Establishing strong 1-boundedness via non-microstates free entropy techniques

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Abstract

We show that, for many choices of finite tuples of generators $\mathbf{X} = (x_1, \ldots, x_d)$ of a tracial von Neumann algebra (M, τ) satisfying certain decomposition properties (non-primeness, possessing a Cartan subalgebra, or property Γ), one can find a diffuse, hyperfinite subalgebra $N \subseteq W^*(\mathbf{X})^\omega$ (often in $W^*(\mathbf{X})$ itself), such that

$$W^*(N, \mathbf{X} + \sqrt{t}\mathbf{S}) = W^*(N, \mathbf{X}, \mathbf{S})$$

for all $t > 0$. (Here $\mathbf{S}$ is a free semicircular family, free from $\{\mathbf{X} \cup N\}$). This gives a short 'non-microstates' proof of strong 1-boundedness for such algebras.

1 Introduction

Voiculescu's free entropy dimension [Voi96] is defined for n-tuples of self-adjoint elements in a tracial von Neumann algebra and is related to the short-term behavior of free entropy under semicircular perturbations. Since the free entropy dimension of a generating set of a diffuse amenable von Neumann algebra is always 1 (cf. [Jun03b]), it makes sense to consider a kind of relative quantity (see [Shl02; Jun07; JP24]). A strengthened version of the vanishing of such a relative quantity is called strong 1-boundedness and was defined by Jung [Jun07] for finite tuples of self-adjoint elements in a tracial von Neumann algebra (see also [Ge96; HS07; Hay18]). Being strongly 1-bounded is an invariant of the von Neumann algebra that the tuple generates, and is implied by that algebra possessing a Cartan subalgebra, property Γ, tensor product decomposition, and most instances of Property (T) ([Voi96; Jun07; JS07; Hay18; HJK25]). On the other hand, Connes-embeddable von Neumann algebras that are free products (e.g. free group factors) are not strongly 1-bounded.

Voiculescu also introduced a non-microstates free entropy χ^* which is defined via the free Fisher information Φ^* ([Voi98b]). There are conditions on χ^* similar to strong 1-boundedness, which can be stated in terms of Φ^* (see e.g. [Shl21] for some applications to von Neumann algebras of groups). In this paper we will be considering a version of these conditions relative to a diffuse abelian sub-algebra.

Suppose that $\mathbf{X}$ is a self-adjoint d-tuple in a tracial von Neumann algebra, N is some fixed von Neumann subalgebra, and $\mathbf{S}$ is a free $(0,1)$-semicircular d-tuple, free from $N \cup \{\mathbf{X}\}$. Let $\Phi^*(\cdots : N)$ denote Voiculescu's free Fisher relative to N ([Voi98b]). Then

$$\Phi^*(\mathbf{X} + \sqrt{\epsilon}\mathbf{S} : N) \leq \frac{d}{\epsilon} \quad \forall\, \epsilon > 0. \tag{1}$$

with equality iff $N \vee W^*(\mathbf{X} + \sqrt{\epsilon}\mathbf{S}) = N \vee W^*(\mathbf{X}, \mathbf{S})$, i.e., iff $\mathbf{X} \subset N \vee W^*(\mathbf{X} + \sqrt{\epsilon}\mathbf{S})$.

Thus if e.g. $\mathbf{X} = \mathbf{0}$ (or more generally $\mathbf{X} \subset N$), equality in (1) holds trivially for all $\epsilon > 0$.

Our main result is that this phenomenon also occurs when $\mathbf{X}$ is one of many choices of generators for the key examples of strongly 1-bounded algebras. We show that in these cases one can find a diffuse abelian subalgebra N of an ultrapower $W^*(\mathbf{X})^\omega$ (an often even of $W^*(\mathbf{X})$ itself) such that $\mathbf{X} \subset N \vee W^*(\mathbf{X} + \sqrt{\epsilon}\mathbf{S})$ for all $\epsilon > 0$. The key ingredient is the consideration of $N \vee W^*(\mathbf{X} + \sqrt{\epsilon}\mathbf{S})$ as an N, N bimodule, and the identification of $\mathbf{X}$ as the "non-coarse part" of the vector associated to $\mathbf{X} + \sqrt{\epsilon}\mathbf{S}$. We then show that such an estimate on the free Fisher information (we can loosen the conditions to require only that N be diffuse and hyperfinite) implies the strong 1-boundedness of $W^*(\mathbf{X})$. Indeed, we estimate the relative non-microstates free entropy $\chi^*(\cdots : N)$, and appeal to the results of [JP24] to establish strong enough bounds on the microstates free entropy χ. This provides a quick proof that $W^*(\mathbf{X})$ is strongly 1-bounded, indeed has non-positive 1-bounded entropy.

