Consistency of some well-posed five-field theories of dissipative relativistic fluid dynamics

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Abstract

Within the FTBDNK family of formulations of relativistic Navier-Stokes (H. Freistühler and B. Temple, *Proc. R. Soc. A* **470**, 20140055 (2014), *Proc. R. Soc. A* **473** (2017), 20160729; F. S. Bemfica, M. Disconzi, and J. Noronha, *Phys. Rev. D* **98**,104064 (2018), *Phys. Rev. D* **100**, 104020 (2019); P. Kovtun, *J. High Energy Phys.* **2019**, 034 (2019)), this paper collects some consistency properties for certain causal hyperbolic five-field theories obtained from the Landau-Lifshitz formulation via Eulerian gradient shifts, a family, $\mathcal{E}_5(L)$, of models that slightly generalize a class identified in H. Freistühler, *J. Math. Phys.* **61**, 033101 (2020). With ϵ the magnitude of the dissipation coefficients that quantify viscosity and heat conduction, the paper shows that any element of $\mathcal{E}_5(L)$ is $O(\epsilon^2)$ equivalent to the Landau-Lifshitz formulation, has an $O(\epsilon^3)$ excess entropy production, represents heterogeneous local thermodynamic equilibria cleanly, and admits regular heteroclinic profiles for all shock waves of sufficiently small amplitude.

1 Introduction

To implement causality and hyperbolicity, B. Temple and this author proposed in [9, 10] a novel four-field formulation of relativistic Navier-Stokes for the dissipative ultrarelativistic fluid (radiation dominated matter, or 'pure radiation') and a corresponding five-field formulation for non-barotropic fluids. To justify these formulations, we considered (quoting from

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[10]:) "the space $\mathcal{F}_5$ of all pairs of linear gradient forms

$$\begin{aligned} \Delta T^{\alpha\beta} &= T_U^{\alpha\beta\gamma\delta}\frac{\partial U_\gamma}{\partial x^\delta} + T_\rho^{\alpha\delta}\frac{\partial \rho}{\partial x^\delta} + T_n^{\alpha\delta}\frac{\partial n}{\partial x^\delta}, \\ \Delta N^{\beta} &= N_U^{\gamma\delta}\frac{\partial U_\gamma}{\partial x^\delta} + N_\rho^{\delta}\frac{\partial \rho}{\partial x^\delta} + N_n^{\delta}\frac{\partial n}{\partial x^\delta}, \end{aligned} \tag{1.1}$$

and express the smallness of dissipation by giving them a common small factor $\epsilon > 0$, i.e., we consider $(\Delta T^{\alpha\beta}, \Delta N^\beta) \in \mathcal{F}_5$ as representing the five-field theory

$$\begin{aligned} \frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \epsilon\Delta T^{\alpha\beta}\right) &= 0, \\ \frac{\partial}{\partial x^\beta}(N^{\beta} + \epsilon\Delta N^{\beta}) &= 0. \end{aligned} \tag{1.2}$$

We characterize a group of transformations that establishes formal equivalences between different elements of $\mathcal{F}_5$ up to $O(\epsilon^2)$, and then show that our theory lies in the same equivalence class as Eckart's and Landau's."

The group of *first-order equivalence transformations* characterized in [10] is generated by three kinds of elements that we called velocity shifts, thermodynamic shifts, and Eulerian gradient reexpressions, respectively. In this paper, we will restrict attention to a subgroup $\mathcal{E}_5$ that exclusively consists of Eulerian gradient reexpressions which at the same time are (velocity or thermodynamic) shifts and consider the elements of $\mathcal{E}_5(L)$, the orbit of the Landau-Lifshitz description under the action of $\mathcal{E}_5$. We notably show the following three facts.

