Texture-zeros in minimal seesaw from non-invertible symmetry fusion rules

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Abstract

The Z_2 gauging of Z_N symmetry can enforce certain elements of the fermion Yukawa couplings to vanish. We have performed a systematical study of texture zero patterns of lepton mass matrices in the minimal seesaw model, and we present all the possible patterns of the charged lepton Yukawa coupling Y_E, neutrino Yukawa coupling Y_ν, right-handed neutrino mass matrix M_R and the light neutrino mass matrix M_ν which can be derived from the Z_2 gauging of Z_N symmetry. The realization of the textures with the maximum number of zeros and the second maximum number of zeros from non-invertible symmetry is studied, and the phenomenological implications in neutrino oscillation are discussed.

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1 Introduction

It is well-known that the masses and flavor mixing of quarks and leptons exhibit a peculiar hierarchical pattern [1]. The mass spectrum spans more than twelve orders of magnitude, from sub-eV neutrinos to the heavy top quark with a mass about 173 GeV. The quark flavor mixing is described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix which is parameterized by three mixing angles θ^q, θ^q_{13}, θ^q_{23} and one CP-violating phase δ^q_{CP}. It is known experimentally that the three quark angles are small and hierarchical with $\theta^q_{12} \sim \lambda$, $\theta^q_{23} \sim \lambda^2$, $\theta^q_{13} \sim \lambda^3$, where $\lambda \approx 0.225$. In analogy the lepton flavor mixing is described by Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix which are parameterized by three lepton mixing angles θ^ℓ_{12}, θ^ℓ_{13}, θ^ℓ_{23}, one Dirac CP violation phase δ^ℓ_{CP} and two additional Majorana CP phases α_{21} and α_{31} if neutrinos are their own antiparticles [1]. Neutrino oscillation experiments reveal two large mixing angles $\theta^\ell_{12} \approx 33.4^\circ$, $\theta^\ell_{23} \approx 43.5^\circ$ and one relatively small angle $\theta^\ell_{13} \approx 8.6^\circ$ [2]. Similar results are reached in other global analysis of neutrino oscillation data [3, 4]. Understanding the above patterns of masses and mixing angles of the quarks and leptons, along with the sources and magnitude of CP violation, is still a big challenge. The fundamental principle behind the structure of fermion masses and mixing is still unknown, see Refs. [5–8] for recent review and Ref. [9] for a pedagogic introduction on this topic.

Within the Standard Model (SM), the fermion masses and mixing arise from the Yukawa couplings which are free parameters. In the absence of a complete flavor theory, a popular phenomenological approach is to assume specific texture of quark and lepton mass matrices. The simplest attempt is to assume some entries of the fermion mass matrix vanishing in a specific weak-interaction basis [10, 11], it is called texture zero in the literature. The assumption of texture zero reduces the number of free parameters, leading to testable relations among fermion masses, mixing angles, and CP-violating phases. For instance, the Fritzsch six-zero ansatz for the quark mass matrices can give rise to the famous relation $\sin\theta_c \approx \sqrt{m_d/m_s}$, which elegantly related mass ratio of down-quark and strange quark to the Cabibbo angle θ_{C}. Likewise the texture zero of neutrino mass matrix in the charged lepton diagonal basis has been extensively studied [12, 13]. The vanishing entries yield enough constraints so that ones can not only explain the neutrino oscillation data but also make predictions for yet-undetermined parameters such as the absolute neutrino mass scale, the ordering of neutrino masses or the lepton CP violating phases [14, 15].

From a theoretical standpoint, the texture zeros may originate from the spontaneous or explicit breaking of Abelian [16, 17] or non-Abelian flavor symmetries [18–21], or from the structure of higher-dimensional operators in effective field theories. The texture zero may reflect some underlying selection rules. In the last several years, the concept of symmetry was generalized [22] and the applications of non-invertible symmetries to particle physics have been extensively discussed, see Refs. [23–27] for recent review on non-invertible symmetries. It was found that the fusion rules of the non-invertible symmetry can naturally give rise to texture zeros in Yukawa couplings [28–30]. The quark Yukawa textures from non-invertible symmetry has been explored in [28,29]. It is notable that one can use non-invertible symmetry allows to construct certain texture-zero pattern for quark mass matrices, which can resolve the strong CP problem without invoking the axion [31, 32]. The possible textures of neutrino and charged lepton mass matrices are studied under the assumption that the light neutrino masses are described by the Weinberg operator [30].

The discovery of neutrino oscillations shows that neutrinos have masses. To account for the tiny neutrino masses, the most straightforward and elegant way is to extend the SM by introducing right-handed neutrinos which are singlets of SM. Although the number of right-handed neutrinos is arbitrary in principle, at least two are necessary to explain the two mass-squared differences observed by solar and atmospheric neutrino oscillation experiments. This is the so-called minimal seesaw model [33,34] in which the lightest neutrino is massless, namely $m_1 = 0$ for normal neutrino mass ordering (NO) and $m_3 = 0$ for inverted neutrino mass ordering (IO). Although the minimal seesaw model can accommodate tiny neutrino masses by extending the SM at the least cost, it can not predict neither the lepton mixing parameters nor the neutrino mass spectrum. The number of free parameters in the minimal seesaw model is still larger than the number of lower energy observables. One way of reducing the number of free parameters and increasing the predictive power is to assume texture zero in the lepton Yukawa couplings and mass matrix of right-handed neutrinos, which could possibly be realized by imposing $U(1)$ or Z_n Abelian flavor symmetry [16,35,36].

In the present work, by using Z_2 gauging of Z_N symmetries, we shall study the patterns of the texture zeros of the lepton Yukaka matrices and the right-handed neutrino mass matrix in the framework of minimal seesaw model. It is notable that there are some texture zero patterns which can not be derived from the conventional Abelian flavor symmetry. The corresponding phenomenological predictions for lepton mixing parameters are studied.

The remaining part of this paper is organized as follows. We recapitulate the formalism of the Z_2 gauging of Z_N symmetries in section 2, and the non-invertible selection rules are presented. In section 3, we perform a comprehensive study of texture zero of charged lepton mass matrix, Dirac neutrino mass matrix and right-handed neutrino mass matrix which can be obtained from Z_2 gauging of Z_N symmetries with $N = 3, 4, 5$. Furthermore, we identify the viable textures with the maximum number of zeros and the second maximum number of zeros in section 4, and the phenomenological implications in neutrino oscillation experiments are discussed. The new textures of the charged lepton and neutrino Yukawa couplings, which can be generated from the non-invertible Z_N symmetry for $N \geq 5$, are discussed. Finally we summarize our main conclusion in section 5. We list the possible textures of the neutrino Yukawa coupling Y_ν, right-handed neutrino mass matrix M_R and the corresponding light neutrino mass matrix M_ν which can be derived from the Z_2 gauging of Z_N symmetry for $N = 4, 5$ in Appendix A. Similarly the Z_2 gauging of Z_4 and Z_5 symmetries restricts the allowed forms of the charged lepton Yukawa coupling Y_E which are presented in Appendix B. The texture zeros of the Yukawa couplings Y_E and Y_ν derived from Z_2 gauging of Z_N symmetries for $N = 6, 7$ are collected in Appendix C.

2 Non-invertible selection rules

In ordinary group-theoretic symmetries, the multiplication law takes the form

$$ab = c \tag{2.1}$$

where a, b and c are elements of a group G. In the case that G is a Abelian group, if a field ϕ_a transforms in the representation corresponding to a, and similarly for ϕ_b and ϕ_c, then the process $\phi_a + \phi_b \to \phi_c$ is allowed when $ab = c$. Otherwise, it is forbidden. Different fields may correspond to the same group element, in which case the same selection rule applies. This yields unique and deterministic coupling constraints since the right-hand side of a product is fixed.

Non-invertible symmetries generalize this notion by replacing the unique group multiplication with fusion rules of the form

$$ab = \sum_c N_{ab}^c c \tag{2.2}$$

where a, b and c are symmetry operators and $N_{ab}^c \in \mathbb{Z}_{\geq 0}$ denoted fusion coefficients. A process $\phi_a + \phi_b \to \phi_c$ is allowed if $N_{ab}^c \neq 0$, and forbidden otherwise. Crucially, the product ab need not yield a single output. The right-hand side may contain multiple operators, reflecting the non-invertibility of the symmetry. The richer algebraic structure leads to selection rules that differ qualitatively from those of ordinary group theory.

A concrete and physically motivated realization of such fusion rules in four dimensional quantum field theories arises from gauging outer automorphisms of a symmetry group [37,38]. In particular, explicit models can be constructed by performing the simplest Z_2 gauging of a Z_N symmetry [28,39]. Let g be a generator of Z_N, so that $g^N = \mathbb{1}$. We consider the Z_2 outer automorphism

$$eg^k e^{-1} = g^k\,, \qquad rg^k r^{-1} = g^{-k}\,. \tag{2.3}$$

Using this automorphism, one defines equivalence classes

$$[g^k] = \{g^k, g^{-k}\} \qquad k = 0, 1, \cdots \left\lfloor \frac{N}{2} \right\rfloor\,, \tag{2.4}$$

where $\lfloor x \rfloor$ denotes the floor function, i.e., the greatest integer less than or equal to x. In the following, we use the $[k]$ as a shorthand notation for the conjugacy class $[g^k]$, for simplicity. The fusion rules among these classes take the form

$$[k][k'] = [k + k'] + [k - k']\,. \tag{2.5}$$

Which is precisely the algebra obtained by Z_2 gauging of Z_N. It can be observed that the group Z_N with its outer automorphisms $\{e, r\}$ is isomorphic to the dihedral group $D_M \simeq Z_N \rtimes \mathbb{Z}_2$. The above fusion rules form a subset of the fusion rules among the conjugacy classes of the dihedral group D_M [40]. Such Z_2 gauging of Z_N can be engineered in higher-dimensional field theories in string compactifications. For instance, magnetized compactifications naturally yield Z_N flavor symmetries.

Coupling selection rules now follow directly from the fusion algebra. A two point coupling $\phi_{k_1}\phi_{k_2}$ is allowed if $[0]$ appears on the right-hand side of the fusion of $[k_1]$ and $[k_2]$, which occurs when

$$[0] \in \{[k_1 + k_2], [k_1 - k_2]\} \tag{2.6}$$

which is equivalent to the condition

$$k_1 \pm k_2 = 0 \pmod N\,. \tag{2.7}$$

This implies, in particular, that two-point couplings, including mass terms and kinetic terms, are permitted only between fields in the same class. For the three-point couplings, a term of the form $\phi_{k_1}\phi_{k_2}\phi_{k_3}$ is allowed if

$$[0] \in \{[k_1+k_2+k_3], [k_1+k_2-k_3], [k_1-k_2+k_3], [k_1-k_2-k_3]\} \tag{2.8}$$

which is equivalent to

$$k_1 \pm k_2 \pm k_3 = 0 \pmod M \,. \tag{2.9}$$

More generally, an n-point coupling $\phi_{k_1}\cdots\phi_{k_n}$ is allowed if

$$\sum_{i=1}^{n} \pm k_i = 0 \pmod M \,. \tag{2.10}$$

These generalized selection rules reflect the non-invertible nature of the symmetry. They are not simply additive charge conservation laws, but are instead dictated by multi-valued fusion structure originating from Z_2 gauging.

3 Minimal seesaw model and texture zeros of lepton mass matrices from non-invertible selection rules

In the present work, we focus on the minimal seesaw model featuring only two generations of right-handed neutrinos denoted as ν_{R1} and ν_{R2} [33,34], see Ref. [41] for a review. The gauge invariant Lagrangian for the charged lepton and neutrino masses can be written as

$$\mathcal{L}_{\text{lepton}} = -(Y_E)_{ij}\overline{\ell_{Li}}HE_{Rj} - (Y_\nu)_{ij}\overline{\ell_{Li}}\widetilde{H}\nu_{Rj} - \frac{1}{2}(M_R)_{ij}\overline{\nu^C_{Ri}}\nu_{Rj}\,, \tag{3.1}$$

where $i,j=1,2,3$ are the flavor indices, ℓ_L and H stand for the SM left-handed lepton doublets and Higgs doublet respectively, E_R denotes the right-handed charged lepton fields. We have defined $\widetilde{H}=i\sigma_2H^*$, with σ_2 the second Pauli matrix, and $\nu^C_{Ri}=\mathcal{C}\overline{\nu_{Ri}}^T$ with $\mathcal{C}=i\gamma^0\gamma^2$ being the charge conjugation matrix. Moreover, the charged lepton Yukawa coupling matrix Y_E and the neutrino Yukawa coupling matrix Y_ν are 3×3 and 3×2 general complex matrix, and the right-handed neutrino mass M_R is 2×2 symmetric and complex matrix.