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2 Main Results

This section presents self-contained arguments for our main theorems, and we defer the free probabilistic preliminaries to the latter half of the paper.

Throughout, we fix $\mathbf{X} = (x_1, \ldots, x_d)$ a self-adjoint tuple of random variables in a tracial von Neumann algebra (M, τ), and assume $W^*(\mathbf{X}) = M$. Let further $\mathbf{S} = (s_1, \ldots, s_d)$ be a free semicircular family, free from $W^*(\mathbf{X})$. Also, N will always denote a diffuse hyperfinite von Neumann algebra, most often of M, but we will on occasion take it to be in M^ω, for a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. We denote also $\widetilde{M} = W^*(\mathbf{X}, \mathbf{S})$, and $M_t := W^*(N, \mathbf{X} + \sqrt{t}\mathbf{S}) \subseteq \widetilde{M}^\omega$ for $t \geq 0$. It will be clear from context when N is in the ultrapower as opposed to M itself.

Denote by N^{op} the opposite algebra of N, and by $N \odot N^{\mathrm{op}}$ the algebraic tensor product of N and N^{op}. For $\beta = \sum_{i=1}^n a_i \odot b_i^{\mathrm{op}} \in N \odot N^{\mathrm{op}}$ and $y \in M$, we write $\beta \# y$ to mean $\sum_{i=1}^n a_i y b_i$.

Theorem 2.1. *Suppose that one can find a free ultrafilter ω on a countably infinite set and $N \subseteq W^*(\mathbf{X})^\omega$ diffuse, separable, and abelian, such that $L^2(N\mathbf{X}N) \perp L^2(N) \otimes L^2(N)$ as $N-N$ bimodules. Then*

$$M_t = N \vee W^*(\mathbf{X} + \sqrt{t}\mathbf{S}) = W^*(N, \mathbf{X}, \mathbf{S}) = N \vee \widetilde{M} \tag{2}$$

for all $t > 0$. Equivalently,

$$\mathbf{X} \subset W^*(N, \mathbf{X} + \sqrt{t}\mathbf{S}). \tag{3}$$

Proof. Consider $N \vee \widetilde{M}$ with its canonical trace, which we also denote by τ. There is an atomless standard probability space $(\mathcal{X}, \mu)$ such that $(L^\infty(\mathcal{X}, \mu), \int (\cdot)\, d\mu) \cong (W^*(\mathbf{X}), \tau)$ (here we use the separability assumption on N).

By a classical construction (see e.g. [Voi96], Section 7), each $\xi \in L^2(P,\tau)$ determines a measure ν_ξ on $\mathcal{X}$. For each s_i, we have $\nu_{s_i} = \mu \otimes \mu$, while the condition $L^2(N\mathbf{X}N) \perp L^2(N) \otimes L^2(N)$ is equivalent to $\nu_{x_i} \perp \mu \otimes \mu$ (i.e. the measures are mutually singular).

In this case, there is a net $\{\alpha_\lambda\}_{\lambda\in\Lambda} \in N \odot N^{\mathrm{op}}$, where $\alpha_\lambda = \sum_{n=1}^{N(\lambda)} p_n^{(1)} \otimes p_n^{(2)}$, with each $p_n^{(i)}$ a projection, so that $\|\alpha_\lambda \# s_i - s_i\|_2 \to 0$ and $\|\alpha_\lambda \# x_i\|_2 \to 0$ for each i (see again [Voi96], or [Hay18]). Denote by $E_t : P \to M_t$ the trace-preserving conditional expectation. Recall that it is $N-N$ bimodular. So, we have

$$\sqrt{t}s_i = \lim_\lambda \alpha_\lambda \#(x_i + \sqrt{t}s_i) = \lim_\lambda \alpha_\lambda \# E_t[x_i + \sqrt{t}s_i] = \lim_\lambda E_t[\alpha_\lambda \#(x_i + \sqrt{t}s_i)] = \sqrt{t}E_t(s_i).$$

Hence $E_t(s_i) = s_i$, $1 \le i \le d$, and so $\mathbf{S} \subset M_t$. Since $\mathbf{X} + \sqrt{t}\mathbf{S} \subset M_t$, we get (3). Thus $N \vee \widetilde{M} \supset M_t \supset N \vee \widetilde{M}$ giving us (2). $\square$

Remark 2.1. The ultrapower played no role in the argument. We include it in the statement of the theorem to emphasize that its conclusion remains valid when N is in an ultrapower.

