Spinning Mellin amplitudes

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Abstract: We propose a definition of Mellin amplitudes for conformal correlators involving arbitrary spinning operators in tensor representations of the Lorentz group. These representations cover all bosonic local operators. Our strategy is to perform discrete Mellin transforms on all scalar products involving polarization vectors, so that each polarization vector can be interpreted as the position of a fictitious scalar operator. We establish the general pole structures and factorization properties of these spinning Mellin amplitudes. We also provide a systematic algorithm to derive factorization formulas with arbitrary spinning exchanges, yielding new explicit results up to spin-4. To illustrate the practicality of our formalism, we bootstrap the 3- and 4-point current correlators in a 4d $\mathcal{N}=2$ superconformal field theory, which are dual to gluon scattering amplitudes in $\mathrm{AdS}_5\times\mathrm{S}^3$. The results agree with the snowflake channel of 6- and 8-point scalar supergluon amplitudes in the literature.

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1 Introduction

Since its introduction by Mack [1], the Mellin formalism has proven to be a natural language for studying correlation functions in conformal field theories (CFTs), particularly those with holographic duals [2–4]. In this formalism, correlation functions transform into Mellin amplitudes, which usually exhibit simpler analytic structures compared to the position space functions: operators exchanged in the operator product expansion (OPE) appear as poles in Mellin amplitudes, and the residues factorize neatly into products of sub-amplitudes [5]. When a CFT has a large central charge and is dual to a perturbative quantum field theory in anti-de Sitter space (AdS), exchanged operators (and thus the poles in Mellin amplitudes) correspond to particles propagating in the AdS bulk [6, 7], and Mellin amplitudes act as scattering amplitudes in AdS. Such similarity to flat-space scattering amplitudes makes Mellin formalism especially suitable for the analytic bootstrap of holographic correlators, yielding profound results over the past decade (see [8, 9] for reviews). Recent studies of non-perturbative Mellin amplitudes [10] have further revealed deep connections between the Lorentzian inversion formula [11, 12] and Mellin space dispersion relations [13].

Despite its success, the Mellin formalism remains mostly limited to scalar correlators, largely due to the absence of a definition for Mellin amplitudes of generic spinning correlators. Solving this long-standing problem is crucial for studying scattering processes in AdS, as many particles of interest, such as gluons and gravitons, carry spin. Motivation also comes from recent research on five- and higher-point functions [14–22]. To fully exploit the factorization properties of Mellin amplitudes, it is sometimes necessary to consider multi-channel factorization, which generally involves sub-amplitudes with multiple spinning operators and requires a consistent definition of spinning Mellin amplitudes.

In this paper, we propose a definition for Mellin amplitudes with arbitrary spinning operators in tensor representations of the Lorentz group, which include all bosonic local operators. We mostly focus on traceless symmetric tensors for simplicity. The basic idea is straightforward: spinning operators can be described by their position X^M and polarization Z^M in embedding space [23–25]:

$$\mathcal{O}(X,Z) = Z^{M_1}\cdots Z^{M_J}\mathcal{O}_{M_1\cdots M_J}(X), \tag{1.1}$$

and the correlation functions $\langle\mathcal{O}(X_1,Z_1)\cdots\mathcal{O}(X_N,Z_N)\rangle$ depend on three kinds of scalar products $X_p\cdot X_q$, $Z_p\cdot X_q$, and $Z_p\cdot Z_q$. In the conventional definition of scalar Mellin amplitudes, $X_p\cdot X_q$ is associated with the Mellin variables γ_{pq} via a (continuous) Mellin transform. For spinning Mellin amplitudes, we simply perform additional discrete Mellin transforms on the new scalar products $Z_p\cdot X_q$ and $Z_p\cdot Z_q$:

$$\begin{aligned}\langle\mathcal{O}(X_1,Z_1)\cdots\mathcal{O}(X_N,Z_N)\rangle = \sum_{\eta\in\mathbb{Z}}\int[\mathrm{d}\gamma]\,\mathcal{M}(\gamma_{pq},\eta_{p'q},\eta_{p'q'})\prod_{\substack{p,q=1\\p<q}}^{N}\frac{\Gamma(\gamma_{pq})}{(-2X_p\cdot X_q)^{\gamma_{pq}}}\\ \times\prod_{\substack{p,q=1\\p\neq q}}^{N}\frac{(2Z_p\cdot X_q)^{\eta_{p'q}}}{\eta_{p'q}!}\prod_{\substack{p,q=1\\p<q}}^{N}\frac{(2Z_p\cdot Z_q)^{\eta_{p'q'}}}{\eta_{p'q'}!}.\end{aligned} \tag{1.2}$$

The Mellin amplitudes now depend on two kinds of Mellin variables[1], the continuous γ and the discrete η. The primes in the indices indicate which labels are tied to the polarization Z.

The key advantage of our definition is that it treats all scalar products $X \cdot X$, $Z \cdot X$, and $Z \cdot Z$ on an equal footing. As a result, the factorization properties of spinning Mellin amplitudes follow directly from those of scalar Mellin amplitudes. A simple way to see this is that the continuous and discrete Mellin transforms are related through an identity (2.40) (and (6.1)), which allows us to "Wick rotate" all discrete variables into continuous ones, and treat $\eta = -\gamma$ when analyzing factorization. From this perspective, the polarization Z_q can be viewed as the position $X_{p'}$ of a fictitious scalar operator, and the factorization formula for N-point spinning Mellin amplitudes then follows directly from the factorization formula of $2N$-point scalar Mellin amplitudes. The above argument is confirmed by a detailed investigation of the OPE and the factorization properties of these spinning Mellin amplitudes.

We remark that there have been attempts to define spinning Mellin amplitudes in the literature[2]. The first approach [28–30] keeps all tensor indices manifest. This makes it easier to translate position space Feynman rules of Witten diagrams to Mellin space, but the resulting Mellin amplitude is cluttered with indices, which complicates the computation. The second approach [31, 32], restricted to four-point correlators, chooses to perform discrete Mellin transform on the independent tensor structures $V_{i,jk}$ and H_{ij} involving polarization. While these structures manifest the embedding-space property $\mathcal{O}(X, Z + \beta X) = \mathcal{O}(X, Z)$, they mix Z and X in a nontrivial manner. As a result, spinning Mellin amplitudes defined in this way have significantly more complicated factorization properties, which makes the computation impractical in higher-point generalizations. Our definition avoids these shortcomings, treating spinning and scalar operators on an equal footing and manifesting simple factorization properties at any multiplicity.

Finally, we emphasize that although we mostly focus on traceless symmetric tensors, our definition naturally extends to operators with arbitrary mixed-symmetry tensors. Such operators can be described by introducing additional polarization vectors, and the corresponding Mellin amplitude is defined by performing Mellin transforms over all scalar products. An operator described by h polarization vectors effectively corresponds to $h+1$ scalar operators in the Mellin amplitude.

This paper is organized as follows. In section 2, we introduce our definition of spinning Mellin amplitudes. In section 3, we relate the multi-particle OPE to the pole structure of spinning Mellin amplitudes, generalizing the argument in [3]. In section 4, we discuss the general strategy to obtain factorization formulas by solving a Casimir equation recursively. In section 5, we demonstrate the practical utility of our definition by bootstrapping the three-point gluon amplitudes in AdS_{d+1} dual to generic CFTs as well as some four-point gluon amplitudes in $\mathrm{AdS}_5 \times \mathrm{S}^3$ in a specific $\mathcal{N} = 2$ superconformal field theory (SCFT). Finally, in Section 6, we conclude and discuss some future directions.

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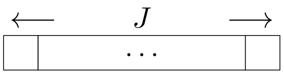
[1] Similar discrete Mellin transforms have been used to deal with the R-symmetry structures in the $\mathrm{AdS}\times\mathrm{S}$ formalism [26].

[2] See also [27] for Mellin amplitudes involving fermionic operators.

2 Definition of spinning Mellin amplitudes

2.1 Spinning operators and the embedding formalism

We consider CFTs in dimension $d \geq 3$ and work in Euclidean signature. In this signature, the metric is given by $g_{\mu\nu} = \delta_{\mu\nu}$, and the corresponding conformal group is $SO(d+1,1)$. Local operators in CFTs are classified by their conformal dimensions Δ and their irreducible representations of the Lorentz group $SO(d)$. The simplest cases are scalar operators $\mathcal{O}(x)$, which transform trivially under rotations. Other representations carry non-trivial spins and can be written as tensors[3] whose indices satisfy certain symmetries. These tensors can be described by Young diagrams. The most common spinning operators are rank-J traceless symmetric tensors $O_{\mu_1\cdots\mu_J}(x)$ ("spin-J" operators), which correspond to single-row Young diagrams with J boxes:



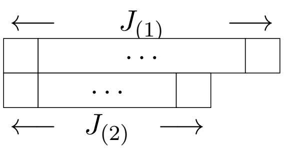
(2.1)

To keep track of all the components of these tensors, we can write them in index-free notation [33] using a null polarization vector z^μ:

$$\mathcal{O}(x,z) = z^{\mu_1}\cdots z^{\mu_J}\mathcal{O}_{\mu_1\cdots\mu_J}(x), \quad z^2 = 0, \tag{2.2}$$

so that the symmetry property is automatic and the tracelessness is ensured by the null condition of the polarization.

In general, all irreducible tensor representations of $SO(d)$ can be written as traceless tensors with mixed symmetries. These operators correspond to multi-row Young diagrams and carry multiple spin labels. For example, the two-row Young diagram ("spin-$(J_{(1)}, J_{(2)})$" operators)



(2.3)

implies that $\mathcal{O}_{\mu_1\cdots\mu_{J_{(1)}},\nu_1\cdots\nu_{J_{(2)}}}(x)$ is first symmetrized under $\mu_i \leftrightarrow \mu_j$, $\nu_i \leftrightarrow \nu_j$ and then anti-symmetrized under $\mu_i \leftrightarrow \nu_i$. The index-free notation [34] now assigns different polarizations for each row:

$$\mathcal{O}(x, z_{(1)}, z_{(2)}) = \mathcal{O}_{\mu_1\cdots\mu_{J_{(1)}},\nu_1\cdots\nu_{J_{(2)}}} z_{(1)}^{\mu_1}\cdots z_{(1)}^{\mu_{J_1}} z_{(2)}^{\nu_1}\cdots z_{(2)}^{\nu_{J_2}}, \tag{2.4}$$

with

$$z_{(1)}^2 = z_{(2)}^2 = z_{(1)}\cdot z_{(2)} = 0. \tag{2.5}$$

The anti-symmetry under $\mu_i \leftrightarrow \nu_i$ implies that replacing any one of the $z_{(2)}$ with $z_{(1)}$ will yield 0, which is equivalent to the property

$$z_{(1)}\cdot\frac{\partial}{\partial z_{(2)}}\mathcal{O}(x,z_{(1)},z_{(2)}) = 0 \quad \Longrightarrow \quad \mathcal{O}(x,z_{(1)},z_{(2)}+\beta z_{(1)}) = \mathcal{O}(x,z_{(1)},z_{(2)}). \tag{2.6}$$

[3] We only consider bosonic operators in this paper. In general, spinors are needed for fermionic operators.

Similarly, for Young diagrams with h rows, we have operators that depend on h polarizations:

$$\mathcal{O}(x, z_{(1)}, \cdots, z_{(h)}), \quad z_{(i)}^2 = z_{(i)} \cdot z_{(j)} = 0, \tag{2.7}$$

and satisfying

$$\mathcal{O}(x, \cdots, z_{(i)} + \sum_{j<i} \beta_{ij} z_{(j)}, \cdots) = \mathcal{O}(x, \cdots, z_{(i)}, \cdots). \tag{2.8}$$

Using the embedding formalism [23–25], we can describe these spinning operators and their correlation functions in the embedding space $\mathbb{R}^{d+1,1}$, where the conformal group is linearly realized as $SO(d+1,1)$ rotations. The spacetime coordinate x^μ is lifted to a projective null ray $X^M \sim \lambda X^M$ passing through the origin in $\mathbb{R}^{d+1,1}$:

$$X^M = (X^0, X^\mu, X^{d+1}), \quad X \cdot X = -(X^0)^2 + (X^1)^2 + \cdots + (X^{d+1})^2 = 0, \tag{2.9}$$

so the CFT and all the operators are located on the light-cone $X \cdot X = 0$. We can return to the d-dimensional space by taking the Poincaré section

$$(X^+, X^\mu, X^-) = (1, x^\mu, x \cdot x), \quad \text{where } X^\pm := X^0 \pm X^{d+1}. \tag{2.10}$$

The spinning operators are promoted to transverse tensors on the light-cone, and the polarizations $z_{(i)}^\mu$ are similarly lifted to $Z_{(i)}^M$, satisfying

$$Z_{(i)} \cdot Z_{(i)} = Z_{(i)} \cdot Z_{(j)} = 0, \quad Z_{(i)} \cdot X = 0 \tag{2.11}$$

due to the tracelessness and transversality. These polarizations are related to $z_{(i)}^\mu$ through the parameterization

$$(Z_{(i)}^+, Z_{(i)}^\mu, Z_{(i)}^-) = (0, z_{(i)}^\mu, 2x \cdot z_{(i)}), \quad \text{where } Z_{(i)}^\pm := Z_{(i)}^0 \pm Z_{(i)}^{d+1}. \tag{2.12}$$

The uplift of operators from d-dimensional space to the embedding space can be achieved as follows:

$$\mathcal{O}(X, Z_{(1)}, \cdots, Z_{(h)}) = (X^+)^{-\Delta} \mathcal{O}\left(x^\mu = \frac{X^\mu}{X^+}, \cdots, z_{(i)}^\mu = Z_{(i)}^\mu - \frac{Z_{(i)}^+}{X^+} X^\mu, \cdots \right). \tag{2.13}$$

These spinning operators in the embedding space now satisfy the following equations

$$\mathcal{O}(\lambda X, \cdots, \alpha_i Z_{(i)}, \cdots) = \lambda^{-\Delta} \alpha_i^{J_i} \mathcal{O}(X, \cdots, Z_{(i)}, \cdots), \tag{2.14}$$

$$\mathcal{O}(X, \cdots, Z_{(i)} + \beta X, \cdots) = \mathcal{O}(X, \cdots, Z_{(i)}, \cdots), \tag{2.15}$$

$$\mathcal{O}(X, \cdots, Z_{(i)} + \sum_{j<i} \beta_{ij} Z_{(j)}, \cdots) = \mathcal{O}(X, \cdots, Z_{(i)}, \cdots). \tag{2.16}$$

Note that the last two equations can be combined into one as

$$\mathcal{O}(X, \cdots, Z_{(i)} + \beta_{i0} X + \sum_{j<i} \beta_{ij} Z_{(j)}, \cdots) = \mathcal{O}(X, \cdots, Z_{(i)}, \cdots). \tag{2.17}$$

This indicates that the position X of the operators can be viewed as a kind of "polarization" $Z_{(0)}$ with spin $-\Delta$ in the embedding space, and vice versa. This idea serves as the core of our definition of spinning Mellin amplitudes. Later, we will make this statement more precise. In particular, for the p-th operator $\mathcal{O}(X_p, Z_{p,(1)}, \cdots, Z_{p,(h)})$ in the correlation function, we would like to denote

$$X_{p'} := Z_{p,(1)}, \quad \Delta_{p'} := -J_{p,(1)}, \tag{2.18a}$$

$$X_{p''} := Z_{p,(2)}, \quad \Delta_{p''} := -J_{p,(2)}, \tag{2.18b}$$

and similarly for the other polarizations and spin labels.

We will mainly focus our discussion on the traceless symmetric operators, where there is only one polarization Z. Explicitly, these operators satisfy

$$\mathcal{O}(\lambda X, \alpha Z) = \lambda^{-\Delta} \alpha^J \mathcal{O}(X, Z), \tag{2.19}$$

$$\mathcal{O}(X, Z + \beta X) = \mathcal{O}(X, Z) \iff (X \cdot \partial_Z)\mathcal{O}(X, Z) = 0. \tag{2.20}$$

2.2 Scalar Mellin amplitudes

The Mellin representation [1–3] provides a nice representation for scalar correlators that manifests conformal invariance. The Mellin amplitude $\mathcal{M}(\gamma)$ is defined as the Mellin transform of the (connected) correlation function:

$$G(X_1, \cdots, X_N) = \int [\mathrm{d}\gamma]\, \mathcal{M}(\gamma) \prod_{\substack{p,q=1 \\ p<q}}^{N} \frac{\Gamma(\gamma_{pq})}{(-X_{pq})^{\gamma_{pq}}}, \tag{2.21}$$

where $X_{pq} := 2X_p \cdot X_q$. The Mellin variables γ_{pq} are symmetric in their indices and satisfy the constraints

$$\forall\, 1 \leq p \leq N: \quad \sum_{\substack{q=1 \\ q \neq p}}^{N} \gamma_{pq} = \Delta_p. \tag{2.22}$$

As a result, there are $\frac{1}{2}N(N-3)$ independent Mellin variables, and the integration in (2.21) is over any set of independent Mellin variables. The freedom of choosing independent Mellin variables allows us to write the amplitudes in many different ways. Therefore, it is useful to think of the Mellin amplitudes are defined on the support of (2.22). It is also very helpful to think of the Mellin variables as products of auxiliary momenta $\mathtt{k}_p$ that satisfy momentum conservation and on-shell conditions:

$$\gamma_{pq} = \mathtt{k}_p \cdot \mathtt{k}_q, \quad \sum_{p=1}^{N} \mathtt{k}_p = 0, \quad \mathtt{k}_p^2 = -\Delta_p. \tag{2.23}$$

This notation automatically solves (2.22). For this reason, we also call (2.22) as the "momentum conservation" constraints. These auxiliary momenta offer further insight into the pole structures of Mellin amplitudes, which we discuss below.

The Mellin amplitude $\mathcal{M}(\gamma)$ is a meromorphic function[4] with poles corresponding to operators exchanged in the OPE. We will discuss this in more detail in section 3. For now,

[4]We restrict to perturbative Mellin amplitudes in this paper. In a non-perturbative CFT, the location of $\mathcal{M}(\gamma)$ could have accumulating points due to the large-spin operators [10], spoiling meromorphicity at these accumulating points.

we simply note that the OPE channel

$$\langle \mathcal{O}(X_1)\cdots\mathcal{O}(X_\ell)|\mathcal{O}_{\Delta,J}|\mathcal{O}(X_{\ell+1})\cdots\mathcal{O}(X_N)\rangle \tag{2.24}$$

for the exchange of an operator $\mathcal{O}_{\Delta,J}$ with dimension Δ and spin J corresponds to a sequence of poles in the Mellin amplitude [5]

$$\mathcal{M}(\gamma) \sim \frac{\mathcal{Q}_m}{\gamma_{LR}-\tau-2m}, \quad m=0,1,2,\cdots, \quad \tau := \Delta - J. \tag{2.25}$$

Here,

$$\gamma_{LR} := \sum_{a\in\mathcal{L}}\sum_{i\in\mathcal{R}}\gamma_{ai} = \sum_{a\in\mathcal{L}}\mathtt{k}_a\cdot\sum_{i\in\mathcal{R}}\mathtt{k}_i \tag{2.26}$$

can be interpreted as the Mandelstam invariant of the auxiliary momentum flow across the OPE channel, where the sets $\mathcal{L}=\{1,\cdots,\ell\}$ and $\mathcal{R}=\{\ell+1,\cdots,N\}$ contain all external labels on the left and right, respectively. In the following, we will always use $a,b,\cdots$ and $i,j,\cdots$ to denote external labels on the left and right, and omit the range specifications $a,b\in\mathcal{L}$ and $i,j\in\mathcal{R}$ in the sums. The residues $\mathcal{Q}_m$ are given by the product of the left and right half-amplitudes $\mathcal{M}_\mathcal{L}$ and $\mathcal{M}_\mathcal{R}$ under certain shifts. For example, if the exchanged operator is a scalar ($J=0$), we have (see (87) in [5])

$$\mathcal{Q}_m = \frac{-2\Gamma(\Delta)m!}{(1+\Delta-\frac{d}{2})_m}\left(\frac{1}{2^m m!}\hat{\mathtt{x}}^m\circ\mathcal{M}_\mathcal{L}\right)\left(\frac{1}{2^m m!}\hat{\mathtt{x}}^m\circ\mathcal{M}_\mathcal{R}\right), \tag{2.27}$$

where $(1+\Delta-\frac{d}{2})_m$ is the Pochhammer symbol, defined as

$$(\gamma)_m := \gamma(\gamma+1)\cdots(\gamma+m-1). \tag{2.28}$$

The shift operator $\hat{\mathtt{x}}$ is defined by[5]

$$\hat{\mathtt{x}}\circ\mathcal{M}_\mathcal{L} = \sum_{\substack{a,b\\a\neq b}}\gamma_{ab}[\mathcal{M}_\mathcal{L}]^{ab}, \quad \hat{\mathtt{x}}\circ\mathcal{M}_\mathcal{R} = \sum_{\substack{i,j\\i\neq j}}\gamma_{ij}[\mathcal{M}_\mathcal{R}]^{ij}, \tag{2.29}$$

where we have introduced the abbreviation

$$[\mathcal{M}]^{pq} := \mathcal{M}(\gamma_{pq}\mapsto\gamma_{pq}+1), \quad [\mathcal{M}]_{pq} := \mathcal{M}(\gamma_{pq}\mapsto\gamma_{pq}-1). \tag{2.30}$$

Explicit formulas for the exchange of spin-1 and spin-2 operators can also be found in [5].

In general, the residue $\mathcal{Q}_m$ can be determined by studying the Casimir equation for the specific OPE channel (2.24) in Mellin space. Consider the Casimir operator $\mathcal{C}$ acting on the left half of the external operators:

$$\mathcal{C} := \frac{1}{2}\left[\sum_a\left(X_{aM}\frac{\partial}{\partial X_a^N} - X_{aN}\frac{\partial}{\partial X_a^M}\right)\right]^2. \tag{2.31}$$

[5]Strictly speaking, this formula can only be used after we have substituted the Mellin variables involving the exchanged operator in terms of γ_{ab} and γ_{ij} in $\mathcal{M}_{\mathcal{L}/\mathcal{R}}$ using momentum conservation. We will discuss this subtle issue in section 4.