After the electroweak gauge symmetry spontaneous breaking by the vacuum expectation value of the Higgs field $\langle H\rangle = \left(0, v/\sqrt{2}\right)^T$ with $v \simeq 246$ GeV, the charged lepton and Dirac neutrino mass matrices are obtained as

$$M_E = \frac{1}{\sqrt{2}}Y_E v\,,\quad M_D = \frac{1}{\sqrt{2}}Y_\nu v\,. \tag{3.2}$$

In the limit $M_R \gg v$, integrating out the heavy right-handed neutrinos, the effective light neutrino mass matrix is given by the seesaw formula

$$M_\nu = -M_D M_R^{-1} M_D^T = -\frac{1}{2}v^2 Y_\nu M_R^{-1} Y_\nu^T\,. \tag{3.3}$$

Both M_E and M_ν can be diagonalized by bi-unitary transformations,

$$U_{\ell_L}^\dagger M_E U_{E_R} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad U_{\nu_L}^\dagger M_\nu U_{\nu_L}^* = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{3.4}$$

The mismatch between the flavor eigenstates and mass eigenstates results in the lepton mixing matrix in the charged current interaction, i.e.

$$U = U_{\ell_L}^\dagger U_{\nu_L}. \tag{3.5}$$

Here U is the so-called Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, and it is parameterized as [1]

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{pmatrix}, \tag{3.6}$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, $\theta_{ij} \in [0, \pi/2)$ are the three lepton mixing angles, $\delta_{CP} \in [0, 2\pi)$ is the Dirac CP violation phase, and α_{21}, α_{31} are two Majorana CP violation phases. In the minimal seesaw model, the lightest neutrino is massless so that we have $m_1 = 0$ for normal ordering (NO) neutrino mass spectrum and $m_3 = 0$ for inverted ordering (IO) neutrino mass spectrum [33, 34]. As a consequence, α_{21} is the unique physical Majorana CP phase and α_{31} can be absorbed by field redefinition.

In the following, we study which texture zero patterns of the Yukawa matrices Y_E, Y_ν and the mass matrix M_R can be derived from the Z_2 gauging of Z_N symmetries with $N = 3, 4, 5$. The three generations of lepton fields ℓ_{Li}, E_{Ri} $(i = 1, 2, 3)$, and the two right-handed neutrinos ν_{Rj} $(j = 1, 2)$ as well as the Higgs field H are assigned to certain classes. Then the textures of the lepton mass matrices can be straightforwardly obtained by applying the non-invertible selection rules described in section 2. Since none of the charged lepton masses is vanishing, the rank of the Y_E should be equal to three. We are interested in the lepton mass matrices with vanishing entries, all entries are non-zero or at least one column of the mass matrix are vanishing if three generations of left-handed doublets ℓ_L correspond to the same class. Consequently we discard this kind of assignment in the following. Analogously we neglect the cases that either the three right-handed fields E_{Ri} are assigned to the same class.

3.1 $N = 3$

When $N = 3$, there are two classes, [0] and [1]. From Eq. (2.5), we see that the relevant fusion rules are

$$[0] \times [0] = [0], \quad [0] \times [1] = [1], \quad [1] \times [1] = [0] + [1]. \tag{3.7}$$

In order to obtain a full-rank charged-lepton mass matrix containing texture zeros, the three generations of left-handed lepton doublets cannot simultaneously correspond to the same conjugacy class. Thus we only have the following possible assignments,

$$\ell_L \sim ([0],[0],[1]) \quad \text{or} \quad ([1],[1],[0]) \tag{3.8}$$

or their permutations. The above constraint equally applies to the three generations of right-handed charged leptons, which means

$$E_R \sim ([0],[0],[1]) \quad \text{or} \quad ([1],[1],[0]) \tag{3.9}$$

or their permutations. For simplicity, the two generations of right-handed neutrino are assigned to the following classes

$$\nu_R \sim ([0],[1])\,, \tag{3.10}$$

including their permutations, consequently the right-handed neutrino mass matrix M_R is diagonal, the off-diagonal entry is constrained to be vanishing. The Higgs field can correspond to either [0] or [1]. Then, using the fusion rule of Eq. (3.7), we can determine the textures of Y_E, M_R, Y_ν and $M_\nu = -M_D M_R^{-1} M_D^T$, and the results are shown in table 1 and table 2. It is obvious that the light neutrino mass matrix M_ν is block diagonal or none entry is equal to zero. From table 1, we see that four types of full-rank Y_E can be obtained as follow,

$$Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \quad \text{for} \quad (\ell_L; E_R; H) \sim ([0],[0],[1];[0],[0],[1];[0])\,, \quad ([1],[1],[0];[1],[1],[0];[0])\,, \tag{3.11}$$

$$Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{pmatrix} \quad \text{for} \quad (\ell_L; E_R; H) \sim ([0],[0],[1];[1],[1],[0];[1])\,, \tag{3.12}$$

$$Y_E = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & 0 & \times \end{pmatrix} \quad \text{for} \quad (\ell_L; E_R; H) \sim ([1],[1],[0];[0],[0],[1];[1])\,, \tag{3.13}$$

$$Y_E = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & 0 \end{pmatrix} \quad \text{for} \quad (\ell_L; E_R; H) \sim ([1],[1],[0];[1],[1],[0];[1])\,, \tag{3.14}$$

including their permutations of rows and columns. In the simplest scenario of Z_N traditional flavor symmetry, Q_ψ denotes the Z_N charge of the field ψ. Obviously the Z_N charges of lepton fields satisfy the following identity,

$$\left(Q_{\bar{\ell}_{Li}} + Q_{E_{Ri}}\right) + \left(Q_{\bar{\ell}_{Lj}} + Q_{E_{Rj}}\right) = \left(Q_{\bar{\ell}_{Li}} + Q_{E_{Rj}}\right) + \left(Q_{\bar{\ell}_{Lj}} + Q_{E_{Ri}}\right) \tag{3.15}$$

for any flavor indices i, j. As a consequence, if the off-diagonal (ij) and (ji) entries are non-vanishing, the diagonal entries (ii) and (jj) would be zero or non-zero simultaneously. The same holds true if both (ij) and (ji) entries are vanishing. Thus the sub-blocks $\begin{pmatrix} \times & \times \\ \times & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & \times \\ \times & \times \end{pmatrix}$, $\begin{pmatrix} \times & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & \times \end{pmatrix}$ in the Yukawa matrix is forbidden in the simplest realization of Z_N flavor symmetry[1]. It is notable that the Z_2 gauging of Z_N symmetry can give rise to certain textures which are inaccessible within the conventional Z_N symmetry. For instance, the textures in Eqs. (3.12, 3.13, 3.14) cannot be produced from the Z_N symmetry.

	$Y_E\,,\ H \sim [0]$	$Y_E\,,\ H \sim [1]$
$\ell_L \sim ([0],[0],[1])$ $E_R \sim ([0],[0],[1])$	$\begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & \times \end{pmatrix}$
$\ell_L \sim ([0],[0],[1])$ $E_R \sim ([1],[1],[0])$	$\begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix}$	$\begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{pmatrix}$
$\ell_L \sim ([1],[1],[0])$ $E_R \sim ([0],[0],[1])$	$\begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix}$	$\begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & 0 & \times \end{pmatrix}$
$\ell_L \sim ([1],[1],[0])$ $E_R \sim ([1],[1],[0])$	$\begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}$	$\begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & 0 \end{pmatrix}$

Table 1: The possible patterns of the charged lepton Yukawa matrix Y_E for different assignments of the fields ℓ_L, E_R and H in the Z_2 gauging of Z_3 symmetry, where the symbols "$\times$" denotes non-vanishing elements.

[1]Notice that these textures could be achievable if ones introduce multiple Higgs fields charged under the Abelian flavor symmetry [16] or there are some flavons in the context of Supersymemtry [35].

		M_R	Y_ν	M_ν
$\ell_L \sim ([0],[0],[1])$ $\nu_R \sim ([0],[1])$	$H \sim [0]$	$\begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix}$	$\begin{pmatrix} \times & 0 \\ \times & 0 \\ 0 & \times \end{pmatrix}$	$\begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}$
	$H \sim [1]$		$\begin{pmatrix} 0 & \times \\ 0 & \times \\ \times & \times \end{pmatrix}$	$\begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}$
$\ell_L \sim ([1],[1],[0])$ $\nu_R \sim ([0],[1])$	$H \sim [0]$	$\begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix}$	$\begin{pmatrix} 0 & \times \\ 0 & \times \\ \times & 0 \end{pmatrix}$	$\begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}$
	$H \sim [1]$		$\begin{pmatrix} \times & \times \\ \times & \times \\ 0 & \times \end{pmatrix}$	$\begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}$

Table 2: The possible patterns of Y_ν, M_R and M_ν for different assignments of ℓ_L, ν_R and H in the Z_2 gauging of Z_3 symmetry, where "$\times$" denotes non-zero entry.

3.2 $N = 4$

When $N = 4$, there are three classes, [0], [1] and [2]. The fusion rules are

$$\begin{aligned} &[0]\times[0]=[0]\,, \quad [0]\times[1]=[1]\,, \quad [0]\times[2]=[2]\,, \\ &[1]\times[1]=[0]+[2]\,, \quad [1]\times[2]=[1]\,, \quad [2]\times[2]=[0]\,. \end{aligned} \tag{3.16}$$

The three generations of left-handed lepton doublets can be assigned to the following classes:

$$\begin{aligned} \ell_L \sim \; & ([0],[1],[2])\,, \quad ([0],[0],[1])\,, \quad ([0],[0],[2])\,, \quad ([1],[1],[0])\,, \\ & ([1],[1],[2])\,, \quad ([2],[2],[0])\,, \quad ([2],[2],[1])\,, \end{aligned} \tag{3.17}$$

up to their permutations. Similarly, the right-handed charged-leptons can correspond to

$$\begin{aligned} E_R \sim \; & ([0],[1],[2])\,, \quad ([0],[0],[1])\,, \quad ([0],[0],[2])\,, \quad ([1],[1],[0])\,, \\ & ([1],[1],[2])\,, \quad ([2],[2],[0])\,, \quad ([2],[2],[1])\,, \end{aligned} \tag{3.18}$$

including their permutations. The two generations of right-handed neutrino are assigned to the non-invertible classes in the following patterns:

$$\nu_R \sim ([0],[1])\,, \quad ([0],[2])\,, \quad ([1],[2])\,, \tag{3.19}$$

or their permutations. The Higgs field can correspond to either [0], [1] or [2]. The possible patterns of Y_ν, M_R and M_ν are summarized in table 4 and table 5 respectively. In a similar way, the textures of the charged lepton Yukawa matrix Y_E can be reached[2]. After enumerating

[2]The possible textures of charged-lepton Yukawa couplings from non-invertible Z_4 symmetry have been studied in [30]

all possible assignments of left- and right-handed lepton fields and Higgs ($7 \times 7 \times 3 = 147$ assignments), we have found two distinct full-rank Y_E textures:

$$Y_E = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \tag{3.20}$$

up to independent permutations of rows and columns. The corresponding transformations of lepton fields ℓ_L, E_R and Higgs field H under non-invertible symmetry are given in Appendix B. Note that the above two textures in Eq. (3.20) can also be produced by conventional Z_N symmetry. As regard the neutrino sector, the textures of Y_ν, M_R and M_ν for different assignments of ℓ_L, ν_R and H are collected in Appendix A.

3.3 $N = 5$

When $N = 5$, there are three classes [0], [1] and [2]. The fusion rules are

$$\begin{aligned} [0] \times [0] = [0]\,, \quad [0] \times [1] = [1]\,, \quad [0] \times [2] = [2]\,, \\ [1] \times [1] = [0] + [2]\,, \quad [1] \times [2] = [1] + [2]\,, \quad [2] \times [2] = [0] + [1]\,. \end{aligned} \tag{3.21}$$

It can be observed that the fusion algebra for $N = 5$ possesses a symmetry. Specifically, the fusion rules remain invariant under the exchange $[1] \leftrightarrow [2]$. This invariance arises because the group Z_5 admits four distinct non-trivial automorphisms, but only a Z_2 subgroup of the full automorphism group is gauged.