Corollary 2.1. *Let (M,τ) be a finitely-generated diffuse tracial von Neumann algebra. If (a) M has a Cartan subalgebra, or if (b) M has property Γ (in particular, if M has diffuse center or is hyperfinite), then there is a diffuse abelian N as in Theorem 2.1 such that (2) holds for any finite set of generators $\mathbf{X}$ for M.*

Proof. a): Let $N \le M$ be a Cartan subalgebra, so that $M = \mathcal{N}_M(N)''$. So, any s.a. tuple of generators generates a bimodule disjoint from $L^2(N) \otimes L^2(N)$, i.e. is singular with respect to N in the sense of [Voi93], and this means the conditions for (2) to hold are satisfied.

b): In this case, there is an $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and a diffuse abelian $N \le M' \cap M^\omega$ by [Dix69]. By taking an element $a \in N$ with diffuse spectrum and replacing N by $W^*(a)$, we may assume N is separable. It follows that $M \le \mathcal{N}_{M^\omega}(N)''$ and we may conclude as before. $\square$

We now give some examples where (2) is satisfied only for certain choices of generators.

Corollary 2.2.
1. *Suppose M is non-prime, i.e. $M \cong M_1 \overline{\otimes} M_2$ with M_1, M_2 diffuse, and fix $N_1 \le M_1, N_2 \le M_2$ diffuse abelian subalgebras. Then, (2) holds when the generating set $\mathbf{X}$ is chosen from $M_1 \overline{\otimes} N_2 \cup N_1 \overline{\otimes} M_2$.*
2. *If M is generated by unitaries $v_1, \dots, v_k$ which have Property C' in the sense of [GP15] (see proof for description), then one can find N depending on the v_i such that (2) holds.*

Proof. First, suppose M is non-prime; write $M \cong M_1 \overline{\otimes} M_2$, and fix $N_1 \le M_1$, $N_2 \le M_2$ diffuse abelian algebras. Observe that $M_1 \overline{\otimes} N_2 \le \mathcal{N}_M(\mathbb{C}1 \overline{\otimes} N_2)''$, and similarly $N_1 \overline{\otimes} M_2 \le \mathcal{N}_M(N_1 \overline{\otimes} \mathbb{C}1)''$. Then $N = N_1 \overline{\otimes} N_2$ works.

Next, suppose $M = W^*(v_1, \dots, v_k)$ for unitaries $v_1, \dots, v_k$ with Property C'. This means that the v_i have diffuse spectrum and there are mutually commuting unitaries $u_1, \dots, u_k \in M^\omega$ for some free ultrafilter ω, also with diffuse spectrum, such that $[v_j, u_j] = 0$ for $1 \le j \le k$. Let $N = W^*(u_1, \dots, u_k)$, which is abelian.

Note, though, that N may not be separable. However, each $W^*(u_j)$ is, and so we may repeat the argument of Theorem 2.1 to find, for each $1 \leq j \leq k$, a net $\{\beta^{(j)}\}_{i\in I}$ in $W^*(u_j) \odot W^*(u_j)^{\mathrm{op}} \subseteq N \odot N^{\mathrm{op}}$ so that $\beta_i^{(j)} \# s_j \to s_j$ and $\beta_i^{(j)} \# x_j \to 0$. $\square$

2.1 Non-microstates free entropy

Let $N \subseteq M$ be a unital $*$-subalgebra, $\mathbf{S} = (s_1, \ldots, s_d)$ a free $(0,1)$-semicircular family, free from $N \cup \mathbf{X}$; denote by $E_\epsilon : M \to W^*(N, \mathbf{X} + \sqrt{\epsilon}\mathbf{S})$ the trace-preserving conditional expectation.