Theorem 1. *An Eulerian gradient shift is a* second-order *equivalence transformation.*

Theorem 2. *On* any *gradient, the difference between the entropy production of an element of $\mathcal{E}_5(L)$ and the entropy production of the Landau-Lifshitz formulation L itself is of third order in ϵ.*

Theorem 3. *Every element of $\mathcal{E}_5(L)$ respects arbitrary, in particular heterogeneous, thermodynamic equilibria.*

Note that the essential idea of Eulerian gradient shifts (together with a version of Theorem 2!) appeared already in [7], though the variant we use here has an additional degree of freedom (cf. Remark 1 below). Theorems 1, 2, and 3 are proved in the next section, while Secs. 3 to 5 gather additional aspects of the FTBDNK[1] family.

2 Eulerian gradient shifts

To recapitulate and state things concisely, we write (1.2) shortly as

$$\frac{\partial}{\partial x^\beta}\left(T^{a\beta} + \epsilon\Delta T^{a\beta}\right) = 0, \tag{2.1}$$

[1]This acronym was kindly suggested by Marcelo Disconzi (private communication, February 2025).

where the index a runs from 0 to 4, $T^{0\beta} \equiv N^\beta, \Delta T^{0\beta} \equiv \Delta N^\beta$, and (1.1) as

$$\Delta T^{a\beta} = B^{a\beta c\delta} \frac{\partial \psi_c}{\partial x^\delta}. \tag{2.2}$$

In [10] I used gradient transforms

$$\psi^a = \tilde{\psi}^a + \epsilon(\Delta\tilde{\psi})^a \quad \text{with} \quad (\Delta\tilde{\psi})^a = S^{a\beta c} \frac{\partial \tilde{\psi}_c}{\partial x^\beta} \tag{2.3}$$

to rewrite the total energy-momentum-mass tensor $T^{a\beta} + \epsilon \Delta T_L^{a\beta}$ as

$$\begin{aligned}
T^{a\beta} + \epsilon \Delta T_L^{a\beta} &= T^{a\beta}(\psi^e) + \epsilon B_L^{a\beta c\delta}(\psi^e) \frac{\partial \psi^c}{\partial x^\delta} \\
&= T^{a\beta}(\tilde{\psi}^e) + \epsilon \left(B_L^{a\beta c\delta}(\tilde{\psi}^e) \frac{\partial \tilde{\psi}^c}{\partial x^\delta} + \frac{\partial T^{a\beta}}{\partial \psi^f}(\tilde{\psi}^e) S^{f\delta c}(\tilde{\psi}^e) \frac{\partial \tilde{\psi}_c}{\partial x^\delta} \right) \\
&\quad + \epsilon^2 \left(\frac{\partial^2 T^{a\beta}}{\partial \psi^f \partial \psi^g}(\tilde{\psi}^e)(\Delta\tilde{\psi})^f(\Delta\tilde{\psi})^g + \frac{\partial B_L^{a\beta c\delta}}{\partial \psi^f}(\tilde{\psi}^e)(\Delta\tilde{\psi})^f \frac{\partial \psi_c}{\partial x^\delta} + B_L^{a\beta c\delta}(\tilde{\psi}^e) \frac{\partial (\Delta\tilde{\psi})_c}{\partial x^\delta} \right) \\
&\quad + O(\epsilon^3).
\end{aligned} \tag{2.4}$$

We abbreviate this, keeping the order of terms, as

$$T^{a\beta} + \epsilon \Delta T^{a\beta} = \tilde{T}^{a\beta} + \epsilon \left(\Delta\tilde{T}^{a\beta} + \tilde{\Delta}\tilde{T}^{a\beta} \right) + \epsilon^2 R^{a\beta} + O(\epsilon^3). \tag{2.5}$$

The idea was to use

$$\widetilde{\Delta T}^{a\beta} = \Delta\tilde{T}^{a\beta} + \tilde{\Delta}\tilde{T}^{a\beta} = \tilde{B}^{a\beta c\delta} \frac{\partial \tilde{\psi}_c}{\partial x^\delta} \tag{2.6}$$

with

$$\tilde{B}^{a\beta c\delta} = B_L^{a\beta c\delta} + \delta B^{a\beta c\delta} \quad \text{where} \quad \delta B^{a\beta c\delta} = \frac{\partial T^{a\beta}}{\partial \psi^f} S^{f\delta c} \tag{2.7}$$

as new formulation of the dissipation tensor, this being justified by the fact that, as equation (2.5) shows,

$$\frac{\partial}{\partial x^\beta} (\tilde{T}^{a\beta} + \epsilon \widetilde{\Delta T}^{a\beta}) = 0 \tag{2.8}$$

agrees with (2.1) to first order in ϵ.

