

Can One Model Gravitational Nonlinear Memory with Fractional Derivative Operators?

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Abstract

We investigate whether fractional calculus, with its intrinsic long-tailed memory and nonlocal features, can provide a viable model for gravitational-wave memory effects. We consider two toy constructions: (i) a fractional modification of the linearized Einstein field equations using a sequential Caputo operator, and (ii) a fractionalized quadrupole formula where the source moment is acted upon by the same operator. Both constructions yield history-dependent responses with small memory-like offsets. However, in all cases, the signal decays to zero at late times, failing to reproduce the permanent displacement predicted by GR. Our results, therefore, constitute a no-go demonstration: naive fractionalization is insufficient to model nonlinear gravitational memory. We argue that any successful model must incorporate fractional kernels directly into the hereditary flux-balance integral of General Relativity, while preserving gauge invariance and dimensional consistency. We also discuss possible connections to modified gravity and the absence of memory in the spacetime with dimensions $D > 4$.

1 Introduction

The nonlinear gravitational wave (GW) memory effect is fundamentally hereditary [1–4]: the permanent offset in the metric is obtained by integrating the flux of radiated energy to null infinity. This hereditary nature is reminiscent of the long-tailed memory kernels that naturally arise in fractional calculus [5, 6]. Unlike integer-order differential operators, fractional operators encode a continuum of scales, making them attractive candidates for modeling gravitational systems with nonlocal, history-dependent responses. The central question of this work is whether fractional derivatives can provide a more natural or flexible mathematical structure to capture such hereditary effects compared to the classical integral formulations of General Relativity (GR).

Gravitation with fractional calculus has been studied in several works [7–10]. For modifications of the Einstein field equations proposed in these papers, one improvement would be to use a dimensionally consistent form of the fractional models (see Sec. 3.1). To our knowledge, there are no prior studies that model gravitational memory with fractional derivatives.

The linear “ordinary” memory effect [11] arises in linearized GR from the flux of unbound matter escaping to null infinity. The nonlinear “null” memory effect [1, 3] is sourced by the GWs themselves: the energy carried by the radiation produces a permanent change in the gravitational

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field after the burst passes. Thorne [4] provided a useful physical interpretation of the nonlinear memory: the gravitons emitted by the source can be regarded as the effective unbound radiation that produces the memory.

Recent surge of activity in the theory side of the memory effect is mainly due to its connection to the infrared structure of GR. The soft graviton theorem [12] shows that adding a low-energy graviton to a scattering process is equivalent to a Ward identity associated with the asymptotic symmetries at null infinity. These symmetries form the Bondi-van der Burg-Metzner-Sachs (BMS) group. One class of these symmetries, the supertranslations, appears physically as the permanent displacement in position that detectors would measure after a GW passes [13]. In this way, memory effects, soft theorems, and asymptotic symmetries constitute the “infrared triangle”. In addition to the displacement memory, several other memory effects were discovered, one of which is the spin memory that is related to superrotations and subleading soft theorems [14]. Memory has also been studied in linearized massive gravity and in higher curvature, and higher-dimensional gravity theories. For example, the relation between the memory effect and graviton mass has been investigated in [15], which shows that the observations of GW memory can strictly bound the graviton mass or potentially be incompatible with a massive graviton. In higher-dimensional spacetimes with an even number of dimensions, memory is non-existent [16]. From the computational point of view, an elegant derivation of memory was given in [17]: it was shown that gravitational wave memory arises as a consequence of the fact that the Riemann tensor obeys a wave equation (sometimes called the Penrose equation) for all Riemannian spacetimes, including the Einstein metrics.

In this work, we focus on the nonlinear memory contribution, which in the far zone can be expressed as [2, 4]

$$\Delta h_{ij}^{TT} = \frac{4}{r} \int_{-\infty}^{\infty} du \left[\int \frac{dE^{\text{gw}}}{du d\Omega'} \frac{n'_j n'_k}{1 - \mathbf{n}' \cdot \mathbf{N}} d\Omega' \right]^{TT}, \quad (1)$$

where u is the retarded time, Ω' is the solid angle, $\mathbf{n}'$ is the unit vector in the direction Ω' , and $\mathbf{N}$ is the direction of the detector.