The *free Fisher information relative to N*, $\Phi^*((\cdot) : N)$, is defined in terms of the conjugate variables $\mathcal{J}(x_i : N)$. We will not need the precise definition of either quantity. Instead, we record that Corollary 3.9 in [Voi98b] shows $\mathcal{J}(x_i + \sqrt{\epsilon} s_i : N) = \frac{1}{\sqrt{\epsilon}} E_\epsilon(s_i)$, and so by the definition (see Definition 6.1 in [Voi98b]) of the free Fisher information, we have

$$\Phi^*(\mathbf{X} + \sqrt{\epsilon}\mathbf{S} : N) = \sum_{i=1}^{N} \left\| \mathcal{J}(x_i + \sqrt{\epsilon} s_i : N) \right\|_2^2 = \frac{1}{\epsilon} \sum_{1 \leq i \leq d} \| E_\epsilon(s_i) \|_2^2 .$$

We then define the *non-microstates free entropy of* $\mathbf{X}$ *relative to* N by

$$\chi^*(\mathbf{X} : N) := \frac{1}{2} \int_0^\infty \left[\frac{d}{1+t} - \Phi^*(\mathbf{X} + \sqrt{t}\mathbf{S} : N) \right] dt + \frac{d}{2} \log(2\pi e).$$

Using the Kaplansky density theorem, these formulae hold with N replaced by $W^*(N)$ ([Voi98b]). We will thus without comment take N to be a von Neumann subalgebra of M.

Now we record the implication (2) has for non-microstates free entropy.

Corollary 2.3. *If* $\mathbf{X} = (x_1, \ldots, x_d)$ *and* N *satisfy (2), and* $\mathbf{S}$ *is a free semicircular family, free from* $\mathbf{X} \cup N$*, then*

$$\Phi^*(\mathbf{X} + \sqrt{\epsilon}\mathbf{S} : N) = \frac{d}{\epsilon}, \ \textit{and} \ \chi^*(\mathbf{X} + \sqrt{\epsilon}\mathbf{S} : N) = \frac{d}{2}[\log(2\pi e) + \log \epsilon]$$

for all $\epsilon > 0$.

Proof. The first assertion follows from the formula given for the free Fisher information above and the fact that, in our case, $E_\epsilon = \mathrm{id}|_{W^*(N, \mathbf{X}, \mathbf{S})}$ for all $\epsilon > 0$.

For the statement about χ^*, let $\mathbf{S}, \mathbf{S}', \mathbf{S}''$ be free semicircular families, free from each other as well as $\mathbf{X} \cup N$. Observe that $(\sqrt{\delta_1 + \delta_2}) s_j$ and $\sqrt{\delta_1} s_j' + \sqrt{\delta_2} s_j'$ have the same $*$-distribution; in particular, the latter are semicircular variables, free from $\mathbf{X} \cup N$. Then the above arguments easily show

$$\Phi^*(\mathbf{X} + (\sqrt{\delta_1 + \delta_2})\mathbf{S} : N) = \Phi^*(\mathbf{X} + \sqrt{\boldsymbol{\delta_1}}\mathbf{S}' + \sqrt{\boldsymbol{\delta_2}}\mathbf{S}'' : N),$$

and hence

$$\chi^*(\mathbf{X} + \sqrt{\epsilon}\mathbf{S} : N) = \frac{1}{2} \int_0^\infty \left[\frac{d}{1+t} - \frac{d}{\epsilon + t} \right] dt + \frac{d}{2} \log(2\pi e) = \frac{d}{2} \log(2\pi e \epsilon).$$

$\square$

2.2 Conditional Microstates Free Entropy

We briefly review (conditional) microstates free entropy. For microstates free entropy, we refer to [Voi93; Voi94; Voi96], while for the conditional version, to [Shl02; JP24]. In this paper, we use the notation of [JP24].

For $m, k \in \mathbb{N}, \gamma > 0$, Voiculescu's microstate spaces for $\mathbf{X}$ are defined as

$$\begin{aligned}\Gamma_R(\mathbf{X}; m, \gamma, k) := \{(X_1^{(k)}, \ldots, X_d^{(k)}) \in (M_k(\mathbb{C})_{\text{s.a.}})^d : \left\|X_j^{(k)}\right\| \leq R \; \forall \, 1 \leq j \leq d, \text{ and} \\ \left|\operatorname{tr}_k(X_{i_1} X_{i_2} \cdots X_{i_p}) - \tau(x_{i_1} x_{i_2} \cdots x_{i_p})\right| < \gamma \; \forall \, (i_1, \ldots, i_m) \in [d]^p, \; 1 \leq p \leq m\}.\end{aligned}$$

We identify isometrically $(M_k(\mathbb{C}))^d_{\text{s.a.}}$, equipped with the Hilbert-Schmidt metric, with $\mathbb{R}^{dk^2}$, equipped with the Euclidean metric, so that we can take the Lebesgue volume of the former space.

