Linear Lie Groups All of Whose Irreducible Finite-Dimensional Not Necessarily Unitary Representations Are of Bounded Dimension and Separate the Points of the Group

A. I. Shtern

10.10.2025

Lomonosov Moscow State University,
Department of Mechanics and Mathematics;

NRC "Kurchatov Institute" - SRISA

E-mail: rrowwmail.ru, aishternmtu-net.ru

2020 *Mathematics Subject Classification.* Primary 22A99, Secondary 22A25 22E25.

Abstract

We prove that all linear Lie groups satisfying the conditions listed in the title are finite extensions of commutative Lie groups.

Keywords. Lie groups, finite-dimensionally representable topological group.

1 Introduction

In this note, we prove that every Lie group satisfying the conditions listed in the title is a finite extension of a commutative Lie group admitting the set of continuous characters that separates the elements of the group.

2 Preliminaries

Recall that a locally compact group (in particular, a Lie group) all of whose continuous unitary representations are finite-dimensional and their dimensions are jointly bounded is a finite extension of a commutative locally compact (Lie) group [1], [2].

3 Main Results

Theorem 1 *A linear Lie group for which all its irreducible continuous finite-dimensional not necessarily unitary linear representations are of bounded degree and separate the points of the group is a finite extension of a commutative Lie group.*

Proof. Let G be a Lie group satisfying the conditions of the theorem.

Let G_0 be the connected component of the identity of G. The group G_0 obviously satisfies the conditions of the theorem. Let

$$G_0 = LR$$

be a Levi decomposition of G, where L is a semisimple subgroup of G_0 and R is the radical of G_0.

Then L is the identity group, because otherwise its adjoint group is a quotient group of G_0 which is a nontrivial semisimple Lie group, and hence has continuous finite-dimensional representations of arbitrary large finite dimensions, which contradicts the assumption.

Therefore, G_0 is solvable. A continuous irreducible representation of a connected solvable Lie group is one-dimensional by the Lie theorem. Hence, all irreducible representations of G_0 are one-dimensional. If the connected solvable group G_0 is not commutative, then G_0 has finite-dimensional representations, which contradicts the conditions of the theorem that hold for G_0. Thus, G_0 is commutative.

The quotient group G/G_0 is discrete and satisfies the conditions of the theorem. Then the group algebra $l^1(G/G_0)$ satisfies the identity (1) of Subsec. 3.6.1 in [3] for $r = r(n)$, where the dimensions of the representations mentioned in the condition of the theorem do not exceed n and $r(n)$ is the least value of r for which this identity holds.

By the Gel'fand–Raikov theorem, this group has a set of irreducible representations in Hilbert spaces that separates the points of G/G_0.

As was in fact proved in Proposition 3.6.3 of [3], the dimensions of these representations are bounded by n and, by Thoma's theorem [4], G/G_0 is a finite extension of a commutative normal subgroup M of G/G_0.

The group M is discrete, since G/G_0 is, and hence the (commutative) closure of M with respect to the topology of G coincides with M.

Since the extension H of G_0 by M is a (disconnected) Lie group, it follows that a theorem of Calvin Moore [5] (concerning locally compact groups all of whose continuous unitary representations are finite-dimensional and have bounded dimensions) holds for H.

Hence, if the extension H of G_0 by M is not a finite extension of a commutative Lie group G_0, then the family of irreducible finite-dimensional continuous unitary representations of H either does not have bounded dimensions or does not separate the points of the group, which contradicts the conditions of the theorem for G, since they obviously remain valid for the subgroup H of G.

Thus, H is a finite extension of a commutative Lie group and, therefore, so is G.

Since the maximal commutative Lie subgroup of H also has the properties imposed on G, it follows that this commutative group is finite-dimensionally representable, and hence has the set of continuous characters separating the elements of the group.

This completes the proof of the theorem.

4 Discussion

In [5], we proved a theorem for Hausdorff topological groups all of whose irreducible unitary finite-dimensional representations are finite-dimensional and their dimensions are jointly bounded for which the family of these finite-dimensional representations separates the points of the group: the theorem claims that every group of this kind is a finite extension of a commutative topological group whose continuous characters separate the points of the group.

Funding

The research was supported by the Scientific Research Institute for System Analysis of the National Research Centre "Kurchatov Institute" according to the project FNEF-2024-0001.

References

[1] C. C. Moore, Groups with finite dimensional irreducible representation, *Trans. Amer. Math. Soc.*, **166**, 401–410 (1972).

[2] A. I. Štern, Locally bicompact groups with finite-dimensional irreducible representations, *Math. USSR, Sb.*, **19** (1973), 85–94 (1974).

[3] J. Dixmier, *Les C^*-algébres et leurs représentations*, Gauthier-Villars & Cie, Éditeur-Imprimeur, Paris, 1964.

[4] E. Thoma, Eine Charakterisierung diskreter Gruppen vom Typ I, *Invent. Math.*, **6**, 190–196 (1968).

[5] A. I. Shtern, Topological groups with representations of bounded dimension, *Izvestiya RAN Ser. Mat.* (Russian) (to appear).