The three generations of left-handed lepton doublets and right-handed charged-lepton correspond to the classes $[k_1]$, $[k_2]$, $[k_3]$ with $k_1 \neq k_2 \neq k_3$, there are totally seven possibilities as follows,

$$\begin{aligned} \ell_L, E_R \sim \; & ([0],[1],[2])\,, \quad ([0],[0],[1])\,, \quad ([0],[0],[2])\,, \quad ([1],[1],[0])\,, \\ & ([1],[1],[2])\,, \quad ([2],[2],[0])\,, \quad ([2],[2],[1])\,, \end{aligned} \tag{3.22}$$

up to permutations. The two generations of right-handed neutrino can be assigned to:

$$\nu_R \sim ([0],[1])\,, \quad ([0],[2])\,, \quad ([1],[2])\,, \tag{3.23}$$

including their permutations. The Higgs field H can be [0], [1] or [2]. Then, we can examine which entries of M_R, Y_ν and $M_\nu = -M_D M_R^{-1} M_D^T$ are allowed by our selection rules. The possible textures of Y_ν, M_R and M_ν are shown in table 6 and table 7 in Appendix A. Meanwhile, after enumerating all possible cases of left- and right-handed charged-lepton and higgs ($7 \times 7 \times 3 = 147$ assignments)[3], we have found eight distinct full-rank Y_E textures:

$$Y_E = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix}, \quad \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \times \\ \times & \times & 0 \\ \times & \times & \times \end{pmatrix},$$

[3]The possible textures of charged-lepton Yukawa couplings non-invertible Z_5 symmetry have been studied in [30]

$$\begin{pmatrix} 0 & \times & \times \\ 0 & \times & \times \\ \times & 0 & \times \end{pmatrix}, \quad \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & 0 & \times \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & 0 \end{pmatrix}, \tag{3.24}$$

up to permutations of rows and columns. Notice that the second texture in the first line of Eq. (3.24) is the so-called nearest neighbor interaction texture [42–44]. The Appendix B enumerates all possible assignments of the fields ℓ_L, E_R, H which are capable of generating these textures. One sees that the first and third textures in Eq. (3.24) can be produced by the traditional Z_N symmetry, while the other textures cannot.

4 Textures of lepton mass matrices and phenomenological implications

In this section, we study the phenomenological implications of texture zeros in Y_E, Y_ν, and M_R which are derived from the fusion rules of the Z_2 gauging of the Z_N symmetry with $N = 3, 4, 5$. We are interested in the textures of the lepton mass matrices with maximum number of zeros and the second maximum number of zeros, and the charged-lepton masses and neutrino oscillation data are required to be accommodated.

4.1 Texture zeros of neutrino mass matrix with diagonal charged-lepton mass matrix

Since none of the three charged leptons are massless, the Yukawa matrix Y_E should be of full-rank. As a result, Y_E can contain at most six texture zeros and it can be taken to be diagonal by using the freedom of field redefinition of lepton fields ℓ_L and E_R, i.e.

$$Y_E = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}. \tag{4.1}$$

The textures of Y_ν with three or more zeros for a generic right-handed neutrino mass matrix M_R will lead to vanishing lepton mixing angles or two massless neutrinos, being therefore excluded experimentally. Hence Y_ν can have at most two zero elements. Moreover, textures with two zeros placed in the same row (or column) in Y_ν would lead to two (or one) vanishing lepton mixing angle, they are also excluded by present neutrino data. Thus there are only six Y_ν textures with two zeros [36]:

$$T_1 = \begin{pmatrix} 0 & \times \\ \times & 0 \\ \times & \times \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & \times \\ \times & \times \\ \times & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} \times & \times \\ 0 & \times \\ \times & 0 \end{pmatrix},$$

$$T_4 = \begin{pmatrix} \times & 0 \\ 0 & \times \\ \times & \times \end{pmatrix}, \quad T_5 = \begin{pmatrix} \times & 0 \\ \times & \times \\ 0 & \times \end{pmatrix}, \quad T_6 = \begin{pmatrix} \times & \times \\ \times & 0 \\ 0 & \times \end{pmatrix}. \tag{4.2}$$

From table 2 and table 4, we can see that the non-invertible Z_3 and Z_4 symmetry can not produce any of these six textures $T_1 \sim T_6$.

As regards the Z_2 gauging of Z_5 symmetry, the corresponding fusion rules in Eq. (3.21) allow the following three-point couplings,

$$\phi_0\phi_0\phi_0\,, \quad \phi_0\phi_1\phi_1\,, \quad \phi_0\phi_2\phi_2\,, \quad \phi_1\phi_1\phi_2\,, \quad \phi_1\phi_2\phi_2\,, \tag{4.3}$$

where ϕ_k corresponds to the class $[k]$. As shown in table 6, all the six textures $T_1 \sim T_6$ can be generated from the non-invertible symmetry of $N = 5$ for the following assignments of lepton fields and Higgs field,

$$\begin{aligned}
T_1 &: \begin{pmatrix} 0 & \times \\ \times & 0 \\ \times & \times \end{pmatrix} \quad \text{for} \quad (\ell_L; \nu_R; H) \sim ([0],[1],[2];[2],[1];[1])\,,\ ([0],[2],[1];[1],[2];[2])\,, \\
T_2 &: \begin{pmatrix} 0 & \times \\ \times & \times \\ \times & 0 \end{pmatrix} \quad \text{for} \quad (\ell_L; \nu_R; H) \sim ([0],[2],[1];[2],[1];[1])\,,\ ([0],[1],[2];[1],[2];[2])\,, \\
T_3 &: \begin{pmatrix} \times & \times \\ 0 & \times \\ \times & 0 \end{pmatrix} \quad \text{for} \quad (\ell_L; \nu_R; H) \sim ([2],[0],[1];[2],[1];[1])\,,\ ([1],[0],[2];[1],[2];[2])\,, \\
T_4 &: \begin{pmatrix} \times & 0 \\ 0 & \times \\ \times & \times \end{pmatrix} \quad \text{for} \quad (\ell_L; \nu_R; H) \sim ([0],[1],[2];[1],[2];[1])\,,\ ([0],[2],[1];[2],[1];[2])\,, \\
T_5 &: \begin{pmatrix} \times & 0 \\ \times & \times \\ 0 & \times \end{pmatrix} \quad \text{for} \quad (\ell_L; \nu_R; H) \sim ([0],[2],[1];[1],[2];[1])\,,\ ([0],[1],[2];[2],[1];[2])\,, \\
T_6 &: \begin{pmatrix} \times & \times \\ \times & 0 \\ 0 & \times \end{pmatrix} \quad \text{for} \quad (\ell_L; \nu_R; H) \sim ([2],[0],[1];[1],[2];[1])\,,\ ([1],[0],[2];[2],[1];[2])\,.
\end{aligned} \tag{4.4}$$

Notice that that the same texture of Y_ν is obtained under the interchange of [1] and [2], since the fusion rules in Eq. (3.21) for $N = 5$ enjoy this symmetry.

Then we proceed to discuss the texture of the right-handed neutrino mass matrix M_R. It is notable that only two textures can be obtained from a generic Z_2 gauging of the Z_N symmetries:

$$R_1 = \begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix} \quad \text{for} \quad \nu_R \sim ([k_1], [k_2])\,, \quad k_1 \pm k_2 \neq 0\,,$$
$$R_2 = \begin{pmatrix} \times & \times \\ \times & \times \end{pmatrix} \quad \text{for} \quad \nu_R \sim ([k_1], [k_2])\,, \quad k_1 \pm k_2 = 0\,. \tag{4.5}$$

From table 6, we observe that the texture R_1 of the right-handed neutrino mass matrix can really be produced from non-invertible Z_5 symmetry for the following assignments of ν_R:

$$R_1 = \begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix} \text{ for } \nu_R \sim ([0],[1])\,, ([0],[2])\,, ([1],[2])\,, ([1],[0])\,, ([2],[0])\,, ([2],[1])\,. \tag{4.6}$$

However, the second texture R_2, when combined with $T_1 \sim T_6$, would result in a light neutrino mass matrix $M_\nu = -v^2 Y_\nu M_R^{-1} Y_\nu^T$ which doesn't have vanishing entry. Consequently it is uninteresting to us in light of texture zero.

From Eq. (B.4), we see that the desired diagonal charged-lepton Yukawa matrix Y_E can also be obtained from non-invertible Z_5 symmetry as:

$$Y_E = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \quad \text{when} \quad (\ell_L; E_R; H) \sim ([0],[1],[2];[0],[1],[2];[0])\,. \tag{4.7}$$

The Yukawa coupling Y_E is also diagonal if one interchanges the assignments for the three generations of ℓ_L and E_R simultaneously. Note that diagonal charged-lepton Yukawa coupling Y_E requires the Higgs to correspond to the class $[0]$, while the neutrino Yukawa coupling $T_1 \sim T_6$ demands the Higgs in the class $[1]$ or $[2]$. As a result, one could work in the framework of supersymmetric standard model, in which two Higgs fields H_u and H_d are needed for holomorphicity of supersymmetry and the cancellation of chiral anomalies. H_u and H_d couple with the neutrino and changed lepton Yukawa couplings respectively, and they can correspond to different classes of non-invertible symmetry.

For the texture R_1 of M_R and the six textures $T_1 \sim T_6$ of Y_ν, using the seesaw formula $M_\nu = -\frac{1}{2} v^2 Y_\nu M_R^{-1} Y_\nu^T$ in Eq. (3.3), we find that the resulting light neutrino mass matrix has one zero element:

$$Y_\nu = T_1 \text{ or } T_4, \quad M_R = R_1, \quad M_\nu = \begin{pmatrix} \times & 0 & \times \\ 0 & \times & \times \\ \times & \times & \times \end{pmatrix}\,, \tag{4.8a}$$

$$Y_\nu = T_2 \text{ or } T_5, \quad M_R = R_1, \quad M_\nu = \begin{pmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}, \tag{4.8b}$$

$$Y_\nu = T_3 \text{ or } T_6, \quad M_R = R_1, \quad M_\nu = \begin{pmatrix} \times & \times & \times \\ \times & \times & 0 \\ \times & 0 & \times \end{pmatrix}. \tag{4.8c}$$

One see that the off-diagonal entry (12), (13) or (23) of M_ν is vanishing[4]. In all these cases, the charged lepton mass matrix is diagonal so that the lepton mixing completely arises from the neutrino sector with $U = U_{\nu_L}$. From Eq. (3.4), we obtain

$$M_\nu = U\,\mathrm{diag}(m_1, m_2, m_3)U^T\,, \tag{4.9}$$

which implies

$$(M_\nu)_{\alpha\beta} = \sum_{i=1,2,3} m_i U_{\alpha i} U_{\beta i}\,. \tag{4.10}$$

Consequently the condition of vanishing entry $(M_\nu)_{\alpha\beta} = 0$ imposes relations among the neutrino parameters. To be more specific, we have

$$\begin{aligned} \text{NO}: \quad & \frac{m_2}{m_3} = -\frac{U_{\alpha 3}U_{\beta 3}}{U_{\alpha 2}U_{\beta 2}}\,, \\ \text{IO}: \quad & \frac{m_1}{m_2} = -\frac{U_{\alpha 2}U_{\beta 2}}{U_{\alpha 1}U_{\beta 1}}\,, \end{aligned} \tag{4.11}$$

in minimal seesaw model, where we have taken into account $m_1 = 0$ for NO and $m_3 = 0$ for IO neutrino masses. It is known that the neutrino oscillation experiments can only measure the neutrino mass squared differences $\Delta m^2_{21} \equiv m^2_2 - m^2_1$ in solar neutrino experiments and $\Delta m^2_{31} \equiv m^2_3 - m^2_1$ in atmospheric neutrino experiments. Thus we have

$$\begin{aligned} \text{NO}: \frac{\Delta m^2_{21}}{\Delta m^2_{31}} &= \left|\frac{U_{\alpha 3}U_{\beta 3}}{U_{\alpha 2}U_{\beta 2}}\right|^2, \\ \text{IO}: \frac{\Delta m^2_{21}}{|\Delta m^2_{31}|} &= \left|\frac{U_{\alpha 1}U_{\beta 1}}{U_{\alpha 2}U_{\beta 2}}\right|^2 - 1\,. \end{aligned} \tag{4.12}$$

When the lepton mixing parameters and the mass squared differences Δm^2_{21} and Δm^2_{31} are limited in the experimentally allowed 3σ regions [45], only the first two cases in Eqs. (4.8a, 4.8b) are viable and the constraint in Eq. (4.12) can be satisfied if the neutrino masses are IO [36]. The JUNO experiment is expected to be able to determine the neutrino mass ordering at a $3-4\sigma$ significance after six years running [46], thus the prediction for IO ordering can be tested in near future. With the standard parametrization of the lepton mixing matrix

[4] The light neutrino mass matrix M_ν is a symmetric matrix satisfying $(M_\nu)_{ij} = (M_\nu)_{ji}$.