In Mellin space, it becomes a difference operator $\hat{\mathbb{C}}$ [3, 5] that acts on the amplitude as

$$\hat{\mathbb{C}}\mathcal{M} = \gamma_{LR}(d-\gamma_{LR})\mathcal{M} + \sum_{\substack{a,b \\ a\neq b}}\sum_{\substack{i,j \\ i\neq j}} \Big(\gamma_{ai}\gamma_{bj}(\mathcal{M} - [\mathcal{M}]^{ai,bj}_{aj,bi}) + \gamma_{ab}\gamma_{ij}[\mathcal{M}]^{ab,ij}_{ai,bj}\Big)\,. \tag{2.32}$$

Following a similar trick in the derivation of conformal blocks, acting $\hat{\mathbb{C}}$ on the left half of the external operators is actually equivalent to simply acting a Casimir on the exchanged operator $\mathcal{O}_{\Delta,J}$, with eigenvalue $-c_{\Delta,J}$. This gives rise to the Casimir equation of the OPE channel (2.24) in Mellin space

$$\hat{\mathbb{C}}\mathcal{M} = -c_{\Delta,J}\mathcal{M}, \quad c_{\Delta,J} := -\Delta(d-\Delta) + J(d-2+J), \tag{2.33}$$

and the corresponding residues $\mathcal{Q}_m$ can then be solved recursively from this equation. We will discuss how to solve the Casimir equation in detail in section 4.

2.3 Discrete Mellin transform

Before we move on to the definition of spinning Mellin amplitudes, let us introduce the important notion of the discrete Mellin transform. In correlation functions of spinning operators, one encounters not only scalar products $X_p \cdot X_q$, but also terms like $Z_p \cdot X_q$ and $Z_p \cdot Z_q$. The dependence of the correlator on the polarizations is always polynomial. Such polynomials do not admit a well-defined Mellin amplitude under (2.21). To deal with polynomials we must instead work with a discrete version of the Mellin transform.

Let $F(Z_1, \ldots, Z_N)$ be a polynomial constructed from products $Z_p \cdot Z_q$, with scaling behavior $F \sim Z_p^{J_p}$ for each p. Its discrete Mellin transform is defined by

$$F(Z_1, \cdots, Z_N) = \sum_{\eta\in\mathbb{Z}} \mathcal{M}(\eta) \prod_{\substack{p,q=1 \\ p<q}}^{N} \frac{(Z_{pq})^{\eta_{p'q'}}}{\eta_{p'q'}!}, \tag{2.34}$$

where $Z_{pq} := 2Z_p \cdot Z_q$. The primes on the labels in η indicate that these variables are the Mellin transform of Z. This notation will prove convenient when we have Mellin transforms involving scalar products of both X and Z. The scaling of F imposes the constraints

$$\forall\, 1 \leq p \leq N: \quad \sum_{\substack{q=1 \\ q\neq p}}^{N} \eta_{p'q'} = J_p, \tag{2.35}$$

and the Mellin amplitudes are defined on the support of the above constraints.

When performing the sum in (2.34), the factorials $\eta_{p'q'}!$ in the denominators force $\eta_{p'q'}$ to be non-negative integers (for negative values, $\eta_{p'q'}!$ diverges, so $1/\eta_{p'q'}!$ vanishes). The constraints (2.35) therefore reduce the summation range to a finite number of lattice points. As a result, (2.34) becomes a finite sum and always produces a polynomial. It is also important to note that only the values of $\mathcal{M}(\eta)$ at these finitely many lattice points are meaningful. Consequently, there is an additional level of ambiguity in presenting the discrete Mellin amplitude, apart from the "momentum conservation" ambiguity of (2.35). For example, consider the polynomial

$$F(Z_1, Z_2, Z_3, Z_4) = Z_{13}Z_{24} + Z_{14}Z_{23} - Z_{12}Z_{34}. \tag{2.36}$$

Taking the amplitude to be

$$\mathcal{M}(\eta) = \eta_{1'3'}\eta_{2'4'} + \eta_{1'4'}\eta_{2'3'} - \eta_{1'2'}\eta_{3'4'} \tag{2.37}$$

correctly reproduces F after summing over all the contributing lattice points

$$(\eta_{1'2'}, \eta_{1'3'}, \eta_{1'4'}, \eta_{2'3'}, \eta_{2'4'}, \eta_{3'4'}) = (1,0,0,0,0,1),\ (0,1,0,0,1,0),\ (0,0,1,1,0,0). \tag{2.38}$$

However, the alternative form

$$\mathcal{M}(\eta) = \frac{1-(-1)^{\eta_{1'3'}}}{2}\eta_{2'4'} + 42\,\eta_{1'3'}\eta_{1'4'} + \frac{\eta_{1'4'}\eta_{2'3'}}{\eta_{1'2'}+1} - \frac{1}{2}\eta_{1'2'}\eta_{3'4'}(\eta_{3'4'}+1) \tag{2.39}$$

evaluates to the same values at the lattice points in (2.38) and thus reproduces the same polynomial. Both presentations are equivalent, though clearly the former is more natural than the latter. In subsection 2.5, we will impose further restrictions on the form of $\mathcal{M}$ to make it more physically interpretable.

Although the kernel in the discrete Mellin transform looks quite different from that in (2.21), they are in fact related by the simple identity

$$\operatorname*{Res}_{\gamma=-\eta} \frac{\Gamma(\gamma)}{(-P)^{\gamma}} = \frac{(+P)^{\eta}}{\eta!}, \quad \eta = 0,1,2,\cdots. \tag{2.40}$$

This inspires us to adopt the notation

$$\gamma_{p'q} := -\eta_{p'q}, \quad \gamma_{p'q'} := -\eta_{p'q'} \tag{2.41}$$

to write the continuous and discrete Mellin variables in the same manner. This notation will not cause any confusion, since the γ's with primed indices should always be understood as discrete Mellin variables. We will use the two symbols γ and $-\eta$ interchangeably in the rest of the paper.

2.4 Spinning Mellin amplitudes

With both continuous and discrete Mellin transforms, we can now present the definition of spinning Mellin amplitudes. For simplicity, we first focus on traceless symmetric operators. The definition for mixed symmetry operators will be given at the end of this subsection. For a (connected) spinning correlation function, we define

$$\begin{aligned} &G(X_1, Z_1, \cdots, X_N, Z_N) \\ &= \sum_{\eta\in\mathbb{Z}} \int [\mathrm{d}\gamma]\, \mathcal{M}(\gamma,\eta) \prod_{\substack{p,q=1\\p<q}}^{N} \frac{\Gamma(\gamma_{pq})}{(-2X_p\cdot X_q)^{\gamma_{pq}}} \prod_{\substack{p,q=1\\p\neq q}}^{N} \frac{(2Z_p\cdot X_q)^{\eta_{p'q}}}{\eta_{p'q}!} \prod_{\substack{p,q=1\\p<q}}^{N} \frac{(2Z_p\cdot Z_q)^{\eta_{p'q'}}}{\eta_{p'q'}!}, \end{aligned} \tag{2.42}$$

which is the Mellin transform, continuous or discrete, for all the scalar products. The Mellin variables $\gamma_{pq}, \eta_{p'q}, \eta_{p'q'}$ are symmetric in their indices and satisfy the following constraints due to the weight counting (2.19):

$$\forall 1 \le p \le N: \quad \sum_{\substack{q=1\\q\neq p}}^{N}(\gamma_{pq} - \eta_{pq'}) = \Delta_p, \quad \sum_{\substack{q=1\\q\neq p}}^{N}(\eta_{p'q} + \eta_{p'q'}) = J_p. \tag{2.43}$$

Spinning operators also satisfy the "gauge invariance" condition (2.20). In Mellin space, this is reflected in the constraint[6]

$$\forall 1 \leq p \leq N : \quad \sum_{\substack{q=1 \\ q\neq p}}^{N} \left(\gamma_{pq} [\mathcal{M}]^{pq}_{p'q} - \eta_{pq'} [\mathcal{M}]^{pq'}_{p'q'} \right) = 0. \tag{2.44}$$

Note that after performing the shifts, the Mellin variables in the resulting function live on a different support, which satisfy (2.43) with $\Delta_p \mapsto \Delta_p - 1$ and $J_p \mapsto J_p - 1$[7]. We will discuss the support issues in detail in section 4.1.

Using the primed notation (2.18) and (2.41), the definition of the spinning Mellin amplitude can be written in a unified way:

$$G(X_1, X_{1'}, \cdots, X_N, X_{N'}) = \sum_{\eta \in \mathbb{Z}} \int [\mathrm{d}\gamma]\, \mathcal{M}(\gamma, \eta) \prod_{\substack{r,s\in\mathcal{S} \\ r\prec s}} \begin{cases} \dfrac{\Gamma(\gamma_{rs})}{(-X_{rs})^{\gamma_{rs}}}, & r, s \text{ unprimed}, \\ \\ \dfrac{(X_{rs})^{\eta_{rs}}}{\eta_{rs}!}, & \text{otherwise}, \end{cases} \tag{2.45}$$

where $\mathcal{S} = \{1, 1', 2, 2', \cdots, N, N'\}$ is the set of all external labels, and $r \prec s$ means comparing r and s after dropping all primes (e.g., $1' \prec 2$). The constraints on the Mellin variables now combine into one:

$$\forall r \in \mathcal{S} : \quad \sum_{\substack{s\in\mathcal{S} \\ s\neq r}} \gamma_{rs} = \Delta_r, \tag{2.46}$$

and can be written in terms of auxiliary momentum as

$$\gamma_{rs} = \mathtt{k}_r \cdot \mathtt{k}_s, \quad \sum_{s\in\mathcal{S}} \mathtt{k}_s = 0, \quad \mathtt{k}_s^2 = -\Delta_s, \quad \mathtt{k}_p \cdot \mathtt{k}_{p'} = 0. \tag{2.47}$$

The gauge invariance constraint becomes

$$\forall 1 \leq p \leq N : \quad \sum_{\substack{s\in\mathcal{S} \\ s\neq p,p'}} \gamma_{ps} [\mathcal{M}]^{ps}_{p's} = 0. \tag{2.48}$$

One should keep in mind that the unprimed variables γ_{pq}, Δ_p are continuous and can take generic real values, while the primed variables $\gamma_{p'q}, \gamma_{p'q'}, \Delta_{p'}$ are discrete and take only integer values.

Definition (2.45) suggests us to treat the N-point spinning Mellin amplitude as a $2N$-point scalar Mellin amplitude, where the spinning operator $\mathcal{O}(X_p, Z_p) = \mathcal{O}(X_p, X_{p'})$ can be viewed as two fictitious scalars $\mathcal{O}(X_p)$ and $\mathcal{O}(X_{p'})$ at null-separated positions $X_p \cdot X_{p'} = 0$. Such statement can be made rather precise when considering the factorization of spinning

[6] When written in terms of η, the shifts are actually in the opposite direction. For example, $[\mathcal{M}]_{p'q} = \mathcal{M}(\gamma_{p'q} \mapsto \gamma_{p'q} - 1) \equiv \mathcal{M}(\eta_{p'q} \mapsto \eta_{p'q} + 1)$.

[7] One way to understand this is to note that the function $X_p \cdot \partial_{Z_p} G(X_1, Z_1, \cdots, X_N, Z_N)$ in (2.20) has conformal dimension $\Delta_p - 1$ and spin $J_p - 1$ with respect to the p-th operator.

Mellin amplitudes. Consider the Casimir equation for the spinning amplitude in the OPE channel

$$\langle \mathcal{O}(X_1, X_{1'}) \cdots \mathcal{O}(X_\ell, X_{\ell'}) | \mathcal{O}_{\Delta,J} | \mathcal{O}(X_{\ell+1}, X_{(\ell+1)'}) \cdots \mathcal{O}(X_N, X_{N'}) \rangle. \tag{2.49}$$

The corresponding Casimir operator now acts on both the position X and the polarization Z on the left half:

$$\begin{aligned} \mathcal{C} &= \frac{1}{2} \left[\sum_{a=1}^{\ell} \left(X_{aM} \frac{\partial}{\partial X_a^N} - X_{aN} \frac{\partial}{\partial X_a^M} + Z_{aM} \frac{\partial}{\partial z_A^N} - Z_{aN} \frac{\partial}{\partial z_A^M} \right) \right]^2 \\ &= \frac{1}{2} \left[\sum_{a \in \mathcal{L}} \left(X_{aM} \frac{\partial}{\partial X_a^N} - X_{aN} \frac{\partial}{\partial X_a^M} \right) \right]^2, \end{aligned} \tag{2.50}$$

where $\mathcal{L} = \{1, 1', \cdots, \ell, \ell'\}$ contains all external labels on the left (and similarly for $\mathcal{R}$). This Casimir operator acts in the same way on both X_a and $X_{a'}$. As a result, through the relation (2.40) between the Mellin kernels, the Casimir operator in Mellin space takes the same form as in scalar amplitudes (2.32):

$$\hat{\mathbb{C}}\mathcal{M} = \gamma_{LR}(d - \gamma_{LR})\mathcal{M} + \sum_{\substack{a,b \\ a \neq b}} \sum_{\substack{i,j \\ i \neq j}} \left(\gamma_{ai}\gamma_{bj}(\mathcal{M} - [\mathcal{M}]^{ai,bj}_{aj,bi}) + \gamma_{ab}\gamma_{ij}[\mathcal{M}]^{ab,ij}_{ai,bj} \right), \tag{2.51}$$

where the only difference is that $a, b \in \mathcal{L}$ and $i, j \in \mathcal{R}$ now ranges over both unprimed and primed labels. Since the equation is identical to the Casimir equation for $2N$-point scalar Mellin amplitudes, we can directly borrow the factorization formula from the scalar case. For instance, for the exchange of a scalar operator with dimension Δ, the spinning Mellin amplitudes factorize as

$$\mathcal{M}(\gamma, \eta) \sim \frac{\mathcal{Q}_m}{\gamma_{LR} - \tau - 2m}, \quad m = 0, 1, 2, \cdots, \tag{2.52}$$

where the residues are still given by

$$\mathcal{Q}_m = \frac{-2\Gamma(\Delta)m!}{(1 + \Delta - \frac{d}{2})_m} \left(\frac{1}{2^m m!} \hat{\mathbb{x}}^m \circ \mathcal{M}_{\mathcal{L}} \right) \left(\frac{1}{2^m m!} \hat{\mathbb{x}}^m \circ \mathcal{M}_{\mathcal{R}} \right), \tag{2.53}$$

and the "Mandelstam" variable is

$$\gamma_{LR} = \sum_{a,i} \gamma_{ai} = \sum_{b=1}^{\ell} \sum_{j=\ell+1}^{N} (\gamma_{bj} - \eta_{b'j} - \eta_{bj'} - \eta_{b'j'}). \tag{2.54}$$

This is indeed the correct factorization formula for the N-point spinning Mellin amplitudes defined in (2.42)!

We will justify the pole structure (2.52) in section 3 through a careful OPE analysis. The correctness of our spinning factorization formula is also supported by concrete examples in section 5.

Finally, we remark that there is nothing special about symmetric traceless operators in our definition. For general mixed-symmetry correlators, one simply Mellin transforms all scalar products, and a spinning operator with h polarizations in the correlator can be viewed as $h+1$ scalars. For example, for spin-$(J_{(1)}, J_{(2)})$ operators, there are two polarizations

$$X_{p'} := Z_{p,(1)}, \quad X_{p''} := Z_{p,(2)}. \tag{2.55}$$

We define the corresponding Mellin amplitude in a similar way as

$$\begin{aligned} &G(X_1, X_{1'}, X_{1''}, X_2, X_{2'}, X_{2''}, \cdots, X_\ell, X_{\ell'}, X_{\ell''}) \\ &= \sum_{\eta \in \mathbb{Z}} \int [\mathrm{d}\gamma]\, \mathcal{M}(\gamma, \eta) \prod_{\substack{r,s \in \mathcal{S} \\ r \prec s}} \begin{cases} \dfrac{\Gamma(\gamma_{rs})}{(-X_{rs})^{\gamma_{rs}}}, & r, s \text{ unprimed}, \\ \\ \dfrac{(X_{rs})^{\eta_{rs}}}{\eta_{rs}!}, & \text{otherwise}, \end{cases} \end{aligned} \tag{2.56}$$

where $\mathcal{S} = \{1, 1', 1'', 2, 2', 2'', \cdots, \ell, \ell', \ell''\}$ contains all external labels, the Mellin variables are subjected to

$$\forall r \in \mathcal{S}: \quad \sum_{\substack{s \in \mathcal{S} \\ s \neq r}} \gamma_{rs} = \Delta_r, \tag{2.57}$$

and the amplitudes satisfy a series of gauge invariance constraints related to (2.17):

$$\forall 1 \le p \le N: \quad \sum_{\substack{s \in \mathcal{S} \\ s \neq p, p', p''}} \gamma_{ps} [\mathcal{M}]^{ps}_{p's} = \sum_{\substack{s \in \mathcal{S} \\ s \neq p, p', p''}} \gamma_{ps} [\mathcal{M}]^{ps}_{p''s} = \sum_{\substack{s \in \mathcal{S} \\ s \neq p, p', p''}} \gamma_{p's} [\mathcal{M}]^{p's}_{p''s} = 0. \tag{2.58}$$

The factorization of these Mellin amplitudes then resembles that of $3N$-point scalar amplitudes.

2.5 The relation to single-spin Mellin amplitudes

In principle, we can also adapt the factorization formulas for spin-J ($J = 1, 2$) operators in [5] to our spinning Mellin amplitudes. These formulas involve single-spin Mellin amplitudes (the only spinning operator is the exchanged operator). However, our definition of spinning amplitudes, when restricted to single-spin ones, are different from that in [5]. Although in section 4 we will directly derive the factorization formula within our framework, it is still useful to compare these two definitions of single-spin amplitudes.

Consider the single-spin Mellin amplitude in our definition, which will later be identified with the left half-amplitude:

$$\begin{aligned} &G_{\mathcal{L}}(X_1, X_2, \cdots, X_\ell, X_L, Z_L) \\ &= \sum_{\eta \in \mathbb{Z}} \int [\mathrm{d}\gamma]\, \mathcal{M}_{\mathcal{L}}(\gamma, \eta) \prod_{\substack{a,b \\ a<b}} \frac{\Gamma(\gamma_{ab})}{(-X_{ab})^{\gamma_{ab}}} \prod_a \frac{\Gamma(\gamma_{aL})}{(-X_{aL})^{\gamma_{aL}}} \prod_a \frac{(2X_a \cdot Z_L)^{\eta_{aL'}}}{\eta_{aL'}!}. \end{aligned} \tag{2.59}$$

As usual, the range of a, b is over all external labels on the left: $\mathcal{L} = \{1, \cdots, \ell\}$. Suppose the exchanged operator $\mathcal{O}(X_L, Z_L)$ has dimension Δ and spin J. The Mellin variables satisfy the momentum conservation:

$$\sum_a \gamma_{aL} = \Delta, \quad \sum_a \eta_{aL'} = J, \qquad \forall 1 \le a \le \ell: \ \gamma_{aL} - \eta_{aL'} + \sum_{\substack{b \\ b \neq a}} \gamma_{ab} = \Delta_a, \tag{2.60}$$

and the half-amplitude $\mathcal{M}_{\mathcal{L}}(\gamma,\eta)$ satisfies the gauge invariance condition

$$\sum_a \gamma_{aL}[\mathcal{M}_{\mathcal{L}}]^{aL}_{aL'} = 0. \tag{2.61}$$

To relate our definition to that in [5], recall that their single-spin Mellin amplitude $\widetilde{\mathcal{M}}^{c_1\cdots c_J}_{\mathcal{L}}$ has multiple components and is defined for each polarization structures as

$$\begin{aligned} &G_{\mathcal{L}}(X_1,\cdots,X_\ell,X_L,Z_L)\\ &= \sum_{c_1,\cdots,c_J\in\mathcal{L}} (Z_L\cdot X_{c_1})(Z_L\cdot X_{c_2})\cdots(Z_L\cdot X_{c_J})\\ &\qquad\times\int[\mathrm{d}\tilde\gamma]\widetilde{\mathcal{M}}^{c_1\cdots c_J}_{\mathcal{L}}(\tilde\gamma)\prod_{\substack{a,b\\a<b}}\frac{\Gamma(\tilde\gamma_{ab})}{(-X_{ab})^{\tilde\gamma_{ab}}}\prod_a\frac{\Gamma(\tilde\gamma_{aL}+n_a)}{(-X_{aL})^{\tilde\gamma_{aL}+n_a}}. \end{aligned} \tag{2.62}$$

where n_a denotes the number of occurrences of a in the sequence $\{c_1,\cdots,c_J\}$. For this definition, the momentum conservation reads

$$\sum_a \tilde\gamma_{aL} = \Delta - J = \tau, \qquad \forall 1\le a\le \ell:\ \tilde\gamma_{aL}+\sum_{\substack{b\\b\neq a}}\tilde\gamma_{ab} = \Delta_a, \tag{2.63}$$

and the gauge invariance condition is

$$\sum_a(\tilde\gamma_{aL}+n_a)\mathcal{M}^{ac_2\cdots c_J}_{\mathcal{L}} = 0. \tag{2.64}$$

By construction, $\widetilde{\mathcal{M}}^{c_1\cdots c_J}_{\mathcal{L}}$ is symmetric under permuting $\{c_1,\cdots,c_J\}$. Therefore, we can further collect terms in (2.62) that have the same power of $Z_L\cdot X$. For a product with given powers $(Z_L\cdot X_1)^{n_1}(Z_L\cdot X_2)^{n_2}\cdots(Z_L\cdot X_\ell)^{n_\ell}$, there are $\frac{J!}{n_1!\cdots n_\ell!}$ such terms in the sum, and we can rewrite (2.62) as

$$\begin{aligned} &G_{\mathcal{L}}(X_1,\cdots,X_\ell,X_L,Z_L)\\ &= \sum_{\substack{n_1,\cdots,n_\ell\ge 0\\ n_1+\cdots+n_\ell=J}} 2^{-J}\frac{J!}{n_1!\cdots n_\ell!}(2Z_L\cdot X_1)^{n_1}(2Z_L\cdot X_2)^{n_2}\cdots(2Z_L\cdot X_\ell)^{n_\ell}\\ &\qquad\times\int[\mathrm{d}\tilde\gamma]\widetilde{\mathcal{M}}^{(n_1\cdots n_\ell)}_{\mathcal{L}}(\tilde\gamma)\prod_{\substack{a,b\\a<b}}\frac{\Gamma(\tilde\gamma_{ab})}{(-X_{ab})^{\tilde\gamma_{ab}}}\prod_a\frac{\Gamma(\tilde\gamma_{aL}+n_a)}{(-X_{aL})^{\tilde\gamma_{aL}+n_a}}, \end{aligned} \tag{2.65}$$

where

$$\widetilde{\mathcal{M}}^{(n_1\cdots n_\ell)}_{\mathcal{L}} := \widetilde{\mathcal{M}}_{\mathcal{L}}^{\overbrace{1\cdots 1}^{n_1}\cdots\overbrace{\ell\cdots\ell}^{n_\ell}}. \tag{2.66}$$

We have also rescaled $Z_L\cdot X$ to $2Z_L\cdot X$ for convenience. Comparing with our definition (2.59), we can identify $\eta_{aL'}=n_a$ and $\gamma_{aL}=\tilde\gamma_{aL}+n_a$ on both sides, which leads to

$$\widetilde{\mathcal{M}}^{(n_1\cdots n_\ell)}_{\mathcal{L}}(\tilde\gamma_{ab},\tilde\gamma_{aL}) = \frac{2^J}{J!}\mathcal{M}_{\mathcal{L}}(\gamma_{ab}\mapsto\tilde\gamma_{ab},\gamma_{aL}\mapsto\tilde\gamma_{aL}+n_a,\eta_{aL'}\mapsto n_a). \tag{2.67}$$

Using the shift operators, we can also write[8]

$$\widetilde{\mathcal{M}}_{\mathcal{L}}^{c_1\cdots c_J}(\tilde{\gamma}_{ab},\tilde{\gamma}_{aL}) = \frac{2^J}{J!}[\mathcal{M}_{\mathcal{L}}]^{c_1L,\cdots,c_JL}_{c_1L',\cdots,c_JL'}\Big|_{\eta_{aL'}\mapsto 0,\gamma_{ab}\mapsto\tilde{\gamma}_{ab},\gamma_{aL}\mapsto\tilde{\gamma}_{aL}}. \tag{2.68}$$

One can also see that the gauge invariance condition (2.64) is satisfied.