Definition 2.1 ([Voi93; Voi94])**.** The *microstates free entropy* of $\mathbf{X}$ is

$$\chi(\mathbf{X}) = \sup_{R>0} \inf_{m,\gamma} \limsup_{k \to \infty} \left[\frac{1}{k^2} \log \operatorname{vol}_{dk^2}(\Gamma_R(\mathbf{X}; m, \gamma, k)) + \frac{1}{2} d \log k\right].$$

In fact, the supremum over R is superfluous, and one may evaluate the entropy for any fixed choice of R which exceeds $\max_{1 \leq j \leq d}\{\|x_j\|_{\text{op}}\}$.

We now record the definition of conditional microstates free entropy as found in [Shl02] (which is different from Voiculescu's original definition in [Voi94]).

Let $\mathbf{Y} = (y_1, \ldots, y_r)$ be a self-adjoint r-tuple in M with $\max_{1 \leq j \leq r}\{\|y_j\|_{\text{op}}\} < S$. We say that $\mathbf{Y}^{(k)} = (Y_1^{(k)}, \ldots, Y_r^{(k)})_{k \geq 1}$ is a *microstate sequence for* Y if $\mathbf{Y}^{(k)} \in (M_k(\mathbb{C})_{\text{s.a.}})^r$ and, for each $n \in \mathbb{N}, \delta > 0$, there exists an $l \in \mathbb{N}$ so that $\mathbf{Y}^{(l)} \in \Gamma_S(\mathbf{Y}; n, l, \delta)$, and write $\mathbf{Y}^{(k)} \rightsquigarrow \mathbf{Y}$.

With $\mathbf{X}, \mathbf{Y}$ as above, assume without loss of generality that $R > S$. Fix a microstate sequence $(\mathbf{Y}^{(k)})_{k \geq 1}$ for $\mathbf{Y}$ and define

$$\begin{aligned}&\Gamma_R(\mathbf{X} \mid \mathbf{Y}^{(k)} \rightsquigarrow \mathbf{Y}; m, k, \gamma) := \\ &\quad \{(X_1^{(k)}, \ldots, X_d^{(k)}) \in (M_k(\mathbb{C})_{\text{s.a.}})^d : (X_1^{(k)}, \ldots, X_d^{(k)}, \mathbf{Y}^{(k)}) \in \Gamma_R(\mathbf{X}, \mathbf{Y}; m, k, \gamma)\}.\end{aligned}$$

Informally, these are those microstates $(X_1^{(k)}, \ldots, X_d^{(k)})$ for which $(X_1^{(k)}, \ldots, X_d^{(k)}, \mathbf{Y}^{(k)})$ is a microstate for $\{\mathbf{X}, \mathbf{Y}\}$. Accordingly, we make the following:

Definition 2.2 ([Shl02; JP24])**.** The *microstates free entropy of* $\mathbf{X}$ *relative to* $\mathbf{Y}^{(\mathbf{k})} \rightsquigarrow \mathbf{Y}$ is

$$\chi(\mathbf{X} \mid \mathbf{Y}^{(\mathbf{k})} \rightsquigarrow \mathbf{Y}) = \sup_{R>0} \inf_{m,\gamma} \limsup_{k \to \infty} \left[\frac{1}{k^2} \log \operatorname{vol}_{dk^2}(\Gamma_R(\mathbf{X} \mid \mathbf{Y}^{(\mathbf{k})} \rightsquigarrow \mathbf{Y}; m, \gamma, k) + \frac{1}{2} d \log k\right].$$

Again, it is not necessary to take the supremum over the cutoff constant R.

We need to relate non-microstates and microstates free entropies. It was proven in [BCG03] that $\chi \leq \chi^*$, and a conditional variant was obtained in [JP24]. We will use the latter result, the statement of which we have simplified to suit our purposes.