in Eq. (3.6), for IO neutrino mass spectrum, one can express the Dirac CP violation phase $\cos\delta_{CP}$ in terms of lepton mixing angles and the ratio $\Delta m^2_{21}/|\Delta m^2_{31}|$ as follows

$$(M_\nu)_{12} = 0 : \cos\delta_{CP} = \frac{[(r_m+1)s^4_{12} - c^4_{12}]s^2_{13}s^2_{23} + r_m s^2_{12}c^2_{12}c^2_{23}}{2(r_m s^2_{12}+1)s_{12}c_{12}s_{13}s_{23}c_{23}}\,, \tag{4.13a}$$

$$(M_\nu)_{13} = 0 : \cos\delta_{CP} = -\frac{[(r_m+1)s^4_{12} - c^4_{12}]s^2_{13}c^2_{23} + r_m s^2_{12}c^2_{12}s^2_{23}}{2(r_m s^2_{12}+1)s_{12}c_{12}s_{13}s_{23}c_{23}}\,, \tag{4.13b}$$

$$(M_\nu)_{23} = 0 : \cos\delta_{CP} \approx -\frac{[(r_m+1)c^4_{12} - s^4_{12}]c_{12}s_{23}c_{23}}{2s^3_{12}s_{13}\cos 2\theta_{23}}\,, \tag{4.13c}$$

where $r_m \equiv \frac{\Delta m^2_{21}}{|\Delta m^2_{31}|}$. Freely varying θ_{12}, θ_{13}, θ_{23} and r_m within their 3σ experimental ranges [45], we find that the relations in Eqs. (4.13a,4.13b) constrain the Dirac CP violating phase δ_{CP} to lie in the following narrow regions[5]

$$(M_\nu)_{12} = 0 : \quad 266.69^\circ \le \delta_{CP} \le 271.33^\circ\,, \tag{4.14a}$$

$$(M_\nu)_{13} = 0 : \quad 268.40^\circ \le \delta_{CP} \le 272.98^\circ\,. \tag{4.14b}$$

For the texture $(M_\nu)_{23} = 0$, the relation in Eq. (4.13c) leads to $|\cos\delta_{CP}| > 1$ for the experimental values of lepton mixing angles and r_m. Hence this texture is not compatible with neutrino data. The predictions for δ_{CP} in Eqs. (4.14a, 4.14b) can be tested by the planned high-precision measurements of neutrino CP violation at the T2HK [47] and DUNE [48] experiments under construction.

4.2 Texture zeros of neutrino mass matrix with block-diagonal charged-lepton mass matrix

The preceding sections 3.1, 3.2 and 3.3 reveal seven distinct types of non-diagonal Y_E textures which can be constructed from non-invertible $Z_{3,4,5}$ symmetries. In the following, we shall discuss the texture

$$Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}\,, \tag{4.15}$$

and its permutations of rows and columns, which has the next most zeros after the diagonal one.

Since Y_E is block diagonal, the charged lepton sector contributes to one lepton mixing angle. The neutrino Yukawa coupling Y_ν can have at most three zero elements. Textures with three zeros placed in the same column of Y_ν will lead to two massless neutrinos, which is excluded experimentally. If there are two zeros in some row of Y_ν, the light neutrino mass matrix would be block diagonal so that the experimental data of lepton mixing angles can

[5] The relations in Eqs. (4.13a, 4.13b, 4.13c) can not fix the sign of δ_{CP}, it is found that δ_{CP} is limited in the region $\delta_{CP} \in \pm[266.69^\circ, 271.33^\circ]$ for $(M_\nu)_{12} = 0$ and $\delta_{CP} \in \pm[268.40^\circ, 272.98^\circ]$ for $(M_\nu)_{13} = 0$ respectively. Here the parts of negative sign are dropped in light of the global fitting for δ_{CP} [45].

not be accommodated. Therefore there is only one texture of Y_ν with three vanishing entries up to permutations of rows and columns in the minimal seesaw, i.e.

$$Y_\nu^{(3)} = \begin{pmatrix} \times & 0 \\ 0 & \times \\ 0 & \times \end{pmatrix} . \tag{4.16}$$

This texture can be obtained from the Z_2 gauging of Z_N symmetry for $N = 3, 4, 5$, as can be seen from tables 2, 4 and 6. We see that the two columns of $Y_\nu^{(3)}$ have different textures, the two right-handed neutrinos must carry different charges under the non-invertible symmetry, which forces M_R to be diagonal. As a consequence, M_ν is block diagonal as well,

$$\begin{aligned} M_\nu &= -\frac{1}{2}v^2 Y_\nu M_R^{-1} Y_\nu^T \\ &= \begin{pmatrix} \times & 0 \\ 0 & \times \\ 0 & \times \end{pmatrix} \begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix} \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \end{pmatrix} \\ &= \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix} . \end{aligned} \tag{4.17}$$

The block diagonal Y_E and M_ν are excluded by present neutrino data[6].

In the case that Y_ν has two vanishing entries, they can be in the same column or two different columns. Thus Y_ν can be of the following patterns

$$Y_\nu^{(2)} = \begin{pmatrix} \times & \times \\ 0 & \times \\ 0 & \times \end{pmatrix} , \quad Y_\nu^{\prime(2)} = \begin{pmatrix} 0 & \times \\ \times & 0 \\ \times & \times \end{pmatrix} \tag{4.18}$$

up to independent permutations of rows and columns. The texture Y_E in Eq. (4.15) and $Y_\nu^{(2)}$ can accommodate the measured lepton masses and mixing parameters for a diagonal right-handed neutrino mass matrix M_R, and a numerical example in the case of NO is given as follows.

$$Y_E \frac{v}{\sqrt{2}} = \begin{pmatrix} 44.800 + 54.000i & -68.243 & 0 \\ -14.523 - 24.146i & 27.741 + 4.5401i & 0 \\ 0 & 0 & 1776.9 \end{pmatrix} \text{MeV} ,$$

[6] If Y_E provides a rotation matrix in (12)-plane and M_ν provides a rotation matrix in (23) plane, the three lepton mixing angles would be correlated as $\tan\theta_{23}\tan\theta_{12} = \sin\theta_{13}$ which is not consistent with the experimental data [2].

$$Y_\nu M_R^{-1/2}\frac{v}{\sqrt{2}} = \begin{pmatrix} 0.10326+0.039452i & -0.10805 \\ 0 & 0.086538 \\ 0 & 0.16738 \end{pmatrix}\sqrt{\mathrm{eV}}\,. \tag{4.19}$$

Following the procedure outlined in section 3 to diagonalize the corresponding charged lepton mass matrix and light neutrino mass matrix, we obtain the three charged lepton masses as follows,

$$m_e = 0.51119\,\mathrm{MeV}\,,\; m_\mu = 105.66\,\mathrm{MeV}\,,\; m_\tau = 1776.9\,\mathrm{MeV}\,, \tag{4.20}$$

which are in well agreement with their measured values [1], and the light neutrino masses are determined to be

$$m_1 = 0\,,\; m_2 = 8.6544\times 10^{-3}\,\mathrm{eV}\,,\; m_3 = 5.0129\times 10^{-2}\mathrm{eV}\,. \tag{4.21}$$

The numerical values of the unitary transformations U_{ℓ_L} and U_{ν_L} as well as lepton mixing matrix $U = U^\dagger_{\ell_L}U_{\nu_L}$ are also be fixed by the diagonalization, and then we can extract the lepton mixing angles and the CP violating phases as,

$$\theta_{12} = 33.680^\circ\,,\; \theta_{13} = 8.5599^\circ\,,\; \theta_{23} = 43.301^\circ\,,\; \delta_{CP} = 212.00^\circ\,,\; \alpha_{21} = 350.54^\circ\,. \tag{4.22}$$

All the three mixing angles and Dirac CP phase δ_{CP} are compatible with the preferred values from global fitting at the 1σ level [45]. From the view of non-invertible symmetry, the absence of zeros in the second column of $Y_\nu^{(2)}$ requires

$$[k_{\ell_{Li}}] = [k_H + k_{\nu_{R2}}] \;\;\text{or}\;\; [k_H - k_{\nu_{R2}}]\,,\quad i=1,2,3\,, \tag{4.23}$$

where $[k_{\ell_{Li}}]$, $[k_H]$, $[k_{\nu_{R2}}]$ denote the charges of non-invertible symmetry carried by the left-handed lepton ℓ_{Li}, Higgs field H, and right-handed neutrino ν_{R2}, respectively. As a consequence, the three generations of lepton doublets ℓ_{Li} $(i=1,2,3)$ have at most two different transformation rules under non-invertible symmetry. Thus only the unviable textures $Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}$, $Y_\nu = \begin{pmatrix} 0 & \times \\ 0 & \times \\ \times & \times \end{pmatrix}$ and their simultaneous row permutations in this scenario can be produced from the Z_2 gauging of Z_N symmetries, yet the phenomenological viable textures such as these in Eq. (4.19) can not, although either Y_E in Eq. (4.15) and $Y_\nu^{(2)}$ alone can be produced from the non-invertible symmetry.

Now we proceed to consider the remaining alternative structure of the neutrino Yukawa coupling $Y_\nu^{\prime(2)}$. Given the charged lepton Yukawa coupling Y_E in Eq. (4.15), using the freedom of field redefinition, we find that there are six independent textures for the charged lepton and neutrino Yukawa couplings,

$$A_1 : Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}\,,\quad Y_\nu = \begin{pmatrix} 0 & \times \\ \times & 0 \\ \times & \times \end{pmatrix}\,,$$

$$A_2: Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad Y_\nu = \begin{pmatrix} 0 & \times \\ \times & \times \\ \times & 0 \end{pmatrix},$$

$$B_1: Y_E = \begin{pmatrix} \times & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & \times \end{pmatrix}, \quad Y_\nu = \begin{pmatrix} 0 & \times \\ \times & 0 \\ \times & \times \end{pmatrix},$$

$$B_2: Y_E = \begin{pmatrix} \times & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & \times \end{pmatrix}, \quad Y_\nu = \begin{pmatrix} 0 & \times \\ \times & \times \\ \times & 0 \end{pmatrix},$$

$$C_1: Y_E = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}, \quad Y_\nu = \begin{pmatrix} 0 & \times \\ \times & 0 \\ \times & \times \end{pmatrix},$$

$$C_2: Y_E = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}, \quad Y_\nu = \begin{pmatrix} \times & \times \\ 0 & \times \\ \times & 0 \end{pmatrix}. \tag{4.24}$$

Analogous to the discussions in section 4.1, the above textures cannot be generated from the non-invertible symmetry for $N = 3, 4$, while they can be derived from the Z_2 gauging of Z_5 symmetries. From Eqs. (4.4) and (B.6), we see that the textures A_1 and A_2 can be naturally obtained if the lepton and Higgs fields transform in the following fashion under the Z_2 gauging of Z_5 symmetries,

$$\begin{aligned} A_1: \quad \text{for} \quad (\ell_L; \nu_R; E_R; H_u; H_d) \sim \; & ([0],[1],[2];[2],[1];[2],[2],[0];[1];[2]), \\ & ([0],[2],[1];[1],[2];[1],[1],[0];[2];[1]), \\ & ([1],[0],[2];[1],[2];[2],[2],[0];[1];[2]), \\ & ([2],[0],[1];[2],[1];[1],[1],[0];[2];[1]), \end{aligned} \tag{4.25}$$

$$\begin{aligned} A_2: \quad \text{for} \quad (\ell_L; \nu_R; E_R; H_u; H_d) \sim \; & ([0],[2],[1];[2],[1];[1],[1],[0];[1];[1]), \\ & ([0],[1],[2];[1],[2];[2],[2],[0];[2];[2]). \end{aligned} \tag{4.26}$$

The charges of the Higgs H_u anb H_d under the non-invertible symmetry are different for A_1, while they are the same for A_2. Hence the texture A_1 can only be produced within the supersymmetric standard model, while A_2 no longer necessitates supersymmetry. Similarly the textures B_1, B_2, C_1 and C_2 can be derived by permutating the assignments for three generations of leptons, i.e.

$$\begin{aligned} B_1: \quad \text{for} \quad (\ell_L; \nu_R; E_R; H_u; H_d) \sim \; & ([0],[1],[2];[2],[1];[1],[0],[1];[1];[1]), \\ & ([0],[2],[1];[1],[2];[2],[0],[2];[2];[2]), \end{aligned} \tag{4.27}$$

$$
\begin{aligned}
B_2: \quad \text{for} \quad (\ell_L;\nu_R;E_R;H_u;H_d) \sim \; & ([0],[2],[1];[2],[1];[2],[0],[2];[1];[2])\,, \\
& ([0],[1],[2];[1],[2];[1],[0],[1];[2];[1])\,, \\
& ([1],[2],[0];[1],[2];[2],[0],[2];[1];[2])\,, \\
& ([2],[1],[0];[2],[1];[1],[0],[1];[2];[1])\,, \qquad (4.28)
\end{aligned}
$$

$$
\begin{aligned}
C_1: \quad \text{for} \quad (\ell_L;\nu_R;E_R;H_u;H_d) \sim \; & ([1],[0],[2];[1],[2];[0],[1],[1];[1];[1])\,, \\
& ([2],[0],[1];[2],[1];[0],[2],[2];[2];[2])\,, \qquad (4.29)
\end{aligned}
$$

$$
\begin{aligned}
C_2: \quad \text{for} \quad (\ell_L;\nu_R;E_R;H_u;H_d) \sim \; & ([2],[0],[1];[2],[1];[0],[2],[2];[1];[2])\,, \\
& ([1],[0],[2];[1],[2];[0],[1],[1];[2];[1])\,, \\
& ([2],[1],[0];[1],[2];[0],[2],[2];[1];[2])\,, \\
& ([1],[2],[0];[2],[1];[0],[1],[1];[2];[1])\,. \qquad (4.30)
\end{aligned}
$$