To represent our single-spin amplitude $\mathcal{M}_{\mathcal{L}}$ in terms of $\widetilde{\mathcal{M}}_{\mathcal{L}}$, we need to interpolate a function on the lattice points of η, which takes the value $2^{-J}J!\widetilde{\mathcal{M}}_{\mathcal{L}}^{(n_1\cdots n_\ell)}$ at the point $(\eta_{1L'},\cdots,\eta_{\ell L'}) = (n_1,\cdots,n_\ell)$. In principle, there are infinitely many such interpolating functions, as discussed in section 2.3. Among these functions, we prefer the one that has the pole structure (2.52) and depends polynomially on η in the numerators. When further require the amplitude to have the lowest possible powers in η, the interpolating function becomes unique and can be constructed in the following way:

$$\mathcal{M}_{\mathcal{L}}(\gamma_{ab},\gamma_{aL},\eta_{aL}) = \frac{J!}{2^J}\sum_{\substack{n_1,\cdots,n_\ell\geq 0\\ n_1+\cdots+n_\ell=J}}\left(\prod_a \frac{(\eta_{aL'})^{\underline{n_a}}}{n_a!}\right)\widetilde{\mathcal{M}}_{\mathcal{L}}^{(n_1\cdots n_\ell)}\Big|_{\tilde{\gamma}_{ab}\mapsto\gamma_{ab},\tilde{\gamma}_{aL}\mapsto\gamma_{aL}-\tilde{n}_a}, \tag{2.69}$$

where the factorial power is defined as

$$(\eta)^{\underline{n}} := \eta(\eta-1)\cdots(\eta-n+1), \tag{2.70}$$

and $\tilde{n}_a$ stands for $\eta_{aL'}$ when the original $\tilde{\gamma}_{aL}$ appears in the denominators, or n_a when it appears in the numerators. The factor $(\eta_{aL'})^{\underline{n_a}}$ in the formula can be thought of as related to $(2Z_L\cdot X_a)^{n_a}$ in position space.

As an example, consider the correlator $\langle\mathcal{OOOJ}\rangle$ in the holographic model of supergluons in $\mathrm{AdS}_5\times\mathrm{S}^3$, where $\mathcal{O}$ are the scalar supergluons and $\mathcal{J}$ is the spin-1 gluon operator (a brief introduction of this model can be found in section 5.2). The color-ordered Mellin amplitude of this correlator, up to the overall coefficients and R-symmetry structures, is given in [19] by

$$\widetilde{\mathcal{M}}_{\mathcal{L}}^{1} = \widetilde{\mathcal{M}}_{\mathcal{L}}^{(100)} = \frac{1}{\tilde{\gamma}_{12}-1}+\frac{1}{\tilde{\gamma}_{1L}-1}, \tag{2.71}$$

$$\widetilde{\mathcal{M}}_{\mathcal{L}}^{2} = \widetilde{\mathcal{M}}_{\mathcal{L}}^{(010)} = \frac{1}{\tilde{\gamma}_{12}-1}-\frac{1}{\tilde{\gamma}_{1L}-1}, \tag{2.72}$$

$$\widetilde{\mathcal{M}}_{\mathcal{L}}^{3} = \widetilde{\mathcal{M}}_{\mathcal{L}}^{(001)} = -\frac{1}{\tilde{\gamma}_{12}-1}-\frac{1}{\tilde{\gamma}_{1L}-1}. \tag{2.73}$$

Following (2.69), we simply replace $\tilde{\gamma}_{12}\mapsto\gamma_{12}$, and replace $\tilde{\gamma}_{1L}\mapsto\gamma_{1L}-\eta_{1L'}$ since all of them are in the denominators. These amplitudes are then added up together with the prefactor of η. The result is

$$\mathcal{M}_{\mathcal{L}}(\gamma_{ab},\gamma_{aL},\eta_{aL}) = \frac{1}{2}\frac{\eta_{1L'}+\eta_{2L'}-\eta_{3L'}}{\gamma_{12}-1}+\frac{1}{2}\frac{\eta_{1L'}-\eta_{2L'}-\eta_{3L'}}{\gamma_{1L}-\eta_{1L'}-1}, \tag{2.74}$$

which has the correct s- and t-channel pole structures. One can also check that this amplitude satisfy the gauge invariance condition, and correctly reproduce the corresponding $\widetilde{\mathcal{M}}_{\mathcal{L}}$ when we put it into (2.68).

[8] Actually, by momentum conservation, we can only take $\eta_{aL'}=0$ after performing the shifts.

3 Pole structures and residues

When we separate the external operators of a conformal correlator into two groups and let them approach each other, the correlator factorizes into a product for each exchanged operator in the OPE. This factorization is captured by the pole structures in Mellin amplitudes through Mellin transform, and the residues on the poles are factorizable accordingly. In this section, we first show how scaling behaviors in position space are related to poles in Mellin space. We then analyze the scaling behaviors arise in the OPE of generic spinning correlators, which map to corresponding pole structures in spinning Mellin amplitudes. These pole structures and their factorization properties serve as the basis for the derivation of factorization formula in section 4. The main argument in this section is adapted and refined from [3].

3.1 Scales, poles, and pinches

A feature of the (inverse) Mellin transform is that the scales in position space are encoded in the poles of the Mellin amplitudes. This can be illustrated by a toy integral

$$\mathcal{I}_1 = \int_{a-i\infty}^{a+i\infty} \frac{\mathrm{d}\gamma}{2\pi i} f(\gamma) \frac{\Gamma(\gamma)}{x^\gamma}, \quad a > 0. \tag{3.1}$$

Suppose $f(\gamma)$ has no poles and decays sufficiently rapidly at infinity. By residue theorem, we can close the contour to the left, and the integral picks up poles in the Γ function. The result evaluates to

$$\mathcal{I}_1 = \sum_{m=0}^{\infty} \operatorname*{Res}_{\gamma=-m} \left[f(\gamma) \frac{\Gamma(\gamma)}{x^\gamma} \right] = \sum_{m=0}^{\infty} \frac{(-)^m}{m!} f(-m) x^m, \tag{3.2}$$

from which we see that each pole in the integrand corresponds to a x^m in the result. Conversely, when there are terms like $x^{\tau+n}$ appearing in the result, we can deduce that $f(\gamma)$ cannot be regular and must contain poles at $\gamma = -\tau - n$. In this case, one can see that the pole structures and the scaling behaviors are directly related.

For multi-variate Mellin transforms, poles may not be explicitly present in the integrand but can arise from the pinching of other poles. This can be shown by another toy integral

$$\mathcal{I}_2 = \int_{a-i\infty}^{a+i\infty} \frac{\mathrm{d}\gamma}{2\pi i} \frac{\Gamma(\gamma_1+\gamma)\Gamma(\gamma_2-\gamma)}{x^\gamma}, \quad -\gamma_1 < a < \gamma_2, \tag{3.3}$$

which evaluates to

$$\mathcal{I}_2 = \sum_{m=0}^{\infty} \operatorname*{Res}_{\gamma=-\gamma_1-m} \left[\frac{\Gamma(\gamma_1+\gamma)\Gamma(\gamma_2-\gamma)}{x^\gamma} \right] = \sum_{m=0}^{\infty} \frac{(-)^m}{m!} x^{\gamma_1+m} \Gamma(\gamma_1+\gamma_2+m). \tag{3.4}$$

As a function of γ_1 and γ_2, the result has poles located at $\gamma_1+\gamma_2 \in \mathbb{Z}_{\leq 0}$, which shares the same pole structures of $\Gamma(\gamma_1+\gamma_2)$. If $\mathcal{I}_2$ appears as a sub-integral within a multi-variate Mellin transform of γ_1 or γ_2, say

$$\mathcal{J} = \int_{a'-i\infty}^{a'+i\infty} \frac{\mathrm{d}\gamma_1}{2\pi i} \frac{\Gamma(\gamma_1)}{y^{\gamma_1}} \int_{a-i\infty}^{a+i\infty} \frac{\mathrm{d}\gamma}{2\pi i} \frac{\Gamma(\gamma_1+\gamma)\Gamma(\gamma_2-\gamma)}{x^\gamma}, \tag{3.5}$$

then even if the integrand does not explicitly contain a factor of $\Gamma(\gamma_1+\gamma_2)$, effectively there exist "poles" located at $\gamma_1 = -\gamma_2 - m$ for $m = 0, 1, \ldots$. One can see this even before performing the integral: taking $\gamma_1 + \gamma_2 \to -m$ in the integrand of (3.3), the poles of $\Gamma(\gamma_1+\gamma)\Gamma(\gamma_2-\gamma)$ pinch the contour of γ, which give rise to poles in the result. We denote this pole-pinching mechanism by

$$\Gamma(\gamma_1+\gamma)\Gamma(\gamma_2-\gamma) \rightsquigarrow \Gamma(\gamma_1+\gamma_2). \tag{3.6}$$

Generalizing this observation enables us to recursively analyze the scaling dependence of multi-variate Mellin transforms by identifying all possible ways of pinching explicit poles to produce effective "poles". See appendix C of [35] for a systematic discussion with more pedagogical examples and efficient algorithms.

3.2 Two-particle OPE

For simplicity, we will mainly use the usual d-dimensional language instead of embedding formalism to discuss the OPE.

Let us first consider the OPE of two traceless mixed-symmetry tensors $\mathcal{A}_{\alpha_1\cdots\alpha_{J_A}}$ and $\mathcal{B}_{\beta_1\cdots\beta_{J_B}}$, where α, β denote[9] the Lorentz indices and J_A, J_B denote the ranks (namely, the sum of all spins). As both operators are bosonic, only bosonic operators (traceless mixed-symmetry tensors) appear in the OPE:

$$\mathcal{A}_{\alpha_1\cdots\alpha_{J_A}}(x)\mathcal{B}_{\beta_1\cdots\beta_{J_B}}(0) = \sum_{\mathcal{C}} F^{\gamma_1\cdots\gamma_{J_C}}_{\alpha_1\cdots\alpha_{J_A}\beta_1\cdots\beta_{J_B}}(x,\partial)\mathcal{C}_{\gamma_1\cdots\gamma_{J_C}}(0). \tag{3.7}$$

Here ∂ generates the descendant operators when acting on $\mathcal{C}$. We first focus on the primary part of the OPE coefficient $\widetilde{F}(x) := F(x, 0)$, which is a tensor that can only depend on the metric and x.

What is the possible structure of $\widetilde{F}$? Due to the traceless condition of $\mathcal{C}$, the metric $g^{\gamma\gamma}$ cannot appear. Moreover, in index-free notation, the indices $_{\alpha,\beta}$ are dotted into $(z_A)^\alpha, (z_B)^\beta$ respectively[10]. Since $z_A \cdot z_A = z_B \cdot z_B = 0$, the metrics $g_{\alpha\alpha}$, $g_{\beta\beta}$ in $\widetilde{F}$ does not affect the final result and can be safely dropped out. Therefore, $\widetilde{F}$ can only depend on

$$\{x^2, x_\alpha, x_\beta, x^\gamma, g_{\alpha\beta}, \delta_\alpha{}^\gamma, \delta_\beta{}^\gamma\},$$

from which we can write down

$$\widetilde{F} \propto (x^2)^{-k}(x_\alpha)^{m_1}(x_\beta)^{m_2}(x^\gamma)^{m_3}(g_{\alpha\beta})^{n_1}(\delta_\alpha{}^\gamma)^{n_2}(\delta_\beta{}^\gamma)^{n_3}, \tag{3.8}$$

with the powers to be determined later. Now, for simplicity, we restrict to the case where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are all symmetric operators (the generalization to the mixed-symmetry case is straightforward). When we perform the OPE in the correlator, in the Poincaré section

[9] For simplicity, we use α to denote a generic member of $\alpha_1, \cdots, \alpha_{J_A}$. Similarly, $g_{\alpha\alpha}$ represents arbitrary metric with both indices in α, say $g_{\alpha_1\alpha_2}, g_{\alpha_1\alpha_3}, \cdots$ collectively.

[10] Here $(z_A)^\alpha$ denotes arbitrary polarization vectors $(z_{A,(i)})^\alpha$ of the mixed-symmetry operator $\mathcal{A}$. Similarly for $(z_B)^\beta$.

(2.10) it reads $\langle \mathcal{A}(x, z_A)\mathcal{B}(0, z_B)\cdots\rangle$. Dotted with the polarizations, the tensor structures in $\widetilde{F}$ map to scalar products in the discrete Mellin transform:

$$x_\alpha \leftrightarrow z_A \cdot x \propto (Z_A \cdot X_B), \quad x_\beta \leftrightarrow z_B \cdot x \propto (Z_B \cdot X_A), \quad g_{\alpha\beta} \leftrightarrow z_A \cdot z_B \propto (Z_A \cdot Z_B). \tag{3.9}$$

Therefore we can identify

$$m_1 = \eta_{A'B}, \quad m_2 = \eta_{AB'}, \quad n_1 = \eta_{A'B'}. \tag{3.10}$$

Furthermore, counting spins and dimensions on both sides in the OPE, we have

$$m_1 + n_1 + n_2 = J_A, \quad m_2 + n_1 + n_3 = J_B, \quad m_3 + n_2 + n_3 = J_C, \tag{3.11a}$$

$$\Delta_A + \Delta_B = 2k - m_1 - m_2 - m_3 + \Delta_C. \tag{3.11b}$$

From these constraints we can determined all the powers in (3.8). In particular,

$$k = \frac{\tau_A + \tau_B - \tau_C}{2} + \eta_{A'B} + \eta_{AB'} + \eta_{A'B'}, \tag{3.12}$$

where $\tau := \Delta - J$ is the conformal twist of a bosonic local operator (for mixed-symmetry operators the twist is defined by the dimension minus the rank $\Delta - J_{(1)} - \cdots - J_{(h)}$). The scaling behavior of $x^2 \propto (-X_A \cdot X_B)$ corresponds to a pole at $\gamma_{AB} = k$ in the continuous Mellin transform. When $\tau_C - \tau_A - \tau_B \notin 2\mathbb{Z}_{\geq 0}$, this pole cannot come from Γ functions or pinching mechanism, and must present in the Mellin amplitude $\mathcal{M}$. In such case

$$\mathcal{M}(\gamma, \eta) \sim \frac{\text{res.}}{\gamma_{AB} - k} \propto \frac{\text{res.}}{\gamma_{LR} - \tau_C}, \tag{3.13}$$

where

$$\gamma_{LR} = -(\mathtt{k}_A + \mathtt{k}_{A'} + \mathtt{k}_B + \mathtt{k}_{B'})^2 = \tau_A + \tau_B - 2(\gamma_{AB} - \eta_{A'B} - \eta_{AB'} - \eta_{A'B'}) \tag{3.14}$$

is the Mandelstam in the OPE channel we are considering.

When considering the descendants of $\mathcal{C}$, we can expand F as a series of ∂. For the coefficients of each ∂, the counting in (3.10) and (3.11) still applies, but Δ_C and J_C should be replaced by the dimension and spin of the descendant operators. Schematically, these descendants are obtained by acting $(\partial^2)^m(\partial)^\ell$ on $\mathcal{C}$, which has dimension $\Delta_C + 2m + \ell$ and rank $J_C + \ell$. Therefore, they have twist $\tau_C + 2m$, and contribute to poles at $\gamma_{LR} = \tau_C + 2m$ according to our previous discussion. In summary, for the whole conformal multiplet of $\mathcal{C}$, there exist a sequence of poles in the Mellin amplitude:

$$\mathcal{M}(\gamma, \eta) \sim \frac{1}{\gamma_{LR} - \tau_C - 2m}, \quad m = 0, 1, 2, \cdots, \tag{3.15}$$

as we expect in (2.52).

In the perturbation around generalized free field theories (including the large central charge limit of holographic CFTs), the OPE of $\mathcal{A}$ and $\mathcal{B}$ contains a special sector of composite operators, the double-trace operators. They can be viewed as the "double-particle states" composed by $\mathcal{A}$ and $\mathcal{B}$, and have twist $\tau = \tau_A + \tau_B + 2m$ for $m \in \mathbb{Z}_{\geq 0}$. These operators are generally constructed by inserting several derivatives in the normal ordering of $\mathcal{A}$ and $\mathcal{B}$, which we denote schematically as $[\mathcal{A}\partial\cdots\partial\mathcal{B}]$. Since $\tau-\tau_A-\tau_B\in 2\mathbb{Z}_{\geq 0}$, the poles of double-trace operators might not present in the Mellin amplitudes, but come from the Γ functions. Actually, by definition, the three-point function

$$\langle\mathcal{A}(x)\mathcal{B}(0)[\mathcal{A}\partial\cdots\partial\mathcal{B}](y)\rangle\sim\partial\cdots\partial\langle\mathcal{A}(x)\mathcal{A}(y)\rangle\partial\cdots\partial\langle\mathcal{B}(0)\mathcal{B}(y)\rangle. \tag{3.16}$$

can be computed by taking derivatives on the product of two point functions $\langle\mathcal{A}(x)\mathcal{A}(y)\rangle$ and $\langle\mathcal{B}(0)\mathcal{B}(y)\rangle$. This three-point function is completely regular when we take the OPE limit $x\to 0$. As a result, one can show that the corresponding OPE coefficient only contains x^2 with non-negative integer powers, which are precisely captured by the explicit poles of $\Gamma(\gamma_{AB})$ in the Mellin transform.

3.3 Multi-particle OPE

The above discussion readily generalizes to multi-operator OPEs. For example, the three-particle OPE reads, schematically,

$$\mathcal{A}_\alpha(x_1)\mathcal{B}_\beta(x_2)\mathcal{C}_\gamma(x_3)=\sum_{\mathcal{D}}\frac{(g_{\alpha\beta})^{\eta_{1'2'}}(x_{12}^\alpha)^{\eta_{1'2}}\cdots+O(g_{\alpha\alpha},g_{\beta\beta},g_{\gamma\gamma})}{(x_{12}^2)^{\gamma_{12}}(x_{13}^2)^{\gamma_{13}}(x_{23}^2)^{\gamma_{23}}}\mathcal{D}_\delta(0). \tag{3.17}$$

Generically, such scaling behaviors can only arise from explicit poles in $\mathcal{M}(\gamma,\eta)$ of the form

$$\mathcal{M}(\gamma,\eta)\sim\frac{\text{res.}}{\gamma_{LR}-\tau_D-2m}, \tag{3.18}$$

where $\gamma_{LR}=-\left(\sum_{a\in\mathcal{L}}\mathtt{k}_a\right)^2$ is the Mandelstam in the corresponding OPE channel, with $\mathcal{L}=\{1,1',2,2',3,3'\}$ (and may contains indices like $1''$ for mixed-symmetry operators).

