ON THE COMMUTATIVITY OF THE BEREZIN TRANSFORM

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ABSTRACT. We consider the commutativity problem for the Berezin transform on weighted Fock spaces. Given a real number $m > 0$, for every $\alpha > 0$ we denote by B_α the Berezin transform associated to the measure μ_m^α with density proportional to $e^{-\alpha|z|^m}$ with respect to Lebesgue measure on the complex plane and normalized so that $\mu_\phi^\alpha(\mathbb{C}) = 1$. We show that the commutativity relation $B_\alpha B_\beta f = B_\beta B_\alpha f$ holds for all $f \in L^\infty(\mathbb{C})$ and $\alpha, \beta > 0$ if and only if $m = 2$.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

We fix a real number $m > 0$ and for every $\alpha > 0$ we consider the (normalized) measure on the complex plane

$$d\mu_{\alpha,m}(z) = \frac{m\alpha^{2/m}}{2\pi\Gamma(2/m)} e^{-\alpha|z|^m}\, dA(z),$$

where dA is Euclidean area measure on the complex plane $\mathbb{C}$ and Γ is the Euler gamma function. We denote by $L^2_{\alpha,m}(\mathbb{C})$ the space of all square integrable functions with respect to the measure $\mu_{\alpha,m}$ and let

$$F^2_{\alpha,m}(\mathbb{C}) = L^2_{\alpha,m}(\mathbb{C}) \cap \mathrm{Hol}(\mathbb{C})$$

denote the corresponding weighted Fock space. The space $F^2_{\alpha,m}(\mathbb{C})$ is a Hilbert space of entire functions, and we denote by $K_{\alpha,m}(\cdot,\cdot)$ its reproducing kernel,

$$f(z) = \langle f, K_{\alpha,m}(\cdot, z)\rangle, \qquad f \in F^2_{\alpha,m}(\mathbb{C}),\ z \in \mathbb{C}.$$

Let P_+ be the orthogonal projection from $L^2_{\alpha,m}(\mathbb{C})$ onto $F^2_{\alpha,m}(\mathbb{C})$. Given $f \in L^\infty(\mathbb{C})$, we define the Toeplitz operator T_f on $F^2_{\alpha,m}(\mathbb{C})$ by the formula

$$T_f g = P_+(fg), \qquad g \in F^2_{\alpha,m}(\mathbb{C}).$$

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The Berezin transform of T_f (or the Berezin transform of f) is given by

$$\begin{aligned}(B_{\alpha,m}T_f)(z) = (B_{\alpha,m}f)(z) &= \frac{\langle T_f K_{\alpha,m}(\cdot,z), K_{\alpha,m}(\cdot,z)\rangle}{K_{\alpha,m}(z,z)} \\ &= \int_{\mathbb{C}} f(w)\frac{|K_{\alpha,m}(w,z)|^2}{K_{\alpha,m}(z,z)}\,d\mu_m^\alpha(w), \qquad z\in\mathbb{C}.\end{aligned}$$

For the Berezin transform of operators in a more general context and for the Berezin transform of functions see [1,4,5,7].

Since $\|K_{\alpha,m}(\cdot,z)\|^2 = K_{\alpha,m}(z,z)$, it is clear that $B_{\alpha,m}$ is a contraction on $L^\infty(\mathbb{C})$ for every $\alpha > 0$.

In the case of the classical Fock spaces, that is, when $m = 2$, it is known (see, for instance, [7, Chapter 2] that the Berezin transform satisfies the commutativity relation $B_{\alpha,2}B_{\beta,2} = B_{\beta,2}B_{\alpha,2}$ on $L^\infty(\mathbb{C})$.