The starting point of the present paper, indeed already behind [7], is now the following observation.

Lemma 1. *If* (2.3) *is an Eulerian gradient reexpression, i.e.,*

$$S^{a\beta c} = C^a_{\tilde{a}} T^{\tilde{a}\beta c} \quad \text{with } T^{a\beta c} = \partial T^{a\beta} / \partial \psi_c, \tag{2.9}$$

then (2.8) *agrees with* (2.1) *even to* second *order in* ϵ.

Proof. In that case,

$$(\Delta\tilde{\psi})^a = S^{a\beta c}\frac{\partial\tilde{\psi}_c}{\partial x^\beta} = C^a_{\tilde{a}}(\partial\tilde{T}^{\tilde{a}\beta}/\partial x^\beta) = O(\epsilon) \tag{2.10}$$

and thus

$$R^{a\beta} = O(\epsilon) \tag{2.11}$$

in (2.5). □

Definition 1. *(i) A transformation* (2.3) *is called an* Eulerian gradient shift *iff* $S^{a\beta c}$ *is given by* (2.9) *with*

$$C^\alpha_{\tilde{\alpha}} = \frac{\mu}{\theta^2}U^\alpha U_{\tilde{\alpha}} + \frac{\nu}{\theta}\Pi^\alpha_{\tilde{\alpha}} \tag{2.12}$$

$$C^4_4 = \lambda \tag{2.13}$$

$$C^\alpha_4 = C^4_\alpha = 0 \tag{2.14}$$

and coefficients λ, μ, ν.

(ii) We denote by $\mathcal{E}_5$ *the subgroup of* $\mathcal{F}_5$ *that consists of all Eulerian gradient shifts, and by* $\mathcal{E}_5(L)$ *the orbit of the Landau-Lifshitz dissipation tensor under the action of* $\mathcal{E}_5$*.*

Remark 1. *(i) Choice* (2.12) *is very similar to that of Def. 1 in [7], corresponding, for a given fluid* $p = p(\theta,\psi)$*, to*

$$-\tilde{\Delta}T^{\alpha\beta} \equiv U^\alpha U^\beta\tilde{R} + (\tilde{Q}^\alpha U^\beta + U^\alpha\tilde{Q}^\beta) + \Pi^{\alpha\beta}\tilde{P} \tag{2.15}$$

and

$$-\tilde{\Delta}N^\beta \equiv U^\beta\tilde{N} + \frac{1}{h}\tilde{Q}^\beta \tag{2.16}$$

with

$$\tilde{R} = \rho_\theta\Theta + \rho_\psi\Psi, \quad \tilde{P} = p_\theta\Theta + p_\psi\Psi, \quad \tilde{N} = n_\theta\Theta + n_\psi\Psi \tag{2.17}$$

where now

$$\Theta = -\mu U_\epsilon\partial_\delta T^{\epsilon\delta}, \quad \tilde{Q}_\gamma = \nu\Pi_{\gamma\epsilon}\partial_\delta T^{\epsilon\delta}, \quad \Psi = \lambda\partial_\delta N^\delta. \tag{2.18}$$

Concretely, specializing to $\lambda = \mu$ *recovers the dissipation tensors of Definition 4 of [7].*

(ii) The assertion of Lemma 4 in [7] remains valid.