In perturbation theory, single-trace operators correspond to explicit poles of $\mathcal{M}(\gamma,\eta)$, while multi-trace operators correspond to the effective poles arising from pinching $\Gamma(\gamma_{pq})$ among themselves or with explicit poles in $\mathcal{M}(\gamma,\eta)$. For example, the poles of $[\mathcal{ABC}]$ arise from the pinching

$$\Gamma(\gamma_{12})\Gamma(\gamma_{13})\Gamma(\gamma_{23})\rightsquigarrow\Gamma(\gamma_{12}+\gamma_{13}+\gamma_{23}), \tag{3.19}$$

while the poles of $[\mathcal{EC}]$ (suppose $\mathcal{E}$ is a single-trace operator appearing in the OPE of $\mathcal{A}\times\mathcal{B}$) arises from pinching[11]

$$\Gamma\left(\frac{\tau_E-\gamma_{\{AB\}}}{2}\right)\Gamma(\gamma_{13})\Gamma(\gamma_{23})\rightsquigarrow\Gamma\left(\frac{\tau_E-\gamma_{\{AB\}}}{2}+\gamma_{13}+\gamma_{23}\right), \tag{3.20}$$

where $\gamma_{\{AB\}}:=\tau_A+\tau_B-2(\gamma_{12}-\eta_{1'2}-\eta_{12'}-\eta_{1'2'})$ is the Mandelstam in $\mathcal{A}\times\mathcal{B}$ OPE, and the sequence of poles $\Gamma\left(\frac{\tau_E}{2}-\gamma_{\{AB\}}\right)$ arise from the Mellin amplitude $\mathcal{M}$. The pinch leads to the regular behavior $(x_{13}^2)^{-\gamma_{13}}(x_{23}^2)^{-\gamma_{23}}$ with $\gamma_{13},\gamma_{23}\in\mathbb{Z}_{\leq 0}$ as expected, as well as the singular behavior $(x_{12}^2)^{-\gamma_{12}}$ producing $\mathcal{E}$ with

$$\gamma_{12}=\frac{\tau_A+\tau_B-\tau_E}{2}+\eta_{1'2}+\eta_{12'}+\eta_{1'2'}-m. \tag{3.21}$$

[11]We remind that $\Gamma(x)$ does not necessarily mean that there is a Γ function in the integrand, but simply stands for generic functions with a sequence of poles at $x=0,-1,-2,\cdots$

3.4 The Casimir equation

From the previous subsection, we learned that for each single-trace operator $\mathcal{O}_{\Delta,J}$ with dimension Δ and rank J, the OPE channel

$$\langle \mathcal{O}(X_1, Z_{1'}) \cdots \mathcal{O}(X_\ell, Z_{\ell'}) | \mathcal{O}_{\Delta,J} | \mathcal{O}(X_{\ell+1}, Z_{(\ell+1)'}) \cdots \mathcal{O}(X_N, Z_{N'}) \rangle$$

corresponds to a sequence of poles in $\mathcal{M}(\gamma, \eta)$ of the form

$$\mathcal{M} \sim \frac{\mathcal{Q}_m}{\gamma_{LR} - \tau - 2m}, \quad m = 0, 1, 2, \cdots, \quad \gamma_{LR} = \sum_a \sum_i \gamma_{ai}, \tag{3.22}$$

Recall that we assume the summation range of $a, b, \cdots$ and $i, j, \cdots$ are over the sets $\mathcal{L}$ and $\mathcal{R}$ respectively, where $\mathcal{L}$ and $\mathcal{R}$ consist of all external labels on the left and right, including the primed labels from the polarizations.

Following [3, 5] and our previous discussions, the residue $\mathcal{Q}_m$ can be determined by studying the Casimir equation

$$\hat{\mathbb{C}} \mathcal{M} = -c_{\Delta,J} \mathcal{M}, \tag{3.23}$$

where the action of $\hat{\mathbb{C}}$ can be found in (2.51), and $c_{\Delta,J}$ is the eigenvalue of $\mathcal{O}_{\Delta,J}$:

$$c_{\Delta,J} := -\Delta(d - \Delta) + J_{(1)}(d - 2 + J_{(1)}) + \cdots + J_{(h)}(d - 2h + J_{(h)}). \tag{3.24}$$

We mainly focus on the exchange of traceless symmetric operators, for which $c_{\Delta,J}$ reduces to $-\Delta(d - \Delta) + J(d - 2 + J)$. Evaluating the residue of $\mathcal{M}$ at $\gamma_{LR} = \Delta - J + 2m$ in the Casimir equation, we obtain a recurrence relation for $\mathcal{Q}_m$:

$$\begin{aligned} 0 &= \left(2J(\Delta - 1 + 2m) + 2m(d - 2\Delta - 2m)\right) \mathcal{Q}_m \\ &+ \sum_{\substack{a,b \\ a \neq b}} \sum_{\substack{i,j \\ i \neq j}} \Big(\gamma_{ai}\gamma_{bj} (\mathcal{Q}_m - [\mathcal{Q}_m]^{ai,bj}_{aj,bi}) + \gamma_{ab}\gamma_{ij} [\mathcal{Q}_{m-1}]^{ab,ij}_{aj,bi} \Big) . \end{aligned} \tag{3.25}$$

4 Factorization formulas

In this section, we systematically derive the factorization formula for spinning Mellin amplitudes by solving the Casimir recurrence relation (3.25). The techniques employed here largely originate from [5], where the authors derived factorization formulas case by case for spin-0,1,2 exchanges in scalar Mellin amplitudes. We are going to solve the same recurrence relation, but the interpretation changes: the indices of the Mellin variables now include polarizations, and the solution yields the factorization formula for spinning amplitudes. The final results are collected in 4.6.

We improve upon their technique by identifying a closed algebra of so-called covariant shift operators within the Casimir equation, which will be introduced shortly. These covariant shift operators relieve us from keeping track of all the component labels as in [5], and also make the complicated combinatorial relations in the recurrence relation more transparent. In the main part of this section, we assume the exchange operator is a symmetric traceless operator $\mathcal{O}(X, Z)$. The extension to mixed symmetry exchanges will be discussed at the end.

4.1 Support issues and covariant shifts

We first discuss the important issue of support when dealing with shifts in Mellin amplitudes. As mentioned above, the Mellin amplitude is defined on the support of momentum conservation (2.46). For example, the following two amplitudes are equivalent:

$$\mathcal{M} = 0, \qquad \mathcal{M}' = -\Delta_r + \sum_{\substack{s\in\mathcal{S} \\ r\neq s}} \gamma_{rs}. \tag{4.1}$$

However, when acting the shift operator $[\,\boldsymbol{\cdot}\,]^{rs}$ which shifts $\gamma_{rs} \mapsto \gamma_{rs} + 1$ on these two amplitudes, the results appear to be different:

$$[\mathcal{M}]^{rs} = 0, \qquad [\mathcal{M}']^{rs} = -\Delta_r + 1 + \sum_{\substack{s\in\mathcal{S} \\ r\neq s}} \gamma_{rs}. \tag{4.2}$$

This is because the momentum conservation (2.46) also got shifted by the shift operator, and the supports of Mellin amplitudes are changed. Since γ_{rs} appears in the momentum conservation constraints of Δ_r and Δ_s, the operator $[\,\boldsymbol{\cdot}\,]^{rs}$ effectively shifts the support to $\Delta_r \mapsto \Delta_r - 1$ and $\Delta_s \mapsto \Delta_s - 1$ (while $[\,\boldsymbol{\cdot}\,]_{rs}$ shifts the support to $\Delta_r \mapsto \Delta_r + 1$ and $\Delta_s \mapsto \Delta_s + 1$). Note that when r or s are primed indices, the spins also get shifted but in the opposite direction since $\Delta_{r'} \equiv -J_r$. This explains the support of the gauge invariance constraint (2.44), which is on $\Delta_p - 1$ and $J_p - 1$.

Since we will use the shift operators very frequently in the derivation, it is crucial to keep track of the support on which an expression is defined. Especially, we use $\mathscr{M}_{\mathcal{L}}^{\Delta,J}$ to denote the space of left half-amplitudes such that the exchanged operator $\mathcal{O}(X_L, Z_L)$ has dimension Δ and spin J. For instance, the half-amplitude itself $\mathcal{M}_{\mathcal{L}}$ lives in $\mathscr{M}_{\mathcal{L}}^{\Delta,J}$, while the gauge invariance condition (2.61) is an equality in $\mathscr{M}_{\mathcal{L}}^{\Delta-1,J-1}$. We also note that the single-spin amplitudes $\widetilde{\mathcal{M}}_{\mathcal{L}}$ defined in [5] are functions in $\mathscr{M}_{\mathcal{L}}^{\Delta-J,0}$.

In factorization formulas, there are usually shift operators acting on the half-amplitudes (e.g., see (2.27)). However, simple shift operators like $[\,\boldsymbol{\cdot}\,]^{ab}$ may change the supports of Mellin amplitudes and effectively affect the external dimensions and spins ($\Delta_a \mapsto \Delta_a - 1$, $\Delta_b \mapsto \Delta_b - 1$). In order to keep the external dimensions and spins intact in factorization formula, we introduce the following covariant shift operators:

$$\begin{aligned}
&[\,\boldsymbol{\cdot}\,]^{aL}_{aL'} &&: \mathscr{M}_{\mathcal{L}}^{\Delta,J} \to \mathscr{M}_{\mathcal{L}}^{\Delta-1,J-1}, \\
&[\,\boldsymbol{\cdot}\,]^{ab}_{aL,bL} &&: \mathscr{M}_{\mathcal{L}}^{\Delta,J} \to \mathscr{M}_{\mathcal{L}}^{\Delta+2,J}, \\
&[\,\boldsymbol{\cdot}\,]^{ab}_{aL,bL'} &&: \mathscr{M}_{\mathcal{L}}^{\Delta,J} \to \mathscr{M}_{\mathcal{L}}^{\Delta+1,J-1}, \\
&[\,\boldsymbol{\cdot}\,]^{ab}_{aL',bL'} &&: \mathscr{M}_{\mathcal{L}}^{\Delta,J} \to \mathscr{M}_{\mathcal{L}}^{\Delta,J-2}.
\end{aligned} \tag{4.3}$$

These operators (and their inverses) always maintain the supports of external operators, at the cost of shifting the effective dimension and spin of the exchanged operator. This is acceptable and welcome in the factorization formula, where only the twist τ of the exchanged operator appears in the pole structures, and gets shifted to $\tau+2m$ for descendant poles. The covariant shift operators makes it much easier to keep track of the support issues.

As an example, the operator $\hat{\mathbb{x}}$ defined in (2.29) should be understood as

$$\hat{\mathbb{x}} := \sum_{\substack{a,b \\ a\neq b}} \gamma_{ab} [\,\boldsymbol{\cdot}\,]^{ab}_{aL,bL}. \tag{4.4}$$

Acting $\hat{\mathbb{x}}^m$ on $\mathcal{M}_\mathcal{L} \in \mathscr{M}^{\Delta,0}_\mathcal{L}$ indeed brings the result into the function space $\mathscr{M}^{\tau+2m,0}_\mathcal{L}$, compatible with the pole structure.

4.2 Function space in factorization formula

We expect the factorization formula to take the schematic form:

$$\mathcal{Q}_m(\gamma_{ab}, \gamma_{ij}, \gamma_{ai}) \sim (\text{shift operators}) \circ \Big(\mathcal{M}_\mathcal{L}(\gamma_{ab}, \gamma_{aL}, \eta_{aL'})\mathcal{M}_\mathcal{R}(\gamma_{ij}, \gamma_{iR}, \eta_{iR'})\Big), \tag{4.5}$$

We see that it relates the residue $\mathcal{Q}_m(\gamma_{ab}, \gamma_{ij}, \gamma_{ai})$ with the half-amplitudes $\mathcal{M}_\mathcal{L}(\gamma_{ab}, \gamma_{aL}, \eta_{aL'})$ and $\mathcal{M}_\mathcal{R}(\gamma_{ij}, \gamma_{iR}, \eta_{iR'})$, which depend on different sets of Mellin variables at first glance. In order to formulate the factorization formula properly, we must be able to sensibly compare functions depending on $\gamma_{aL}, \eta_{aL'}, \gamma_{iR}, \eta_{iR'}$ with functions depending on γ_{ai}. Motivated by the interpretation of Mellin variables as Lorentz products of auxiliary momenta, we impose the following relations between the variables γ_{aL}, γ_{iR} and γ_{ai}, γ_{LR}:

$$\forall a: \ \ \gamma_{aL} = \sum_i \gamma_{ai}, \tag{4.6a}$$

$$\forall i: \ \ \gamma_{iR} = \sum_a \gamma_{ai}, \tag{4.6b}$$

$$\gamma_{LR} = \sum_a \gamma_{aL} = \sum_i \gamma_{iR} = \sum_{a,i} \gamma_{ai}. \tag{4.6c}$$

We can now present the product $\mathcal{M}_\mathcal{L}\mathcal{M}_\mathcal{R}$ as a function that explicitly involves all the Mellin variables $(\gamma_{ab}, \gamma_{aL}, \eta_{aL'}, \gamma_{ij}, \gamma_{iR}, \eta_{iR'}, \gamma_{ai}, \gamma_{LR})$ through substitution of the above relations. However, $\mathcal{M}_\mathcal{L}\mathcal{M}_\mathcal{R}$ is not a generic function of these variables satisfying these relations. In particular, the value of γ_{LR} is not generic but fixed by the support of $\mathcal{M}_{\mathcal{L}/\mathcal{R}} \in \mathscr{M}^{\Delta,J}_{\mathcal{L}/\mathcal{R}}$ to be $\gamma_{LR} = \Delta$:

$$\forall a: \quad \gamma_{aL} - \eta_{aL'} + \sum_{\substack{b \\ b\neq a}} \gamma_{ab} = \Delta_a, \tag{4.7a}$$

$$\sum_a \gamma_{aL} = \Delta, \quad \sum_a \eta_{aL'} = J, \tag{4.7b}$$

$$\forall i: \quad \gamma_{iR} - \eta_{iR'} + \sum_{\substack{j \\ j\neq i}} \gamma_{ij} = \Delta_i, \tag{4.7c}$$

$$\sum_i \gamma_{iR} = \Delta, \quad \sum_i \eta_{iR'} = J. \tag{4.7d}$$

This suggests we consider the function space $\mathscr{Q}^{\Delta,J}$ of functions

$$\mathcal{Q}(\gamma_{ab}, \gamma_{aL}, \eta_{aL'}, \gamma_{ij}, \gamma_{iR}, \eta_{iR'}, \gamma_{ai}, \gamma_{LR})$$

on the support of (4.6) and (4.7). For example, the product $\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}} \in \mathscr{Q}^{\Delta,J}$. Also, we expect that the shift operators will map the product to a function $\widetilde{\mathcal{Q}}_m \in \mathscr{Q}^{\tau+2m,\text{something}}$, because $\mathcal{Q}_m$ is the residue at $\gamma_{LR} = \tau + 2m$. We will now show that the "something" has to be zero and relate $\widetilde{\mathcal{Q}}_m \in \mathscr{Q}^{\tau+2m,0}$ to the residue $\mathcal{Q}_m$.

First, note that substituting (4.6a) into (4.7a) implies that $\widetilde{\mathcal{Q}}_m$ satisfies

$$\forall a: \quad -\Delta_a + \sum_{\substack{b \\ b\neq a}} \gamma_{ab} + \sum_i \gamma_{ai} = \eta_{aL'},$$

which is not quite compatible with the momentum conservation

$$\forall a: \quad \mathtt{k}_a \cdot \left(\sum_a \mathtt{k}_a + \sum_i \mathtt{k}_i\right) = 0$$

we expect for $\mathcal{Q}_m$ in the full amplitude. On the other hand, $\widetilde{\mathcal{Q}}_m$ still depends on the variables $\eta_{aL'}, \eta_{iR'}$ since $\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}$ depends on them. This is not what expect for the full amplitude, either, because $\mathcal{Q}_m$ should depend on $\gamma_{ab}, \gamma_{ij}, \gamma_{ai}$ only. We can resolve both paradoxes by simply evaluating $\widetilde{\mathcal{Q}}_m$ on the lattice point where $\eta_{aL'} = 0$ and $\eta_{iR'} = 0$, which implies that $\widetilde{\mathcal{Q}}_m \in \mathscr{Q}^{\tau+2m,0}$. In summary,

$$\mathcal{Q}_m = \underbrace{(\text{shift operators}) \circ \Big(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}\Big)}_{\widetilde{\mathcal{Q}}_m \in \mathscr{Q}^{\tau+2m,0}} \Big|_{\substack{\eta_{aL'}=0 \\ \eta_{iR'}=0}}. \tag{4.8}$$

Similar to the covariant shift operators for the function space $\mathscr{M}^{\Delta,J}_{\mathcal{L}/\mathcal{R}}$, we introduce the following covariant shift operators for $\mathscr{Q}^{\Delta,J}$:

$$\begin{aligned}
\langle \mathcal{Q} \rangle^{ai} &:= [\mathcal{Q}]^{aL\,,\,iR,\,ai,LR}_{aL',\,iR'}, & \langle\,\boldsymbol{\cdot}\,\rangle^{ai} &: \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta-1,J-1}, \\
\langle \mathcal{Q} \rangle_{ai} &:= [\mathcal{Q}]^{aL',\,iR'}_{aL\,,\,iR,\,ai,LR}, & \langle\,\boldsymbol{\cdot}\,\rangle_{ai} &: \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta+1,J+1}, \\
\{\mathcal{Q}\}^{ab,ij} &:= [\mathcal{Q}]^{ab,\;ij}_{aL',bL',iR',jR'}, & \{\,\boldsymbol{\cdot}\,\}^{ab,ij} &: \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta,J-2}, \\
\{\mathcal{Q}\}_{ab,ij} &:= [\mathcal{Q}]^{aL',bL',iR',jR'}_{ab,\;ij}, & \{\,\boldsymbol{\cdot}\,\}_{ab,ij} &: \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta,J+2}.
\end{aligned} \tag{4.9}$$

Let us explain our notation, taking the definition of $\langle \mathcal{Q} \rangle^{ai}$ as an example. The superscripts inform us to increment all appearances of $\gamma_{aL}, \gamma_{iR}, \gamma_{ai}, \gamma_{LR}$ in a given presentation of $\mathcal{Q}$, so that the relations (4.6) are preserved. The subscripts increment $\eta_{aL'}$ and $\eta_{iR'}$, restoring the on-shell condition for the external particle a in (4.7a). The net result of these shifts is that the support (4.7b) is effectively shifted as $\Delta \mapsto \Delta - 1$ and $J \mapsto J - 1$. We can now write the Casimir recurrence relation in terms of covariant shifts:

$$\left(2J(\Delta - 1 + 2m) + 2m(d - 2\Delta - 2m)\right)\mathcal{Q}_m + \hat{\mathbb{F}}\mathcal{Q}_m + \hat{\mathbb{X}}\mathcal{Q}_{m-1} = 0, \tag{4.10}$$

where

$$\begin{aligned}
\hat{\mathbb{F}}\mathcal{Q} &:= \sum_{\substack{a,b \\ a\neq b}} \sum_{\substack{i,j \\ i\neq j}} \gamma_{ai}\gamma_{bj}\left(\mathcal{Q} - \langle \mathcal{Q} \rangle^{ai,bj}_{aj,bi}\right), & \hat{\mathbb{F}}&: \; \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta,J}, \\
\hat{\mathbb{X}}\mathcal{Q} &:= \sum_{\substack{a,b \\ a\neq b}} \sum_{\substack{i,j \\ i\neq j}} \gamma_{ab}\gamma_{ij}\left\{\langle \mathcal{Q} \rangle_{aj,bi}\right\}^{ab,ij}, & \hat{\mathbb{X}}&: \; \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta+2,J}.
\end{aligned} \tag{4.11}$$

Finally, let us discuss the important factorizable subspace $\mathscr{M}_{\mathcal{L}}^{\Delta,J} \otimes \mathscr{M}_{\mathcal{R}}^{\Delta,J} \subset \mathscr{Q}^{\Delta,J}$ of functions such as the product $\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}$, which only depend on γ_{ai} through the particular combinations γ_{aL}, γ_{iR}. For a factorizable function, it is always possible to write it in a factorized form where the variables γ_{ai}, γ_{LR} do not appear. As a result, when restricted to the factorizable subspace, the covariant shifts of $\mathscr{Q}^{\Delta,J}$ factorizes:

$$\langle \mathcal{F}_{\mathcal{L}} \otimes \mathcal{F}_{\mathcal{R}} \rangle^{ai} = [\mathcal{F}_{\mathcal{L}}]^{aL}_{aL'} \otimes [\mathcal{F}_{\mathcal{R}}]^{iR}_{iR'}, \quad \{\mathcal{F}_{\mathcal{L}} \otimes \mathcal{F}_{\mathcal{R}}\}^{ab,ij} = [\mathcal{F}_{\mathcal{L}}]^{ab}_{aL',bL'} \otimes [\mathcal{F}_{\mathcal{R}}]^{ij}_{iR',jR'}, \tag{4.12}$$

and similarly for the operators with subscripts.

4.3 Lessons from scalar amplitude factorization

We briefly review the results and conjectures obtained in [5], by simply translating equations in that paper into our definition of spinning Mellin amplitudes using (2.68) and (2.69). This will give us a taste of the general structure of solutions as well as some hints on how to proceed systematically.