The problem we consider here is to find out for which positive real numbers m does the latter commutativity remain true. The main goal of this paper is to establish the following:

Theorem. *Assume that $m > 0$ and consider the test functions $f_\delta(z) := e^{-\delta|z|^m}$, $z\in\mathbb{C}$, $\delta > 0$. Then the following are equivalent*

(1) *The commutativity relation $B_{\alpha,m}B_{\beta,m}f = B_{\beta,m}B_{\alpha,m}f$ holds for all $f\in L^\infty(\mathbb{C})$ and all $\alpha,\beta > 0$.*
(2) *The commutativity relation $B_{\alpha,m}B_{\beta,m}f_\delta = B_{\beta,m}B_{\alpha,m}f_\delta$ holds for all $\alpha,\beta,\delta > 0$.*
(3) *The commutativity relation $(B_{\alpha,m}B_{\beta,m}f_\delta)(0) = (B_{\beta,m}B_{\alpha,m}f_\delta)(0)$ holds for all $\alpha,\beta,\delta > 0$.*
(4) $m = 2$.

The proof of this result relies on expansions of the reproducing kernel using the sequence of the Stieltjes even moments (s_d) of the measure $e^{-\alpha t^m}\,dt$ on $[0,+\infty[$ which will be defined in the sequel. For similar arguments involving the Stieltjes moments see [2,3,6].

2. Proof of the main result

We start with the equality

$$\|z^n\|^2_{\alpha,m} = \frac{\Gamma(2(n+1)/m)}{\Gamma(2/m)\alpha^{2n/m}}, \qquad n \in \mathbb{Z}_+.$$

Therefore, we have

$$K_{\alpha,m}(z,w) = \sum_{n\geq 0} \frac{(z\overline{w})^n}{\|z^n\|^2_{\alpha,m}} = \Gamma(2/m) \sum_{n\geq 0} \frac{(\alpha^{\frac{2}{m}} z\overline{w})^n}{\Gamma(2(n+1)/m)}.$$

Next we define the Stieltjes moments

$$s_n(\alpha,m) := \alpha^{-2n/m}\Gamma(2(n+1)/m),$$

and consider an entire function

$$S_{\alpha,m}(\zeta) := \sum_{n\geq 0} \frac{\zeta^n}{s_n(\alpha,m)}.$$

Then

$$K_{\alpha,m}(z,w) = \Gamma(2/m) S_{\alpha,m}(z\overline{w}).$$

Consequently, the Berezin transform can be expressed as

$$\begin{aligned}(B_{\alpha,m}f)(z) &= \int_{\mathbb{C}} f(w) \frac{|K_{\alpha,m}(w,z)|^2}{K_{\alpha,m}(z,z)}\, d\mu_{\alpha,m}(w)\\ &= \frac{m\alpha^{2/m}}{2\pi S_{\alpha,m}(|z|^2)} \int_{\mathbb{C}} f(w)|S_{\alpha,m}(z\overline{w})|^2 e^{-\alpha|w|^m}\, dA(w).\end{aligned}$$

Since

$$S_{\alpha,m}(0) = s_0(\alpha,m)^{-1} = \Gamma(2/m)^{-1},$$

we have

$$(B_{\alpha,m}f)(0) = \frac{m\alpha^{2/m}}{2\pi\Gamma(2/m)} \int_{\mathbb{C}} f(w) e^{-\alpha|w|^m}\, dA(w). \tag{2.1}$$

Furthermore,

$$|S_{\alpha,m}(\zeta)|^2 = \sum_{k\geq 0} \sum_{0\leq d\leq k} \frac{\zeta^d \overline{\zeta}^{k-d}}{s_d(\alpha,m)s_{k-d}(\alpha,m)}, \qquad \zeta \in \mathbb{C}.$$

Applying the Berezin transform to the test function $f_\delta(w)=e^{-\delta|w|^m}$, $\delta\ge 0$, and using pairwise orthogonality of w^q, $q\in\mathbb{Z}_+$, on the circles $\partial D(0,R)$, $R>0$, we obtain

$$(2.2)\quad (B_{\alpha,m}f_\delta)(z)$$
$$=\frac{m\alpha^{2/m}}{2\pi S_{\alpha,m}(|z|^2)}\sum_{k\ge 0}\sum_{0\le d\le k}\int_{\mathbb{C}} f_\delta(w)\frac{(z\overline{w})^d(w\overline{z})^{k-d}}{s_d(\alpha,m)s_{k-d}(\alpha,m)}e^{-\alpha|w|^m}\,dA(w)$$
$$=\frac{m\alpha^{2/m}}{2\pi S_{\alpha,m}(|z|^2)}\sum_{n\ge 0}|z|^{2n}\int_{\mathbb{C}}\frac{|w|^{2n}}{s_n^2(\alpha,m)}e^{-(\alpha+\delta)|w|^m}\,dA(w).$$