We next consider the entropy production

$$\mathcal{Q} = \frac{\eta}{2\theta}||\mathcal{S}\mathbf{u}||^2 + \frac{\zeta}{\theta}(\nabla\cdot\mathbf{u})^2 + \frac{\kappa}{h^2}|\nabla\psi|^2 + \tilde{\mathcal{Q}}.$$

Lemma 2. *For* (2.1) *with* $\Delta T^{\alpha\beta} \in \mathcal{E}_5(L)$*, the excess entropy production induced by the causalizing approximation*
(i) vanishes on Eulerian gradients,
(ii) is of order $O(\epsilon^3)$ *on general gradients.*

Proof. Computing as in the proof of Lemma 5 in [7], we now find

$$\tilde{\mathcal{Q}} = \mu\theta^{-2}\Theta^2 + \lambda\Psi^2 + \frac{\nu}{\theta(\rho+p)}|\mathbf{q}|^2. \tag{2.19}$$

Thus, (ii) holds as $\Theta, \Psi, \mathbf{q}$ and λ, μ, ν are of $O(\epsilon)$, and (i) follows as $\Theta, \Psi, \mathbf{q}$ vanish on Eulerian gradients. □

In the same way we see

Lemma 3. $\tilde{\Delta} T^{\alpha\beta}$ *vanishes on Eulerian gradients.*

Note finally that Lemma 1 means Theorem 1 and Lemma 2 implies Theorem 2, and as local thermodynamic equilibria, characterized (cf. [4, 12]) by

$$\frac{\partial\psi_\gamma}{\partial x^\delta} \text{ being antisymmetric in } \gamma, \delta, \quad \text{and} \quad \frac{\partial\psi}{\partial x^\delta} = 0, \tag{2.20}$$

of course both are Eulerian gradients and annhilate the Landau-Lifshitz dissipation tensor, Lemma 3 implies Theorem 3.

3 The barotropic case

Barotropic fluids are given by an equation of state $p = p(\theta)$ and one uses only the components of the 4-vector $\psi^\alpha = U^\alpha/\theta$ as Godunov variables. Calling the resulting 4-field theory according to Landau-Lifshitz L_4, one defines barotropic Eulerian gradient shifts analogously to the aboveconsidered non-barotropic ones by suppressing the parts refering to the potential $\psi = g/\theta$ from the above considerations and obtains a barotropic counterpart $\mathcal{E}_4(\mathcal{L}_4)$.

4 Connection with symmetry in Godunov variables

From

$$\Delta T^{\alpha\beta} = B^{\alpha\beta\gamma\delta}\frac{\partial\psi_\gamma}{\partial x^\delta}$$

one sees that symmetry in the sense of Def. 2 (i) in [7] combined with the a priori principle that the energy-momentum tensor must be symmetric in the indices α and β implies that the coefficient field $B^{\alpha\beta\gamma\delta}$ is also symmetric in the indices γ and δ. I. e., the vanishing of $\tilde{\Delta}T^{\alpha\beta}$ on local thermodynamic equilibria, $(2.20)_1$, is also a consequence of symmetry in the sense of Def. 2 (i) in [7].

5 Eulerian gradient shifts of the Eckart description

Returning to the non-barotropic case, we state that Eckart's formulation E, while first-order equivalent with the Landau-Lifshitz description L, is not second-order equivalent with L. The orbit $\mathcal{E}_5(E)$ of E under the action of thr group of Eulerian gradient shifts can be studied analogously to the above considerations. on $\mathcal{E}_5(L)$.