For all the six cases A_1, A_2, B_1, B_2, C_1 and C_2, the charged lepton mass matrix is block diagonal and it can be diagonalized by a unitary rotation in the (12)-plane, (13)-plane or (23)-plane. Consequently the unitary transformation $U_{\ell L}$ can be parametrized as follows[7],

$$
\begin{aligned}
A_1 \text{ and } A_2: \; U_{\ell L} &= \begin{pmatrix} \cos\theta & -\sin\theta\, e^{i\sigma} & 0 \\ \sin\theta\, e^{-i\sigma} & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
B_1 \text{ and } B_2: \; U_{\ell L} &= \begin{pmatrix} \cos\theta & 0 & -\sin\theta\, e^{i\sigma} \\ 0 & 1 & 0 \\ \sin\theta\, e^{-i\sigma} & 0 & \cos\theta \end{pmatrix}, \\
C_1 \text{ and } C_2: \; U_{\ell L} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta\, e^{i\sigma} \\ 0 & \sin\theta\, e^{-i\sigma} & \cos\theta \end{pmatrix}. \qquad (4.31)
\end{aligned}
$$

Since the two columns of $Y_\nu^{'(2)}$ have different patterns of zeros, two right-handed neutrinos should transform differently under the Z_2 gauging of the Z_N symmetries, as can be seen from Eqs. (4.25, 4.26, 4.27, 4.28, 4.29, 4.30) as well. Therefore the non-invertible symmetry enforces the right-handed neutrino mass matrix M_R to be diagonal. Using the seesaw formula of Eq. (3.3), we find that one element of the light neutrino mass matrix is vanishing. To be more specific, we have $(M_\nu)_{12} = (M_\nu)_{21} = 0$ for the cases A_1, B_1, C_1, and $(M_\nu)_{13} = (M_\nu)_{31} = 0$ for the cases A_2, B_2, and $(M_\nu)_{23} = (M_\nu)_{32} = 0$ for the case C_2. The light neutrino mass matrix M_ν can be expressed in terms of light neutrino masses and lepton mixing matrix as

$$
M_\nu = U_{\nu L}\,\mathrm{diag}(m_1, m_2, m_3) U_{\nu L}^T = U_{\ell L} U\,\mathrm{diag}(m_1, m_2, m_3) U^T U_{\ell L}^T\,, \qquad (4.32)
$$

where the identity $U_{\nu L} = U_{\ell L} U$ has been used. In the minimal seesaw model, one has $m_1 = 0$ for NO and $m_3 = 0$ for IO. The vanishing entry $(M_\nu)_{\alpha\beta} = 0$ can correlate neutrino masses

[7]Note that we have dropped a diagonal phase matrix in the left side of $U_{\ell L}$ since it is unphysical.

$\frac{\Delta m^2_{21}}{\lvert\Delta m^2_{31}\rvert}=$	NO	IO
A_1	$\left\lvert\frac{(U^2_{13}-U^2_{23}e^{2i\sigma})\sin 2\theta+2U_{13}U_{23}e^{i\sigma}\cos 2\theta}{(U^2_{12}-U^2_{22}e^{2i\sigma})\sin 2\theta+2U_{12}U_{22}e^{i\sigma}\cos 2\theta}\right\rvert^2$	$\left\lvert\frac{(U^2_{11}-U^2_{21}e^{2i\sigma})\sin 2\theta+2U_{11}U_{21}e^{i\sigma}\cos 2\theta}{(U^2_{12}-U^2_{22}e^{2i\sigma})\sin 2\theta+2U_{12}U_{22}e^{i\sigma}\cos 2\theta}\right\rvert^2-1$
A_2	$\left\lvert\frac{U_{33}(U_{13}\cos\theta-U_{23}e^{i\sigma}\sin\theta)}{U_{32}(U_{12}\cos\theta-U_{22}e^{i\sigma}\sin\theta)}\right\rvert^2$	$\left\lvert\frac{U_{31}(U_{11}\cos\theta-U_{21}e^{i\sigma}\sin\theta)}{U_{32}(U_{12}\cos\theta-U_{22}e^{i\sigma}\sin\theta)}\right\rvert^2-1$
B_1	$\left\lvert\frac{U_{23}(U_{13}\cos\theta-U_{33}e^{i\sigma}\sin\theta)}{U_{22}(U_{12}\cos\theta-U_{32}e^{i\sigma}\sin\theta)}\right\rvert^2$	$\left\lvert\frac{U_{21}(U_{11}\cos\theta-U_{31}e^{i\sigma}\sin\theta)}{U_{22}(U_{12}\cos\theta-U_{32}e^{i\sigma}\sin\theta)}\right\rvert^2-1$
B_2	$\left\lvert\frac{(U^2_{13}-U^2_{33}e^{2i\sigma})\sin 2\theta+2U_{13}U_{33}e^{i\sigma}\cos 2\theta}{(U^2_{12}-U^2_{32}e^{2i\sigma})\sin 2\theta+2U_{12}U_{32}e^{i\sigma}\cos 2\theta}\right\rvert^2$	$\left\lvert\frac{(U^2_{11}-U^2_{31}e^{2i\sigma})\sin 2\theta+2U_{11}U_{31}e^{i\sigma}\cos 2\theta}{(U^2_{12}-U^2_{32}e^{2i\sigma})\sin 2\theta+2U_{12}U_{32}e^{i\sigma}\cos 2\theta}\right\rvert^2-1$
C_1	$\left\lvert\frac{U_{13}(U_{23}\cos\theta-U_{33}e^{i\sigma}\sin\theta)}{U_{12}(U_{22}\cos\theta-U_{32}e^{i\sigma}\sin\theta)}\right\rvert^2$	$\left\lvert\frac{U_{11}(U_{21}\cos\theta-U_{31}e^{i\sigma}\sin\theta)}{U_{12}(U_{22}\cos\theta-U_{32}e^{i\sigma}\sin\theta)}\right\rvert^2-1$
C_2	$\left\lvert\frac{(U^2_{23}-U^2_{33}e^{2i\sigma})\sin 2\theta+2U_{23}U_{33}e^{i\sigma}\cos 2\theta}{(U^2_{22}-U^2_{32}e^{2i\sigma})\sin 2\theta+2U_{22}U_{32}e^{i\sigma}\cos 2\theta}\right\rvert^2$	$\left\lvert\frac{(U^2_{21}-U^2_{31}e^{2i\sigma})\sin 2\theta+2U_{21}U_{31}e^{i\sigma}\cos 2\theta}{(U^2_{22}-U^2_{32}e^{2i\sigma})\sin 2\theta+2U_{22}U_{32}e^{i\sigma}\cos 2\theta}\right\rvert^2-1$

Table 3: The correlation between $\frac{\Delta m^2_{21}}{|\Delta m^2_{31}|}$ and the lepton mixing parameters for the textures A_1, A_2, B_1, B_2, C_1 and C_2, where both NO and IO neutrino mass spectrum are considered.

with mixing parameters as follows

$$\begin{cases} \dfrac{m_2}{m_3} = -\dfrac{(U_{\ell L}U)_{\alpha 3}(U_{\ell L}U)_{\beta 3}}{(U_{\ell L}U)_{\alpha 2}(U_{\ell L}U)_{\beta 2}} & \text{for} \quad \text{NO}\,, \\ \dfrac{m_2}{m_1} = -\dfrac{(U_{\ell L}U)_{\alpha 1}(U_{\ell L}U)_{\beta 1}}{(U_{\ell L}U)_{\alpha 2}(U_{\ell L}U)_{\beta 2}} & \text{for} \quad \text{IO}\,. \end{cases} \tag{4.33}$$

Hence the ratio of the neutrino mass squared differences fulfill the following relations

$$\begin{cases} \dfrac{\Delta m^2_{21}}{\Delta m^2_{31}} = \left|\dfrac{\sum_{i,j}(U_{\ell L})_{\alpha i}U_{i3}(U_{\ell L})_{\beta j}U_{j3}}{\sum_{k,l}(U_{\ell L})_{\alpha k}U_{k2}(U_{\ell L})_{\beta l}U_{l2}}\right|^2 & \text{for} \quad \text{NO}\,, \\ \dfrac{\Delta m^2_{21}}{|\Delta m^2_{31}|} = \left|\dfrac{\sum_{i,j}(U_{\ell L})_{\alpha i}U_{i1}(U_{\ell L})_{\beta j}U_{j1}}{\sum_{k,l}(U_{\ell L})_{\alpha k}U_{k2}(U_{\ell L})_{\beta l}U_{l2}}\right|^2 - 1 & \text{for} \quad \text{IO}\,. \end{cases} \tag{4.34}$$

We list the expressions of $\Delta m^2_{21}/|\Delta m^2_{31}|$ for the six textures A_i, B_i and C_i $(i = 1, 2)$ in table 3. Due to the presence of θ and σ parameterizing the charged lepton mixing, both NO and IO neutrino masses can be accommodated. We plot the allowed regions of θ and σ constrained by Eq. (4.34) in figure 1 and figure 2 for NO and IO respectively, where we have taken several representative values of δ_{CP} in its 3σ region, and the three mixing angles are fixed at their best-fit values, $\Delta m^2_{21}/|\Delta m^2_{31}|$ is limited within its 3σ range [45].

4.3 The possible new texture zeros from higher N

In this section, we shall investigate whether the Z_2 gauging of the Z_N symmetries for larger $N > 5$ can give rise to new textures of Y_E and M_ν. When $N = 6, 7$, there are four classes [0], [1], [2] and [3]. The three generations of left-handed lepton doublets ℓ_L and right-handed charged leptons E_R each admit $= C^3_6 - 4 = 16$ distinct charge assignments, where we exclude the assignment that all the three generations ℓ_L(or E_R) correspond to the same class.

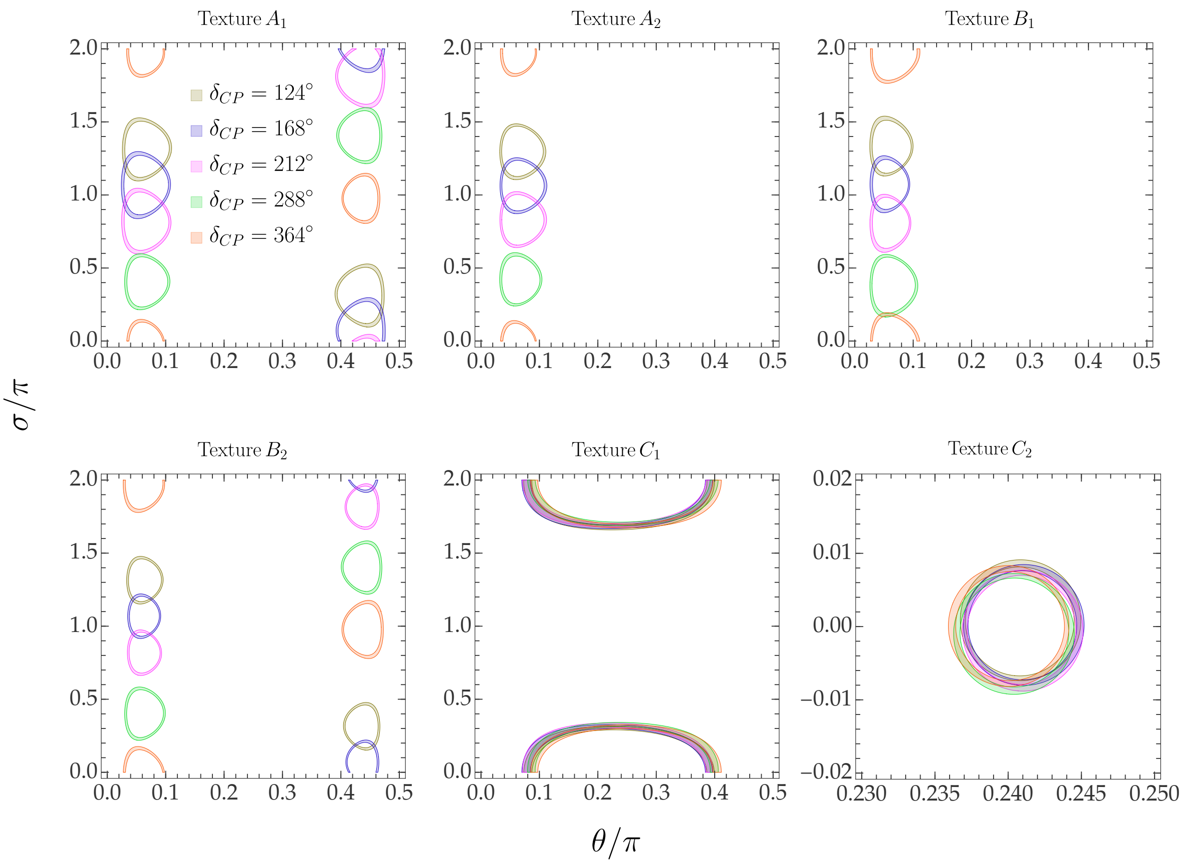

Figure 1: The contour plots of the texture zero constraint Eq. (4.34) in the $\theta - \sigma$ plane for the NO neutrino masses. Five representative values $\delta_{CP} = 124°$, $168°$, $212°$, $288°$, $364°$ are shown in different colors, and the width of the lines arises from the 3σ uncertainty of $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ [45].