Theorem 2.2 ([BCG03; JP24])**.** *Given finite self-adjoint tuples* $\mathbf{X}, \mathbf{Y}$ *in a tracial von Neumann algebra* M*, assume that there is a microstate sequence* $\mathbf{Y}^{(k)} \rightsquigarrow \mathbf{Y}$*. Then*

$$\chi(\mathbf{X} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y}) \leq \chi^*(\mathbf{X} : W^*(\mathbf{Y})).$$

2.3 Strong 1-boundedness and 1-bounded entropy

Strong 1-boundedness was defined by Jung in [Jun07]. We remark that Hadwin and Shen contemporaneously defined the closely related notion of free orbit dimension in [HS07]. We employ the notation and terminology of [Hay18], which made explicit the entropic quantity implicit in [Jun07].

Given a set $S \subseteq \mathbb{R}^n$, denote by $\mathcal{K}_\epsilon(S)$ the minimal number of ϵ-balls (with respect to the Euclidean metric) needed to cover S. Define

$$\mathbb{K}_\epsilon(\mathbf{X}) := \sup_{R>0} \inf_{m,\gamma} \limsup_{k\to\infty} \left[\frac{1}{k^2} \log \mathcal{K}_\epsilon(\Gamma_R(\mathbf{X}; m, \gamma, k))\right],$$

making the analogous definition for $\mathbb{K}_\epsilon(\mathbf{X} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y})$.

Suppose that there is $Y \in M_{\text{s.a.}}$ with diffuse spectrum and $\chi(Y) > -\infty$ so that $\mathbf{X} \cup \{Y\}$ is a finite generating set for (M, τ). In particular, there is a microstates sequence $\mathbf{Y}^{(k)} \to Y$. Jung showed in [Jun07] that if $\mathbf{X}'$ is any other generating set of self-adjoints for M, then

$$\sup_{\epsilon>0} \mathbb{K}_\epsilon(\mathbf{X} \cup \{Y\} \mid \mathbf{Y^{(k)}} \rightsquigarrow Y) = \sup_{\epsilon>0} \mathbb{K}_\epsilon(\mathbf{X'} \mid \mathbf{Y^{(k)}} \rightsquigarrow Y).$$

This motivates the following

Definition 2.3 ([Hay18], see also [Jun07])**.** The *1-bounded entropy* of (M, τ) is

$$h(M) := \sup_{\epsilon>0} \mathbb{K}_\epsilon(\mathbf{X} \mid \mathbf{Y^{(k)}} \rightsquigarrow Y)$$

for $\mathbf{X}, Y$ as in the paragraph above.

We will use the following property of 1-bounded entropy (see [Hay18]): if $N_1 \leq N_2 \leq M_2 \leq M_1$ is diffuse, then $h(N_1 : M_1) \leq h(N_2 : M_2)$.

For any free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, we have $h(M) = h(M : M^\omega)$.

A diffuse, tracial von Neumann algebra (M, τ) is **strongly 1-bounded** if and only if $h(M) < \infty$. We refer the reader to [Hay18; HJK25; Hay+21] for further details and applications. If $\mathbf{X}$ generates a diffuse von Neumann algebra with $\delta_0(\mathbf{X}) > 1$, then $h(W^*(\mathbf{X})) = \infty$ ([Hay18],[Voi96]).

We now state the theorem linking Theorems 2.1 and 2.2 to strong 1-boundedness. In the examples above, when N was diffuse and abelian, we took it to be separable in order to identify it with a standard probability space. In this section, we take N to be separable as well; in the abelian case, this is equivalent to N being singly generated, and is used throughout the proofs and definitions involving 1-bounded entropy.

Lemma 2.1. *If* $\mathbf{X}$ *and* $N = W^*(\mathbf{Y})$ *satisfy (2) with* N *diffuse and hyperfinite, then* $h(W^*(\mathbf{X})) < \infty$*.*

