{s_1,\dots,s_{d-1},s_d}\left(n_1 \cdots n_{d-1}\, n_d^{1-\frac{1}{s_1 \cdots s_{d-1}}}\right).$$

Our approach. Both Bukh's method [5] and ours rely on the fact that non-regular sequences of polynomials form a small subset of all polynomial sequences. However, the way we quantify the smallness of this set differs from the one in [5]. In Bukh's approach, this smallness is measured probabilitically. In contrast, we characterize it using algebro-geometric invariants, showing that this set has bounded degree (Theorem 3.1) and bounded dimension (Theorem 3.3). In particular, Theorem 3.1 provides an effective version of the classical result [28] that these sequences form a proper subvariety. Bounds of this nature are of considerable interest in commutative algebra [2, 4, 17]. We note that bounding the dimension over the whole space $(\dim \mathcal{U}_{\mathbb{P}^N(\mathbb{K})}(m_1, \dots, m_s))$ was computed in [3, Proposition 2.4], using techniques of Hilbert schemes [27, Section 4.6.1]. However, for our purpose, we need to compute in Theorem 3.3 the dimension $(\dim \mathcal{U}_X(m_1, \dots, m_s))$ for a subvariety $X \subseteq \mathbb{P}^N(\mathbb{K})$. The previous approach does not extend to this setting as the theory of Hilbert schemes for arbitrary varieties remains largely undeveloped.

Our approach is based on the equivalence between the regularity of polynomial sequences and the exactness of their corresponding Koszul complexes. While this connection is well known [25], the effective bound is obtained by a more delicate analysis of the Koszul complex. Our application of this bound further hinges on the technique of counting rational points in algebraic varieties.

Organization. Section 2 collects algebraic preliminaries. In Section 3 we develop the non-regular sequence machinery used in our constructions. Section 4 proves Theorem 1.3; Section 5 establishes Theorem 1.5.

2. Preliminaries

Let q be a prime power and let $\mathbb{F}_q$ be the finite field of q elements. In this paper, we reserve $\mathbb{F} = \overline{\mathbb{F}}_q$ for the algebraic closure of $\mathbb{F}_q$, and we use $\mathbb{K}$ for an arbitrary field. We denote by $\mathbb{P}^N(\mathbb{K})$ the N-dimensional projective space over a field $\mathbb{K}$. By definition, $\mathbb{P}^N(\mathbb{K}) = \left(\mathbb{K}^{N+1} \setminus \{0\}\right) / \sim$ where $v \simeq w$ for $v, w \in \mathbb{K}^{N+1} \setminus \{0\}$ if and only if $v = \lambda w$ for some $\lambda \in \mathbb{K} \setminus \{0\}$.

2.1. Commutative algebra.

Let $\mathbb{K}$ be a field and let $R = \mathbb{K}[x_0, \cdots, x_N]$ be the polynomial ring over $\mathbb{K}$ in $N+1$ variables. The ideal generated by $f_1, \dots, f_r \in R$ is denoted by $\langle f_1, \dots, f_r \rangle$. The *height* of a prime ideal $\mathfrak{p} \subseteq R$ is

$$\mathrm{ht}(\mathfrak{p}) = \max\{t : (0) = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_t = \mathfrak{p}, \mathfrak{p}_i \text{ is a prime ideal}, 0 \le i \le t\}.$$

The *height* of an ideal $\mathfrak{a} \subseteq R$ is

$$\mathrm{ht}(\mathfrak{a}) \coloneqq \min\{\mathrm{ht}(\mathfrak{p}) : \mathfrak{a} \subseteq \mathfrak{p}, \mathfrak{p} \text{ is a prime ideal}\}.$$

Given an integer $m \geq 0$, we write R_m for the subspace of R consisting of degree m homogeneous polynomials in R. Consequently, for each homogeneous ideal $\mathfrak{a}$ of R, we have

$$\mathfrak{a} = \oplus_{m=0}^{\infty} \mathfrak{a}_m, \quad \text{where} \quad \mathfrak{a}_m \coloneqq R_m \cap \mathfrak{a}, \quad \text{and} \tag{3}$$

$$R/\mathfrak{a} = \oplus_{m=0}^{\infty} (R/\mathfrak{a})_m, \quad \text{where} \quad (R/\mathfrak{a})_m \coloneqq R_m/\mathfrak{a}_m. \tag{4}$$

For a sequence $f \coloneqq (f_1, \dots, f_r) \in R^r$ of polynomials, let

$$\mathfrak{a}_f^{(i)} \coloneqq \begin{cases} \langle f_1, \dots, f_i \rangle & \text{if } 1 \leq i \leq r, \\ (0), & \text{if } i = 0. \end{cases}$$

We say that f is *regular* if $\mathfrak{a}_f^{(r)} \subsetneq R$, and for each $1 \leq i \leq r$, the image of f_i in $R/\mathfrak{a}_f^{(i-1)}$ is a non-zero divisor.

Associated to every $f \in R^r$, there is a *Koszul complex*:

$$(K_\bullet(f), d_\bullet(f)) : 0 \to \wedge^r R^r \to \wedge^{r-1} R^r \to \cdots \to \wedge^2 R^r \to R^r \to R \to 0.$$

Here for each $0 \leq i \leq r$, $\wedge^i R^r$ is the i-th wedge product of R^r and the differential map $d_i(f) : \wedge^i R^r \to \wedge^{i-1} R^r$ is the R-linear map determined by

$$d_i(f)(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{k=1}^{i} (-1)^{k+1} f_{j_k} e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \wedge \widehat{e}_{j_k} \wedge e_{j_{k+1}} \cdots \wedge e_{j_i}, \quad 1 \leq j_1 < \cdots < j_i \leq r,$$

where $e_1, \dots, e_r$ is a basis of R^r over R and $\widehat{e}_{j_k}$ means that e_{j_k} is omitted in the wedge product.

According to [25, Theorem 14.7] and [7, Lemma 3.2], the regularity of $f \in R^r$ is characterized by the exactness of $(K_\bullet(f), d_\bullet(f))$ and the height of $\mathfrak{a}_f^{(r)}$.

Lemma 2.1 (Criteria for regularity)**.** *Let R be a polynomial ring over a field $\mathbb{K}$. For each $f \in R^r$, the following are equivalent:*

(a) f is a regular sequence;
(b) $\mathrm{Ker}(d_1(f)) = \mathrm{Im}(d_2(f))$;
(c) $\mathrm{ht}(\mathfrak{a}_f^{(r)}) = r$.

We will also need the following fact in computational commutative algebra.

Lemma 2.2 (Bounded generation of kernel)**.** [15, 16] *There exists a function $B_1 : \mathbb{N}^4 \to \mathbb{N}$ with the following property. For any field $\mathbb{K}$ and matrix $A \in R^{a \times b}$, where $R = \mathbb{K}[x_0, \dots, x_N]$ and elements of A are homogeneous polynomials of degree at most m, the R-module $\mathrm{L}(A) \coloneqq \{v \in R^a : vA = 0\}$ is generated by vectors in R^a whose elements are polynomials of degree at most $B_1(N, m, a, b)$.*

2.2. **Algebraic geometry.** The following two facts are standard in algebraic geometry.

Lemma 2.3 (Fiber dimension formula)**.** [13, Proposition 10.6.1] *Assume that X and Y are quasi-projective varieties over $\mathbb{K}$ and $f : X \to Y$ is a regular map. If for any $y \in Y$, $\dim f^{-1}(y) \geq d$ (resp. $\dim f^{-1}(y) \leq d$), then $\dim X \geq d + \dim Y$ (resp. $\dim X \leq d + \dim Y$).*

Lemma 2.4 (Generalized Bezout theorem)**.** [12, Example 12.3.1] *Let X and Y be two quasi-projective subvarieties of $\mathbb{P}^N(\mathbb{K})$, then $\deg(X \cap Y) \leq \deg(X) \deg(Y)$.*

Recall that $\mathbb{F} = \overline{\mathbb{F}}_q$ and $\mathbb{P}^N(\mathbb{K})$ is the N-dimensional projective space over a field $\mathbb{K}$. For each projective subvariety $X \subseteq \mathbb{P}^N(\mathbb{F})$, we define $X(\mathbb{F}_q) \coloneqq X \cap \mathbb{P}^N(\mathbb{F}_q)$. According to the next two lemmas, $|X(\mathbb{F}_q)|$ can be bounded in terms of $\dim(X)$ and $\deg(X)$.