- **Primary factorization of spin-J exchanges**

  In appendix B of [5], the authors conjectured a formula for the primary residue $\mathcal{Q}_0$ that applies to any spin-J exchange:

$$\mathcal{Q}_0 = \kappa_{\Delta J} J! \sum_{\substack{n_{ai}\geq 0 \\ \Sigma_i n_{ai}=n_a \\ \Sigma_a n_{ai}=n_i}} \left(\prod_{a,i} \frac{(\gamma_{ai})_{n_{ai}}}{n_{ai}!} \right) \widetilde{\mathcal{M}}_{\mathcal{L}}^{(n_1\cdots n_\ell)} \widetilde{\mathcal{M}}_{\mathcal{R}}^{(n_{\ell+1}\cdots n_N)}, \tag{4.13}$$

  where

$$\kappa_{\Delta J} = (-2)^{1-J}(\Delta + J - 1)\Gamma(\Delta - 1). \tag{4.14}$$

  These n_{ck} actually count the number of occurrence of pair (c,k) in the sequence $\{(c_1,k_1),\cdots,(c_J,k_J)\}$ appearing in the component amplitudes $\widetilde{\mathcal{M}}_{\mathcal{L}}^{c_1\cdots c_J}\widetilde{\mathcal{M}}_{\mathcal{R}}^{k_1\cdots k_J}$. Using (2.68), it is not hard to show that

$$\mathcal{Q}_0 = \frac{2^J \kappa_{\Delta J}}{(J!)^2}\, \hat{\mathbb{N}}^J \circ (\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}})\Big|_{\substack{\eta_{aL'}=0 \\ \eta_{iR'}=0}}, \tag{4.15}$$

  where

$$\hat{\mathbb{N}}\mathcal{Q} := \sum_{a,i} \gamma_{ai} \langle \mathcal{Q} \rangle^{ai}, \quad \hat{\mathbb{N}}:\ \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta-1,J-1}. \tag{4.16}$$

- **Descendant factorization of scalar exchanges**

  In [5], the authors showed that for scalar exchanges,

$$\mathcal{Q}_m = \kappa_{\Delta 0} \frac{m!}{(1-\frac{d}{2}+\Delta)_m} \widetilde{\mathcal{M}}_{\mathcal{L},m}\widetilde{\mathcal{M}}_{\mathcal{R},m}, \tag{4.17}$$

  where (in the non-covariant notation of that paper)

$$\widetilde{\mathcal{M}}_{\mathcal{L},m} = \frac{1}{2m}\sum_{\substack{a,b \\ a\neq b}} \gamma_{ab}[\widetilde{\mathcal{M}}_{\mathcal{L},m-1}]^{ab}, \quad \widetilde{\mathcal{M}}_{\mathcal{R},m} = \frac{1}{2m}\sum_{\substack{i,j \\ i\neq j}} \gamma_{ij}[\widetilde{\mathcal{M}}_{\mathcal{R},m-1}]^{ij}. \tag{4.18}$$

It is not hard to show that, in terms of covariant shifts, this is equivalent to

$$\mathcal{Q}_m = \frac{\kappa_{\Delta 0}}{4^m m!(1-\frac{d}{2}+\Delta)_m}\, \hat{\mathbb{X}}^m \circ (\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}})\Big|_{\substack{\eta_{aL'}=0 \\ \eta_{iR'}=0}} . \tag{4.19}$$

4.4 A nilpotent algebra of operators

From subsection 4.3, we see that $\hat{\mathbb{N}}$ and $\hat{\mathbb{X}}$ are important operators when constructing solutions of the Casimir recurrence relation (4.10). It is therefore natural to study the commutators of $\hat{\mathbb{N}}, \hat{\mathbb{X}}, \hat{\mathbb{F}}$. It turns out that we only need three groups of operators to obtain a systematic solution of (4.10):

$$\hat{\mathbb{N}}\mathcal{Q} := \sum_{a,i} \gamma_{ai} \langle \mathcal{Q} \rangle^{ai}, \qquad \hat{\mathbb{N}} : \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta-1,J-1}, \tag{4.20a}$$

$$\hat{\mathbb{M}}\mathcal{Q} := \sum_{a,i} \gamma_{aL}\gamma_{iR} \langle \mathcal{Q} \rangle^{ai}, \qquad \hat{\mathbb{M}} : \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta-1,J-1}, \tag{4.20b}$$

$$\hat{\mathbb{X}}\mathcal{Q} := \sum_{\substack{a,b \\ a\neq b}} \sum_{\substack{i,j \\ i\neq j}} \gamma_{ab}\gamma_{ij} \{\langle \mathcal{Q} \rangle_{aj,bi}\}^{ab,ij}, \qquad \hat{\mathbb{X}} : \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta+2,J}, \tag{4.20c}$$

$$\hat{\mathbb{Y}}\mathcal{Q} := \sum_{\substack{a,b \\ a\neq b}} \sum_{\substack{i,j \\ i\neq j}} \gamma_{ab}\gamma_{ij} \{\langle \mathcal{Q} \rangle_{bi}\}^{ab,ij}, \qquad \hat{\mathbb{Y}} : \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta+1,J-1}, \tag{4.20d}$$

$$\hat{\mathbb{Z}}\mathcal{Q} := \sum_{\substack{a,b \\ a\neq b}} \sum_{\substack{i,j \\ i\neq j}} \gamma_{ab}\gamma_{ij} \{\mathcal{Q}\}^{ab,ij}, \qquad \hat{\mathbb{Z}} : \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta,J-2}, \tag{4.20e}$$

$$\hat{\mathbb{F}}\mathcal{Q} := \sum_{\substack{a,b \\ a\neq b}} \sum_{\substack{i,j \\ i\neq j}} \gamma_{ai}\gamma_{bj} (\mathcal{Q} - \langle \mathcal{Q} \rangle^{ai,bj}_{aj,bi}), \qquad \hat{\mathbb{F}} : \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta,J}, \tag{4.20f}$$

$$\hat{\mathbb{G}}\mathcal{Q} := \sum_{\substack{a,b \\ a\neq b}} \sum_{\substack{i,j \\ i\neq j}} \gamma_{ai}\gamma_{bj} (\langle \mathcal{Q} \rangle^{ai} - \langle \mathcal{Q} \rangle^{ai,bj}_{bi}), \qquad \hat{\mathbb{G}} : \mathscr{Q}^{\Delta,J} \to \mathscr{Q}^{\Delta-1,J-1}. \tag{4.20g}$$

It is straightforward to compute their commutation relations. In particular, $\{\hat{\mathbb{N}}, \hat{\mathbb{X}}, \hat{\mathbb{Y}}, \hat{\mathbb{Z}}\}$ close into a peculiar step-3 nilpotent algebra:

$$[\hat{\mathbb{N}}, \hat{\mathbb{X}}] = 2\hat{\mathbb{Y}}, \quad [\hat{\mathbb{N}}, \hat{\mathbb{Y}}] = \hat{\mathbb{Z}}, \quad [\hat{\mathbb{N}}, \hat{\mathbb{Z}}] = 0, \quad \hat{\mathbb{X}}, \hat{\mathbb{Y}}, \hat{\mathbb{Z}} \text{ commute.} \tag{4.21}$$

The commutation relations involving $\hat{\mathbb{M}}$ are not so simple when acting on generic functions in $\mathscr{Q}^{\Delta,J}$. However, its action on factorizable functions is much simpler. In fact,

$\{\hat{\mathbb{M}}, \hat{\mathbb{X}}, \hat{\mathbb{Y}}, \hat{\mathbb{Z}}\}$ all preserve factorizability due to (4.12). Specifically,

$$\hat{\mathbb{m}}\mathcal{M}_{\mathcal{L}} := \sum_{a} \gamma_{aL}[\mathcal{M}_{\mathcal{L}}]^{aL}_{aL'}, \qquad \hat{\mathbb{m}}: \mathscr{M}^{\Delta,J}_{\mathcal{L}} \to \mathscr{M}^{\Delta-1,J-1}_{\mathcal{L}}, \tag{4.22a}$$

$$\hat{\mathbb{x}}\mathcal{M}_{\mathcal{L}} := \sum_{\substack{a,b \\ a\neq b}} \gamma_{ab}[\mathcal{M}_{\mathcal{L}}]^{ab}_{aL,bL}, \qquad \hat{\mathbb{x}}: \mathscr{M}^{\Delta,J}_{\mathcal{L}} \to \mathscr{M}^{\Delta+2,J}_{\mathcal{L}}, \tag{4.22b}$$

$$\hat{\mathbb{y}}\mathcal{M}_{\mathcal{L}} := \sum_{\substack{a,b \\ a\neq b}} \gamma_{ab}[\mathcal{M}_{\mathcal{L}}]^{ab}_{aL',bL}, \qquad \hat{\mathbb{y}}: \mathscr{M}^{\Delta,J}_{\mathcal{L}} \to \mathscr{M}^{\Delta+1,J-1}_{\mathcal{L}}, \tag{4.22c}$$

$$\hat{\mathbb{z}}\mathcal{M}_{\mathcal{L}} := \sum_{\substack{a,b \\ a\neq b}} \gamma_{ab}[\mathcal{M}_{\mathcal{L}}]^{ab}_{aL',bL'}, \qquad \hat{\mathbb{z}}: \mathscr{M}^{\Delta,J}_{\mathcal{L}} \to \mathscr{M}^{\Delta,J-2}_{\mathcal{L}}. \tag{4.22d}$$

Amazingly, these half-operators close into exactly the same step-3 nilpotent algebra:

$$[\hat{\mathbb{m}}, \hat{\mathbb{x}}] = 2\hat{\mathbb{y}}, \quad [\hat{\mathbb{m}}, \hat{\mathbb{y}}] = \hat{\mathbb{z}}, \quad [\hat{\mathbb{m}}, \hat{\mathbb{z}}] = 0, \quad \hat{\mathbb{x}}, \hat{\mathbb{y}}, \hat{\mathbb{z}} \text{ commute.} \tag{4.23}$$

This can also be understood from their action in position space. If $\mathcal{M}_{\mathcal{L}} \leftrightarrow G_{\mathcal{L}}$ are related through the Mellin transform (2.42), we have

$$\hat{\mathbb{m}}\mathcal{M}_{\mathcal{L}} \quad \leftrightarrow \quad -(X_L \cdot \partial_{Z_L})G_{\mathcal{L}}, \tag{4.24a}$$

$$\hat{\mathbb{x}}\mathcal{M}_{\mathcal{L}} \quad \leftrightarrow \quad -2(\partial_{X_L} \cdot \partial_{X_L})G_{\mathcal{L}}, \tag{4.24b}$$

$$\hat{\mathbb{y}}\mathcal{M}_{\mathcal{L}} \quad \leftrightarrow \quad -2(\partial_{X_L} \cdot \partial_{Z_L})G_{\mathcal{L}}, \tag{4.24c}$$

$$\hat{\mathbb{z}}\mathcal{M}_{\mathcal{L}} \quad \leftrightarrow \quad -2(\partial_{Z_L} \cdot \partial_{Z_L})G_{\mathcal{L}}. \tag{4.24d}$$

The nilpotent algebra is nothing but the algebra between these differential operators. In particular, $\hat{\mathbb{m}}$ is precisely the operator needed in the gauge invariance condition (2.61), which reads $\hat{\mathbb{m}}\mathcal{M}_{\mathcal{L}} = 0$. This is crucial when we compute the action of $\hat{\mathbb{M}}, \hat{\mathbb{X}}, \hat{\mathbb{Y}}, \hat{\mathbb{Z}}$, because $\hat{\mathbb{m}}$ can always be commuted to the right to annihilate $\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}$. For example,

$$\begin{aligned}
\hat{\mathbb{M}}\hat{\mathbb{X}}^m(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}) &= (\hat{\mathbb{m}}\hat{\mathbb{x}}^m\mathcal{M}_{\mathcal{L}})(\hat{\mathbb{m}}\hat{\mathbb{x}}^m\mathcal{M}_{\mathcal{R}}) = (2m\hat{\mathbb{y}}\hat{\mathbb{x}}^{m-1}\mathcal{M}_{\mathcal{L}})(2m\hat{\mathbb{y}}\hat{\mathbb{x}}^{m-1}\mathcal{M}_{\mathcal{R}}) \\
&= 4m^2\hat{\mathbb{Y}}\hat{\mathbb{X}}^{m-1}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}), \\
\hat{\mathbb{M}}\hat{\mathbb{X}}\hat{\mathbb{Y}}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}) &= (\hat{\mathbb{m}}\hat{\mathbb{x}}\hat{\mathbb{y}}\mathcal{M}_{\mathcal{L}})(\hat{\mathbb{m}}\hat{\mathbb{x}}\hat{\mathbb{y}}\mathcal{M}_{\mathcal{R}}) = ((\hat{\mathbb{x}}\hat{\mathbb{z}} + 2\hat{\mathbb{y}}^2)\mathcal{M}_{\mathcal{L}})((\hat{\mathbb{x}}\hat{\mathbb{z}} + 2\hat{\mathbb{y}}^2)\mathcal{M}_{\mathcal{R}}) \\
&= \hat{\mathbb{X}}\hat{\mathbb{Z}}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}) + 4\hat{\mathbb{Y}}^2(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}) + 2(\hat{\mathbb{x}}\hat{\mathbb{z}} \otimes \hat{\mathbb{y}}^2 + \hat{\mathbb{y}}^2 \otimes \hat{\mathbb{x}}\hat{\mathbb{z}})(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}).
\end{aligned}$$

Note that cross terms such as $(\hat{\mathbb{x}}\hat{\mathbb{z}} \otimes \hat{\mathbb{y}}^2 + \hat{\mathbb{y}}^2 \otimes \hat{\mathbb{x}}\hat{\mathbb{z}})$ are inevitable, which suggests that the half-operators $\hat{\mathbb{x}}, \hat{\mathbb{y}}, \hat{\mathbb{z}}$ are more fundamental objects.

For the operators $\hat{\mathbb{F}}, \hat{\mathbb{G}}$, it turns out that all we need is their commutator with $\hat{\mathbb{N}}$:

$$[\hat{\mathbb{N}}, \hat{\mathbb{F}}] = 2\hat{\mathbb{G}}, \quad [\hat{\mathbb{N}}, \hat{\mathbb{G}}] = 0. \tag{4.25}$$

The action of $\hat{\mathbb{F}}$ on factorizable functions is even simpler:

$$\hat{\mathbb{F}}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}) = 0, \tag{4.26}$$

and the action of $\hat{\mathbb{G}}$ on factorizable functions reads

$$\hat{\mathbb{G}}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}) = \left(\gamma_{LR}\hat{\mathbb{N}} - \hat{\mathbb{M}}\right)(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}). \tag{4.27}$$

4.5 Systematic algorithm for the factorization formula

With the operators defined above, we are ready to construct solutions $\mathcal{Q}_m$ to the Casimir recurrence (4.10). The general strategy is to act on $\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}$ with various operators. As we have seen, operators have simpler actions on factorizable functions, so we would like to preserve factorizability as much as possible. In practice, this means we should construct solutions by acting with $\hat{\mathbb{M}}, \hat{\mathbb{X}}, \hat{\mathbb{Y}}, \hat{\mathbb{Z}}$ first and $\hat{\mathbb{N}}$ last. On general grounds, for a spin-J exchange, we expect the residue $\mathcal{Q}_m$ to be a degree-J polynomials with respect to the γ_{ai} variables. Since each $\hat{\mathbb{N}}$ increases the degree of γ_{ai} by one, this suggests the following ansatz:

$$\mathcal{Q}_m = \sum_{s=0}^{J} \hat{\mathbb{N}}^s \hat{\mathbb{P}}_{s,m}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}})\Big|_{\substack{\eta_{aL'}=0\\ \eta_{iR'}=0}}. \tag{4.28}$$

where $\hat{\mathbb{P}}_{s,m}(\hat{\mathbb{x}}, \hat{\mathbb{y}}, \hat{\mathbb{z}}) : \mathscr{M}_{\mathcal{L}}^{\Delta,J} \otimes \mathscr{M}_{\mathcal{R}}^{\Delta,J} \to \mathscr{M}_{\mathcal{L}}^{\Delta-J+s+2m,s} \otimes \mathscr{M}_{\mathcal{R}}^{\Delta-J+s+2m,s}$ is some polynomial in the half-operators. We do not need $\hat{\mathbb{m}}$, because it can always be commuted to the end and annihilate $\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}$.

Since $\hat{\mathbb{F}}$ vanishes and $\hat{\mathbb{G}} = \gamma_{LR}\hat{\mathbb{N}} - \hat{\mathbb{M}}$ on factorizable functions,

$$\begin{aligned}\hat{\mathbb{F}}\hat{\mathbb{N}}^s\hat{\mathbb{P}}_{s,m}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}) &= 2s\hat{\mathbb{N}}^{s-1}(\hat{\mathbb{M}} - \gamma_{LR}\hat{\mathbb{N}})\hat{\mathbb{P}}_{s,m}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}})\\ &= 2s\hat{\mathbb{N}}^{s-1}\hat{\mathbb{M}}\hat{\mathbb{P}}_{s,m}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}) - 2s(\Delta - J + s + 2m - 1)\hat{\mathbb{N}}^s\hat{\mathbb{P}}_{s,m}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}).\end{aligned} \tag{4.29}$$

On the other hand,

$$\hat{\mathbb{X}}\hat{\mathbb{N}}^s = \hat{\mathbb{N}}^s\hat{\mathbb{X}} - 2s\hat{\mathbb{N}}^{s-1}\hat{\mathbb{Y}} + s(s-1)\hat{\mathbb{N}}^{s-2}\hat{\mathbb{Z}}. \tag{4.30}$$

Since only $\hat{\mathbb{N}}$ breaks factorizability and depends on γ_{ai} individually, (4.10) should hold for each degree of γ_{ai}. In other words, the coefficient of each $\hat{\mathbb{N}}^s$ should vanish individually. Putting everything together, we obtain

$$\begin{aligned}0 &= \left(2(J-s)(\Delta - 1 + 2m + s) - 2m(2\Delta - d + 2m)\right)\hat{\mathbb{P}}_{s,m}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}})\\ &\quad + 2(s+1)\hat{\mathbb{M}}\hat{\mathbb{P}}_{s+1,m}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}) + \hat{\mathbb{X}}\hat{\mathbb{P}}_{s,m-1}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}})\\ &\quad - 2(s+1)\hat{\mathbb{Y}}\hat{\mathbb{P}}_{s+1,m-1}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}) + (s+2)(s+1)\hat{\mathbb{Z}}\hat{\mathbb{P}}_{s+2,m-1}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}}).\end{aligned} \tag{4.31}$$

This is a double recurrence relation that determines $\hat{\mathbb{P}}_{s,m}$ in terms of $\hat{\mathbb{P}}_{\geq s,m}$ and $\hat{\mathbb{P}}_{s,\leq m}$.

The double recurrence starts from $\hat{\mathbb{P}}_{J,0} : \mathscr{M}_{\mathcal{L}}^{\Delta,J} \otimes \mathscr{M}_{\mathcal{R}}^{\Delta,J} \to \mathscr{M}_{\mathcal{L}}^{\Delta,J} \otimes \mathscr{M}_{\mathcal{R}}^{\Delta,J}$. Since $\hat{\mathbb{P}}_{J,0}$ is a polynomial in $\hat{\mathbb{x}}, \hat{\mathbb{y}}, \hat{\mathbb{z}}$, the only possibility is to take it to be a constant:

$$\hat{\mathbb{P}}_{J,0} = \mathcal{K}_{\Delta J}. \tag{4.32}$$

Next, fixing $s = J$, we solve for $\hat{\mathbb{P}}_{J,m} : \mathscr{M}_{\mathcal{L}}^{\Delta,J} \otimes \mathscr{M}_{\mathcal{R}}^{\Delta,J} \to \mathscr{M}_{\mathcal{L}}^{\Delta+2m,J} \otimes \mathscr{M}_{\mathcal{R}}^{\Delta+2m,J}$. Looking at how $\hat{\mathbb{x}}, \hat{\mathbb{y}}, \hat{\mathbb{z}}$ change the support, we see that the only possibility is

$$\hat{\mathbb{P}}_{J,m} = \mathcal{N}_{J,m}\hat{\mathbb{X}}^m. \tag{4.33}$$

Plugging it into the recurrence relation, we obtain[12]

$$\left(-2m(2\Delta - d + 2m)\right)\mathcal{N}_{J,m} + \mathcal{N}_{J,m-1} = 0 \implies \mathcal{N}_{J,m} = \frac{\mathcal{K}_{\Delta J}}{4^m m!(1 - \frac{d}{2} + \Delta)_m}. \tag{4.34}$$

[12]These types of recurrence relations can be easily solved using `RSolve[]` in Mathematica. In practice, a better method is to plug in many specific values of m and use `FindSequenceFunction[]` to find $\mathcal{N}_{J,m}$.

Although we have only solved the $s = J$ part in the ansatz (4.28), this is actually the complete factorization formula for scalar ($J = 0$) exchanges! And indeed, choosing the normalization $\mathcal{K}_{\Delta 0} = \kappa_{\Delta 0}$ precisely recovers (4.19).

Continuing to $s = J - 1$. The only polynomial for $\hat{\mathbb{P}}_{J-1,m} : \mathscr{M}_{\mathcal{L}}^{\Delta,J} \otimes \mathscr{M}_{\mathcal{R}}^{\Delta,J} \to \mathscr{M}_{\mathcal{L}}^{\Delta-1+2m,J-1} \otimes \mathscr{M}_{\mathcal{R}}^{\Delta-1+2m,J-1}$ is

$$\hat{\mathbb{P}}_{J-1,m} = \mathcal{N}_{J-1,m} \hat{\mathbb{X}}^{m-1} \hat{\mathbb{Y}}. \tag{4.35}$$

Plugging into the recurrence relation, we have

$$\begin{aligned} &(2(\Delta + J - 2 + 2m) - 2m(2\Delta - d + 2m)) \mathcal{N}_{J-1,m} + \mathcal{N}_{J-1,m-1} \\ = &-2J(4m^2 \mathcal{N}_{J,m} - \mathcal{N}_{J,m-1}). \end{aligned} \tag{4.36}$$

Imposing the boundary condition $\mathcal{N}_{J-1,-1} = 0$, we obtain

$$\mathcal{N}_{J-1,m} = \frac{\mathcal{K}_{\Delta J}}{4^m m! (1 - \frac{d}{2} + \Delta)_m} \frac{2Jm(d - 2\Delta)}{\Delta - J - d + 2}. \tag{4.37}$$

An important observation is that although we only demanded $\mathcal{N}_{J-1,-1} = 0$, it turns out that $\mathcal{N}_{J-1,0} = 0$ too. This is a non-trivial consistency check, because $\mathcal{N}_{J-1,0}$ would be the coefficient of $\hat{\mathbb{X}}^{-1}\hat{\mathbb{Y}}$ which contains negative powers. Also, we remark that with the $s = J$ and $s = J - 1$ results at hand, we are able to recover the complete factorization formula for spin $J = 1$ exchanges. Specifically, choosing $\mathcal{K}_{\Delta 1} = 2\kappa_{\Delta 1}$ recovers (97) of [5]:

$$J = 1: \quad \mathcal{Q}_m = \frac{\mathcal{K}_{\Delta 1}}{4^m m! (1 - \frac{d}{2} + \Delta)_m} \left(\hat{\mathbb{N}} \hat{\mathbb{X}}^m + \frac{2m(d - 2\Delta)}{\Delta - d + 1} \hat{\mathbb{X}}^{m-1} \hat{\mathbb{Y}} \right) (\mathcal{M}_{\mathcal{L}} \mathcal{M}_{\mathcal{R}}) \Big|_{\substack{\eta_{aL'}=0 \\ \eta_{iR'}=0}}. \tag{4.38}$$

For $s = J - 2$ and $\hat{\mathbb{P}}_{J-2,m} : \mathscr{M}_{\mathcal{L}}^{\Delta,J} \otimes \mathscr{M}_{\mathcal{R}}^{\Delta,J} \to \mathscr{M}_{\mathcal{L}}^{\Delta-2+2m,J-2} \otimes \mathscr{M}_{\mathcal{R}}^{\Delta-2+2m,J-2}$, we now have three possibilities:

$$\hat{\mathbb{P}}_{J-2,m} = \mathcal{N}^{(1)}_{J-2,m} \hat{\mathbb{X}}^{m-2} \hat{\mathbb{Y}}^2 + \mathcal{N}^{(2)}_{J-2,m} \hat{\mathbb{X}}^{m-1} \hat{\mathbb{Z}} + \mathcal{N}^{(3)}_{J-2,m} \hat{\mathbb{X}}^{m-2} \frac{\hat{\mathbb{x}}\hat{\mathbb{z}} \otimes \hat{\mathbb{y}}^2 + \hat{\mathbb{y}}^2 \otimes \hat{\mathbb{x}}\hat{\mathbb{z}}}{2}. \tag{4.39}$$