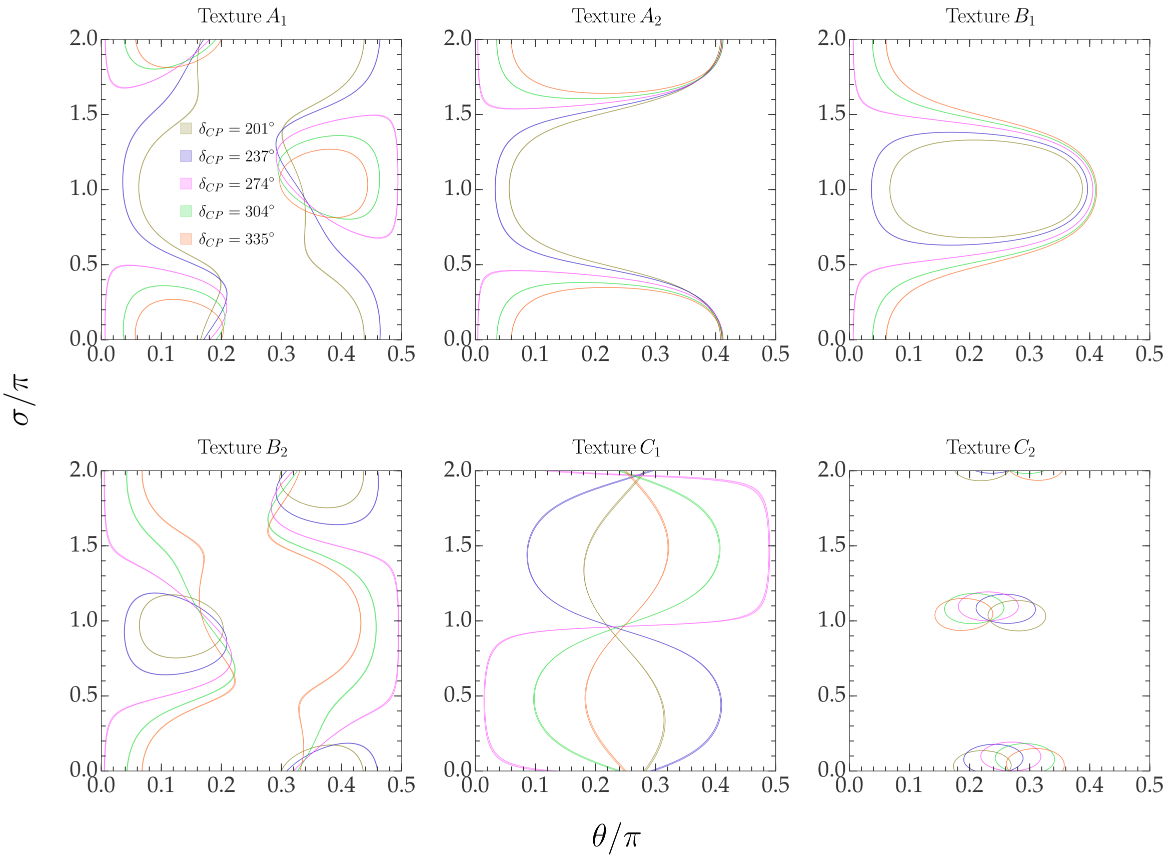

Figure 2: The contour plots of the texture zero constraint in Eq. (4.34) in the $\theta - \sigma$ plane for the IO neutrino masses. Five representative values $\delta_{CP} = 124°,\ 168°,\ 212°,\ 288°,\ 364°$ are shown in different colors, and the width of the lines arises from the 3σ uncertainty of $\frac{\Delta m^2_{21}}{|\Delta m^2_{31}|}$ [45].

Similarly the two generations of right-handed neutrino ν_R admit $C_5^2 - 4 = 6$ distinct charge assignments. The Higgs field H can be in [0], [1], [2] or [3].

From Eq. (2.5), we know that the fusion rules for $N = 6$ read as follows,

$$\begin{gathered}
[0]\times[0]=[0]\,,\quad [0]\times[1]=[1]\,,\quad [0]\times[2]=[2]\,,\quad [0]\times[3]=[3]\,,\\
[1]\times[1]=[0]+[2]\,,\quad [1]\times[2]=[1]+[3]\,,\quad [1]\times[3]=[2]\,,\\
[2]\times[2]=[0]+[2]\,,\quad [2]\times[3]=[1]\,,\quad [3]\times[3]=[0]\,.
\end{gathered}\tag{4.35}$$

The fusion rules for $N = 7$ are different from these of $N = 6$ case,

$$\begin{gathered}
[0]\times[0]=[0]\,,\quad [0]\times[1]=[1]\,,\quad [0]\times[2]=[2]\,,\quad [0]\times[3]=[3]\,,\\
[1]\times[1]=[0]+[2]\,,\quad [1]\times[2]=[1]+[3]\,,\quad [1]\times[3]=[2]+[3]\,,\\
[2]\times[2]=[0]+[3]\,,\quad [2]\times[3]=[1]+[2]\,,\quad [3]\times[3]=[0]+[1]\,.
\end{gathered}\tag{4.36}$$

We see that the above fusion rules for $N = 7$ are invariant under the permutation: $[1] \to [2]$, $[2] \to [3]$, $[3] \to [1]$.

In the case of $N = 6$, the corresponding trilinear couplings would be allowed by non-invertible symmetry if one has the following charge assignments for the fields,

$$\begin{gathered}
[0]\times[0]\times[0]\,,\quad [0]\times[1]\times[1]\,,\quad [0]\times[2]\times[2]\,,\\
[0]\times[3]\times[3]\,,\quad [1]\times[1]\times[2]\,,\quad [1]\times[2]\times[3]\,,\quad [2]\times[2]\times[2]\,.
\end{gathered}\tag{4.37}$$

When $N = 7$, likewise the charge assignments and contractions for the trilinear couplings compatible with non-invertible symmetry are given by

$$\begin{gathered}
[0]\times[0]\times[0]\,,\quad [0]\times[1]\times[1]\,,\quad [0]\times[2]\times[2]\,,\quad [0]\times[3]\times[3]\,,\\
[1]\times[1]\times[2]\,,\quad [1]\times[2]\times[3]\,,\quad [1]\times[3]\times[3]\,,\quad [2]\times[2]\times[3]\,.
\end{gathered}\tag{4.38}$$

From the selection rules of non-invertible symmetry in Eqs. (4.37,4.38), we can read out the admissiable trilinear Yukawa couplings. Here we list the textures of full-rank Y_E,

$$\begin{aligned}
N=6:\ Y_E = &\begin{pmatrix}\times&0&0\\0&\times&0\\0&0&\times\end{pmatrix},\ \begin{pmatrix}\times&\times&0\\0&\times&0\\0&0&\times\end{pmatrix},\ \begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix},\\
&\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&0&0\end{pmatrix},\ \begin{pmatrix}\times&\times&\times\\\times&\times&0\\\times&\times&0\end{pmatrix},\ \begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&0\end{pmatrix},
\end{aligned}\tag{4.39}$$

$$N=7:\ Y_E = \begin{pmatrix}\times&0&0\\0&\times&0\\0&0&\times\end{pmatrix},\ \begin{pmatrix}\times&\times&0\\0&\times&0\\0&0&\times\end{pmatrix},\ \begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix},\ \begin{pmatrix}\times&\times&0\\\times&0&\times\\0&\times&0\end{pmatrix},\ \begin{pmatrix}\times&\times&\times\\\times&\times&0\\0&0&\times\end{pmatrix},$$

$$\begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & 0 & \times \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ \times & \times & 0 \\ \times & \times & 0 \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & 0 \end{pmatrix}, \tag{4.40}$$

where independent permutations of rows and columns should be included as well. Analogously the following textures of neutrino Yukawa coupling Y_ν of rank two can be achieved,

$$N = 6: \; Y_\nu = \begin{pmatrix} \times & 0 \\ 0 & \times \\ 0 & 0 \end{pmatrix}, \; \begin{pmatrix} \times & 0 \\ \times & 0 \\ 0 & \times \end{pmatrix}, \; \begin{pmatrix} \times & \times \\ \times & 0 \\ 0 & 0 \end{pmatrix}, \; \begin{pmatrix} \times & \times \\ \times & 0 \\ \times & 0 \end{pmatrix}, \; \begin{pmatrix} \times & \times \\ \times & \times \\ 0 & 0 \end{pmatrix}, \; \begin{pmatrix} \times & \times \\ \times & \times \\ \times & 0 \end{pmatrix}, \tag{4.41}$$

$$\begin{aligned} N = 7: \; Y_\nu = & \begin{pmatrix} \times & 0 \\ 0 & \times \\ 0 & 0 \end{pmatrix}, \; \begin{pmatrix} \times & 0 \\ \times & 0 \\ 0 & \times \end{pmatrix}, \; \begin{pmatrix} \times & \times \\ \times & 0 \\ 0 & 0 \end{pmatrix}, \; \begin{pmatrix} \times & \times \\ \times & 0 \\ \times & 0 \end{pmatrix}, \\ & \begin{pmatrix} \times & \times \\ \times & \times \\ 0 & 0 \end{pmatrix}, \; \begin{pmatrix} \times & \times \\ \times & 0 \\ 0 & \times \end{pmatrix}, \; \begin{pmatrix} \times & \times \\ \times & \times \\ \times & 0 \end{pmatrix}, \end{aligned} \tag{4.42}$$

up to permutations of rows and columns. In Appendix C, we give an example transformation rule of the fields ℓ_L, E_R, ν_R, H under the Z_2 gauging of Z_N symmetries for each texture in above.

In comparison with table 2, table 4 and table 6 for $N = 3, 4, 5$ respectively, we see that no new texture of Y_ν is obtained from $N = 6, 7$. In fact, there are seven distinct textures of the 3×2 matrix Y_ν with rank 2 and at least one zero entry, up to row and column permutations. All of them can be achieved from the non-invertible Z_5 symmetry, as shown in table 6. Regarding the charged lepton Yukawa coupling, the Z_2 gauging of Z_N symmetries for $N = 6, 7$ give rise to a new texture:

$$Y_E = \begin{pmatrix} \times & \times & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}. \tag{4.43}$$

When further extending the analysis to $N = 9$, one more new texture can be obtained,

$$Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix}. \tag{4.44}$$

In matrix theory, there exist fourteen distinct textures of full-rank 3×3 matrix Y_E containing vanishing elements, up to row and column permutations. Ten of them shown in Eq. (4.40) and Eq. (4.44) can be derived from non-invertible symmetry. The remaining four textures:

$$\begin{pmatrix} \times & 0 & \times \\ \times & 0 & 0 \\ \times & \times & 0 \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ 0 & 0 & \times \\ \times & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ \times & \times & 0 \\ \times & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{pmatrix} \tag{4.45}$$

cannot be generated from the Z_2 gauging of Z_N symmetries, the reason is similar to that discussed in the paragraphs surrounding Eq. (4.23).

5 Conclusion and outlook

The origin of the fermion masses and flavor mixing is a long-standing puzzle in particle physics. Texture zeros refer to vanishing entries in a fermion mass matrix, it is a useful approach to reduce the number of free parameters in mass matrices, thereby enhancing the predictive power of a model. The zero entries in fermion mass matrices could lead to testable relations between fermion mass ratios and mixing parameters. The texture zeros of neutrino mass matrix have been extensively studied to explain observed neutrino oscillation data and to predict yet-undetermined quantities such as the leptonic CP violation phase, the neutrino mass ordering, and the absolute neutrino mass scale [12–15].