Lemma 2.5 (Number of $\mathbb{F}_q$-points I)**.** [18, Theorem 1] *There is a function $C : \mathbb{N}^3 \to \mathbb{N}$ with the following property. For any prime power q and any irreducible projective subvariety $X \subseteq \mathbb{P}^N(\mathbb{F})$ defined over $\mathbb{F}_q$ with $\dim X = n$ and $\deg X = k$, we have*

$$||X(\mathbb{F}_q)| - q^n| \le (k-1)(k-2)q^{n-\frac{1}{2}} + C(n,k,N)q^{n-1}.$$

Lemma 2.6 (Number of $\mathbb{F}_q$-points II)**.** [10, Corollary 3.3] *Let X be a projective subvariety of $\mathbb{P}^N(\mathbb{F})$ with $\dim X = n$ and $\deg X = k$. Then $|X(\mathbb{F}_q)| \le \frac{k(q^{n+1}-1)}{q-1}$.*

Suppose $R = \mathbb{F}[x_0, \ldots, x_N]$ is the polynomial ring over $\mathbb{F}$ in $N+1$ variables. Each homogeneous ideal $\mathfrak{a} = \langle f_1, \ldots, f_r \rangle \subseteq R$ defines a projective subvariety of $\mathbb{P}^N(\mathbb{F})$, denoted as $V(\mathfrak{a})$ or $V(f_1, \ldots, f_r)$. The *homogeneous coordinate ring* of a projective subvariety $X \subseteq \mathbb{P}^N(\mathbb{F})$ is $\mathbb{F}[X] := R/\mathfrak{a}_X$, where $\mathfrak{a}_X$ is the defining ideal of X. Note that $\mathfrak{a}_X$ is a homogeneous ideal. By (4), $\mathbb{F}[X]$ is graded as $\mathbb{F}[X] = \oplus_{m=0}^{\infty} \mathbb{F}[X]_m$. The *Hilbert function* of X is defined as

$$h_X : \mathbb{N} \to \mathbb{N}, \quad h_X(m) = \dim_{\mathbb{F}} \mathbb{F}[X]_m.$$

Lemma 2.7. [14, Remark 13.10] *For any k-dimensional subvariety $X \subseteq \mathbb{P}^N(\mathbb{F})$ and any integer $m \ge 0$, we have $h_X(m) \ge \binom{m+k}{m}$.*

We consider the Veronese map $\nu_m : \mathbb{K}^{N+1} \to \mathbb{K}^{\binom{N+m}{m}}$ defined by

$$\nu_m(u_0, \cdots, u_N) := (u_{i_1} \cdots u_{i_m})_{0 \le i_1 \le \cdots \le i_m \le N}. \tag{5}$$

It is worth noticing that if $X = \{[v_1], \ldots, [v_s]\} \subseteq \mathbb{P}^N(\mathbb{F})$ is a finite set, then we have

$$h_X(m) = \dim \operatorname{span}_{\mathbb{F}} \{\nu_m([v_1]), \ldots, \nu_m([v_s])\}. \tag{6}$$

Thus, $h_X(m)$ measures the linear dependence of $\nu_m([v_1]), \ldots, \nu_m([v_s])$. Although the study of h_X dated back to 1887 [6], the following notion was introduced fairly recently [5].

Definition 2.8 (s-wise m-independence)**.** Let $s, m \ge 0$ be fixed integers and let $X \subseteq \mathbb{P}^N(\mathbb{F})$ be a subset.

- The set X is s-wise m-independent if $h_S(m) = s$ for any $S \subseteq X$ such that $|S| = s$.
- Suppose further that X is a finite set. Then X is minimally m-dependent if X is not $|X|$-wise m-independent, but any proper subset $Y \subsetneq X$ is $|Y|$-wise m-independent.

Given integers $N, t, m \ge 0$, we define

$$X(N,t,m) := \{([v_1], \cdots, [v_t]) \in (\mathbb{P}^N(\mathbb{F}))^t : \{[v_1], \ldots, [v_t]\} \text{ is minimally } m\text{-dependent}\}.$$

By definition, $X(N,t,m)$ is a quasi-projective subvariety of $(\mathbb{P}^N(\mathbb{F}))^t$. As a result, the function $\psi(N,t,m) := \dim X(N,t,m)$ is well-defined. The following lemma states that s-wise m-independent sets exist over sufficiently large fields.

Lemma 2.9 (s-wise m-independent sets over large fields)**.** [5, Lemma 15] *There is a function $q_0 : \mathbb{N}^4 \to \mathbb{N}$ with the following property. If N, s, m, r are positive integers such that $N > r > \max_{2 \le t \le s} \left\{ \frac{\psi(N,t,m)}{t-1} \right\}$, then for any prime power $q \ge q_0(N,s,m,r)$, there exist $f_1, \ldots, f_r \in \mathbb{F}_q[x_0, \ldots, x_N]_m$ such that $V(f_1, \ldots, f_r)(\mathbb{F}_q)$ is s-wise m-independent.*

For ease of reference, we also record the lemma that estimates the value of $\psi(N,t,m)$.

Lemma 2.10 (Upper bound of $\psi(N,t,m)$)**.** [5, Lemma 22] *Given integers $N, t, m \ge 3$, we have*
(a) If $t \le m+1$, then $X(N,t,m)$ is an empty set.
(b) If $m \le t \le N$, then $\psi(N,t,m) \le \lfloor \frac{3t}{m+4} \rfloor (N + 1 + \frac{m-2}{m+4} t)$.

2.3. **An inequality.** We define a function

$$D : \mathbb{N}^2 \to \mathbb{N}, \quad D(r,t) := \min\left\{ m \in \mathbb{N} : \binom{m+r}{r} > t \right\}. \tag{7}$$

The following inequality is observed in [5, Lemma 24].

Lemma 2.11. *For any positive integers r and t, we have $\prod_{i=1}^{r} D(i,t) \le t^{1+\log r} r!$.*

3. The variety of non-regular sequences

The main results of this section are Propositions 3.1 and 3.3. In particular, we consider the set of non-regular sequences of homogeneous polynomials and we shall show that this set is a variety of bounded degree (Theorem 3.1) and dimension (Theorem 3.3).

In what follows, we provide a quantitative strengthening of the well-known fact [28] that a generic sequence of $s \le N+1$ homogeneous polynomials in $\mathbb{K}[x_0, \cdots, x_N]$ is regular.

Proposition 3.1 (Equations for non-regular sequences)**.** *There is a function $B_2 : \mathbb{N}^{s+2} \to \mathbb{N}$ with the following property. Let $\mathbb{K}$ be a field and let X be a projective subvariety of $\mathbb{P}^N(\mathbb{K})$ defined by a regular sequence $f_1, \dots, f_{N-n} \in R := \mathbb{K}[x_0, \dots, x_N]$. Suppose $k := \max_{1 \le i \le N-n}\{\deg f_i\}$. For any integers $0 \le s \le n$ and $0 \le m_1, \dots, m_s$, the set*

$$\mathcal{U}_X(m_1, \dots, m_s) := \left\{ (h_1, \dots, h_s) \in \prod_{i=1}^{s} R_{m_i} : \dim(X \cap V(h_1, \dots, h_s)) \ge n - s + 1 \right\} \tag{8}$$

is a subvariety of $\prod_{i=1}^{s} R_{m_i} \simeq \mathbb{K}^{\sum_{i=1}^{s}\binom{N+m_i}{m_i}}$ defined by at most $B_2(N,k,m_1,\dots,m_s)$ polynomials of degree at most $B_2(N,k,m_1,\dots,m_s)$. In particular, the degree of $\mathcal{U}_X(m_1,\dots,m_s)$ is at most

$$B_3(N,k,m_1,\dots,m_s) := B_2(N,k,m_1,\dots,m_s)^{B_2(N,k,m_1,\dots,m_s)}.$$

Proof. Denote $m_{s+i} := \deg f_i \le k$, $1 \le i \le N-n$. We observe that

$$\mathcal{U}_X(m_1,\dots,m_s) = \mathcal{U}_{\mathbb{P}^N}(m_1,\dots,m_{s+N-n}) \cap \left(\prod_{i=1}^{s} R_{m_i} \times \{(f_1,\dots,f_{N-n})\} \right).$$

Moreover, $\prod_{i=1}^{s} R_{m_i} \times \{(f_1,\dots,f_{N-n})\}$ is an affine linear subspace in $\prod_{i=1}^{s+N-n} R_{m_i}$, which is defined by $\sum_{i=1}^{N-n}\left[\binom{N+m_{s+i}}{m_{s+i}} - 1\right] = O(N(N+k)^k)$ linear polynomials. Henceforth, it is sufficient to assume that $X = \mathbb{P}^N(\mathbb{K})$. In particular, we have $n = N$. In the rest of the proof, we abbreviate $\mathcal{U}_{\mathbb{P}^N}(m_1,\dots,m_s)$ as $\mathcal{U}$.