Note that we require $\{\hat{\mathbb{x}}, \hat{\mathbb{y}}, \hat{\mathbb{z}}\}$ and $\{\hat{\mathbb{x}}, \hat{\mathbb{y}}, \hat{\mathbb{z}}\}$ to appear symmetrically, because the factorization formula should not depend on which side we call left. Plugging into the recurrence relation, we obtain a recurrence relation for each of the three structures. Demanding $\mathcal{N}^{(1,2,3)}_{J-2,-1} = 0$, we obtain a unique solution:

$$\frac{\mathcal{N}^{(1)}_{J-2,m}}{\mathcal{N}_{J,m}} = \frac{2J(J-1)m(m-1)(2\Delta - d)(2\Delta - d + 2)}{(\Delta - J - d + 2)(\Delta - J - d + 3)}, \tag{4.40a}$$

$$\frac{\mathcal{N}^{(2)}_{J-2,m}}{\mathcal{N}_{J,m}} = J(J-1)m\Big[1 + \frac{2m}{\Delta - J - d + 2} - \frac{2(\Delta + J - 3)(\Delta - J - d + m + 2)}{(2J + d - 4)(\Delta - J - d + 3)}\Big], \tag{4.40b}$$

$$\frac{\mathcal{N}^{(3)}_{J-2,m}}{\mathcal{N}_{J,m}} = -\frac{4J(J-1)m(m-1)(2\Delta - d)}{(\Delta - J - d + 2)(\Delta - J - d + 3)}. \tag{4.40c}$$

Again, we see that although we only demanded $\mathcal{N}^{(1,2,3)}_{J-2,-1} = 0$, we still obtain $\mathcal{N}^{(1,3)}_{J-2,m} = 0$ for $m = 0, 1$ and $\mathcal{N}^{(2)}_{J-2,m} = 0$ for $m = 0$, consistent with the fact that $\hat{\mathbb{X}}$ has to appear with non-negative powers. Also, with the $s = J, J-1, J-2$ results at hand, we are able to recover the complete factorization formula for spin $J = 2$ exchanges. Specifically, choosing $\mathcal{K}_{\Delta 2} = \kappa_{\Delta 2}$ recovers (107) of [5], with the precise dictionary given by

$$L_m^{ab} = \frac{1}{2^m m!}[\hat{x}^m \mathcal{M}_{\mathcal{L}}]^{aL,bL}_{aL',bL'}\Big|_{\eta=0}, \quad \dot{L}_m^a = \frac{1}{2^{m-1}(m-1)!}[\hat{x}^{m-1}\hat{y}\mathcal{M}_{\mathcal{L}}]^{aL}_{aL'}\Big|_{\eta=0}, \tag{4.41a}$$

$$\ddot{L}_m = \frac{1}{2^{m-1}(m-1)!}\hat{m}\hat{x}^{m-1}\hat{y}\mathcal{M}_{\mathcal{L}}\Big|_{\eta=0}, \quad \widetilde{L}_m = \frac{1}{2^{m-1}(m-1)!}\hat{x}^{m-1}\hat{z}\mathcal{M}_{\mathcal{L}}\Big|_{\eta=0}, \tag{4.41b}$$

$$h_m^{(3)} = 4^{m-2}((m-2)!)^2\mathcal{N}^{(1)}_{2-2,m}, \quad h_m^{(3)} + 2h_m^{(4)} + h_m^{(5)} = 4^{m-1}((m-1)!)^2\mathcal{N}^{(2)}_{2-2,m}, \tag{4.41c}$$

$$h_m^{(3)} + h_m^{(4)} = 4^{m-2}(m-1)!(m-2)!\mathcal{N}^{(3)}_{2-2,m}. \tag{4.41d}$$

Using the method described above, we can systemaically solve all $\hat{\mathbb{P}}_{s,m}$. Requiring the half-operators to map $\mathscr{M}^{\Delta,J}_{\mathcal{L}/\mathcal{R}} \to \mathscr{M}^{\Delta-J+s+2m,s}_{\mathcal{L}/\mathcal{R}}$, it is not hard to show that there are exactly

$$\left\lceil \frac{J-s+1}{2} \right\rceil = 1, 1, 2, 2, 3, \cdots$$

choices of half-operators. More explicitly, starting at $\hat{\mathbb{x}}^m$ for $s = J$, whenever we decrease s by one, replace one $\hat{\mathbb{x}}$ by $\hat{\mathbb{y}}$ and then replace $\hat{\mathbb{y}}^2 \mapsto \hat{\mathbb{x}}\hat{\mathbb{z}}$ any number of times:

$J-s$	$\mathscr{M}^{\Delta,J}_{\mathcal{L}/\mathcal{R}} \to \mathscr{M}^{\Delta-J+s+2m,s}_{\mathcal{L}/\mathcal{R}}$	number of half-structures
0	$\hat{\mathbb{x}}^m$	1
1	$\hat{\mathbb{x}}^{m-1}\hat{\mathbb{y}}$	1
2	$\hat{\mathbb{x}}^{m-2}\hat{\mathbb{y}}^2,\ \hat{\mathbb{x}}^{m-1}\hat{\mathbb{z}},$	2
3	$\hat{\mathbb{x}}^{m-3}\hat{\mathbb{y}}^3,\ \hat{\mathbb{x}}^{m-2}\hat{\mathbb{y}}\hat{\mathbb{z}},$	2
4	$\hat{\mathbb{x}}^{m-4}\hat{\mathbb{y}}^4,\ \hat{\mathbb{x}}^{m-3}\hat{\mathbb{y}}^2\hat{\mathbb{z}},\ \hat{\mathbb{x}}^{m-2}\hat{\mathbb{z}}^2$	3
$\vdots$		$\vdots$

The possible structures in $\hat{\mathbb{P}}_{s,m}$ can be found by pairing the left and right half-operators symmetrically, leading to a total number

$$\frac{1}{2} \times \left\lceil \frac{J-s+1}{2} \right\rceil \times \left(\left\lceil \frac{J-s+1}{2} \right\rceil + 1 \right) = 1, 1, 3, 3, 6, 6, 10, \cdots$$

For any spin-J exchange, only those structures with $J - s \leq J$ are needed. As a result, there are 1, $2 = 1+1$, $5 = 2+3$, $8 = 5+3$, $14 = 8+6$ terms in total in the spin-0,1,2,3,4 factorization formulas. We provide the explicit factorization formulas up to spin-4 in an ancillary file.

As a final check, consider $\hat{\mathbb{P}}_{s,m=0} : \mathscr{M}^{\Delta,J}_{\mathcal{L}} \otimes \mathscr{M}^{\Delta,J}_{\mathcal{R}} \to \mathscr{M}^{\Delta-J+s,s}_{\mathcal{L}} \otimes \mathscr{M}^{\Delta-J+s,s}_{\mathcal{R}}$. The only polynomial of $\hat{\mathbb{x}}, \hat{\mathbb{y}}, \hat{\mathbb{z}}$ that yields something of this form is a constant, which then implies that $s = J$. In other words, $\mathcal{N}_{J,0} = \mathcal{K}_{\Delta J}$ is sufficient for the primary factorization formula. By choosing

$$\mathcal{K}_{\Delta J} = \frac{2^J \kappa_{\Delta J}}{(J!)^2}, \tag{4.42}$$

we obtain

$$\mathcal{Q}_0 = \frac{2^J \kappa_{\Delta J}}{(J!)^2}\hat{\mathbb{N}}^J(\mathcal{M}_{\mathcal{L}} \otimes \mathcal{M}_{\mathcal{R}}), \tag{4.43}$$

which proves the conjecture (4.15).

4.6 A collection of factorization formulas

We collect all factorization formulas derived in the operator language here for reader's convenience.

- Scalar factorization:

$$\mathcal{Q}_m = \frac{\mathcal{K}_{\Delta 0}}{4^m m!(1-\frac{d}{2}+\Delta)_m} \hat{\mathbb{X}}^m \circ (\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}})\Big|_{\substack{\eta_{aL'}=0\\ \eta_{iR'}=0}} . \tag{4.44}$$

- Spin-1 factorization:

$$\mathcal{Q}_m = \frac{\mathcal{K}_{\Delta 1}}{4^m m!(1-\frac{d}{2}+\Delta)_m} \left(\hat{\mathbb{N}}\hat{\mathbb{X}}^m + \frac{2m(d-2\Delta)}{\Delta-d+1}\hat{\mathbb{X}}^{m-1}\hat{\mathbb{Y}}\right)(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}})\Big|_{\substack{\eta_{aL'}=0\\ \eta_{iR'}=0}} . \tag{4.45}$$

- Spin-2 factorization:

$$\begin{aligned}\mathcal{Q}_m = \frac{\mathcal{K}_{\Delta 2}}{4^m m!(1-\frac{d}{2}+\Delta)_m} \Big[&\hat{\mathbb{N}}^2\hat{\mathbb{X}}^m + A_1\,\hat{\mathbb{N}}\hat{\mathbb{X}}^{m-1}\hat{\mathbb{Y}} + A_2\,\hat{\mathbb{X}}^{m-2}\hat{\mathbb{Y}}^2 + A_3\,\hat{\mathbb{X}}^{m-1}\hat{\mathbb{Z}} \\ &+A_4\,\hat{\mathbb{X}}^{m-2}(\hat{\mathbb{x}}\hat{\mathbb{z}}\otimes\hat{\mathbb{y}}^2+\hat{\mathbb{y}}^2\otimes\hat{\mathbb{x}}\hat{\mathbb{z}})\Big](\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}})\Big|_{\substack{\eta_{aL'}=0\\ \eta_{iR'}=0}} ,\end{aligned} \tag{4.46}$$

  with

$$A_1 = -\frac{4m(2\Delta-2)}{\Delta-d}, \tag{4.47a}$$

$$A_2 = \frac{4m(m-1)(2\Delta-d)(2\Delta-d+2)}{(\Delta-d)(\Delta-d+1)}, \tag{4.47b}$$

$$A_3 = 2m\left[1+\frac{2m}{\Delta-d}-\frac{2(\Delta-1)(\Delta-d+m)}{d(\Delta-d+1)}\right], \tag{4.47c}$$

$$A_4 = -\frac{4m(m-1)(2\Delta-d)}{(\Delta-d)(\Delta-d+1)}. \tag{4.47d}$$

Here, the normalization $\mathcal{K}_{\Delta J}$ is defined by

$$\mathcal{K}_{\Delta J} = \frac{(-1)^{J-1}}{(J!)^2}2(\Delta+J-1)\Gamma(\Delta-1). \tag{4.48}$$

The definitions of these shift operators can be found in (4.9), (4.20) and (4.22).

4.7 Generalization to mixed-symmetry tensors

Finally, we briefly comment on how to generalize to mixed-symmetry tensors. The strategy is exactly the same, although the resulting factorization formulas will be significantly more complicated. For example, if the exchanged operator L/R has spin-$(J_{(1)}, J_{(2)})$, we simply introduce more shift half-operators like $\{\hat{\mathbb{m}}, \hat{\mathbb{x}}, \hat{\mathbb{y}}, \hat{\mathbb{z}}\}$:

$$\hat{\mathbb{m}}_{10\mathcal{L}} \leftrightarrow -(X_L\cdot\partial_{Z_L^{(1)}}),\quad \hat{\mathbb{m}}_{21\mathcal{L}} \leftrightarrow -(Z_L^{(1)}\cdot\partial_{Z_L^{(2)}}),\quad \hat{\mathbb{m}}_{20\mathcal{L}} \leftrightarrow -(X_L\cdot\partial_{Z_L^{(2)}}), \tag{4.49a}$$

$$\hat{\mathbb{y}}_{00\mathcal{L}} \leftrightarrow -2(\partial_{X_L}\cdot\partial_{X_L}),\ \hat{\mathbb{y}}_{01\mathcal{L}} \leftrightarrow -2(\partial_{X_L}\cdot\partial_{Z_L^{(1)}}),\ \cdots,\ \hat{\mathbb{y}}_{22\mathcal{L}} \leftrightarrow -2(\partial_{Z_L^{(2)}}\cdot\partial_{Z_L^{(2)}}). \tag{4.49b}$$

These satisfy a slightly more complicated nilpotent algebra, as can be derived from position space, and $\hat{\mathbb{m}}_{10\mathcal{L}}, \hat{\mathbb{m}}_{21\mathcal{L}}, \hat{\mathbb{m}}_{20\mathcal{L}}$ all annihilate $\mathcal{M}_{\mathcal{L}}$ due to gauge invariance. These shift half-operators can be viewed as the action of shift operators on factorizable functions. For example,

$$\hat{\mathbb{M}}_{20}\mathcal{Q} := \sum_{a,i} \gamma_{aL}\gamma_{iR}[\mathcal{Q}]^{aL,iR,ai,LR}_{aL'',iR''}, \tag{4.50a}$$

$$\hat{\mathbb{M}}_{21}\mathcal{Q} := \sum_{a,i} \gamma_{aL}\gamma_{iR}[\mathcal{Q}]^{aL',iR'}_{aL'',iR''}, \tag{4.50b}$$

$$\vdots$$

$$\hat{\mathbb{Y}}_{02}\mathcal{Q} := \sum_{\substack{a,b\\a\neq b}}\sum_{\substack{i,j\\i\neq j}} \gamma_{ab}\gamma_{ij}[\mathcal{Q}]^{ab,ij}_{bL,iR,bi,LR,aL'',jR''}, \tag{4.50c}$$

$$\hat{\mathbb{Y}}_{12}\mathcal{Q} := \sum_{\substack{a,b\\a\neq b}}\sum_{\substack{i,j\\i\neq j}} \gamma_{ab}\gamma_{ij}[\mathcal{Q}]^{ab,ij}_{bL',iR',aL'',jR''}, \tag{4.50d}$$

$$\vdots$$

Similarly, we can define $\hat{\mathbb{N}}_{10}, \hat{\mathbb{N}}_{21}, \hat{\mathbb{N}}_{20}$ that do not preserve factorizability:

$$\hat{\mathbb{N}}_{20}\mathcal{Q} := \sum_{a,i} \gamma_{ai}[\mathcal{Q}]^{aL,iR,ai,LR}_{aL'',iR''}, \tag{4.51a}$$

$$\hat{\mathbb{N}}_{21}\mathcal{Q} := \sum_{a,i} \gamma_{ai}[\mathcal{Q}]^{aL',iR'}_{aL'',iR''}, \tag{4.51b}$$

$$\vdots$$

It can be checked that $\hat{\mathbb{N}}$ and $\hat{\mathbb{Y}}$ again form the same nilpotent algebra as $\hat{\mathbb{m}}_{\mathcal{L}}$ and $\hat{\mathbb{y}}_{\mathcal{L}}$. In order to work out the factorization formula, simply use the ansatz

$$\mathcal{Q}_m = \sum_{\substack{s_{10},s_{21},s_{20}\geq 0\\ s_{21}+s_{20}=J_{(2)}\\ s_{10}-s_{21}=J_{(1)}}} \hat{\mathbb{N}}^{s_{10}}_{10}\hat{\mathbb{N}}^{s_{21}}_{21}\hat{\mathbb{N}}^{s_{20}}_{20}\hat{\mathbb{P}}_{s_{10},s_{21},s_{20},m}(\mathcal{M}_{\mathcal{L}}\mathcal{M}_{\mathcal{R}})\Big|_{\substack{\eta_{aL'}=\eta_{aL''}=0\\ \eta_{iR'}=\eta_{iR''}=0}}. \tag{4.52}$$

to solve the Casimir recurrence equation.

5 Applications

5.1 Bootstrapping $\langle\mathcal{V}_\Delta\mathcal{V}_\Delta\mathcal{V}_\Delta\rangle$: the AdS on-shell method

We are now ready to put our definition of the spinning Mellin amplitude to use. As a first example, we will bootstrap the three-point function of vector operators $\mathcal{V}_\Delta(X,Z)$ with scaling dimension Δ:

$$\langle\mathcal{V}_\Delta(X_1,Z_1)\mathcal{V}_\Delta(X_2,Z_2)\mathcal{V}_\Delta(X_3,Z_3)\rangle.$$

In the AdS/CFT correspondence, $\mathcal{V}_\Delta$ is generally dual to a massive spin-1 particle in the bulk. When $\Delta = d-1$, this operator becomes a conserved current, and the corresponding bulk particle is a gluon.

By definition, the Mellin amplitude depends on 3 continuous variables $\gamma_{12}, \gamma_{13}, \gamma_{23}$ and 9 discrete variables $\eta_{1'2}, \eta_{1'3}, \eta_{2'1}, \eta_{2'3}, \eta_{3'1}, \eta_{3'2}, \eta_{1'2'}, \eta_{1'3'}, \eta_{2'3'}$, subject to the constraints:

$$\gamma_{21} - \eta_{2'1} + \gamma_{31} - \eta_{3'1} = \Delta, \qquad \eta_{21'} + \eta_{2'1'} + \eta_{31'} + \eta_{3'1'} = 1, \tag{5.1a}$$

$$\gamma_{12} - \eta_{1'2} + \gamma_{32} - \eta_{3'2} = \Delta, \qquad \eta_{12'} + \eta_{1'2'} + \eta_{32'} + \eta_{3'2'} = 1, \tag{5.1b}$$

$$\gamma_{13} - \eta_{1'3} + \gamma_{23} - \eta_{2'3} = \Delta, \qquad \eta_{13'} + \eta_{1'3'} + \eta_{23'} + \eta_{2'3'} = 1. \tag{5.1c}$$

We can solve $\gamma_{12}, \gamma_{13}, \gamma_{23}$ in terms of the discrete η-variables, and after imposing the form (2.69) that the Mellin amplitude be multi-linear with respect to the labels $1', 2', 3'$, we obtain the following ansatz:

$$\begin{aligned}\mathcal{M} &= c_1\eta_{1'2'}\eta_{3'1} + c_2\eta_{1'2'}\eta_{3'2} + c_3\eta_{2'3'}\eta_{1'2} + c_4\eta_{2'3'}\eta_{1'3} + c_5\eta_{3'1'}\eta_{2'3} + c_6\eta_{3'1'}\eta_{2'1} \\ &\quad + c_7\eta_{1'2}\eta_{2'3}\eta_{3'1} + c_8\eta_{1'2}\eta_{2'3}\eta_{3'2} + c_9\eta_{1'2}\eta_{2'1}\eta_{3'1} + c_{10}\eta_{1'2}\eta_{2'1}\eta_{3'2} \\ &\quad + c_{11}\eta_{1'3}\eta_{2'3}\eta_{3'1} + c_{12}\eta_{1'3}\eta_{2'3}\eta_{3'2} + c_{13}\eta_{1'3}\eta_{2'1}\eta_{3'1} + c_{14}\eta_{1'3}\eta_{2'1}\eta_{3'2}.\end{aligned} \tag{5.2}$$

The Mellin amplitude satisfy the gauge-invariance condition (2.48) of all three external particles. For example, consider the gauge-invariance of particle 3:

$$\gamma_{13}[\mathcal{M}]^{13}_{13'} - \eta_{1'3}[\mathcal{M}]^{1'3}_{1'3'} + \gamma_{23}[\mathcal{M}]^{23}_{23'} - \eta_{2'3}[\mathcal{M}]^{2'3}_{2'3'} = 0, \tag{5.3}$$

which holds on a different support with respect to the operator 3 due to the shifts:

$$\gamma_{13} - \eta_{1'3} + \gamma_{23} - \eta_{2'3} = \Delta - 1, \qquad \eta_{13'} + \eta_{1'3'} + \eta_{23'} + \eta_{2'3'} = 0. \tag{5.4}$$

Since the discrete η variables are non-negative, this corresponds to 5 lattice points. Solving (5.3) on these lattice points, we obtain

$$\begin{gathered} c_2 = -c_1, \quad c_{10} = -c_9, \quad c_5 = -c_4 + \frac{\Delta+1}{2}c_{11} + \frac{\Delta+1}{2}c_{12}, \\ c_7 = \frac{2}{\Delta-1}c_3 - \frac{\Delta+1}{\Delta-1}c_8, \quad c_{14} = \frac{2}{\Delta-1}c_6 - \frac{\Delta+1}{\Delta-1}c_{13}. \end{gathered} \tag{5.5}$$

Similarly, we can impose the gauge-invariance conditions of particles 1 and 2. Together, they are able to fix 10 of the 14 unknown coefficients in the ansatz, which leads to

$$\begin{aligned}\mathcal{M} &= \frac{c_8}{2}(\eta_{1'2} - \eta_{1'3})((\Delta+1)\eta_{2'3'} + 2\eta_{2'3}\eta_{3'2}) \\ &\quad + \frac{c_{11}}{2}(\eta_{2'3} - \eta_{2'1})((\Delta+1)\eta_{1'3'} + 2\eta_{1'3}\eta_{3'1}) \\ &\quad + \frac{c_9}{2}(\eta_{3'1} - \eta_{3'2})((\Delta+1)\eta_{1'2'} + 2\eta_{1'2}\eta_{2'1}) \\ &\quad + \frac{c_7}{2}((\Delta-1)(\eta_{1'2'}\eta_{3'1} - \eta_{1'2'}\eta_{3'2} + \eta_{2'3'}\eta_{1'2} - \eta_{2'3'}\eta_{1'3} + \eta_{3'1'}\eta_{2'3} - \eta_{3'1'}\eta_{2'1}) \\ &\qquad + 2(\eta_{1'2}\eta_{2'3}\eta_{3'1} - \eta_{1'3}\eta_{2'1}\eta_{3'2})).\end{aligned} \tag{5.6}$$

Vector operators satisfying the unitarity bound $\Delta = d - 1$ are conserved currents. In the embedding formalism, the conservation condition translates to [5, 24]:

$$\forall 1 \leq k \leq 3: \quad \frac{\partial}{\partial X_k} \cdot \frac{\partial}{\partial Z_k} \langle \mathcal{V}_\Delta(X_1, Z_1)\mathcal{V}_\Delta(X_2, Z_2)\mathcal{V}_\Delta(X_3, Z_3)\rangle = 0. \tag{5.7}$$