From the theoretical viewpoint, the texture zeros may arise from some continuous or discrete flavor symmetries [5–8], they may reflect some underlying selection rules. Recently it was found that the Z_2 gauging of discrete Z_N symmetries can constrains the structure of the quark and lepton mass matrices, leading to specific patterns of vanishing entries [28–32]. In this paper, we have performed a systematical study of the texture zeros of the lepton mass matrices in the context of minimal seesaw model in which only two right-handed neutrinos are introduced to generate tiny neutrino masses. We study the textures of the charged lepton Yukawa coupling Y_E, the neutrino Yukawa coupling Y_ν, the right-handed neutrino mass matrix M_R and the light neutrino mass matrix M_ν which can be derived from the Z_2 gauging of Z_N symmetry. We are particularly interested in the phenomenologically viable textures with the maximum number of zeros and the second maximum number of zeros.

In the charged lepton diagonal basis, the neutrino Yukawa coupling Y_ν can have at most two zero elements to be compatible with experimental data and there are totally six distinct textures $T_{1,2,3,4,5,6}$, as shown in Eq. (4.4). For proper assignments of matter fields, the Z_2 gauging of Z_5 symmetry can give rise to all the six textures $T_{1,2,3,4,5,6}$ together with a diagonal Y_E, and the right-handed neutrino mass matrix M_R is enforced to be diagonal. As a result, one entry of the light neutrino mass matrix M_ν is vanishing. This zero entry allows to relate the Dirac CP phase δ_{CP} with the lepton mixing angles and the ratio $r_m \equiv \Delta m^2_{21}/|\Delta m^2_{31}|$. It turns out that only the textures T_1, T_2, T_4 and T_5 are compatible with the current neutrino oscillation data, and δ_{CP} is constrained in narrow regions shown in Eqs. (4.14a, 4.14b).

Besides the diagonal charged lepton Yukawa coupling, the Z_2 gauging of Z_N symmetry can also generate block diagonal Y_E. Taking into account the neutrino Yukawa coupling Y_ν, six textures of Y_E and Y_ν named as $A_{1,2}$, $B_{1,2}$ and $C_{1,2}$ can be obtained from non-invertible Z_5 symmetry, as shown in Eqs. (4.25,4.26, 4.27, 4.28, 4.29, 4.30). In this scenario, the charged lepton sector contributes a unitary rotation in the (ij)-plane ($i, j = 1, 2, 3$ and $i < j$) to the lepton mixing, consequently both NO and IO neutrino mass spectrums can be accommodated. Furthermore, the analysis is extended to the large N case. It is found that two more new textures $Y_E = \begin{pmatrix} \times & \times & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}$ and $Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix}$ can be generated from the non-invertible $Z_{6,7}$ and Z_9 symmetries respectively. Notice that the former pattern of Y_E has five zero entries. However, no new texture of Y_ν can be produced anymore.

It is intriguing that the Z_2 gauging of Z_N symmetry can naturally constrain certain elements of the quark and lepton mass matrices to vanish, resulting in predictive textures which can be tested against the more precise experimental data in near future. In particularly, it can generate some textures which can not be obtained from the traditional Z_N or $U(1)$ abelian flavor symmetry. It is known that the quarks and leptons fields are embedded into few gauge multiplets in Grand Unification Theory so that the quark and lepton mass matrices are related to each other. It is interesting to study what kinds of textures of fermion mass matrices can be generated from the non-invertible Z_N symmetry in the context of Grand Unification Theory, we expect to get more predictive textures which are constrained by the abundant experimental data of both neutrino oscillation and rare meson decays. It is also quite interesting to explore the phenomenological implications of the texture zeros derived from non-invertible symmetry in neutrinoless double decay, strong CP problem and baryon asymmetry of the Universe and so on. All these are left for future.

Acknowledgements

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A Possible textures of Y_ν, M_R and M_ν for $N = 4, 5$

A.1 $N = 4$

Based on the Z_2 gauging of Z_4 symmetry in minimal seesaw model, we report the textures of the neutrino Yukawa matrix Y_ν and the right-handed neutrino mass matrix M_R in table 4. The different cases are classified according to the transformation of ℓ_L, ν_R and H under the non-invertible symmetry. The corresponding patterns of the light neutrino mass matrix M_ν are listed in table 5. Notice that interchanging the assignment of ℓ_L leads to row permutation of Y_ν and M_ν, while interchanging the assignment of ν_R leads to column permutation of Y_ν and M_R.

	$Y_\nu\,,\ H \sim [0]$	$Y_\nu\,,\ H \sim [1]$	$Y_\nu\,,\ H \sim [2]$	M_R
$\ell_L \sim ([0],[1],[2])$ $\nu_R \sim ([0],[1])$	$\begin{pmatrix} \times & 0 \\ 0 & \times \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \times \\ \times & 0 \\ 0 & \times \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & \times \\ \times & 0 \end{pmatrix}$	$\begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix}$
$\ell_L \sim ([0],[1],[2])$ $\nu_R \sim ([0],[2])$	$\begin{pmatrix} \times & 0 \\ 0 & 0 \\ 0 & \times \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ \times & \times \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \times \\ 0 & 0 \\ \times & 0 \end{pmatrix}$	$\begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix}$
$\ell_L \sim ([0],[1],[2])$ $\nu_R \sim ([1],[2])$	$\begin{pmatrix} 0 & 0 \\ \times & 0 \\ 0 & \times \end{pmatrix}$	$\begin{pmatrix} \times & 0 \\ 0 & \times \\ \times & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \times \\ \times & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix}$
$\ell_L \sim ([0],[0],[1])$ $\nu_R \sim ([0],[1])$	$\begin{pmatrix} \times & 0 \\ \times & 0 \\ 0 & \times \end{pmatrix}$	$\begin{pmatrix} 0 & \times \\ 0 & \times \\ \times & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \times \end{pmatrix}$	$\begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix}$
$\ell_L \sim ([0],[0],[1])$ $\nu_R \sim ([0],[2])$	$\begin{pmatrix} \times & 0 \\ \times & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \times & \times \end{pmatrix}$	$\begin{pmatrix} 0 & \times \\ 0 & \times \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix}$
$\ell_L \sim ([0],[0],[1])$ $\nu_R \sim ([1],[2])$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \times & 0 \end{pmatrix}$	$\begin{pmatrix} \times & 0 \\ \times & 0 \\ 0 & \times \end{pmatrix}$	$\begin{pmatrix} 0 & \times \\ 0 & \times \\ \times & 0 \end{pmatrix}$	$\begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix}$
$\ell_L \sim ([0],[0],[2])$ $\nu_R \sim ([0],[1])$	$\begin{pmatrix} \times & 0 \\ \times & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \times \\ 0 & \times \\ 0 & \times \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \times & 0 \end{pmatrix}$	$\begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix}$

	$Y_\nu\,,\ H\sim[0]$	$Y_\nu\,,\ H\sim[1]$	$Y_\nu\,,\ H\sim[2]$	M_R
$\ell_L\sim([0],[0],[2])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([0],[0],[2])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ \times & 0\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & \times\\ \times & \times\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & \times\\ \times & \times\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$

	$Y_\nu\,,\ H\sim[0]$	$Y_\nu\,,\ H\sim[1]$	$Y_\nu\,,\ H\sim[2]$	M_R
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0&\times\\0&\times\\0&0\end{pmatrix}$	$\begin{pmatrix}\times&0\\\times&0\\\times&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\\0&\times\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0&0\\0&0\\0&\times\end{pmatrix}$	$\begin{pmatrix}0&\times\\0&\times\\\times&0\end{pmatrix}$	$\begin{pmatrix}\times&0\\\times&0\\0&\times\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0&\times\\0&\times\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\\\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&0\\\times&0\\0&0\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0&\times\\0&\times\\\times&0\end{pmatrix}$	$\begin{pmatrix}\times&0\\\times&0\\0&\times\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\\\times&0\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$

Table 4: Y_ν and M_R matrices in minimal seesaw from Z_2 gauging of Z_4 symmetries, where the crosses represent non-zero matrix elements.

	$M_\nu\,,\ H\sim[0]$	$M_\nu\,,\ H\sim[1]$	$M_\nu\,,\ H\sim[2]$
$\ell_L\sim([0],[1],[2])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times&0&0\\0&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&0&\times\\0&\times&0\\\times&0&\times\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([0],[1],[2])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&0&0\\0&0&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([0],[1],[2])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0&0&0\\0&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&0&\times\\0&\times&0\\\times&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&0&0\\0&\times&0\\0&0&0\end{pmatrix}$
$\ell_L\sim([0],[0],[1])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$

	$M_\nu\,,\ H\sim[0]$	$M_\nu\,,\ H\sim[1]$	$M_\nu\,,\ H\sim[2]$
$\ell_L\sim([0],[0],[1])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$
$\ell_L\sim([0],[0],[1])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([0],[0],[2])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([0],[0],[2])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([0],[0],[2])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$

	$M_\nu\,,\ H\sim[0]$	$M_\nu\,,\ H\sim[1]$	$M_\nu\,,\ H\sim[2]$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$

Table 5: The light neutrino mass matrix M_ν in minimal seesaw from Z_2 gauging of Z_4 symmetry, where the crosses represent non-zero matrix elements.

A.2 $N = 5$

The possible patterns of Y_ν and M_R which can be derived from the Z_2 gauging of Z_5 symmetry are collected in table 6, and the corresponding light neutrino mass matrix $M_\nu = -\frac{1}{2}v^2 Y_\nu M_R^{-1} Y_\nu^T$ is listed in table 7. Similar to $N = 4$, here Y_ν, M_R and M_ν are determined up to row and column permutations.

	$Y_\nu\,,\ H\sim[0]$	$Y_\nu\,,\ H\sim[1]$	$Y_\nu\,,\ H\sim[2]$	M_R
$\ell_L\sim([0],[1],[2])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times & 0\\ 0 & \times\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & \times\\ \times & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([0],[1],[2])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times & 0\\ 0 & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ \times & \times\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([0],[1],[2])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0 & 0\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\\ \times & \times\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ \times & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([0],[0],[1])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([0],[0],[1])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ \times & \times\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([0],[0],[1])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([0],[0],[2])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ \times & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([0],[0],[2])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([0],[0],[2])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ \times & 0\\ \times & \times\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ \times & 0\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$

	$Y_\nu\,,\ H\sim[0]$	$Y_\nu\,,\ H\sim[1]$	$Y_\nu\,,\ H\sim[2]$	M_R
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0&0\\0&0\\\times&0\end{pmatrix}$	$\begin{pmatrix}\times&\times\\\times&\times\\0&0\end{pmatrix}$	$\begin{pmatrix}0&\times\\0&\times\\0&\times\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times&0\\\times&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&\times\\0&\times\\\times&0\end{pmatrix}$	$\begin{pmatrix}\times&\times\\\times&\times\\0&\times\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0&\times\\0&\times\\0&0\end{pmatrix}$	$\begin{pmatrix}\times&0\\\times&0\\0&\times\end{pmatrix}$	$\begin{pmatrix}0&\times\\0&\times\\\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0&0\\0&0\\0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times\\\times&\times\\0&\times\end{pmatrix}$	$\begin{pmatrix}0&\times\\0&\times\\\times&0\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times&0\\\times&0\\0&\times\end{pmatrix}$	$\begin{pmatrix}0&\times\\0&\times\\\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times\\\times&\times\\\times&0\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0&0\\0&0\\\times&0\end{pmatrix}$	$\begin{pmatrix}0&\times\\0&\times\\0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times\\\times&\times\\0&0\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0&\times\\0&\times\\\times&0\end{pmatrix}$	$\begin{pmatrix}0&\times\\0&\times\\0&0\end{pmatrix}$	$\begin{pmatrix}\times&0\\\times&0\\0&\times\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0&\times\\0&\times\\0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times\\\times&\times\\\times&0\end{pmatrix}$	$\begin{pmatrix}\times&0\\\times&0\\0&\times\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0&0\\0&0\\0&\times\end{pmatrix}$	$\begin{pmatrix}0&\times\\0&\times\\\times&0\end{pmatrix}$	$\begin{pmatrix}\times&\times\\\times&\times\\0&\times\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0&\times\\0&\times\\0&0\end{pmatrix}$	$\begin{pmatrix}0&\times\\0&\times\\\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&0\\\times&0\\0&\times\end{pmatrix}$	$\begin{pmatrix}\times&0\\0&\times\end{pmatrix}$

	$Y_\nu\,,\ H\sim[0]$	$Y_\nu\,,\ H\sim[1]$	$Y_\nu\,,\ H\sim[2]$	M_R
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0 & \times\\ 0 & \times\\ \times & 0\end{pmatrix}$	$\begin{pmatrix}\times & \times\\ \times & \times\\ 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ \times & 0\\ \times & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0\\ 0 & \times\end{pmatrix}$

Table 6: Y_ν and M_R matrices in minimal seesaw from Z_2 gauging of Z_4 symmetry, where the crosses represent non-zero matrix elements.