According to Lemma 2.1, an element $h := (h_1,\dots,h_s) \in \prod_{i=1}^{s} R_{m_i}$ lies in $\mathcal{U}$ if and only if $\operatorname{Im}(d_2(h)) \subsetneq \operatorname{Ker}(d_1(h))$. Here $d_1(h) : R^s \to R$ and $d_2(h) : \wedge^2 R^s \to R^s$ are R-linear maps obtained by linearly extending

$$d_1(h)(e_i) = h_i, \quad d_2(h)(e_i \wedge e_j) = h_i e_j - h_j e_i, \quad 1 \le i,j \le s$$

and $e_1,\dots,e_s$ is a basis of R^s. By definition, we have

$$\operatorname{Im}(d_2(h)) \subseteq \operatorname{Ker}(d_1(h)) \subseteq \bigoplus_{l=0}^{\infty} \prod_{i=1}^{s} R_{l-m_i}.$$

Furthermore, Lemma 2.2 provides a function $B_1 : \mathbb{N}^4 \to \mathbb{N}$ such that $\operatorname{Ker}(d_1(h))$ is generated by some elements of $\bigoplus_{l=0}^{B_1(N,m,s,1)} \prod_{i=1}^{s} R_{l-m_i}$. Since $s \le n = N$, there is a function $B_1' : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $B_1(N,m,s,1) \le B_1'(N,m)$.

If we identify $\wedge^2 R^s$ with $R^{\binom{s}{2}}$, then for each integer l, only elements in $\prod_{1\le i<j\le s} R_{l-m_i-m_j}$ can be mapped into $\prod_{i=1}^s R_{l-m_i}$ by $d_2(h)$. Consequently, $h \in \mathcal{U}$ if and only if the following complex fails to be exact:

$$\bigoplus_{l=0}^{B_1'(N,k)} \prod_{1\le i<j\le s} R_{l-m_i-m_j} \xrightarrow{d_2(h)} \bigoplus_{l=0}^{B_1'(N,k)} \prod_{i=1}^{s} R_{l-m_i} \xrightarrow{d_1(h)} \bigoplus_{l=0}^{B_1'(N,k)} R_l. \tag{9}$$

Since each R_l in (9) is a finite dimensional $\mathbb{K}$-vector space and there are finitely many of them, (9) can be written as a complex of finite dimensional $\mathbb{K}$-vector spaces:

$$\mathbb{K}^{N_3} \xrightarrow{H_2} \mathbb{K}^{N_2} \xrightarrow{H_1} \mathbb{K}^{N_1},$$

where N_1, N_2, N_3 are some positive integers only depending on N and k, and $H_1 \in \mathbb{K}^{N_3\times N_2}$ (resp. $H_2 \in \mathbb{K}^{N_2\times N_1}$) is the matrix of $d_2(h)$ (resp. $d_1(h)$). Correspondingly, the non-exactness of (9) is equivalent to the condition that $\operatorname{rank} H_1 + \operatorname{rank} H_2 \le N_2 - 1$, which is further equivalent to the condition that for each $1 \le i \le N_2 - 1$, either $\operatorname{rank} H_1 \le i-1$ or $\operatorname{rank} H_2 \le N_2 - i - 1$. Therefore, the non-exactness of (9) is defined by the ideal $\mathfrak{d} = \sum_{i=1}^{N_2-1} \mathfrak{a}_i \otimes \mathfrak{b}_{N_2-i}$, where $\mathfrak{a}_i$ (resp. $\mathfrak{b}_{N_2-i}$) is the ideal generated by $i\times i$ (resp. $(N_2-i)\times(N_2-i)$) minors of $N_2 \times N_1$ (resp. $N_3 \times N_2$) matrices.

Noticing that N_1, N_2, N_3 only depends on $N, m_1, \ldots, m_s$ and k, it is clear that $\mathfrak{d}$ is generated by at most $B_2(N, k, m_1, \ldots, m_s)$ polynomials of degree at most $B_2(N, k, m_1, \ldots, m_s)$, for some function $B_2 : \mathbb{N}^{s+2} \to \mathbb{N}$. By Lemma 2.4, the degree of $\mathcal{U}_X(m_1, \ldots, m_s)$ is bounded above by $B_3(N, k, m_1, \ldots, m_s)$. □

Remark 3.2. *Given $h := (h_1, \ldots, h_s) \in \prod_{i=1}^s R_{m_i}$, h lies in $\mathcal{U}_X(m_1, \ldots, m_s)$ if and only if the image of h in $\mathbb{K}[X]$ is a non-regular sequence. Here $\mathbb{K}[X]$ denotes the homogeneous coordinate ring of X. In other words, $\mathcal{U}_X(m_1, \ldots, m_s)$ is the variety consisting of sequences of s homogeneous polynomials, of degrees $m_1, \ldots, m_s$ respectively, which fail to extend $f_1, \ldots, f_{N-n}$ to a regular sequence.*

Next, we estimate the dimension of $\mathcal{U}_X(m_1, \ldots, m_s)$ in terms of $m_1, \ldots, m_s$, N and k.

Proposition 3.3 (Dimension of the variety of non-regular sequences)**.** *Suppose $X \subseteq \mathbb{P}^N(\mathbb{K})$ is a projective subvariety defined by $(N-n)$ homogeneous polynomials in $R := \mathbb{K}[x_0, \ldots, x_N]$, which form a regular sequence. Given positive integers $m_1, \ldots, m_s$, we have*

$$\dim \mathcal{U}_X(m_1, \ldots, m_s) \le \sum_{i=1}^{s} \binom{N+m_i}{N} - \min_{1\le i\le s} \binom{n-i+1+m_i}{m_i}.$$

Here $\mathcal{U}_X(m_1, \ldots, m_s)$ is the variety defined in (8). In particular, for $m_1 = \cdots = m_s = m$, we obtain

$$\dim \mathcal{U}_X(m, \ldots, m) \le s\binom{N+m}{N} - \binom{n-s+1+m}{m}.$$

Proof. Denote $\mathcal{U} := \mathcal{U}_X(m_1, \ldots, m_s)$. We consider for each $1 \le i \le s$ a subset

$$\mathcal{U}_i := \Big\{(h_1, \ldots, h_i) \in \prod_{j=1}^{i} R_{m_j} : \dim\big(X \cap V(h_1, \ldots, h_i)\big) = n-i+1,$$
$$\dim\big(X \cap V(h_1, \ldots, h_{i-1})\big) = n-i+1\Big\}.$$

By Krull's principal ideal theorem, $\mathcal{U}_i$ is a quasi-projective variety and $\mathcal{U} = \cup_{i=1}^s (\mathcal{U}_i \times \prod_{j=i+1}^s R_{m_j})$. We also notice that the Zariski closure of $\mathcal{U}_i$ is

$$\overline{\mathcal{U}}_i := \left\{ (h_1, \cdots, h_i) \in \prod_{j=1}^{i} R_{m_j} : \dim(X \cap V(h_1, \ldots, h_i)) \ge n-i+1 \right\} = \bigcup_{j=1}^{i} (\mathcal{U}_j \times \prod_{l=j+1}^{i} R_{m_l}).$$

We consider the projection map

$$\pi : \mathcal{U}_{i+1} \to \left(\prod_{j=1}^{i} R_{m_j} \right) \setminus \overline{\mathcal{U}}_i, \quad \pi(h_1, \dots, h_{i+1}) = (h_1, \dots, h_i).$$

Given $(h_1, \dots, h_i) \in \left(\prod_{j=1}^{i} R_{m_j} \right) \setminus \overline{\mathcal{U}}_i$ and $h_{i+1} \in R_{m_{i+1}}$, we observe that $(h_1, \dots, h_i, h_{i+1}) \in \mathcal{U}_{i+1}$ if and only if $h_{i+1} \in \cup_{\mathfrak{p} \in \min(\mathfrak{a})} \mathfrak{p}_{m_{i+1}}$, where $\mathfrak{a}$ is the homogeneous ideal generated by $(N-n)$ defining equations of X and $h_1, \dots, h_i$, $\min(\mathfrak{a})$ is the set of all minimal prime ideals $\mathfrak{p}$ containing $\mathfrak{a}$ such that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{a})$, and $\mathfrak{p}_{m_{i+1}}$ is the degree m_{i+1} part of $\mathfrak{p}$. We recall that $\min(\mathfrak{a})$ is finite. Thus, $\pi^{-1}(h_1, \dots, h_i) = \cup_{\mathfrak{p} \in \min(\mathfrak{a})} \mathfrak{p}_{m_{i+1}}$ is a union of finitely many $\mathbb{K}$-linear subspaces. This implies

$$\dim \pi^{-1}(h_1, \dots, h_i) = \max_{\mathfrak{p} \in \min(\mathfrak{a})} \left\{ \binom{N + m_{i+1}}{N} - H_{V(\mathfrak{p})}(m_{i+1}) \right\} \le \binom{N + m_{i+1}}{N} - \binom{n - i + m_{i+1}}{n - i},$$

where the inequality follows from Lemma 2.7, as $\dim V(\mathfrak{p}) = N - \mathrm{ht}(\mathfrak{p}) = N - \mathrm{ht}(\mathfrak{a}) = n - i$. According to Lemma 2.3, we obtain

$$\dim \mathcal{U}_{i+1} \le \sum_{j=1}^{i} \binom{N + m_j}{N} + \binom{N + m_{i+1}}{N} - \binom{n - i + m_{i+1}}{n - i}$$

Henceforth, we have

$$\dim \mathcal{U} = \max_{1 \le i \le s} \left\{ \dim \mathcal{U}_i + \sum_{j=i+1}^{s} \binom{N + m_j}{N} \right\} \le \sum_{j=1}^{s} \binom{N + m_j}{N} - \min_{1 \le i \le s} \binom{n - i + 1 + m_i}{m_i}. \qquad \square$$

Proposition 3.1 and Proposition 3.3 imply that, over any field, the set $\mathcal{U}_X(m_1, \dots, m_s)$ of non-regular sequences is a subvariety of $\prod_{i=1}^{s} R_{m_i}$. Over finite fields, it may happen that $\mathcal{U}_X(m_1, \dots, m_s)$ equals the entire space $\prod_{i=1}^{s} R_{m_i}$, although its dimension is strictly smaller. However, combining Proposition 3.3 with Lemma 2.6 we see that regular sequences always exist over sufficiently large fields.