By (2.1) and (2.2) we have

$$(B_{\beta,m}B_{\alpha,m}f_\delta)(0)=\frac{m\beta^{2/m}}{2\pi\Gamma(2/m)}\int_{\mathbb{C}}(B_{\alpha,m}f_\delta)(z)e^{-\beta|z|^m}\,dA(z)$$
$$=\frac{m^2(\alpha\beta)^{2/m}}{4\pi^2\Gamma(2/m)}\sum_{n\ge 0}\int_{\mathbb{C}}\frac{|z|^{2n}}{S_{\alpha,m}(|z|^2)}e^{-\beta|z|^m}\,dA(z)\ \int_{\mathbb{C}}\frac{|w|^{2n}}{s_n^2(\alpha,m)}e^{-(\alpha+\delta)|w|^m}\,dA(w)$$
$$=\frac{m(\alpha\beta)^{2/m}}{2\pi\Gamma(2/m)}\sum_{n\ge 0}\frac{1}{s_n^2(\alpha,m)}\left(\frac{1}{\delta+\alpha}\right)^{\frac{2n+2}{m}}\Gamma\left(\frac{2n+2}{m}\right)\int_{\mathbb{C}}\frac{|z|^{2n}e^{-\beta|z|^m}}{S_{\alpha,m}(|z|^2)}\,dA(z)$$
$$=\frac{m(\alpha\beta)^{2/m}}{2\pi\Gamma(2/m)}\sum_{n\ge 0}\frac{\alpha^{4n/m}}{\Gamma\left(\frac{2n+2}{m}\right)}\left(\frac{1}{\delta+\alpha}\right)^{\frac{2n+2}{m}}\int_{\mathbb{C}}\frac{|z|^{2n}e^{-\beta|z|^m}}{S_{\alpha,m}(|z|^2)}\,dA(z).$$

(Here we use that f_δ, $B_{\alpha,m}f_\delta$ are positive and bounded so that one can apply the Foubini theorem.)

Thus,

$$(2.3)\quad (B_{\beta,m}B_{\alpha,m}f_\delta)(0)$$
$$=\frac{m(\alpha\beta)^{2/m}}{2\pi\Gamma(2/m)}\sum_{n\ge 0}\left(\frac{1}{\delta+\alpha}\right)^{(2n+2)/m}\frac{\alpha^{4n/m}}{\Gamma(\frac{2n+2}{m})}\int_{\mathbb{C}}\frac{|z|^{2n}e^{-\beta|z|^m}}{S_{\alpha,m}(|z|^2)}\,dA(z).$$

Set

$$U_{\alpha,\beta}(n)=\frac{\alpha^{4n/m}}{\Gamma(\frac{2n+2}{m})}\int_{\mathbb{C}}\frac{|z|^{2n}e^{-\beta|z|^m}}{S_{\alpha,m}(|z|^2)}\,dA(z).$$

Since the series in (2.3) converges for $\delta=0$, we have

$$(2.4)\qquad U_{\alpha,\beta}(n)\le C\alpha^{2n/m},\qquad n\ge 0.$$

Now we rewrite (2.3) as

$$(B_{\beta,m}B_{\alpha,m}f_\delta)(0) = \frac{m(\alpha\beta)^{2/m}}{2\pi\Gamma(2/m)} \sum_{n\geq 0} \left(\frac{1}{\delta+\alpha}\right)^{(2n+2)/m} U_{\alpha,\beta}(n). \tag{2.5}$$