In Mellin space this becomes an equation on the Mellin amplitude $\mathcal{M}$. For example, conservation of operator 3 reads[13]

$$\sum_{\substack{a,b\in\{1,1',2,2'\}\\ a\neq b}} \gamma_{ab}[\mathcal{M}]^{ab}_{a3',b3} = 0, \tag{5.8}$$

which holds on the following support with respect to the operator 3 due to the shifts:

$$\gamma_{13} - \eta_{1'3} + \gamma_{23} - \eta_{2'3} = \Delta + 1, \qquad \eta_{13'} + \eta_{1'3'} + \eta_{23'} + \eta_{2'3'} = 0. \tag{5.9}$$

Since the discrete η variables are non-negative, this corresponds to 5 lattice points. Solving (5.8) on these lattice points, we obtain $c_8 = c_{11}$ in (5.6). Similarly, we can impose conservation of all three operators. Together, they fix $c_8 = c_9 = c_{11}$. In the end, we obtain

$$\begin{aligned}
\mathcal{M} = c\Big(&\eta_{1'2'}(\eta_{3'1} - \eta_{3'2}) + \eta_{2'3'}(\eta_{1'2} - \eta_{1'3}) + \eta_{3'1'}(\eta_{2'3} - \eta_{2'1}) \\
&+ \frac{2}{\Delta - 1}(\eta_{1'2}\eta_{2'3}\eta_{3'1} - \eta_{1'3}\eta_{2'1}\eta_{3'2})\Big) \\
- c'\Big(&(\eta_{1'2} - \eta_{1'3})(\eta_{2'3} - \eta_{2'1})(\eta_{3'1} - \eta_{3'2}) \\
&+ \frac{2}{\Delta - 1}(\eta_{1'2}\eta_{2'3}\eta_{3'1} - \eta_{1'3}\eta_{2'1}\eta_{3'2})\Big).
\end{aligned} \tag{5.10}$$

Several remarks are in order:

- The spacetime dimension d does not appear (5.6) and (5.10), which manifests the view that Mellin formalism is a d-independent representation of conformal field theories [1].
- Although not imposed manually, the kinematic dependence of (5.10) is automatically antisymmetric under permuting the external operators $1, 2, 3$. Since the correlator itself is permutation invariant, the coupling constants c, c' must be antisymmetric as well. This is expected, as the coupling constants are nothing but the structure constants f^{abc} of a global symmetry group, under which these Noether currents transform in the adjoint representation.
- The fact that there are two independent solutions (5.10) for the three-point gluon amplitude is not surprising. This is because the effective bulk Lagrangian can have F^2 and F^3 interactions[14]. The precise values of c, c' depend on the theory considered. For example, in pure Yang-Mills theory, (6.13) of [28] implies $c'/c = \frac{11d-18}{2d(2d-3)}$.
- Note that up to the correction terms proportional to $\frac{2}{\Delta-1}$, the AdS amplitude (5.10) takes a similar form as the flat-space amplitude

$$\begin{aligned}
\mathcal{M}^{\text{flat}} &= g_{F^2}\left((\mathtt{e}_1\cdot\mathtt{e}_2)\mathtt{e}_3\cdot(\mathtt{k}_1 - \mathtt{k}_2) + \text{cyclic}\right) \\
&\quad + g_{F^3}\left(\mathtt{e}_1\cdot(\mathtt{k}_2 - \mathtt{k}_3)\mathtt{e}_2\cdot(\mathtt{k}_3 - \mathtt{k}_1)\mathtt{e}_3\cdot(\mathtt{k}_1 - \mathtt{k}_2)\right).
\end{aligned} \tag{5.11}$$

  This suggests the identification of $\eta_{p'q} \sim \mathtt{e}_p\cdot\mathtt{k}_q$ and $\eta_{p'q'} \sim \mathtt{e}_p\cdot\mathtt{e}_q$ in the flat-space limit.

---

[13] This is in fact $\hat{\mathtt{y}}\mathcal{M} = 0$ if we treat 3 as the "exchanged operator" L. For higher-spin conserved currents, the condition reads $(d-4+2J)\hat{\mathtt{y}}\mathcal{M} + \sum_c \sum_{a\neq b} \gamma_{cL'}\gamma_{ab}[\mathcal{M}]^{ab,cL'}_{aL',bL',cL} = 0$.

[14] Moreover, the four solutions in (5.6) could be thought of as arising from F^3 and $A^\mu(A^\nu \overleftrightarrow{\nabla}_\mu A_\nu)$.

5.2 Case study: $\mathcal{N}=2$ theory of supergluons

Our definition of (multi-)spinning Mellin amplitudes greatly facilitates the study of spinning correlators in specific theories. In this subsection, we showcase this by considering supergluons in $\mathrm{AdS}_5 \times \mathrm{S}^3$ [36–38] (see [15, 17–20, 39–50] for recent progress in bootstrapping this theory). From the boundary point of view, this theory is an $\mathcal{N}=2$ SCFT in $d=4$ dimensions. We restrict our attention to the half-BPS supermultiplet, where the superprimary operator

$$\mathcal{O}^a(x,v)=\mathcal{O}^{a;\alpha_1\alpha_2}(x)v^{\beta_1}v^{\beta_2}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2} \tag{5.12}$$

is a scalar with dimension $\Delta=2$. Here, $a=1,\cdots,\dim G_F$ is the adjoint index of a flavor group G_F, which we take to be $G_F=SU(N_f)$, and v^β is an auxiliary $SU(2)_R$ spinor which ensures $\mathcal{O}^{a;\alpha_1\alpha_2}$ lives in the R-spin-1 representation. Another important primary operator in the supermultiplet is the conserved current $\mathcal{J}^a_\mu(x)$ of the flavor group G_F, which is an $SU(2)_R$ singlet and has dimension $\Delta=3$. This theory is dual to a $\mathcal{N}=1$ super-Yang-Mills theory on $\mathrm{AdS}_5\times S^3$ in the bulk, where $\mathcal{O}^a$ corresponds to a scalar field (supergluon) and $\mathcal{J}^a_\mu$ corresponds to a gauge field (gluon). When considering the correlators of $\mathcal{O}$ and $\mathcal{J}$ at tree level[15], the only single-particle states propagating in the bulk are $\mathcal{O}$ and $\mathcal{J}$, and there are no contributions from other operators in the factorization.

Denoting the two operators collectively by $\mathcal{X}=\mathcal{O},\mathcal{J}$, we shall study their connected correlators. At tree level in the bulk, we can perform color decomposition in the same way as in flat space [51]:

$$\langle\mathcal{X}^{a_1}(X_1,Z_1)\cdots\mathcal{X}^{a_N}(X_N,Z_N)\rangle=\sum_{\sigma\in S_{N-1}}\mathrm{tr}(T^{a_1}T^{a_{2^\sigma}}\cdots T^{a_{N^\sigma}})\langle\mathcal{X}_1\mathcal{X}_{2^\sigma}\cdots\mathcal{X}_{N^\sigma}\rangle, \tag{5.13}$$

where σ denotes a permutation of the labels $\{2,\cdots,N\}$. In particular, if all N operators are of the same type, we will denote $G^{\mathcal{X}}_{1\sigma}:=\langle\mathcal{X}_1\mathcal{X}_{2^\sigma}\cdots\mathcal{X}_{N^\sigma}\rangle$. Cyclic and reflection symmetry of the traces implies that

$$G^{\mathcal{X}}_{12\cdots N}=G^{\mathcal{X}}_{2\cdots N1}=(-)^N G^{\mathcal{X}}_{N\cdots 21}. \tag{5.14}$$

We focus on $G^{\mathcal{X}}_{12\cdots N}$ since any other color ordering can then be obtained by relabeling.

At tree level, the correlator $\mathcal{G}^{\mathcal{O}}_{12\cdots N}$ of any multiplicity N can be constructed recursively [18, 19] with explicit results up to 8 points. The recursion generates all single-gluon amplitudes $\langle\mathcal{O}_1\cdots\mathcal{O}_{N-1}\mathcal{J}_N\rangle$ as a byproduct. However, there is no explicit results for the complete amplitude involving two or more gluons. Even for the four-point function, where one could in principle use superspace methods to compute correlators such as $\langle\mathcal{J}\mathcal{J}\mathcal{O}\mathcal{O}\rangle$ and $\langle\mathcal{J}\mathcal{J}\mathcal{J}\mathcal{J}\rangle$ from $\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$, the currently available results [42] are limited to the so-called "orthogonal frame", where $Z_p\cdot X_q=0$, due to the complicated polarization structure. Below, we remedy this by bootstrapping the complete results of $\langle\mathcal{J}\mathcal{J}\mathcal{O}\mathcal{O}\rangle$ and $\langle\mathcal{J}\mathcal{J}\mathcal{J}\mathcal{J}\rangle$ with a simple calculation, which matches their results when reduced to the orthogonal frame.

[15]More precisely, we are considering the probe limit, which is the leading order in $N_c\to\infty$ while keeping N_f finite. In this case, gravity decouples and we are essentially scattering (super)gluons on the fixed AdS background.

Let us first recall the important consequences of color-ordering. First, it induces a natural set of planar Mandelstam variables in the amplitude. The momentum flowing through a propagator in the amplitude can only be the sums of consecutive external momenta. This leads us to introduce planar Mellin variables for the total momentum between the p-th and $(q-1)$-th operator:

$$\chi_{p,q} := \sum_{\substack{p\leq r<q \\ q\leq s<p}} (\gamma_{rs} - \eta_{r's} - \eta_{rs'} - \eta_{r's'}) = -\left(\sum_{p\leq r<q} (\mathtt{k}_r + \mathtt{k}_{r'})\right)^2 . \tag{5.15}$$

Here, all the ranges of labels should be understood as modulo N. For example, the range $p \leq r < q$ means $r \in \{p, p+1, \cdots, q-2, q-1\}$ when $p < q$, and $r \in \{p, p+1, \cdots, N, 1, \cdots, q-2, q-1\}$ when $p > q$. Color-ordered amplitudes only have poles of the form $\frac{1}{\chi_{p,q}-\tau-2m}$. In total, we have $\frac{N(N-3)}{2}$ planar variables, which can be used to solve all continuous Mellin variables γ_{pq}:

$$\gamma_{pq} = \eta_{p'q} + \eta_{pq'} + \eta_{p'q'} + \frac{\chi_{p,q} - \chi_{p,q+1} - \chi_{p+1,q} + \chi_{p+1,q+1}}{2}, \tag{5.16}$$

where we define $\chi_{p,p+1} := \tau_p$ and $\chi_{p,p} := 0$. The color-ordering also induces a nice basis of R-structures. Denoting $\langle pq\rangle := v_p^\alpha v_q^\beta \epsilon_{\alpha\beta}$, the color-ordered amplitudes can be expanded onto products of non-crossing cycles $V_{p_1p_2\cdots p_n} := \langle p_1p_2\rangle\langle p_2p_3\rangle\cdots\langle p_np_1\rangle$, where each v_p^α represents the R-polarization of the p-th operator.

In the $\mathcal{N} = 2$ supergluon theory we are interested in, both $\mathcal{O}$ and $\mathcal{J}$ have $\tau = 2$. Therefore, these two operators contribute to the same series of poles $\frac{1}{\chi-2-2m}$ in the factorization and their contributions mix together in the residues. However, since the two operators have different $SU(2)_R$ charges, we can separate their individual contributions by projecting out the R-spin-1 and R-spin-0 part [18, 19]. We can thus define the gluon residue

$$\overset{(\mathcal{J})}{\underset{\chi_{p,q}=\tau+2m}{\mathrm{Res}}}\mathcal{M} = \underset{\chi_{p,q}=\tau+2m}{\mathrm{Res}}\mathcal{M}\Big|_{V_{i\cdots jk\cdots l}\mapsto\frac{1}{2}V_{i\cdots j}V_{k\cdots l}}, \quad \forall p\leq i<j<q,\ q\leq k<l<p, \tag{5.17}$$

which extracts the contribution of $\mathcal{J}$ in the channel $\frac{1}{\chi_{p,q}-\tau-2m}$, and define the scalar residue

$$\overset{(\mathcal{O})}{\underset{\chi_{p,q}=\tau+2m}{\mathrm{Res}}}\mathcal{M} = \underset{\chi_{p,q}=\tau+2m}{\mathrm{Res}}\mathcal{M} - \overset{(\mathcal{J})}{\underset{\chi_{p,q}=\tau+2m}{\mathrm{Res}}}\mathcal{M} \tag{5.18}$$

by subtracting the gluon contributions. As an example, from the residue of four-point amplitude

$$\underset{\chi_{1,3}=2}{\mathrm{Res}}\mathcal{M}^{\mathcal{O}}_{1234} = -2V_{1234} - V_{12}V_{34}(\chi_{2,4}-4), \tag{5.19}$$

we can read off the gluon and scalar contributions:

$$\overset{(\mathcal{J})}{\underset{\chi_{1,3}=2}{\mathrm{Res}}}\mathcal{M}^{\mathcal{O}}_{1234} = -V_{12}V_{34}(\chi_{2,4}-3), \quad \overset{(\mathcal{O})}{\underset{\chi_{1,3}=2}{\mathrm{Res}}}\mathcal{M}^{\mathcal{O}}_{1234} = -2\left(V_{1234} - \frac{1}{2}V_{12}V_{34}\right). \tag{5.20}$$

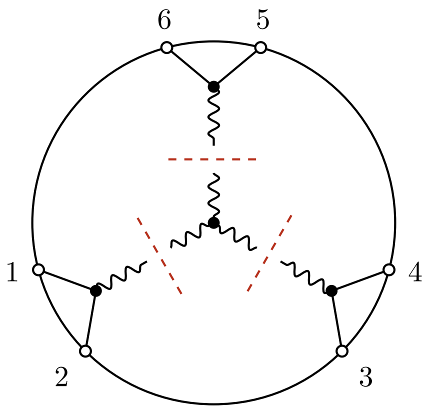


Figure 1: The factorization of $\mathcal{M}^{\mathcal{O}}_{123456}$ on gluons in the snowflake channel. The residue is related to the product of $\mathcal{M}_{\mathcal{J}\mathcal{J}\mathcal{J}}$ and three $\mathcal{M}_{\mathcal{O}\mathcal{O}\mathcal{J}}$.

The Mellin amplitudes for $\langle \mathcal{O}_1 \cdots \mathcal{O}_N \rangle$ $(N \leq 8)$ and $\langle \mathcal{O}_1 \cdots \mathcal{O}_{N-1}\mathcal{J}_N \rangle$ $(N \leq 6)$ can be obtained from `this URL`. The conventions of the data file were described in detail in [18][16]. In our formalism,

$$\mathcal{M}^{\mathcal{O}}_{123} = V_{123}, \quad \mathcal{M}_{\mathcal{O}_1\mathcal{O}_2\mathcal{J}_3} = \frac{i}{2\sqrt{3}} V_{12}(\eta_{13'} - \eta_{23'}). \tag{5.21}$$

The planar variables in the ancillary files are related to the planar variables defined in this paper through:

$$\chi_{p,q} = 2 - \texttt{xx[}p\texttt{,}q\texttt{]}. \tag{5.22}$$

5.2.1 $\langle \mathcal{J}\mathcal{J}\mathcal{J} \rangle$

As discussed in the previous subsection, gauge-invariance and unitarity bound dictate that the three-point gluon Mellin amplitude $\mathcal{M}^{\mathcal{J}}_{123}$ must be a linear combination of two structures. Plugging in $\Delta = 3$, we have

$$\begin{aligned}\mathcal{M}^{\mathcal{J}}_{123} = c\Big(&\eta_{1'2'}(\eta_{3'1} - \eta_{3'2}) + \eta_{2'3'}(\eta_{1'2} - \eta_{1'3}) + \eta_{3'1'}(\eta_{2'3} - \eta_{2'1}) \\ &+ (\eta_{1'2}\eta_{2'3}\eta_{3'1} - \eta_{1'3}\eta_{2'1}\eta_{3'2})\Big) \\ - c'\Big(&(\eta_{1'2} - \eta_{1'3})(\eta_{2'3} - \eta_{2'1})(\eta_{3'1} - \eta_{3'2}) \\ &+ (\eta_{1'2}\eta_{2'3}\eta_{3'1} - \eta_{1'3}\eta_{2'1}\eta_{3'2})\Big).\end{aligned} \tag{5.23}$$

To determine the values of c, c' in this theory, we can glue it with three $\mathcal{M}_{\mathcal{O}\mathcal{O}\mathcal{J}}$ amplitudes in (5.21), and compare the result with the contribution from three gluon exchanges in the snowflake channel in $\mathcal{M}^{\mathcal{O}}_{123456}$, as is shown in figure 1.

[16] Be sure to use the published version or the latest arXiv version (v4), as previous preprint versions contain typos in the ancillary file.

First, using the factorization formula derived in Section 4, we can glue $\mathcal{M}^{\mathcal{J}}_{12L}\otimes\mathcal{M}_{\mathcal{O}_3\mathcal{O}_4\mathcal{J}_R}$ to obtain[17]

$$
\begin{aligned}
&\left(i\sqrt{3}V_{34}\right)^{-1}\underset{\chi_{1,3}=2}{\overset{(\mathcal{J})}{\mathrm{Res}}}\mathcal{M}_{\mathcal{J}_1\mathcal{J}_2\mathcal{O}_3\mathcal{O}_4}=\left(i\sqrt{3}V_{34}\right)^{-1}\mathcal{K}_{3,1}\hat{\mathbb{N}}(\mathcal{M}^{\mathcal{J}}_{12L}\mathcal{M}_{\mathcal{O}_3\mathcal{O}_4\mathcal{J}_R})\Big|_{\substack{\eta_{aL'}=0\\ \eta_{iR'}=0}}\\
&=c\Big(\chi_{2,4}(2\eta_{1'2'}+\eta_{1'2}\eta_{2'\{34\}}+\eta_{2'1}\eta_{1'\{34\}})\\
&\qquad-2(3\eta_{1'2'}+2(\eta_{1'2}\eta_{2'3}+\eta_{2'1}\eta_{1'4})+\eta_{1'2}\eta_{2'4}+\eta_{2'1}\eta_{1'3}+\eta_{1'3}\eta_{2'4}-\eta_{1'4}\eta_{2'3})\Big)\\
&+c'\Big(\chi_{2,4}(2\eta_{1'2}\eta_{2'1}+2\eta_{1'\{34\}}\eta_{2'\{34\}}-3\eta_{1'2}\eta_{2'\{34\}}-3\eta_{2'1}\eta_{1'\{34\}})\\
&\qquad-2(3\eta_{1'2}\eta_{2'1}+2\eta_{1'3}\eta_{2'4}+3\eta_{1'3}\eta_{2'3}+3\eta_{1'4}\eta_{2'4}\\
&\qquad\quad+4\eta_{1'4}\eta_{2'3}-4\eta_{1'2}\eta_{2'4}-4\eta_{2'1}\eta_{1'3}-5\eta_{1'2}\eta_{2'3}-5\eta_{2'1}\eta_{1'4})\Big).
\end{aligned}
$$

Here, we have used short-hand notations such as $\eta_{1'\{34\}}:=\eta_{1'3}+\eta_{1'4}$. Gluing another two copies of $\mathcal{M}_{\mathcal{O}\mathcal{O}\mathcal{J}}$, we obtain

$$
\begin{aligned}
&\left(-6i\sqrt{3}V_{12}V_{34}V_{56}\right)^{-1}\underset{\chi_{1,3}=2}{\overset{(\mathcal{J})}{\mathrm{Res}}}\underset{\chi_{3,5}=2}{\overset{(\mathcal{J})}{\mathrm{Res}}}\underset{\chi_{1,5}=2}{\overset{(\mathcal{J})}{\mathrm{Res}}}\mathcal{M}^{\mathcal{O}}_{123456}\\
&=c\Big(\chi_{1,4}\chi_{2,5}\chi_{3,6}-36+4(\chi_{1,4}+\chi_{2,5}+\chi_{3,6})+6(\chi_{2,4}+\chi_{4,6}+\chi_{2,6})\\
&\qquad+(\chi_{1,4}\chi_{2,5}+\chi_{2,5}\chi_{3,6}+\chi_{3,6}\chi_{1,4})-2(\chi_{1,4}\chi_{2,6}+\chi_{2,5}\chi_{4,6}+\chi_{3,6}\chi_{2,4})\Big)\\
&-c'\Big(144-5\chi_{1,4}\chi_{2,5}\chi_{3,6}-40(\chi_{1,4}+\chi_{2,5}+\chi_{3,6})-6(\chi_{2,4}+\chi_{4,6}+\chi_{2,6})\\
&\qquad+13(\chi_{1,4}\chi_{2,5}+\chi_{2,5}\chi_{3,6}+\chi_{3,6}\chi_{1,4})+2(\chi_{1,4}\chi_{2,6}+\chi_{2,5}\chi_{4,6}+\chi_{3,6}\chi_{2,4})\Big).
\end{aligned}
$$

The correct result can be extracted from $\mathcal{M}^{\mathcal{O}}_{123456}$ using (5.17):

$$
\begin{aligned}
&(2V_{12}V_{34}V_{56})^{-1}\underset{\chi_{1,3}=2}{\overset{(\mathcal{J})}{\mathrm{Res}}}\underset{\chi_{3,5}=2}{\overset{(\mathcal{J})}{\mathrm{Res}}}\underset{\chi_{1,5}=2}{\overset{(\mathcal{J})}{\mathrm{Res}}}\mathcal{M}^{\mathcal{O}}_{123456}\\
&=6(\chi_{2,4}+\chi_{4,6}+\chi_{2,6})-5(\chi_{1,4}+\chi_{2,5}+\chi_{3,6})-9\\
&+2(\chi_{1,4}\chi_{2,5}+\chi_{2,5}\chi_{3,6}+\chi_{3,6}\chi_{1,4})-2(\chi_{1,4}\chi_{2,6}+\chi_{2,5}\chi_{4,6}+\chi_{3,6}\chi_{2,4}).
\end{aligned}
$$

Comparing the two results, we fix the unknown coupling constants in this theory to be

$$
c=\frac{5i}{12\sqrt{3}},\quad c'=\frac{i}{12\sqrt{3}}. \tag{5.24}
$$

5.2.2 $\langle\mathcal{J}\mathcal{J}\mathcal{J}\mathcal{J}\rangle$

With the three-gluon amplitude $\mathcal{M}^{\mathcal{J}}_{123}$ fixed, we can now proceed to bootstrap the four-gluon amplitude $\mathcal{M}^{\mathcal{J}}_{1234}$. Note that both exchange diagrams and contact diagrams contribute to this amplitude, as illustrated in figure 2.