Corollary 3.4. *Given a positive integer s, there is a function $q_0' : \mathbb{N}^{s+2} \to \mathbb{N}$ with the following property. For any positive integers $N, n, m_1, \dots, m_s$ and any prime power $q > q_0'(N, n, m_1, \dots, m_s)$, we have $\mathcal{U}_X(m_1, \dots, m_s) \subsetneq \prod_{i=1}^{s} R_{m_i}$. Here $\mathcal{U}_X(m_1, \dots, m_s)$ is defined as in (8), and $X \subseteq \mathbb{P}^N(\mathbb{K})$ is any projective subvariety defined by $(N-n)$ homogeneous polynomials in $R := \mathbb{F}_q[x_0, \dots, x_N]$, which form a regular sequence.*

In the rest of this section, we derive two more corollaries of Propositions 3.1 and 3.3, which will play a crucial role in our proof of Theorem 1.3.

We need a basic property of s-wise m-independent subsets.

Lemma 3.5. *If $\{[v_1], \dots, [v_s]\} \subseteq \mathbb{P}^N(\mathbb{K})$ is s-wise m-independent, then there exists a $\mathbb{K}$-basis $f_1, \dots, f_{\binom{N+m}{m}}$ of $\mathbb{K}[x_0, \dots, x_N]_m$ such that $f_i(v_j) = \delta_{ij}$ for each $1 \le i \le \binom{N+m}{m}$ and $1 \le j \le s$. Here δ is the Kronecker delta function.*

Proof. For simplicity, we denote $M := \binom{N+m}{m}$. We consider the Veronese map $\nu_m : \mathbb{K}^{N+1} \to \mathbb{K}^M$ defined in (5). By assumption and (6), $\nu_m(v_1), \dots, \nu_m(v_s)$ are linearly independent. We extend $\nu_m(v_1), \dots, \nu_m(v_s)$ to a $\mathbb{K}$-basis of $\mathbb{K}^M$ and let $\ell_1, \dots, \ell_M$ be its dual basis. If we take $f_j := \ell_j \circ \nu_m$ for each $1 \le j \le M$, then $\{f_1, \dots, f_M\}$ is a desired basis of $\mathbb{K}[x_0, \dots, x_N]_m$. $\square$

Corollary 3.6. *There is a function $B_4 : \mathbb{N}^5 \to \mathbb{N}$ with the following property. Suppose $X \subseteq \mathbb{P}^N(\mathbb{F})$ is a projective subvariety defined by a regular sequence of $(N-n)$ homogeneous polynomials in $R := \mathbb{F}_q[x_0, \dots, x_N]$ of degree at most k and E is a subset of $[s_1] \times [s_2] \times \cdots \times [s_{d-1}]$. Let $Y_i \subseteq \mathbb{P}^N(\mathbb{F})$ be an s_i-wise m-independent projective subvariety of degree at most κ, for each $i \in [d-1]$. Let*

$Y \subseteq \prod_{i=1}^{d-1} Y_i^{s_i}$ *consist of points* $y := (y_{1,1}, \ldots, y_{1,s_1}, \ldots, y_{d-1,1}, \ldots, y_{d-1,s_{d-1}})$ *such that* $y_{i,u} \neq y_{i,v}$ *for any* $i \in [d-1]$ *and* $1 \leq u \neq v \leq s_i$, *and* $Z_{E,g,y} \subseteq \mathbb{P}^N$ *be the projective subvariety defined as*

$$Z_{E,g,y} := \left\{ z \in \mathbb{P}^N(\mathbb{F}_q) : g(y_{1,i_1}, \ldots, y_{d-1,i_{d-1}}, z) = 0, \quad \forall (i_1, \cdots, i_{d-1}) \in E \right\}. \tag{10}$$

If $|E| \leq n$ *and* $\sum_{i=1}^{d-1} s_i \dim Y_i \leq \binom{n-|E|+1+m}{m} - 1$, *then*

$$\left| \left\{ g \in R_m^{\otimes d} : \dim(X \cap Z_{E,g,y}) \geq \dim X - |E| + 1 \text{ for some } y \in Y \right\} \right|$$
$$\leq B_4(N, m, s_1 \cdots s_{d-1}, k, \kappa) q^{\binom{N+m}{m}^d - 1}.$$

Proof. Let $y := (y_{1,1}, \ldots, y_{1,s_1}, \ldots, y_{d-1,1}, \ldots, y_{d-1,s_{d-1}}) \in Y$ be fixed. Denote $M := \binom{N+m}{m}$. For each $i \in [d-1]$, we apply Lemma 3.5 to find an $\mathbb{F}_q$-linear basis $\alpha_{i,1}, \ldots, \alpha_{i,M}$ of R_m such that $\alpha_{i,j}(y_{i,l}) = \delta_{jl}$ for any $(j,l) \in [M] \times [s_i]$. Additionally, we fix an $\mathbb{F}_q$-linear basis $\alpha_{d,1}, \ldots, \alpha_{d,M}$ of R_m. Therefore, each $g \in R_m^{\otimes d}$ can be written as

$$g = \sum_{j_1, \ldots, j_d = 1}^{M} \lambda_{j_1, \ldots, j_d} \otimes_{i=1}^{d} \alpha_{i,j_i}. \tag{11}$$

By the choice of $\alpha_{i,j}$'s, the defining equations of $Z_{E,g,y}$ are

$$h_{i_1, \ldots, i_{d-1}}(z) := g(y_{1,i_1}, \ldots, y_{d-1,i_{d-1}}, z) = \sum_{j_d=1}^{M} \lambda_{i_1, \ldots, i_{d-1}, j_d} \alpha_{d,j_d}(z) = 0, \tag{12}$$

where $(i_1, \cdots, i_{d-1}) \in E$. Denote $h := (h_{i_1, \ldots, i_{d-1}})_{(i_1, \cdots, i_{d-1}) \in E} \in R_m^{|E|}$. Then $\dim(X \cap Z_{E,g,y}) \geq \dim X - |E| + 1$ if and only if $h \in \mathcal{U}_X(m, \ldots, m)$. Here $\mathcal{U}_X(m, \ldots, m) \subseteq R_m^{|E|}$ is defined in (8). According to Propositions 3.1 and 3.3, we have

$$\dim \mathcal{U}_X(m, \ldots, m) \leq |E|M - \binom{n - |E| + 1 + m}{m}, \quad \deg \mathcal{U}_X(m, \ldots, m) \leq B_3(N, k, m, \ldots, m),$$

where B_3 is the function in Proposition 3.1.