Lemma 2.1. *Let $m, \alpha, \beta > 0$. If the commutativity relation*

$$(B_{\beta,m}B_{\alpha,m}f_\delta)(0) = (B_{\alpha,m}B_{\beta,m}f_\delta)(0)$$

holds for every $\delta > 0$, then

$$U_{\alpha,\beta}(n) = U_{\beta,\alpha}(n), \qquad 0 \leq n < \frac{m}{2}. \tag{2.6}$$

Proof. Suppose that for every $\delta > 0$ we have $(B_{\beta,m}B_{\alpha,m}f_\delta)(0) = (B_{\alpha,m}B_{\beta,m}f_\delta)(0)$. Then by (2.5) we obtain

$$\sum_{n\geq 0} \left(\frac{1}{\delta+\alpha}\right)^{\frac{2n+2}{m}} U_{\alpha,\beta}(n) = \sum_{n\geq 0} \left(\frac{1}{\delta+\beta}\right)^{\frac{2n+2}{m}} U_{\beta,\alpha}(n), \quad \delta > 0. \tag{2.7}$$

We set $t = \delta^{-2/m}$ and rewrite (2.7) as

$$\sum_{n\geq 0} \frac{t^{n+1}}{(1+\alpha t^{m/2})^{(2n+2)/m}} U_{\alpha,\beta}(n) = \sum_{n\geq 0} \frac{t^{n+1}}{(1+\beta t^{m/2})^{(2n+2)/m}} U_{\beta,\alpha}(n), \qquad t > 0. \tag{2.8}$$

Then, using (2.4), we get

$$\sum_{n\geq 0} t^n (U_{\alpha,\beta}(n) - U_{\beta,\alpha}(n)) = O(t^{m/2}), \qquad t \to 0,$$

and hence,

$$U_{\alpha,\beta}(n) = U_{\beta,\alpha}(n), \qquad 0 \leq n < \frac{m}{2}.$$

□

Lemma 2.2. *Let $m \notin 2\mathbb{Z}_+$. If $\alpha \neq \beta$, then the commutativity relation*

$$(B_{\beta,m}B_{\alpha,m}f_\delta)(0) = (B_{\alpha,m}B_{\beta,m}f_\delta)(0), \quad \delta > 0, \tag{2.9}$$

does not hold.

Proof. By (2.8) and (2.6), relation (2.9) implies that

$$\begin{aligned}\Big(\frac{1}{(1+\alpha t^{m/2})^{2/m}}&-\frac{1}{(1+\beta t^{m/2})^{2/m}}\Big)U_{\alpha,\beta}(0)\\&=\sum_{0<n<\frac{m}{2}}\Big(\frac{t^n}{(1+\beta t^{m/2})^{(2n+2)/m}}-\frac{t^n}{(1+\alpha t^{m/2})^{(2n+2)/m}}\Big)U_{\alpha,\beta}(n)\\&+\sum_{n>\frac{m}{2}}\Big(\frac{t^n}{(1+\beta t^{m/2})^{(2n+2)/m}}U_{\beta,\alpha}(n)-\frac{t^n}{(1+\alpha t^{m/2})^{(2n+2)/m}}U_{\alpha,\beta}(n)\Big)\\&=o(t^{m/2}),\qquad t\to 0,\end{aligned}$$

and hence,

$$(\beta-\alpha)U_{\alpha,\beta}(0)=o(1),\qquad t\to 0,$$

which is impossible unless $\alpha=\beta$. □

Lemma 2.3. *Let m be an even positive integer. Suppose that the commutativity relation $(B_{\beta,m}B_{\alpha,m}f_\delta)(0)=(B_{\alpha,m}B_{\beta,m}f_\delta)(0)$ holds for every $\delta>0$ and every $\alpha,\beta>0$. Then $m=2$.*

Proof. Let $m=2p$. By (2.8) we have

$$\sum_{n\ge 0}\frac{t^n}{(1+\alpha t^p)^{(n+1)/p}}U_{\alpha,\beta}(n)=\sum_{n\ge 0}\frac{t^n}{(1+\beta t^p)^{(n+1)/p}}U_{\beta,\alpha}(n),\qquad t>0.$$