	$M_\nu\,,\ H\sim[0]$	$M_\nu\,,\ H\sim[1]$	$M_\nu\,,\ H\sim[2]$
$\ell_L\sim([0],[1],[2])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times & 0 & 0\\ 0 & \times & 0\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}\times & 0 & \times\\ 0 & \times & 0\\ \times & 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0\\ 0 & \times & \times\\ 0 & \times & \times\end{pmatrix}$
$\ell_L\sim([0],[1],[2])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0\\ 0 & \times & \times\\ 0 & \times & \times\end{pmatrix}$	$\begin{pmatrix}\times & \times & 0\\ \times & \times & 0\\ 0 & 0 & \times\end{pmatrix}$
$\ell_L\sim([0],[1],[2])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0 & 0 & 0\\ 0 & \times & 0\\ 0 & 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & 0 & \times\\ 0 & \times & \times\\ \times & \times & \times\end{pmatrix}$	$\begin{pmatrix}\times & \times & 0\\ \times & \times & \times\\ 0 & \times & \times\end{pmatrix}$
$\ell_L\sim([0],[0],[1])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times & \times & 0\\ \times & \times & 0\\ 0 & 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & \times & 0\\ \times & \times & 0\\ 0 & 0 & \times\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \times\end{pmatrix}$
$\ell_L\sim([0],[0],[1])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times & \times & 0\\ \times & \times & 0\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & \times & \times\\ \times & \times & \times\\ \times & \times & \times\end{pmatrix}$
$\ell_L\sim([0],[0],[1])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & \times & 0\\ \times & \times & 0\\ 0 & 0 & \times\end{pmatrix}$	$\begin{pmatrix}\times & \times & \times\\ \times & \times & \times\\ \times & \times & \times\end{pmatrix}$
$\ell_L\sim([0],[0],[2])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times & \times & 0\\ \times & \times & 0\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}\times & \times & \times\\ \times & \times & \times\\ \times & \times & \times\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \times\end{pmatrix}$

	$M_\nu\,,\ H\sim[0]$	$M_\nu\,,\ H\sim[1]$	$M_\nu\,,\ H\sim[2]$
$\ell_L\sim([0],[0],[2])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([0],[0],[2])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[0])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([1],[1],[2])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$

	$M_\nu\,,\ H\sim[0]$	$M_\nu\,,\ H\sim[1]$	$M_\nu\,,\ H\sim[2]$
$\ell_L\sim([2],[2],[0])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([0],[1])$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([0],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$
$\ell_L\sim([2],[2],[1])$ $\nu_R\sim([1],[2])$	$\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$	$\begin{pmatrix}\times&\times&\times\\\times&\times&\times\\\times&\times&\times\end{pmatrix}$

Table 7: The light neutrino mass matrix M_ν in minimal seesaw from Z_2 gauging of Z_4 symmetry, where the crosses represent non-zero matrix elements.

B The full-rank textures of Y_E and the assignments of matter fields for $N = 4, 5$

From the Z_2 gauging of Z_4 symmetry, one can only obtain two textures of the charged lepton Yukawa coupling Y_E with rank three up to permutations of rows and columns, as shown in Eq. (3.20). The first pattern of Y_E is

$$Y_E=\begin{pmatrix}\times&0&0\\0&\times&0\\0&0&\times\end{pmatrix}\ \text{for}\ (\ell_L;E_R;H)\sim([0],[1],[2];[0],[1],[2];[0])\,,\ ([0],[1],[2];[2],[1],[0];[2])\,.\tag{B.1}$$

The second pattern is

$$Y_E=\begin{pmatrix}\times&\times&0\\\times&\times&0\\0&0&\times\end{pmatrix}\,,\tag{B.2}$$

which can be achieved when the lepton fields and Higgs transform in the following way

$$(\ell_L;E_R;H)\sim([0],[2],[1];[1],[1],[0];[1])\,,\ ([0],[2],[1];[1],[1],[2];[1])\,,([0],[0],[1];[0],[0],[1];[0])\,,$$

$$([0],[0],[1];[1],[1],[0];[1])\,,\ ([0],[0],[1];[1],[1],[2];[1])\,,\ ([0],[0],[1];[2],[2],[1];[2])\,,$$
$$([0],[0],[2];[0],[0],[2];[0])\,,\ ([0],[0],[2];[2],[2],[0];[2])\,,\ ([1],[1],[0];[1],[1],[0];[0])\,,$$
$$([1],[1],[0];[0],[2],[1];[1])\,,\ ([1],[1],[0];[0],[0],[1];[1])\,,\ ([1],[1],[0];[2],[2],[1];[1])\,,$$
$$([1],[1],[0];[1],[1],[2];[2])\,,\ ([1],[1],[2];[1],[1],[2];[0])\,,\ ([1],[1],[2];[0],[2],[1];[1])\,,$$
$$([1],[1],[2];[0],[0],[1];[1])\,,\ ([1],[1],[2];[2],[2],[1];[1])\,,\ ([1],[1],[2];[1],[1],[0];[2])\,,$$
$$([2],[2],[0];[2],[2],[0];[0])\,,\ ([2],[2],[0];[0],[0],[2];[2])\,,\ ([2],[2],[1];[2],[2],[1];[0])\,,$$
$$([2],[2],[1];[1],[1],[0];[1])\,,\ ([2],[2],[1];[1],[1],[2];[1])\,,\ ([2],[2],[1];[0],[0],[1];[2])\,, \tag{B.3}$$

By considering all possible assignments of fields, we find that the eight textures of Y_E in Eq. (3.24) can be derived from the Z_2 gauging of Z_5 symmetry,

$$Y_E = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([0],[1],[2];[0],[1],[2];[0])\,, \tag{B.4}$$

$$Y_E = \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([0],[1],[2];[0],[1],[2];[1])\,, \tag{B.5}$$

$$\begin{aligned} Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim\ & ([0],[2],[1];[1],[1],[0];[1])\,,\ ([0],[0],[1];[0],[0],[1];[0])\,, \\ & ([0],[0],[1];[1],[1],[0];[1])\,,\ ([0],[0],[1];[1],[1],[2];[1])\,, \\ & ([1],[1],[0];[1],[1],[0];[0])\,,\ ([1],[1],[0];[0],[2],[1];[1])\,, \\ & ([1],[1],[0];[0],[0],[1];[1])\,,\ ([1],[1],[0];[2],[2],[1];[1])\,, \\ & ([1],[1],[2];[1],[1],[2];[0])\,,\ ([1],[1],[2];[0],[0],[1];[1])\,, \\ & ([1],[1],[2];[2],[2],[0];[2])\,, \end{aligned} \tag{B.6}$$

$$Y_E = \begin{pmatrix} 0 & 0 & \times \\ \times & \times & 0 \\ \times & \times & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([0],[2],[1];[1],[1],[2];[2])\,,\ ([1],[0],[2];[1],[1],[2];[1])\,, \tag{B.7}$$

$$Y_E = \begin{pmatrix} 0 & \times & \times \\ 0 & \times & \times \\ \times & 0 & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([1],[1],[2];[0],[2],[1];[2])\,,\ ([1],[1],[2];[1],[0],[2];[1])\,, \tag{B.8}$$

$$\begin{aligned} Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim\ & ([0],[0],[1];[2],[2],[1];[2])\,,\ ([1],[1],[2];[1],[1],[0];[2])\,, \\ & ([1],[1],[2];[2],[2],[1];[1])\,, \end{aligned} \tag{B.9}$$

$$Y_E = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & 0 & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([1],[1],[2];[0],[0],[2];[1]) \,,\ ([1],[1],[0];[1],[1],[2];[2]) \,,$$
$$([2],[2],[1];[1],[1],[2];[1]) \,, \tag{B.10}$$
$$Y_E = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & 0 \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([1],[1],[0];[2],[2],[1];[2]) \,,\ ([1],[1],[2];[1],[1],[2];[2]) \,,$$
$$([1],[1],[2];[2],[2],[0];[1]) \,. \tag{B.11}$$

Notice that we can obtain the same textures by exchanging $[1] \leftrightarrow [2]$ in the field assignment, since the fusion rules are invariant under the permutation $[1] \leftrightarrow [2]$ for $N = 5$.

C The full-rank textures of Y_E and Y_ν for $N = 6, 7$

In this section, we report the texture zeros of the Yukawa couplings Y_E and Y_ν which can be derived from the Z_2 gauging of Z_N symmetries for $N = 6, 7$. Generally there are plenty of assignments for the matter fields which are capable of producing the same texture, it is too lengthy to present all of them. Consequently we give a representative assignment for each texture.

From the Z_2 gauging of Z_6 symmetry, one can obtain the following textures of the charged lepton Yukawa coupling Y_E with rank three up to permutations of rows and columns,

$$Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([0],[0],[1];[0],[0],[1];[0]) \,,$$
$$Y_E = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([2],[1],[0];[2],[1],[0];[0]) \,,$$
$$Y_E = \begin{pmatrix} \times & \times & \times \\ \times & \times & 0 \\ \times & \times & 0 \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([2],[0],[0];[1],[1],[3];[1]) \,,$$
$$Y_E = \begin{pmatrix} \times & \times & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([2],[0],[1];[3],[1],[0];[1]) \,,$$

$$
Y_E = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & 0 \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([2],[2],[0];[1],[1],[3];[1]) ,
$$

$$
Y_E = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & 0 & 0 \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([2],[2],[0];[1],[3],[3];[1]) , \tag{C.1}
$$

which are shown in Eq. (4.39). The above textures of Y_E can also be derived from the Z_2 gauging of Z_7 symmetry with the same assignment, and one can obtain three more textures as follows,

$$
Y_E = \begin{pmatrix} \times & \times & \times \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([2],[0],[3];[1],[1],[3];[1]) ,
$$

$$
Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & 0 & \times \\ 0 & \times & 0 \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([2],[3],[0];[3],[1],[2];[1]) ,
$$

$$
Y_E = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & 0 & \times \end{pmatrix} \text{ when } (\ell_L; E_R; H) \sim ([1],[1],[3];[2],[0],[3];[1]) . \tag{C.2}
$$

For the Dirac neutrino Yukawa matrix Y_ν, from the Z_2 gauging of Z_6 symmetry, one can obtain the textures of Y_ν with rank two up to permutations of rows and columns as follows,

$$
Y_\nu = \begin{pmatrix} \times & 0 \\ \times & 0 \\ 0 & \times \end{pmatrix} \text{ when } (\ell_L; \nu_R; H) \sim ([0],[0],[1];[0],[1];[0]) ,
$$

$$
Y_\nu = \begin{pmatrix} \times & 0 \\ 0 & \times \\ 0 & 0 \end{pmatrix} \text{ when } (\ell_L; \nu_R; H) \sim ([1],[0],[2];[1],[0];[0]) ,
$$

$$
Y_\nu = \begin{pmatrix} \times & \times \\ \times & 0 \\ \times & 0 \end{pmatrix} \text{ when } (\ell_L; \nu_R; H) \sim ([2],[0],[0];[1],[3];[1]) ,
$$

$$Y_\nu = \begin{pmatrix} \times & \times \\ \times & \times \\ 0 & 0 \end{pmatrix} \text{ when } (\ell_L; \nu_R; H) \sim ([1], [1], [0]; [2], [0]; [1]) \,,$$

$$Y_\nu = \begin{pmatrix} \times & \times \\ \times & 0 \\ 0 & 0 \end{pmatrix} \text{ when } (\ell_L; \nu_R; H) \sim ([2], [0], [1]; [1], [3]; [1]) \,,$$

$$Y_\nu = \begin{pmatrix} \times & \times \\ \times & \times \\ \times & 0 \end{pmatrix} \text{ when } (\ell_L; \nu_R; H) \sim ([2], [2], [0]; [1], [3]; [1]) \,, \tag{C.3}$$

which are shown in Eq. (4.41). Analogously the above six textures of Y_ν can also be obtained from the Z_2 gauging of Z_7 symmetry and the transformation rules of matter fields under non-invertible symmetry are kept intact. What's more, a new texture can be derived from the non-invertible Z_7 symmetry,

$$Y_\nu = \begin{pmatrix} \times & \times \\ \times & 0 \\ 0 & \times \end{pmatrix} \text{ when } (\ell_L; \nu_R; H) \sim ([2], [3], [0]; [3], [1]; [1]) \,. \tag{C.4}$$

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