Let

$$G_y := \left\{ g \in R_m^{\otimes d} : \dim(X \cap Z_{E,g,y}) \geq \dim X - |E| + 1 \right\}.$$

By comparing λ's in (11) and (12), we conclude that

$$\dim G_y \leq M^d - \binom{n - |E| + 1 + m}{m}, \quad \text{and} \quad \deg G_y \leq B_3(N, k, m, \ldots, m).$$

By Lemma 2.6, we have

$$|G_y| \leq 2B_3(N, m, k) q^{M^d - \binom{n-|E|+1+m}{m}}, \quad |Y_i| \leq 2\kappa q^{n_i}$$

where $n_i := \dim Y_i$, $i \in [d-1]$. This implies

$$|Y| \leq \prod_{i=1}^{d-1} |Y_i| \leq (2\kappa)^{|E|} q^{\sum_{i=1}^{d-1} n_i s_i} \leq (2\kappa)^{|E|} q^{\binom{n-|E|+1+m}{m} - 1}.$$

Consequently, writing $G := \cup_{y \in Y} G_y = \left\{ g \in R_m^{\otimes d} : \dim(X \cap Z_{E,g,y}) \geq \dim X - |E| + 1 \text{ for some } y \in Y \right\}$, we obtain

$$|G| \leq \sum_{y \in Y} |G_y| \leq 2^{|E|+1} \kappa^{|E|} q^{\binom{n-|E|+1+m}{m} - 1} B_3(N, k, m, \ldots, m) q^{M^d - \binom{n-|E|+1+m}{m}}$$
$$= B_4(N, m, s_1 \cdots s_{d-1}, k, \kappa) q^{M^d - 1},$$

where B_4 is the function defined by $B_4(N, m, s_1 \cdots s_{d-1}, k, \kappa) := (2\kappa)^{N+1} B_3(N, k, m, \ldots, m)$. $\square$

Corollary 3.7. *There is a function $q_1 : \mathbb{N}^{t+2} \to \mathbb{N}$ with the following property. Let $t, m, r, s_1, \cdots, s_{d-1}$ be positive integers, and E be a subset of $[s_1] \times [s_2] \times \cdots \times [s_{d-1}]$. Denote $N := t + r + |E|$, $l := \left(\sum_{i=1}^{d-1} s_i\right)(|E| + t)$, $s := \max_{1\le i\le d-1}\{s_i\}$, and $m_j := D(t - j + 1, l)$ for $1 \le j \le t$, where $D(\cdot,\cdot)$ is defined in (7). For each $\sigma \in \mathfrak{S}_d$, let $Y^\sigma \subseteq \prod_{i=1}^{d-1} Y_{\sigma(i)}^{s_{\sigma(i)}}$ be the set consisting of all*

$$y^\sigma := (y_{\sigma(1),1}, \dots, y_{\sigma(1),s_{\sigma(1)}}, \dots, y_{\sigma(d-1),1}, \dots, y_{d-1,s_{\sigma(d-1)}})$$

such that $y_{\sigma(i),u} \neq y_{\sigma(i),v}$ for any $1 \le i \le d-1$ and $1 \le u \neq v \le s_{\sigma(i)}$, $g^\sigma \in R^{\otimes d}$ is the element obtained by permuting d factors of g by σ. For any $y \in Y$,

$$y^\sigma := (y_{\sigma(1),1}, \dots, y_{\sigma(1),s_{\sigma(1)}}, \dots, y_{\sigma(d-1),1}, \dots, y_{\sigma(d-1),s_{\sigma(d-1)}}),$$

and Z_{E,g^σ,y^σ} is the variety defined by (10). If

$$3 \le m, \quad \max_{2\le u\le s}\left\{\frac{\psi(N,u,m)}{u-1}\right\} + 1 \le r, \quad l + 1 \le \binom{t+1+m}{m},$$

then for any prime power $q \ge q_1(N, m, m_1, \dots, m_t)$, there exists a sequence of homogeneous polynomials:

$$\{g\} \times (f_{k,1}, \dots, f_{k,r}, h_{k,1}, \dots, h_{k,t})_{1\le k\le d} \in R_m^{\otimes d} \times \prod_{k=1}^{d}\left(R_m^r \times \prod_{j=1}^{t} R_{m_j}\right)$$

where $R := \mathbb{F}_q[x_0, \dots, x_N]$ such that

(a) For each $1 \le k \le d$, $Y_k := V(f_{k,1}, \dots, f_{k,r})(\mathbb{F}_q)$ is s-wise m-independent, and the dimension of $V(f_{k,1}, \dots, f_{k,r}, h_{k,1}, \dots, h_{k,t})$ is $|E|$.

(b) For each $\sigma \in \mathfrak{S}_d$ and $y \in Y$ we have

$$\dim\left(V(f_{\sigma(d),1}, \dots, f_{\sigma(d),r}, h_{\sigma(d),1}, \dots, h_{\sigma(d),t}) \cap Z_{E,g^\sigma,y^\sigma}\right) = 0, \tag{13}$$

Proof. We define $q_1 : \mathbb{N}^{t+2} \to \mathbb{N}$ as

$$\begin{aligned} q_1(N, m, m_1, \dots, m_t) \quad &:= \max\Big\{q_0(N, N, m, N),\ q_0'(N, N, \underbrace{m, \dots, m}_{r \text{ times}}), \\ &\qquad dB_4(N, m, N, m, m^N),\ d\,2^N m^{N^2} B_3(N, m, m_1, \dots, m_t)\Big\}, \end{aligned}$$

where q_0, q_0' B_3, B_4 are functions in Lemma 2.9, Corollary 3.4, Proposition 3.1 and Corollary 3.6, respectively. Moreover, we observe that varieties $Y_{\sigma(d)}$, $V(h_{\sigma(d),1}, \dots, h_{\sigma(d),t})$ and Z_{E,g^σ,y^σ} actually only depend on the value of $\sigma(d)$. Let q be a prime power such that $q > q_1(N, m, r, m_1, \dots, m_t)$.

By the choice of q, we have $q > q_0(N, s, m, r)$ and $q > q_0'(N, N, m, \dots, m)$. Lemma 2.9 and Corollary 3.4 imply that there are $f_{k,1}, \dots, f_{k,r} \in R_m$ such that $Y_k = V(f_{k,1}, \dots, f_{k,r})(\mathbb{F}_q)$ is s-wise m-independent and each sequence $f_{k,1}, \dots, f_{k,r}$ is regular for each $k \in [d]$. In particular, Y_i is s_i-wise m-independent for each $i \in [d-1]$, as $s_i \le s$. By Lemmas 2.1 and 2.4, we have

$$\dim Y_k = N - r = t + |E|, \quad \deg Y_k \le m^r.$$

Since $l < \binom{t+1+m}{m}$, Corollary 3.6 ensures the existence of a function $B_4 : \mathbb{N}^5 \to \mathbb{N}$ such that apart from a subset of cardinality at most $dB_4(N, m, N, m, m^r)q^{\binom{N+m}{m}^d - 1}$, every $g \in R_m^{\otimes d}$ satisfies $\dim(Y_{\sigma(d)} \cap Z_{E,g^\sigma,y^\sigma}) = t$ for any $\sigma \in \mathfrak{S}_d$ and $y \in Y$. Since $q > dB_4(N, m, N, m, m^r)$, the desired $g \in R_m^{\otimes d}$ must exist.

Next, given each $\{g\} \times (f_{k,1}, \dots, f_{k,r})_{k\in[d]} \in R_m^{\otimes d} \times \prod_{k=1}^{d} R_m^r$ with aforementioned properties, we prove the existence of $(h_{k,1}, \dots, h_{k,t})_{k\in[d]} \in \prod_{k=1}^{d}\prod_{j=1}^{t} R_{m_j}$ such that (a) and (b) hold. We claim that, except a subset of cardinality at most $2^N m^{(N-t)N} B_3(N,m,m_1,\ldots,m_t) q^{\sum_{j=1}^t \binom{N+m_j}{N}-1}$, every $(h_1,\cdots,h_t) \in \prod_{i=1}^t R_{m_i}$ satisfies

$$\dim\left(Y_d \cap V(h_1,\ldots,h_t) \cap Z_{E,g,y}\right) = 0$$

for any $y \in Y$. If the claim holds, then for each $k \in [d]$, there exists $(h_{k,1},\ldots,h_{k,t}) \in \prod_{j=1}^t R_{m_j}$ such that $\dim\left(Y_k \cap V(h_{k,1},\ldots,h_{k,t}) \cap Z_{E,g^\sigma,y^\sigma}\right) = 0$ for any $\sigma \in \mathfrak{S}_d$ such that $\sigma(d) = k$. In particular, this implies $\dim\left(Y_k \cap V(h_{k,1},\ldots,h_{k,t})\right) = |E|$ and the proof is complete since $q > d2^N m^{(N-t)N} B_3(N,m,m_1,\ldots,m_t)$.