By the binomial formula,

$$(1+x)^q=\sum_{s\ge 0}c_{q,s}x^s,\qquad 0\le x<1,$$

where

$$c_{q,0}=1,\quad c_{q,1}=q,\quad c_{q,2}=q(q-1)/2,$$

we obtain

$$\begin{aligned}\sum_{n,s\ge 0}&t^{n+sp}\alpha^s c_{-(n+1)/p,s}U_{\alpha,\beta}(n)\\&=\sum_{n,s\ge 0}t^{n+sp}\beta^s c_{-(n+1)/p,s}U_{\beta,\alpha}(n),\qquad 0<t<1,\end{aligned}\tag{2.10}$$

with both series converging absolutely.

Hence, for every $b \in \mathbb{Z}_+$ we have

$$\sum_{n+sp=b} \alpha^s c_{-(n+1)/p,s} U_{\alpha,\beta}(n) = \sum_{n+sp=b} \beta^s c_{-(n+1)/p,s} U_{\beta,\alpha}(n).$$

By Lemma 2.1 we know already that

$$U_{\alpha,\beta}(0) = U_{\beta,\alpha}(0),. \tag{2.11}$$

Taking the values $b = p, 2p$ we obtain

$$U_{\alpha,\beta}(p) - \frac{\alpha}{p} U_{\alpha,\beta}(0) = U_{\beta,\alpha}(p) - \frac{\beta}{p} U_{\beta,\alpha}(0), \tag{2.12}$$

and

$$\begin{aligned} U_{\alpha,\beta}(2p) - \frac{\alpha(p+1)}{p} U_{\alpha,\beta}(p) + \frac{\alpha^2(p+1)}{2p^2} U_{\alpha,\beta}(0) \\ = U_{\beta,\alpha}(2p) - \frac{\beta(p+1)}{p} U_{\beta,\alpha}(p) + \frac{\beta^2(p+1)}{2p^2} U_{\beta,\alpha}(0). \end{aligned} \tag{2.13}$$

By (2.11) and (2.12) we obtain

$$U_{\alpha,\beta}(p) - U_{\beta,\alpha}(p) = \frac{\alpha - \beta}{p} U_{\alpha,\beta}(0). \tag{2.14}$$

By (2.11)–(2.13) we obtain

$$\begin{aligned} & U_{\alpha,\beta}(2p) - U_{\beta,\alpha}(2p) \\ & \qquad = \frac{(\alpha-\beta)(p+1)}{p} U_{\alpha,\beta}(p) - \frac{(\alpha-\beta)^2(p+1)}{2p^2} U_{\alpha,\beta}(0). \end{aligned} \tag{2.15}$$

Next we use that

$$\begin{aligned} \frac{\partial}{\partial\beta} U_{\alpha,\beta}(n) &= \frac{\partial}{\partial\beta} \frac{\alpha^{2n/p}}{\Gamma(\frac{n+1}{p})} \int_{\mathbb{C}} \frac{|z|^{2n} e^{-\beta|z|^{2p}}}{S_{\alpha,2p}(|z|^2)} \, dA(z) \\ &= -\frac{\alpha^{2n/p}}{\Gamma(\frac{n+1}{p})} \int_{\mathbb{C}} \frac{|z|^{2n+2p} e^{-\beta|z|^{2p}}}{S_{\alpha,2p}(|z|^2)} \, dA(z) \\ &= -\frac{n+1}{\alpha^2 p} \cdot \frac{\alpha^{(2n+2p)/p}}{\Gamma(\frac{n+p+1}{p})} \int_{\mathbb{C}} \frac{|z|^{2n+2p} e^{-\beta|z|^{2p}}}{S_{\alpha,2p}(|z|^2)} \, dA(z) \\ &= -\frac{n+1}{\alpha^2 p} U_{\alpha,\beta}(n+p), \qquad n \geq 0. \end{aligned} \tag{2.16}$$