6 Shock profiles

Theorem 4. *For almost any choice of the coefficients λ, μ, ν, every Lax shock*

$$\psi_c(x) = \begin{cases} \psi_c^-, & x^\beta \xi_\beta < 0, \\ \psi_c^+, & x^\beta \xi_\beta > 0, \end{cases} \qquad \xi^\beta \xi_\beta = 1, \tag{6.21}$$

of sufficiently small amplitude possesses a dissipation profile w. r. t. (2.8), (2.6), (2.7), (2.9), *i.e., the ODE system*

$$\xi_\beta \xi_\delta \tilde{B}^{a\beta c\delta}(\psi^e)\psi_c' = \xi_\beta T^{a\beta}(\psi) - q^a, \quad q^a := \xi_\beta T^{a\beta}(\psi_\pm^e) \tag{6.22}$$

has a solution $\hat{\psi}_c$ on $\mathbb{R}$ which is heteroclinic to the states forming the shock,

$$\hat{\psi}^c(-\infty) = \psi_-^c, \quad \hat{\psi}^c(+\infty) = \psi_+^c. \tag{6.23}$$

Proof. Assume for concreteness and w. l. o. g. that $\xi^\beta = \delta^{\beta 1}$, i. e., (6.22) reads

$$\tilde{B}^{a1c1}(\psi^e)\psi_c' = T^{a1}(\psi^e) - q^a. \tag{6.24}$$

Choosing ψ_*^e such that the Jacobian matrix $A^{a1c} = \dfrac{\partial T^{a1}}{\partial \psi_c}(\psi_*^e)$ satisfies

$$A^{a1c} r_c = 0 \tag{6.25}$$

with some $r_c \neq 0$, we see that at ψ_*^e, also

$$\delta B^{a1c1}\ r_c = C_{fg} \frac{\partial T^{a1}}{\partial \psi_f} \frac{\partial T^{g1}}{\partial \psi_c} r_c = 0$$

and thus

$$\tilde{B}^{a1c1}r_c = \tilde{B}_L^{a1c1}r_c. \tag{6.26}$$

To find profiles for small shock waves, we consider this situation with r_c the characteristic direction of the acoustic mode. As the acoustic mode is extreme, we have the property

$$\forall v_c : \quad v_a^* A^{a1c} v_c = 0 \Rightarrow v_c \in \mathbb{C} r_c. \tag{6.27}$$

With A and B denoting the matrices $(A^{a1c}(\psi_*^e))_{a,c}$ and $\tilde{B}(^{a1c1}(\psi_*^e))_{a,c}$, the linearization of (6.24) at ψ_*^e reads

$$v' = B^{-1}Av. \tag{6.28}$$

As A and B are symmetric and 0 is a simple eigenvalue of A, 0 is generically also simple as an eigenvalue of $B^{-1}A$, and using (6.27), we see that the other eigenvalues of $B^{-1}A$ are also all real. Therefore, (6.24) possesses a onedimensional center manifold $\mathcal{C}(q^a)$ near ψ_*^e that depends regularly on the parameter q^a, for values from a neigborhood of $q_*^a = T^{a1}(\psi_*^e)$. For an open set of such values, $\mathcal{C}(q^a)$ contains exactly two rest points of (6.24): the boundary states ψ_-^c and ψ_+^c of a small shock wave. We wish to conclude, similarly to [15], that the open segment $\mathcal{O}$ of $\mathcal{C}(q^a)$ between ψ_-^c and ψ_+^c is the desired dissipation profile. As, in view of the geometry of the Rankine-Hugoniot conditions and genuine nonlinearity, no further rest point is around nearby, $\mathcal{O}$ is a profile heteroclinic to ψ_-^c and ψ_+^c. It only remains to show that

$$\psi_-^c \text{ is the } \alpha\text{-limit and } \psi_+^c \text{ is the } \omega\text{-limit of } \mathcal{O} \tag{6.29}$$

and not vice versa. However, as the tangent space of $\mathcal{C}(q^a)$ converges to $\mathbb{R}r^a$ as $q^a \to q_*^a$, this follows from the facts that, by virtue of (6.26), the dynamics on $\mathcal{C}(q^a)$ is better and better approximated by

$$\tilde{B}_L^{a1c1}(\psi^e)\psi_c' = T^{a1}(\psi^e) - q^a \tag{6.30}$$

in this limit, and that for the Landau formulation, (6.29) holds. □

Remark 2. *For related results on shock profiles cf. [6, 8, 1, 17, 16, 18].*

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