It is left to prove the claim. To this end, we need to bound $|\cup_{y\in Y}\mathcal{U}_{X_y}|$. For any $y \in Y$, applying Propositions 3.1 and 3.3 to $X_y := Y_d \cap Z_{E,g,y}$, we conclude that

$$\begin{aligned}
\deg \mathcal{U}_{X_y}(m_1,\ldots,m_t) &\le B_3(N,m,m_1,\ldots,m_t),\\
\dim \mathcal{U}_{X_y}(m_1,\ldots,m_t) &\le \sum_{j=1}^t \binom{N+m_j}{N} - \min_{j\in[t]}\left\{\binom{t-j+1+m_j}{m_j}\right\}.
\end{aligned}$$

Thus, Lemma 2.6 leads to

$$|\mathcal{U}_{X_y}(m_1,\ldots,m_t)| \le 2B_3(N,m,m_1,\ldots,m_t) q^{\sum_{j=1}^t \binom{N+m_j}{N} - \min_{j\in[t]}\left\{\binom{t-j+1+m_j}{m_j}\right\}}.$$

Since $|Y_k| \le 2m^r q^{t+|E|}$, we have $|Y| \le \prod_{i=1}^{d-1}|Y_i|^{s_i} \le (2m^r)^{\sum_{i=1}^{d-1} s_i} q^l$. As a consequence, we obtain

$$\begin{aligned}
|\cup_{y\in Y}\mathcal{U}_{X_y}| &\le 2B_3(N,m,m_1,\ldots,m_t) q^{\sum_{j=1}^t \binom{N+m_j}{N} - \min_{j\in[t]}\left\{\binom{t-j+1+m_j}{m_j}\right\}} (2m^r)^{\sum_{i=1}^{d-1} s_i} q^l\\
&= 2^{1+\sum_{i=1}^{d-1} s_i} B_3(N,m,m_1,\ldots,m_t) m^{r\sum_{i=1}^{d-1} s_i} q^{\sum_{j=1}^t \binom{N+m_j}{N} - \min_{j\in[t]}\left\{\binom{t-j+1+m_j}{m_j}\right\}+l}\\
&\le 2^N m^{(N-t)N} B_3(N,m,m_1,\ldots,m_t) q^{\sum_{j=1}^t \binom{N+m_j}{N}-1}.
\end{aligned}$$

The last inequality follows from the constraint on l and the definition of N and m_j, $j \in [t]$. $\square$

4. Hypergraph Turán number

This section is devoted to the proof of Theorem 1.3. To this end, we first establish the following lemma.

Lemma 4.1. *There is a function $B_5 : \mathbb{N}^{d+3} \to \mathbb{N}$ with the following property. Let $N, m, m_1, \ldots, m_d$ be positive integers and let $q > B_5(N, m_1, \ldots, m_d, m, d)$ be a prime power. Denote $R := \mathbb{F}_q[x_0,\cdots,x_N]$. Suppose for each $1 \le i \le d$, V_i is a projective subvariety of $\mathbb{P}^N(\mathbb{F}_q)$ defined by a regular sequence $g_{i,1},\ldots,g_{i,N-k}$ of degree at most m_i. Then for any $g \in R_m^{\otimes d}$, we have $|W_g| \ge q^{dk-1}/2$, where $W_g := \left\{([y_1],\ldots,[y_d]) \in \prod_{i=1}^d V_i(\mathbb{F}_q) : g([y_1],\ldots,[y_d]) = 0\right\}$.*

Proof. By Krull's principal ideal theorem, we have $\dim n := W_g \ge \sum_{i=1}^d \dim V_i - 1 = dk-1$. We consider the Segre embedding $\mathrm{Seg} : (\mathbb{P}^n(\mathbb{F}_q))^d \to \mathbb{P}^{(N+1)^d-1}(\mathbb{F}_q)$ defined by $\mathrm{Seg}([v_1],\ldots,[v_d]) = [v_1 \otimes \cdots \otimes v_d]$. We notice that $\mathrm{Seg}(W_g)$ is a projective subvariety of $\mathbb{P}^{(N+1)^d-1}(\mathbb{F}_q)$ defined by homogeneous quadratic polynomials defining $\mathrm{Seg}\left((\mathbb{P}^n(\mathbb{F}_q))^d\right)$, homogeneous polynomials of degree at most $\max_{1\le i\le d}\{m_i\}$ induced by g_{ij} where $1 \le i \le d$ and $1 \le j \le N-k$, and the homogeneous polynomial of degree m induced by g. This together with Lemma 2.4 implies that $\kappa := \deg \mathrm{Seg}(W_g) \le C_1(N, m, \max_{1\le j\le d}\{m_j\}, d)$ for some function $C_1 : \mathbb{N}^4 \to \mathbb{N}$. By Lemma 2.5, we have

$$|W_g| = |\mathrm{Seg}(W_g)| \ge \frac{1}{2}q^n \ge \frac{1}{2}q^{dk-1}$$

if $q > B_5(N, m_1, \ldots, m_d, m, d) := 4\left(C(dk, C_1(N, m, \max_{1\le j\le d}\{m_j\}, d), N) + (k-1)^2(k-2)^2\right)$ where C is the function in Lemma 2.5. $\square$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\mathcal{H}$ be a $(d-1)$-partite $(d-1)$-uniform hypergraph on vertex set $V(\mathcal{H}) = [s_1] \cup \cdots \cup [s_{d-1}]$. Let

$$\beta := (d^2+4d-5)^{\frac{1}{3}}, \quad s := \max_{1\le i\le d-1}\{s_i\}, \quad t := \left\lceil \beta(e(\mathcal{H})s)^{\frac{1}{3}} \right\rceil.$$

We shall prove that if

$$s_d > \left[\left(\frac{3(d-1)}{\beta^2}\right)^{1+\log t} t^{3(1+\log t)} 3^{t+3} t!\right] 9^{e(\mathcal{H})},$$

then there is an $\mathcal{H}(s_d)$-free d-partite d-uniform hypergraph $\mathcal{G}$ with $\Omega_{s_1,\ldots,s_{d-1}}(n^{d-\frac{1}{e(\mathcal{H})}})$ edges, i.e.

$$\mathrm{ex}(n,\mathcal{H}(s_d)) = \Omega_{s_1,\ldots,s_{d-1}}\left(n^{d-\frac{1}{e(\mathcal{H})}}\right).$$

To construct the desired $\mathcal{G}$, we need some further parameters. Let $S := e(\mathcal{H})$, $r := S+t+3$, $l := (S+t)\sum_{i=1}^{d-1} s_i$, and $N := S+t+r = 2r-3$. Firstly, it is straightforward to verify that $\left\lfloor \frac{3u}{7} \right\rfloor \le c(u-1)$ where $c = 1/2$ if $2 \le u \le 7$ and $c = 7/15$ if $8 \le u$. By Lemma 2.10, we have

$$\psi(N,u,3) \le \left\lfloor \frac{3u}{7} \right\rfloor \left(N+1+\frac{u}{7}\right) \le c(u-1)\left(N+1+\frac{u}{7}\right)$$

for any $2 \le u$. For $u \le 7$, we have $N+1+u/7 \le 2r-1$, whereas for $8 \le u \le r$, we have $N+1+u/7 \le 2r-2+r/7$. Thus, we may derive that $\psi(N,u,3) \le r(u-1)$ for any $2 \le u \le r$.

Next, we notice that

$$\binom{t+4}{3} > \frac{t^3+9t^2}{6} \ge (d-1)sS + \frac{\frac{1}{6}\beta^3-(d-1)}{\beta} t\,(sS)^{\frac{2}{3}} + \frac{3}{2}\beta t\,(sS)^{\frac{1}{3}}.$$

By the AM-GM inequality, we obtain

$$\begin{aligned}
\binom{t+4}{3} - l &> (d-1)sS + \frac{\frac{1}{6}\beta^3-(d-1)}{\beta} t\,(sS)^{\frac{2}{3}} + \frac{3}{2}\beta t\,(sS)^{\frac{1}{3}} - (S+t)\sum_{i=1}^{d-1} s_i \\
&\ge \frac{\frac{1}{6}\beta^3-(d-1)}{\beta} t\,(sS)^{\frac{2}{3}} + \frac{3}{2}\beta t\,(sS)^{\frac{1}{3}} - (d-1)ts \\
&\ge 2t\left[\frac{3}{2}\left(\frac{1}{6}\beta^3-(d-1)\right)\right]^{1/2} (sS)^{1/2} - (d-1)ts \\
&\ge ts(d-1)\left[2\left(\frac{3}{2}\left(\frac{1}{6}\beta^3-(d-1)\right)\right)^{1/2} - (d-1)\right] \\
&= 0.
\end{aligned}$$

Hence t, r satisfy conditions in Corollary 3.7 with $m = 3$. Let q_1, B_5 be functions in Corollary 3.7 and Lemma 4.1, respectively. Suppose $m_j := D(t-j+1, l)$, $1 \le j \le t$, where $D(\cdot,\cdot)$ is the function defined in (7). Let n be an integer such that

$$n > C2^S \max\left\{q_1(N,3,m_1,\ldots,m_t)^S, B_5(N,3,m_1,\ldots,m_t)^S\right\}, \quad \text{where } C := d3^r \prod_{j=1}^{t} m_j.$$

By Bertrand's postulate [21, Theorem 2.4], there exists a prime p such that $Cp^S \le n \le C(2p)^S$. In particular, we must have

$$p > \max\{q_1(N,3,m_1,\cdots,m_t), B_5(N,3,m_1,\cdots,m_t)\}.$$

Thus, there exists

$$\{g\} \times (f_{k,1}, \ldots, f_{k,r}, h_{k,1}, \ldots, h_{k,t})_{1\le k\le d} \in R_3^{\otimes d} \times \prod_{k=1}^{d} \left(R_3^r \times \prod_{j=1}^{t} R_{m_j} \right)$$

such that (a) and (b) of Theorem 3.7 hold with $E = E(\mathcal{H})$, where $R \coloneqq \mathbb{F}_p[x_0, \ldots, x_N]$. Denote

$$V_k \coloneqq V(f_{k,1}, \ldots, f_{k,r}, h_{k,1}, \ldots, h_{k,t}), \quad 1 \le k \le d$$
$$W_g \coloneqq \left\{ (v_1, \ldots, v_d) \in \prod_{k=1}^{d} V_k(\mathbb{F}_p) : g(v_1, \ldots, v_d) = 0 \right\}.$$

Then Lemma 4.1 implies $|W_g| \ge \frac{1}{2}p^{dS-1}$ as $r + t = N - S$. Meanwhile, Lemmas 2.6 and 2.4 imply $|V_k(\mathbb{F}_p)| \le 3^r \prod_{j=1}^{t} m_j p^S = Cp^S/d$ and $n > \sum_{k=1}^{d} |V_k(\mathbb{F}_p)|$.