By analogy,

$$\frac{\partial}{\partial\alpha}U_{\beta,\alpha}(n) = -\frac{n+1}{\beta^2 p}U_{\beta,\alpha}(n+p), \qquad n \geq 0. \tag{2.17}$$

As a consequence, we obtain

$$\begin{cases} \dfrac{\partial}{\partial\alpha}U_{\beta,\alpha}(0) = -\dfrac{1}{\beta^2 p}U_{\beta,\alpha}(p), \\ \dfrac{\partial}{\partial\beta}U_{\alpha,\beta}(0) = -\dfrac{1}{\alpha^2 p}U_{\alpha,\beta}(p), \\ \dfrac{\partial}{\partial\alpha}U_{\beta,\alpha}(p) = -\dfrac{p+1}{\beta^2 p}U_{\beta,\alpha}(2p), \\ \dfrac{\partial}{\partial\beta}U_{\alpha,\beta}(p) = -\dfrac{p+1}{\alpha^2 p}U_{\alpha,\beta}(2p). \end{cases} \tag{2.18}$$

Now, let us evaluate

$$H = \frac{\partial^2}{\partial\alpha\partial\beta}(\alpha^2\beta^2 U_{\alpha,\beta}(0)).$$

First, by (2.11), (2.14), and (2.18) we have

$$\begin{aligned} H &= \frac{\partial}{\partial\alpha}\Big[2\alpha^2\beta U_{\alpha,\beta}(0) - \frac{\beta^2}{p}U_{\alpha,\beta}(p)\Big] \qquad (2.19)\\ &= \frac{\partial}{\partial\alpha}\Big[2\alpha^2\beta U_{\beta,\alpha}(0) - \frac{\beta^2}{p}U_{\beta,\alpha}(p) - \frac{(\alpha-\beta)\beta^2}{p^2}U_{\beta,\alpha}(0)\Big] \\ &= 4\alpha\beta U_{\beta,\alpha}(0) - \frac{2\alpha^2}{\beta p}U_{\beta,\alpha}(p) + \frac{p+1}{p^2}U_{\beta,\alpha}(2p) - \frac{\beta^2}{p^2}U_{\beta,\alpha}(0) + \frac{\alpha-\beta}{p^3}U_{\beta,\alpha}(p). \end{aligned}$$

By (2.11), H is stable under permutation of α and β,

$$H = \frac{\partial^2}{\partial\beta\partial\alpha}(\beta^2\alpha^2 U_{\beta,\alpha}(0)),$$

and, hence,

$$\begin{aligned} &4\alpha\beta U_{\beta,\alpha}(0) - \frac{2\alpha^2}{\beta p}U_{\beta,\alpha}(p) + \frac{p+1}{p^2}U_{\beta,\alpha}(2p) - \frac{\beta^2}{p^2}U_{\beta,\alpha}(0) + \frac{\alpha-\beta}{p^3}U_{\beta,\alpha}(p) \\ &= 4\alpha\beta U_{\alpha,\beta}(0) - \frac{2\beta^2}{\alpha p}U_{\alpha,\beta}(p) + \frac{p+1}{p^2}U_{\alpha,\beta}(2p) - \frac{\alpha^2}{p^2}U_{\alpha,\beta}(0) + \frac{\beta-\alpha}{p^3}U_{\alpha,\beta}(p). \end{aligned}$$