Proof. Fix a microstates sequence $\mathbf{Y}^{(k)} \rightsquigarrow \mathbf{Y}$. The sets $\Gamma_R(\mathbf{X}+\sqrt{\epsilon}\mathbf{S} : \mathbf{S}; m, k, \gamma)$ will denote microstate spaces for $\mathbf{X}+\sqrt{\epsilon}\mathbf{S}$ in the presence of $\mathbf{S}$; see [Voi96]. Now define

$$\begin{aligned}\Gamma_R(\mathbf{X}+\sqrt{\epsilon}\mathbf{S} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y} : \mathbf{S}; m, k, \gamma) &:= \\ &\Gamma_R(\mathbf{X}+\sqrt{\epsilon}\mathbf{S} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y}; m, k, \gamma) \cap \Gamma_R(\mathbf{X}+\sqrt{\epsilon}\mathbf{S} : \mathbf{S}; m, k, \gamma),\end{aligned}$$

and the auxiliary quantity $\chi(\mathbf{X}+\sqrt{\epsilon}\mathbf{S} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y} : \mathbf{S})$ the same way as all other forms of microstates entropy discussed at the beginning of this section. Clearly,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n^2} \log \operatorname{vol}\left(\Gamma_R(\mathbf{X}+\sqrt{\epsilon}\mathbf{S} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y} : \mathbf{S}; m, k, \gamma)\right)& \\ \le \limsup_{n\to\infty} \frac{1}{n^2} \log \operatorname{vol}\left(\Gamma_R(\mathbf{X}+\sqrt{\epsilon}\mathbf{S} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y}; m, k, \gamma)\right)&\end{aligned}$$

which, in conjunction with Theorem 2.1 and Corollary 2.3, implies that

$$\begin{aligned}\chi(\mathbf{X}+\sqrt{\epsilon}\mathbf{S} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y} : \mathbf{S}) &\le \chi(\mathbf{X}+\sqrt{\epsilon}\mathbf{S} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y}) \\ &\le \chi^*(\mathbf{X}+\sqrt{\epsilon}\mathbf{S} : N) \le \frac{d}{2}[\log(2\pi e) + \log\epsilon].\end{aligned}$$

On the other hand, repeating verbatim the (first half of the) proof of Lemma 2.2 in [Jun03a] with the set $\Gamma_R(\mathbf{X} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y}; m, k, \gamma)$ replacing $\Gamma_R(\mathbf{X}; m, k, \gamma)$ shows that

$$\underline{\chi}(\mathbf{S}) + \frac{d}{2}\log\epsilon + \mathbb{K}_{\epsilon'}(\mathbf{X} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y}) \le \chi(\mathbf{X}+\sqrt{\epsilon}\mathbf{S} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y} : \mathbf{S}) \le \frac{d}{2}\log\epsilon + \frac{d}{2}\log(2\pi e)$$

Here $\epsilon' = \Theta(\sqrt{\epsilon})$, with the proportionality constants depending only on $\mathbf{X}, \mathbf{Y}$, while $\underline{\chi}$ is the microstates free entropy with $\liminf_k$ replacing the $\limsup_k$ in its definition. While in general it is not known when $\underline{\chi} = \chi$, it was shown in [Voi98a] that $\underline{\chi}(\mathbf{S}) = \chi(\mathbf{S})$; in our case, they are both equal to $\frac{d}{2}\log(2\pi e)$. It follows that $\mathbb{K}_{\epsilon'}(\mathbf{X} \mid \mathbf{Y^{(k)}} \rightsquigarrow \mathbf{Y}) \le 0$.

Now, the definition of 1-bounded entropy requires that we fix microstates relative to a single self-adjoint, as opposed to the more general situation of a hyperfinite subalgebra, but the proof of lemmata A.4 and A.5 in [Hay18] (really, the fact that all embeddings of a hyperfinite von Neumann algebra into $\mathcal{R}^\omega$ are unitarily conjugate; see [Jun03b]) shows that $h(W^*(\mathbf{X}, \mathbf{Y})) \le 0$. Then, we compute $h(W^*(\mathbf{X})) = h(W^*(\mathbf{X}) : W^*(\mathbf{X})^\omega) \le h(W^*(\mathbf{X}) : W^*(\mathbf{X}, \mathbf{Y})) \le h(W^*(\mathbf{X}, \mathbf{Y})) \le 0$. □

If $\delta_0(\mathbf{X}) > 1$, $h(W^*(\mathbf{X})) = \infty$. Thus we have in particular the following

Corollary 2.4. *There does not exist N as in Theorem 2.1 for any set of generators of $L(\mathbb{F}_r)$ $(r > 1)$, or a free product of diffuse, Connes-embeddable von Neumann algebras, so that (2) would hold.*

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