Let $\mathcal{G}_0$ be the d-partite, d-uniform hypergraph with the vertex set $V(\mathcal{G}_0) = \sqcup_{k=1}^{d} V_k(\mathbb{F}_p)$ and the hyperedge set $E(\mathcal{G}_0) = W_g$. Adding $n - \sum_{k=1}^{d} |V_k(\mathbb{F}_p)|$ isolated vertices to $\mathcal{G}_0$, and denoting the new hypergraph by $\mathcal{G}$. Then $|V(\mathcal{G})| = n \le d(2p)^S$, from which we may conclude that $e(\mathcal{G}) = \Omega_{d,s_1,\ldots,s_{d-1}}(n^{d-\frac{1}{S}})$ since

$$e(\mathcal{G}) = |W_g| \ge \frac{1}{2}p^{dS-1} = n^{(dS-1)\log_n p - \log_n 2} \ge n^{d-\frac{1}{S}} n^{-(d-\frac{1}{S})\log_n C - dS \log_n 2} = 2^{-dS} C^{-d+\frac{1}{S}} n^{d-\frac{1}{S}}.$$

We claim that $\mathcal{G}$ does not contain $\mathcal{H}(s_d)$. Otherwise, we may assume that vertices of the k-th part of $\mathcal{H}(s_d)$ are in $V_k(\mathbb{F}_p)$, which is the k-th part of $\mathcal{G}_0$. This together with the construction of $\mathcal{G}_0$ implies that there is a point $y \coloneqq (y_{1,1}, \ldots, y_{1,s_1}, \ldots, y_{d-1,1}, \ldots, y_{d-1,s_{d-1}}) \in \prod_{i=1}^{d-1} V_i^{s_i}$ such that $y_{i,1}, \ldots, y_{i,s_i}$ are distinct for each $1 \le i \le d-1$, and $|Z_{E,g,y} \cap V_k| \ge s_d$, where $Z_{E,g,y}$ is the set defined in (10). Note that $\dim(Z_{E,g,y} \cap V_k) = 0$, so $|Z_{E,g,y} \cap V_k| = \deg Z_{E,g,y} \le 3^{r+S} \prod_{j=1}^{t} m_j \le 3^{2S+t+3} l^{1+\log t} t!$. Note that

$$l = (S+t)\sum_{i=1}^{d-1} s_i \le (d-1)s(S+t) = (d-1)sS + (d-1)s(\beta+1)(sS)^{1/3} \le (d-1)(\beta+2)sS.$$

Thus we obtain a contradiction that

$$s_d \le |Z_{E,g,y} \cap V_k| \le 3^{2S+t+3} \left[(d-1)(\beta+2)sS\right]^{1+\log t} t! \le 3^{2S+t+3} t^{(1+\log t)\log_t \left[(d-1)(3\beta)\left(\frac{t}{\beta}\right)^3\right]} t! < s_d.$$

□

5. Hypergraph Zarankiewicz number

In this section, we prove Theorem 1.5. We need the following lemma.

Lemma 5.1. *Let $d \ge 2$ be a fixed integer. Assume $s_1, \cdots, s_d, t, r, m$ are positive integers and $\mathcal{H}$ is a $(d-1)$-partite $(d-1)$-uniform hypergraph whose $(d-1)$ parts have $s_1, \cdots, s_{d-1}$ vertices. such that*

$$\binom{r+m+1}{m} > 2te(\mathcal{H}), \quad s_d > m^{e(\mathcal{H})} \prod_{j=1}^{r} D(r-j+1, 2s_1 \cdots s_{d-1}t + 1).$$

For any positive integers $n_1, \ldots, n_d$ satisfying $\log_{n_d}(n_1^{s_1} \cdots n_{d-1}^{s_{d-1}}) \le t$, We have

$$\mathrm{z}(n_1, \ldots, n_d, \mathcal{H}(s_d)) = \Omega_{s_1,\ldots,s_{d-1},r,t}\left(n_1 \cdots n_{d-1} n_d^{1-\frac{1}{e(\mathcal{H})}}\right).$$

Proof. Denote $S := e(\mathcal{H})$, $D := \prod_{i=1}^{r} D(r-j+1, 4St+1)^{1/S}$, $N := r+S$ and $n := \max\{n_1, \dots, n_{d-1}\}$. For each $1 \le j \le r$, we define $\mu_j := D(r-j+1, 2St+1)$. We define

$$\begin{aligned} B_6(s_1, \dots, s_{d-1}, t, r) \quad &:= \max\Big\{ 4D^2,\ 2DB_3(N, m, \mu_1, \dots, \mu_r), \\ &\qquad 4DB_3(N, 0, \underbrace{m, \dots, m}_{S \text{ times}}),\ 4DC(S-1, m\mu_1 \cdots \mu_r, N) \Big\}. \end{aligned}$$

Here B_3 and C are functions in Lemmas 3.1 and 2.5, respectively. Let n_d be an integer such that $n_d > B_6(s_1, \dots, s_{d-1}, t, r)^S$ and by assumption we have $n_d \ge n_1^{s_1/t} \cdots n_{d-1}^{s_{d-1}/t}$. By the choice of n_d, we have $n_d \ge (2D)^S$. Hence by the same argument in the proof Theorem 1.3, we may choose a prime number p such that $(pD)^S \le n_d \le (2pD)^S$. Note that $n_d > (2D)^{2S}$ implies $p > 2D$. Therefore, we obtain

$$\log_p(n_1^{s_1} \cdots n_{d-1}^{s_{d-1}}) = \log_p(2pD) \sum_{i=1}^{d-1} s_i \log_{2pD}(n_i) \le (1 + \log_p(2D)) S \sum_{i=1}^{d-1} s_i \log_{n_d}(n_i) < 2St.$$

Consequently, we have

$$m_j := D(r-j+1, \log_p\left(n_1^{s_1} \cdots n_{d-1}^{s_{d-1}}\right) + 1) < \mu_j, \quad 1 \le j \le r.$$

By (7), we observe that

$$m_j^{r-j+1} \le (r-j+1)! \binom{m_j - 1 + r - j + 1}{r-j+1} \le (r-j+1)! \left(\log_p(n_1^{s_1} \cdots n_{d-1}^{s_{d-1}}) + 1\right).$$

The rest of the proof proceeds in three steps.

(1) We set $R' := \mathbb{F}_p[x_0, \dots, x_n]$ and $R := \mathbb{F}_p[x_0, \dots, x_N]$. Note that $m_j < \mu_j$. For each $1 \le i \le d-1$, we let Y_i be a subset of $\mathbb{P}^n(\mathbb{F}_p)$ consisting of n_i linearly independent points $y_{i,1}, \dots, y_{i,n_i} \in Y_i$. It is clear that Y_i is n_i-wise m-independent for any integer $m \ge 1$. In particular, Y_i is s_i-wise m-independent. Assume $Y \subseteq \prod_{i=1}^{d-1} Y_i^{s_i}$ is the subset consisting of points $y := (y_{1,1}, \dots, y_{1,s_1}, \dots, y_{d-1,1}, \dots, y_{d-1,s_{d-1}})$ such that $y_{i,u} \ne y_{i,v}$ for any $1 \le i \le d-1$ and $1 \le u \ne v \le s_i$. Let $E = E(\mathcal{H}) \subset [s_1] \times \cdots \times [s_{d-1}]$ Given $y \in Y$, we consider

$$G_y := \left\{ g \in R_m'^{\otimes(d-1)} \otimes R_m : \dim Z_{E,g,y} \ge N - S + 1 \right\},$$

where $Z_{E,g,y}$ is defined by (10). By the same argument in the proof of Corollary 3.6, we may conclude that

$$\dim G_y \le \binom{n+m}{m}^{d-1} \binom{N+m}{m} - \binom{r+m+1}{m}, \quad \deg G_y \le B_3(N, 0, m, \dots, m).$$

Hence by Lemma 2.6, we have

$$|G_y(\mathbb{F}_p)| \le 2B_3(N, 0, m, \dots, m) p^{\binom{n+m}{m}^{d-1} \binom{N+m}{m} - \binom{r+m+1}{m}}.$$

This implies

$$\begin{aligned} |\cup_{y \in Y} G_y| &\le 2B_3(N, 0, m, \dots, m) p^{\binom{n+m}{m}^{d-1} \binom{N+m}{m} - \binom{r+m+1}{m}} \prod_{i=1}^{d-1} n_i^{s_i} \\ &\le \left(2B_3(N, 0, m, \dots, m) p^{\binom{n+m}{m}^{d-1} \binom{N+m}{m} - 1} \right) \left(p^{-2St} \prod_{i=1}^{d-1} n_i^{s_i} \right) \\ &< p^{\binom{n+m}{m}^{d-1} \binom{N+m}{m}}. \end{aligned}$$

The last inequality is because $2pD > 2DB_3(N, m, \mu_1, \cdots, \mu_r)$. Thus, there exists some $g \in R'^{\otimes d-1}_m \otimes R_m$ such that $\dim Z_{g,y} = N - S$ for any $y \in Y$.