[17] We have simplified the result a bit by dropping terms of the form η^2 because both operators 1 and 2 have spin-1.

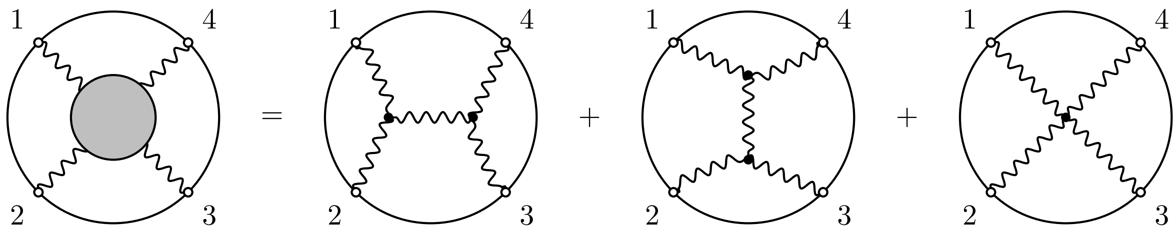


Figure 2: Possible diagrams in the four-gluon amplitude $\mathcal{M}^{\mathcal{J}}_{1234}$.

For the exchange diagrams, we can simply glue two three-gluon amplitudes together. Note that we have to take into account all descendant contributions, namely,

$$\begin{aligned}(\mathcal{M}^{\mathcal{J}}_{1234})_{\text{exchange}} = &\sum_{m=0}^{\infty} \frac{\frac{\mathcal{K}_{3,1}}{4^m m!(m+1)!}\hat{\mathbb{N}}\Big[(\hat{\mathbb{x}}^m\mathcal{M}^{\mathcal{J}}_{12R})(\hat{\mathbb{x}}^m\mathcal{M}^{\mathcal{J}}_{L34})\Big]}{\chi_{1,3}-2-2m}\Bigg|_{\eta_{aL'}=\eta_{iR'}=0} \\ &+\sum_{m=0}^{\infty} \frac{\frac{\mathcal{K}_{3,1}}{4^m m!(m+1)!}\hat{\mathbb{N}}\Big[(\hat{\mathbb{x}}^m\mathcal{M}^{\mathcal{J}}_{23R})(\hat{\mathbb{x}}^m\mathcal{M}^{\mathcal{J}}_{L41})\Big]}{\chi_{2,4}-2-2m}\Bigg|_{\eta_{aL'}=\eta_{iR'}=0}.\end{aligned} \tag{5.25}$$

However, it can be easily checked that $\hat{\mathbb{x}}^2\mathcal{M}^{\mathcal{J}}_{12R}=0$, which implies that the descendant poles truncate to only $m=0,1$. The result is too lengthy to write down explicitly. Here, we merely point out that the result of $(\mathcal{M}^{\mathcal{J}}_{1234})_{\text{exchange}}$ is already gauge-invariant up to contact terms:

$$\hat{\mathbb{m}}(\mathcal{M}^{\mathcal{J}}_{123R})_{\text{exchange}} = \text{contact terms, i.e., no poles of the form } \frac{1}{\chi-\#}.$$

This suggests that we should use gauge-invariance to constrain the contact terms.

To do this, we write down an ansatz for $(\mathcal{M}^{\mathcal{J}}_{1234})_{\text{contact}}$ that is a polynomial[18] of η's and multi-linear in the labels $1',2',3',4'$. The ansatz has 138 unknown coefficients, and dihedral symmetry[19] (5.14) reduces the number of unknowns to 27. We then impose the gauge-invariance condition, which reads

$$\hat{\mathbb{m}}(\mathcal{M}^{\mathcal{J}}_{123R})_{\text{exchange}} + \hat{\mathbb{m}}(\mathcal{M}^{\mathcal{J}}_{123R})_{\text{contact}} = 0, \quad \text{on lattice points.} \tag{5.26}$$

Remarkably, this uniquely fixes all unknown coefficients! The complete result of $\mathcal{M}^{\mathcal{J}}_{1234}$ is recorded in an ancillary file, which agrees with the snow-flake channel of $\mathcal{M}^{\mathcal{O}}_{12345678}$ when glued with four copies of $\mathcal{M}_{\mathcal{O}\mathcal{O}\mathcal{J}}$.

The result of $\mathcal{M}^{\mathcal{J}}_{1234}$ in the orthogonal frame has been obtained in (8.20) in [42]. The orthogonal frame is the special configuration $Z_p\cdot X_q=0$ of the correlator, which is

[18] In principle, we could also write a larger ansatz that also depends on γ's. However, since $(\mathcal{M}^{\mathcal{J}}_{1234})_{\text{exchange}}\sim\chi^0$ as $\chi\to\infty$, we expect $(\mathcal{M}^{\mathcal{J}}_{1234})_{\text{contact}}$ to also obey this power-counting, which restricts it to be a polynomial of η's only. This will hopefully be justified by a careful analysis of the flat-space limit of spinning Mellin amplitudes, which we leave for future work.

[19] The exchange contributions automatically satisfies the dihedral symmetry, as can be checked explicitly.

equivalent to taking all $\eta_{p'q}=0$ in the spinning Mellin amplitudes. In our notation, their result reads:

$$\begin{aligned}\langle\mathcal{J}\mathcal{J}\mathcal{J}\mathcal{J}\rangle_{\text{ortho}} = Z_{12}Z_{34}\int \mathrm{d}\gamma\, \widetilde{\mathcal{M}}^{(1)}(\gamma) \prod_{1\le p<q\le 4}\frac{\Gamma(\gamma_{pq})}{(-X_{pq})^{\gamma_{pq}}} \\ + Z_{14}Z_{23}\int \mathrm{d}\gamma\, \widetilde{\mathcal{M}}^{(2)}(\gamma) \prod_{1\le p<q\le 4}\frac{\Gamma(\gamma_{pq})}{(-X_{pq})^{\gamma_{pq}}} \\ + Z_{13}Z_{24}\int \mathrm{d}\gamma\, \widetilde{\mathcal{M}}^{(3)}(\gamma) \prod_{1\le p<q\le 4}\frac{\Gamma(\gamma_{pq})}{(-X_{pq})^{\gamma_{pq}}}.\end{aligned} \tag{5.27}$$

These Mellin variables satisfy

$$\gamma_{12}+\gamma_{13}+\gamma_{14}=\Delta_1=3, \quad \text{and cyclic,} \tag{5.28}$$

which relate to the Mellin variables $\mathrm{s},\mathrm{t},\mathrm{u}$ in [42] by

$$\mathrm{s}=\Delta_1+\Delta_2-2\gamma_{12}=6-2\gamma_{12}, \quad \mathrm{t}=6-2\gamma_{23}, \quad \mathrm{u}=6-2\gamma_{13}, \quad \mathrm{s}+\mathrm{t}+\mathrm{u}=12. \tag{5.29}$$

On the support of $\eta_{p'q}=0$, these are related to our χ variables through:

$$\chi_{1,3}=\tau_1+\tau_2-2\gamma_{12}+2\eta_{1'2'}=\mathrm{s}+2\eta_{1'2'}-2=\begin{cases}\mathrm{s}, & \text{in } \widetilde{\mathcal{M}}^{(1)},\\ \mathrm{s}-2, & \text{in } \widetilde{\mathcal{M}}^{(2,3)},\end{cases} \tag{5.30a}$$

$$\chi_{2,4}=\tau_2+\tau_3-2\gamma_{23}+2\eta_{2'3'}=\mathrm{t}+2\eta_{2'3'}-2=\begin{cases}\mathrm{t}, & \text{in } \widetilde{\mathcal{M}}^{(2)},\\ \mathrm{t}-2, & \text{in } \widetilde{\mathcal{M}}^{(1,3)}.\end{cases} \tag{5.30b}$$

The Mellin amplitudes read[20]

$$\begin{gathered}\widetilde{\mathcal{M}}^{(1)}=\mathsf{c}_s\frac{25(\mathrm{t}-\mathrm{u})}{2(\mathrm{s}-2)}+\mathsf{c}_s\frac{4(\mathrm{t}-\mathrm{u})}{2(\mathrm{s}-4)}-\mathsf{c}_t\frac{\mathrm{s}+21}{\mathrm{t}-4}+\mathsf{c}_u\frac{\mathrm{s}+21}{\mathrm{u}-4}-\frac{33}{2}(\mathsf{c}_t-\mathsf{c}_u),\\ \widetilde{\mathcal{M}}^{(2)}=-\widetilde{\mathcal{M}}^{(1)}\big|_{\substack{\mathrm{s}\leftrightarrow\mathrm{t}\\ \mathsf{c}_s\leftrightarrow\mathsf{c}_t}}, \quad \widetilde{\mathcal{M}}^{(3)}=-\widetilde{\mathcal{M}}^{(1)}\big|_{\substack{\mathrm{s}\leftrightarrow\mathrm{u}\\ \mathsf{c}_s\leftrightarrow\mathsf{c}_u}}.\end{gathered} \tag{5.31}$$

Here, the color factors are related to traces through

$$\begin{aligned}\mathsf{c}_s=f^{a_1a_2b}f^{b\,a_3a_4}\propto\ &\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})-\mathrm{tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})\\ &-\mathrm{tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2})+\mathrm{tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2}),\end{aligned} \tag{5.32a}$$

$$\mathsf{c}_t=f^{a_1a_4b}f^{b\,a_2a_3}=-\mathsf{c}_s\big|_{a_2\leftrightarrow a_4}, \qquad \mathsf{c}_u=f^{a_1a_3b}f^{b\,a_4a_2}=-\mathsf{c}_s|_{a_2\leftrightarrow a_3}. \tag{5.32b}$$

To extract the partial amplitude we are considering, one can simply take

$$\mathsf{c}_s\mapsto 1, \qquad \mathsf{c}_t\mapsto -1, \qquad \mathsf{c}_u\mapsto 0, \tag{5.33}$$

[20]Note that the coefficient of $\mathsf{c}_t-\mathsf{c}_u$ in $\widetilde{\mathcal{M}}^{(1)}$ in that paper contains a typo. It should be $-\frac{33}{2}$ instead of -33.

which gives the coefficient of $\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ (up to an overall coefficient). With these dictionaries, we can easily check that our results agree with that of [42] up to an overall normalization:

$$\begin{aligned}
-36\mathcal{M}^{\mathcal{J}}_{1234}\Big|_{\eta_{1'2'}=\eta_{3'4'}=1} &= \frac{25(\chi_{2,4}-3)}{\chi_{1,3}-2}+\frac{8(\chi_{2,4}-2)}{\chi_{1,3}-4}+\frac{\chi_{1,3}+21}{\chi_{2,4}-2}+33 \\
&= \frac{25(\mathrm{t}-5)}{\mathrm{s}-2}+\frac{8(\mathrm{t}-4)}{\mathrm{s}-4}+\frac{\mathrm{s}+21}{\mathrm{t}-4}+33 \\
&= \widetilde{\mathcal{M}}^{(1)}\Big|_{\substack{\mathrm{u}=12-\mathrm{s}-\mathrm{t}\\ \mathtt{c}_s=+1,\mathtt{c}_t=-1,\mathtt{c}_u=0}},
\end{aligned} \tag{5.34}$$

and

$$-36\mathcal{M}^{\mathcal{J}}_{1234}\Big|_{\eta_{1'4'}=\eta_{2'3'}=1} = \widetilde{\mathcal{M}}^{(2)}\Big|_{\substack{\mathrm{u}=12-\mathrm{s}-\mathrm{t}\\ \mathtt{c}_s=+1,\mathtt{c}_t=-1,\mathtt{c}_u=0}}, \tag{5.35}$$

$$-36\mathcal{M}^{\mathcal{J}}_{1234}\Big|_{\eta_{1'3'}=\eta_{2'4'}=1} = \widetilde{\mathcal{M}}^{(3)}\Big|_{\substack{\mathrm{u}=12-\mathrm{s}-\mathrm{t}\\ \mathtt{c}_s=+1,\mathtt{c}_t=-1,\mathtt{c}_u=0}}, \tag{5.36}$$

as can be checked.

5.2.3 $\langle\mathcal{J}\mathcal{J}\mathcal{O}\mathcal{O}\rangle$

We have also computed[21] $\langle\mathcal{J}_1\mathcal{J}_2\mathcal{O}_3\mathcal{O}_4\rangle$ and $\langle\mathcal{J}_1\mathcal{O}_2\mathcal{J}_3\mathcal{O}_4\rangle$ using the same strategy. Since $\hat{\mathbb{x}}\mathcal{M}_{\mathcal{O}_1\mathcal{O}_2\mathcal{J}_R}=0$ and $\hat{\mathbb{x}}^2\mathcal{M}_{\mathcal{O}_1\mathcal{O}_R\mathcal{J}_3}=0$, the exchange contributions have truncated poles:

$$\begin{aligned}
&(\mathcal{M}_{\mathcal{J}_1\mathcal{J}_2\mathcal{O}_3\mathcal{O}_4})_{\text{exchange}} \\
&= \mathcal{K}_{2,0}\frac{\mathcal{M}_{\mathcal{O}_4\mathcal{J}_1\mathcal{O}_R}\mathcal{M}_{\mathcal{J}_2\mathcal{O}_3\mathcal{O}_L}}{\chi_{2,4}-2}+\frac{\mathcal{K}_{2,0}}{4}\frac{(\hat{\mathbb{x}}\mathcal{M}_{\mathcal{O}_4\mathcal{J}_1\mathcal{O}_R})(\hat{\mathbb{x}}\mathcal{M}_{\mathcal{J}_2\mathcal{O}_3\mathcal{O}_L})}{\chi_{2,4}-4}\Bigg|_{\eta_{aL'}=\eta_{iR'}=0} \\
&+ \mathcal{K}_{3,1}\frac{\hat{\mathbb{N}}\Big[\mathcal{M}^{\mathcal{J}}_{12R}\mathcal{M}_{\mathcal{O}_3\mathcal{O}_4\mathcal{J}_L}\Big]}{\chi_{1,3}-2}\Bigg|_{\eta_{aL'}=\eta_{iR'}=0},
\end{aligned} \tag{5.37a}$$

$$\begin{aligned}
&(\mathcal{M}_{\mathcal{J}_1\mathcal{O}_2\mathcal{J}_3\mathcal{O}_4})_{\text{exchange}} \\
&= \mathcal{K}_{2,0}\frac{\mathcal{M}_{\mathcal{O}_4\mathcal{J}_1\mathcal{O}_R}\mathcal{M}_{\mathcal{O}_L\mathcal{O}_2\mathcal{J}_3}}{\chi_{2,4}-2}+\frac{\mathcal{K}_{2,0}}{4}\frac{(\hat{\mathbb{x}}\mathcal{M}_{\mathcal{O}_4\mathcal{J}_1\mathcal{O}_R})(\hat{\mathbb{x}}\mathcal{M}_{\mathcal{O}_L\mathcal{O}_2\mathcal{J}_3})}{\chi_{2,4}-4}\Bigg|_{\eta_{aL'}=\eta_{iR'}=0} \\
&+ \mathcal{K}_{2,0}\frac{\mathcal{M}_{\mathcal{J}_1\mathcal{O}_2\mathcal{O}_R}\mathcal{M}_{\mathcal{J}_3\mathcal{O}_4\mathcal{O}_L}}{\chi_{1,3}-2}+\frac{\mathcal{K}_{2,0}}{4}\frac{(\hat{\mathbb{x}}\mathcal{M}_{\mathcal{J}_1\mathcal{O}_2\mathcal{O}_R})(\hat{\mathbb{x}}\mathcal{M}_{\mathcal{J}_3\mathcal{O}_4\mathcal{O}_L})}{\chi_{1,3}-4}\Bigg|_{\eta_{aL'}=\eta_{iR'}=0}.
\end{aligned} \tag{5.37b}$$

Again, it can be checked that the exchange contribution is already gauge-invariant up to contact terms. The ansatz for the contact contribution can be uniquely fixed by gauge-invariance and dihedral symmetry, and the final results are recorded in an ancillary file. The orthogonal frame result can be extracted by setting $\eta_{p'q}=0$, which agrees with (8.11) of [42] up to an overall normalization.

[21]Due to color-ordering, there are two inequivalent color-ordered amplitudes that we need to compute.

6 Summary and outlook

In this paper, we have proposed a definition of spinning Mellin amplitudes for any number of external spinning operators. Compared with previous attempts in the literature, our definition is easy to use and particularly suitable for the computation/bootstrap of higher-point spinning correlators. Moreover, our definition highlights the similarity between a spinning operator with multiple (two, in the case of symmetric traceless tensors) scalar operators, which generalizes the factorization property of scalar Mellin amplitudes to spinning Mellin amplitudes in a straightforward manner. Following the strategy proposed in [5] of solving Casimir recurrence relations, we have developed a systematic algorithm to obtain the factorization formula for the exchange of spinning operators at any descendant level. We have demonstrated the power of our formalism by bootstrapping three- and four-point gluon amplitudes, with complete polarization dependence, in the model of supergluons in $\mathrm{AdS}_5\times\mathrm{S}^3$. Now that we have a framework to deal with spinning correlators, several natural questions can be addressed.

First, it will be very important to investigate the structure and properties of spinning Mellin amplitudes in more details. Our bootstrap calculations in section 5 suggest that there could be a direct relation between the effective bulk Lagrangian and spinning Mellin amplitudes. It will be very interesting to clarify this relation, or even to find a set of Feynman rules in Mellin space, similar to the scalar Feynman rules in [28]. It is also interesting to explore the question of uniqueness in Mellin space: are certain three-point spinning correlators uniquely characterized by certain fundamental principles such as gauge-invariance or unitary, similar to flat-space amplitudes [52]? Previous studies in position space include [53–56] in specific dimensions and [57–61] in general dimensions. The Mellin formalism may provide a natural language to discuss this problem further.

In addition, a careful analysis of contact Witten diagrams along the lines of [2, 28] could elucidate the correct prescription of taking flat-space limits of spinning Mellin amplitudes. In particular, it will be highly non-trivial to see the appearance of gauge invariance $\mathtt{e}\sim\mathtt{e}+\mathtt{k}$ in flat space. The fact that N-point spinning Mellin amplitudes look like $2N$-point scalar Mellin amplitudes reminds us of the recently introduced "scaffolding" perspective on flat-space spinning amplitudes [62], which could play an important role in understanding the flat-space limit of spinning Mellin amplitudes. Hopefully, the flat-space limit will also be understood physically, as in [4, 63] (also see [64] for a literature review on the flat-space limit in other representations). In practice, a good knowledge of the flat-space limit dictates the correct power-counting of Mellin amplitudes, which will be very helpful in the bootstrap of higher-point correlators.

We are optimistic that our spinning Mellin amplitudes will facilitate the bootstrap of higher-point correlators and the study of their properties in specific theories. On one hand, sub-amplitudes with multiple spinning correlators can now be computed independently, which can be glued together to obtain high-multiplicity amplitudes. On the other hand, we are finally in a position to explore the structure of higher-point spinning Mellin amplitudes. For example, it may be possible to reconcile the BCJ relations observed for spinning correlators in differential representations [65, 66] and the nonexistence of BCJ relations for higher-point scalar correlators in Mellin space [15]. It will also potentially shed new light on AdS double-copy relations [40] beyond four point.

We also hope to generalize our discussion to correlators involving fermions, as was also explored in [27]. This will complete people's understanding of Mellin formalism for arbitrary conformal correlators, and provide the necessary foundation to manifest supersymmetry in Mellin space. The definition would certainly depend on the spacetime dimension d. The most interesting case would be $d = 4$, which relates to a series of important holographic models in AdS_5, including the famous $\mathrm{AdS}_5 \times \mathrm{S}^5$ type IIB supergravity. Both Mellin formalism and supersymmetry play important roles in these models, and it would be nice to find a way to combine these two.

Finally, let us comment on the relation of operators with different spins in our formalism. On the most basic level, the possible kinematic dependence of spinning correlators can be classified using group-theoretical methods, and the building blocks at different values of the spin can be related by weight-shifting operators [67]. These may help clarify the role of $\hat{\mathbb{N}}$ in the factorization formula, which seems to "lower" the spin of the exchanged operator. On a higher and more interesting level, it turns out that mathematically we can relate the N-point spinning Mellin amplitude to a $2N$-point "scalar correlator" H, where we no longer view these scalars auxiliary but actual. Using the pole-pinching mechanism reviewed in subsection 3.1, we have[22]

$$G(X_1, X_{1'}, \cdots, X_N, X_{N'}) = \underset{\substack{\Delta_{1'}=-J_1 \\ \cdots \\ \Delta_{N'}=-J_N}}{\mathrm{Res}'} H(X_1, X_{1'}, \cdots, X_N, X_{N'})\Big|_{X_{p'}\cdot X_p=0}, \tag{6.1a}$$

$$H(X_1, X_{1'}, \cdots, X_N, X_{N'}) = \int [\mathrm{d}\gamma]\, \mathcal{M}(\gamma) \prod_{\substack{u,v\in\mathcal{S} \\ u\prec v}} \frac{\Gamma(\gamma_{uv})}{(-2X_u\cdot X_v)^{\gamma_{uv}}}, \tag{6.1b}$$

$$\forall v \in \mathcal{S} = \{1, 1', \cdots, N, N'\}: \quad \sum_{\substack{u\in\mathcal{S} \\ u\neq v}} \gamma_{uv} = \Delta_r. \tag{6.1c}$$

In other words, integer-spin operators seem to arise as residues of a family of continuous-spin operators constructed out of pairs of null-separated scalars. This looks surprisingly similar to the properties of light-ray operators [68], which suggests the exciting possibility of studying spin-analyticity [11] or energy correlators [69] in the spinning Mellin formalism.

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[22] We make two remarks regarding this relation. First, we should define $\mathcal{G}$ for generic $X_p, X_{p'}$ and only impose $X_{p'}\cdot X_p = 0$ after taking the residues $\Delta_{p'} = -J_p$. Second, the operation Res' takes $(N-3)$ residues and then evaluate the remaining 3 at $\Delta_{p'} = -J_p$. The result is independent of how we split $\{1', \cdots, N'\}$ into $(N-3)$ and 3 variables.

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