We obtain that

$$\frac{p+1}{p^2}(U_{\alpha,\beta}(2p) - U_{\beta,\alpha}(2p)) + \frac{\beta^2}{p^2}U_{\beta,\alpha}(0) - \frac{\alpha^2}{p^2}U_{\alpha,\beta}(0) \\ + \frac{2\alpha^2}{\beta p}U_{\beta,\alpha}(p) - \frac{2\beta^2}{\alpha p}U_{\alpha,\beta}(p) + \frac{\beta-\alpha}{p^3}U_{\alpha,\beta}(p) + \frac{\beta-\alpha}{p^3}U_{\beta,\alpha}(p) = 0,$$

and by (2.11) and (2.15) we get

$$\frac{(\alpha-\beta)(p+1)^2}{p^3}U_{\alpha,\beta}(p) - \frac{(\alpha-\beta)^2(p+1)^2}{2p^4}U_{\alpha,\beta}(0) + \frac{\beta^2-\alpha^2}{p^2}U_{\alpha,\beta}(0) \\ + \frac{2\alpha^2}{\beta p}U_{\beta,\alpha}(p) - \frac{2\beta^2}{\alpha p}U_{\alpha,\beta}(p) + \frac{\beta-\alpha}{p^3}(U_{\alpha,\beta}(p) + U_{\beta,\alpha}(p)) = 0,$$

By (2.14) we obtain

$$\frac{(\alpha-\beta)(p+1)^2}{p^3}U_{\alpha,\beta}(p) - \frac{(\alpha-\beta)^2(p+1)^2}{2p^4}U_{\alpha,\beta}(0) - \frac{\alpha^2-\beta^2}{p^2}U_{\alpha,\beta}(0) \\ - \frac{2\alpha^2(\alpha-\beta)}{\beta p^2}U_{\alpha,\beta}(0) + \Big(\frac{2\alpha^2}{\beta p} - \frac{2\beta^2}{\alpha p}\Big)U_{\alpha,\beta}(p) \\ - \frac{2(\alpha-\beta)}{p^3}U_{\alpha,\beta}(p) + \frac{(\alpha-\beta)^2}{p^4}U_{\alpha,\beta}(0) = 0,$$

Dividing by $\alpha-\beta$ we conclude that for $\alpha \neq \beta$ we have

$$\Big(\frac{(p+1)^2}{p^3} + \frac{2(\alpha^2+\alpha\beta+\beta^2)}{\alpha\beta p} - \frac{2}{p^3}\Big)U_{\alpha,\beta}(p) \\ = \Big(\frac{(\alpha-\beta)(p+1)^2}{2p^4} + \frac{\alpha+\beta}{p^2} + \frac{2\alpha^2}{\beta p^2} - \frac{\alpha-\beta}{p^4}\Big)U_{\alpha,\beta}(0).$$

Fix $\alpha = 1$. Then

$$\Big(\frac{2\beta}{p} + O(1)\Big)U_{\alpha,\beta}(p) = \Big(\Big(\frac{1}{p^2} + \frac{1}{p^4} - \frac{(p+1)^2}{2p^4}\Big)\beta + O(1)\Big)U_{\alpha,\beta}(0) \\ = \Big(\frac{(p-1)^2}{2p^4}\beta + O(1)\Big)U_{\alpha,\beta}(0), \qquad \beta \to \infty.$$

Suppose now that $m > 2$, that is, $p > 1$. Then for some $C > 0$ and for all $\beta \geq \beta_0$ we have

$$\int_{\mathbb{C}} \frac{|z|^{2p}e^{-\beta|z|^{2p}}}{S_{1,2p}(|z|^2)}\,dA(z) \geq C\int_{\mathbb{C}} \frac{e^{-\beta|z|^{2p}}}{S_{1,2p}(|z|^2)}\,dA(z).$$

Since

$$\int_{\mathbb{C}} \frac{e^{-\beta|z|^{2p}}}{S_{1,2p}(|z|^2)}\,dA(z) \ge C\beta^{-1/p}, \qquad \beta\to\infty,$$
$$\int_{\mathbb{C}} \frac{|z|^{2p}e^{-\beta|z|^{2p}}}{S_{1,2p}(|z|^2)}\,dA(z) = O(\beta^{-(p+1)/p}), \qquad \beta\to\infty,$$

we obtain a contradiction. Thus, $m=2$. □

Proof of Theorem A. The implication $(4) \implies (1)$ is contained in the book of Zhu [7, Section 3.2]. The proof of the remaining parts follows from Lemmata 2.2 and 2.3. □

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