(2) For a fixed $y \in Y$, we consider

$$\mathcal{U}_{Z_{E,g,y}}(m_1, \dots, m_r) := \left\{ (h_1, \dots, h_r) \in \prod_{j=1}^{r} R_{m_j} : \dim\left(Z_{E,g,y} \cap V(h_1, \dots, h_r)\right) \geq 1 \right\}.$$

By Propositions 3.1 and 3.3, we derive that $\deg \mathcal{U}_{Z_{E,g,y}} \leq B_3(N, 3, m_1, \dots, m_r)$ and

$$\dim \mathcal{U}_{Z_{E,g,y}} \leq \sum_{j=1}^{r} \binom{N+m_j}{N} - \min_{1\leq j\leq r} \binom{r-j+1+m_j}{m_j}.$$

Lemma 2.1 implies that

$$\begin{aligned}|\cup_{y\in Y} \mathcal{U}_{Z_{g,y}}(m_1, \dots, m_r)| &\leq B_3(N, m, m_1, \dots, m_r) p^{\sum_{j=1}^{r} \binom{N+m_j}{N} - \min_{1\leq j\leq r} \binom{r-j+1+m_j}{m_j}} |Y| \\ &\leq B_3(N, m, m_1, \dots, m_r) p^{\sum_{j=1}^{r} \binom{N+m_j}{N} - \min_{1\leq j\leq r} \binom{r-j+1+m_j}{m_j}} \prod_{i=1}^{d-1} n_i^{s_i} \\ &< p^{\sum_{j=1}^{r} \binom{N+m_j}{N}}.\end{aligned}$$

The last inequality follows from the definition of $m_1, \dots, m_r$, and our choice of p. Consequently, there is some $(h_1, \dots, h_r) \in \prod_{j=1}^{r} R_{m_j}$ such that $\dim\left(Z_{E,g,y} \cap V(h_1, \dots, h_r)\right) = 0$ for any $y \in Y$.

(3) Lemmas 2.6 and 2.4 lead to

$$|V(h_1, \dots, h_r)(\mathbb{F}_p)| \leq 2p^{N-r} \prod_{j=1}^{r} m_j = 2p^S \prod_{j=1}^{r} m_j \leq n_d.$$

We add $n_d - |V(h_1, \dots, h_r)(\mathbb{F}_p)|$ distinct points to $V(h_1, \dots, h_r)(\mathbb{F}_p)$ and denote the new set by Y_d. Let $\mathcal{G}$ be the d-partite d-uniform hypergraph defined as follows. The vertex set $V(\mathcal{G})$ of $\mathcal{G}$ is $\sqcup_{k=1}^{d} Y_k$ and the k-th part is Y_k; The edge set $E(\mathcal{G})$ of $\mathcal{G}$ consists of $(y_1, \dots, y_d) \in \left(\prod_{i=1}^{d-1} Y_i\right) \times V(h_1, \dots, h_r)(\mathbb{F}_p)$ such that $g(y_1, \dots, y_d) = 0$. For fixed $(y_1, \dots, y_{d-1}) \in \prod_{i=1}^{d-1} Y_i$, we have $\dim Z = S - 1$ and $\deg Z \leq m \prod_{j=1}^{r} m_j$ where

$$Z := \{z \in \mathbb{P}^N : g(y_1, \dots, y_{d-1}, z) = h_1(z) = \cdots = h_r(d) = 0\}.$$

By Lemma 2.5 and the choice of p, $|Z(\mathbb{F}_p)| \geq p^{S-1}/2$, from which we obtain

$$e(\mathcal{G}) \geq \frac{p^{S-1}}{2} \prod_{i=1}^{d-1} n_i = \Omega(n_1 \cdots n_{d-1} n_d^{1-1/S}).$$

If the k-th part of $\mathcal{G}$ contains that of $\mathcal{K}$ for each $1 \leq k \leq d$, then there is some $y = (y_{1,1}, \dots, y_{s_1}, \dots, y_{d-1,1}, \dots, y_{d-1,s_{d-1}}) \in Y$ such that

$$\dim\left(Z_{E,g,y} \cap V(h_1, \dots, h_r)\right) = 0, \quad |Z_{E,g,y} \cap V(h_1, \dots, h_r)(\mathbb{F}_p)| \geq s_d.$$

However, this leads to a contradiction:

$$s_d > m^S \prod_{j=1}^{r} D(r-j+1, 2St+1) > m^S \prod_{j=1}^{r} m_j \geq |Z_{E,g,y} \cap V(h_1, \dots, h_r)(\mathbb{F}_p)| \geq s_d. \qquad \square$$

Now we are ready to complete the proof of Theorem 1.5.

Proof of Theorem 1.5. Denote $S := e(\mathcal{H})$, $r := \lceil \sqrt{S} \rceil$. For each integer $m > 0$, we consider

$$t := t(m) = \max\left\{u \in \mathbb{N} : 2Su \le \binom{r+m+1}{m} - 1\right\}.$$

Then $\binom{r+m+1}{m} - 1 - 2S < 2St \le \binom{r+m+1}{m} - 1$. Clearly, there are constants c_1, c_2, depending only on S, such that

$$c_1 m^{r+1} < t < c_2 m^{r+1}. \tag{14}$$

According to Lemma 2.11, we have

$$\prod_{j=1}^{r} D(r-j+1, 2St+1) \le (2St+1)^r r! \le (3rSt)^r,$$

from which we obtain

$$m^S \prod_{j=1}^{r} D(r-j+1, 2St+1) \le m^S (3rSt)^r \le c_1^{-\sqrt{S}} t^{\sqrt{S}} (3rSt)^r \tag{15}$$

Let $c_3 := c_1^{-\sqrt{S}}(3Sr)^r$. We define

$$m_0 := \max\{m \in \mathbb{N} : c_3(c_2 m^{r+1})^{2\sqrt{S}+1} < s_d\}.$$

By definition, we have

$$c_4 s_d \le c_3(c_1 m_0^{r+1})^{2\sqrt{S}+1} < c_3(c_2 m_0^{r+1})^{2\sqrt{S}+1} < s_d$$

for some constant c_4 depending on S. Thus, $c_4 s_d < c_3 t_0^{2\sqrt{S}+1}$ for $t_0 := t(m_0)$. Taking $C := (c_3^{-1} c_4)^{1/(2\sqrt{S}+1)}$, we deduce that $s_d \le (C^{-1} t_0)^{2\sqrt{S}+1}$. If $\log_{n_d}(n_1^{s_1} \cdots n_{d-1}^{s_{d-1}}) \le C s_d^{1/(2\sqrt{S}+1)}$, then we conclude that

$$\log_{n_d}(n_1^{s_1} \cdots n_{d-1}^{s_{d-1}}) \le t_0.$$

By (14) and (15), we also have

$$m_0^S \prod_{j=1}^{r} D(r-j+1, 2St+1) \le c_1^{-\sqrt{S}} t_0^{\sqrt{S}} (3rSt_0)^r \le c_3 t_0^{2\sqrt{S}+1} < c_3(c_2 m_0^{r+1})^{2\sqrt{S}+1} < s_d.$$

Hence positive integers $s_1, \ldots, s_d, t_0, r, m_0$ satisfy conditions in Lemma 5.1, and this implies

$$\mathrm{z}(n_1, \ldots, n_d, \mathcal{H}(s_d)) = \Omega_{s_1,\ldots,s_{d-1},s_d}(n_1, \ldots, n_{d-1} n_d^{1-\frac{1}{e(\mathcal{H})}}). \qquad \square$$

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