From an observational perspective, the detection of GW memory is anticipated in the near future with third-generation ground-based detectors (such as the Einstein Telescope and Cosmic Explorer) and with the space-based mission LISA [18, 19]. The observation of memory would provide a unique confirmation of the nonlinear nature of Einstein’s equations in the radiative regime, offering direct evidence for the flux–balance laws associated with asymptotic symmetries [1, 12, 13]. In addition, measuring memory could yield new constraints on modifications of gravity in the strong-field, radiative sector, beyond what is accessible from the oscillatory components of the waveform alone [15, 16]. Thus, theoretical models that attempt to capture memory effects—whether in standard GR or in generalized frameworks such as fractional calculus—are of immediate relevance to the upcoming era of gravitational astronomy. Importantly, even a negative result is valuable: it delineates which kinds of nonlocal or fractional structures cannot reproduce the nonlinear memory, thereby sharpening our understanding of which mathematical frameworks remain compatible with the fundamental predictions of GR.

The layout of the paper is as follows: In Sec. 2, we briefly review the definition and a working classification of fractional operators. In Sec. 3 we investigate two fractional modifications to assess their ability to reproduce memory. In Sec. 3.1, we analyze a fractional linearized Einstein field equation with a sequential Caputo time-fractional derivative, while Sec. 3.2 applies the same operator to the quadrupole moment. Details of the numerical scheme are provided in the Appendix.

2 Definitions and Classification

The starting point for modeling physical systems with fractional derivatives (FD) is the definition that should be chosen for the model. Many definitions for fractional derivatives exist in the literature; see [20] for a review and a brief historical discussion. Before deciding on a definition which is well suited for our model, we briefly explain a proposed criterion for "what makes an operator a fractional derivative" and a working classification of such operators.

In [21] a criterion for deciding whether a given operator is a fractional derivative was proposed. They refine the earlier proposal and introduce two notions: the wide-sense criterion and the strict-sense criterion. In their proposal, fractional derivatives should satisfy linearity, an identity property, backward compatibility, the index law (semi-group property [6]), and an appropriate generalized Leibniz rule. They also have shown that the Grünwald-Letnikov, Riemann-Liouville, and Caputo fractional definitions satisfy these properties in the wide sense.

In [22] a classification of the definitions of fractional derivatives was proposed. They divide fractional derivatives into four classes.

Table 1: Classification of Fractional Derivative Operators

Class	Type of Operator	Description
F1	Classical Derivatives	Standard integer-order derivatives
F2	Modified Derivatives	Variants of classical definitions
F3	Local Operators	Operators with local behavior
F4	Operators with Non-Singular Kernel	Fractional operators with smooth kernels

There are a few properties that, in some sense, characterize the fractional derivative operators in terms of the laws the FD violates. However, new definitions were proposed in [23, 24], which retain some classical properties of the ordinary derivative. Tarasov argued that these definitions are not fractional derivatives, since they can be represented by a finite set of integer order derivatives [25]. Moreover, the well-defined fractional derivatives were shown to necessarily be nonlocal and do not obey the classical chain rule [26, 27]. Consistent with this view, we exclude local operators (F3 class) to model the nonlinear memory effect.

Since our models involve the wave equation and the quadrupole moment and require physically standard initial conditions, we use the left-sided Caputo fractional derivative [28]:

$${}_a^C D_t^\alpha f(t) := \frac{1}{\Gamma(m-\alpha)} \int_{t_a}^t (t-\xi)^{m-\alpha-1} f^{(m)}(\xi) d\xi, \quad m-1 < \alpha < m, \quad m \in \mathbb{N}, \quad (2)$$

where $\Gamma(\cdot)$ is the gamma function and $m = \lceil \alpha \rceil$. The left-sided form is causal and appropriate for the classical initial conditions [6, 21].

Throughout this work, we restrict to $0 < \alpha < 1$ (i.e. $m = 1$) and set the lower terminal to $t_a = 0$ where $t_a < 0$ causes singularity issues for equations (7) and (8). We then use the shorthand notation

$$\partial_t^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} f'(\xi) d\xi, \quad 0 < \alpha < 1, \quad (3)$$

for the Caputo time-fractional derivative used in our models.

3 Discussion and Main Results

In Sec. 2 we restricted the candidate definitions of the fractional derivative to a smaller, well-motivated set. Nevertheless, many definitions remain applicable, and the literature does not offer any specific recipe for which definition must be chosen for the physical system. We adopt the left-sided Caputo derivative because it preserves causality and standard initial conditions.

The reader may find a review of how fractional derivatives are used in [6, 29, 30]. Such models are typically used when the classical local theories fail to capture the observed physical behavior. Nonlocality and long-time memory are the fundamental features of fractional models.

Although nonlinear memory has not been observed yet, GR predicts both linear and nonlinear memory effects. Thus, any viable fractional model must reproduce the GR predictions at least in an appropriate limit. Hence, we compare our results with GR by qualitative features. After consistency with GR is established, fractional formulations may offer new insights about memory effects.

3.1 Modification of the linearized Einstein field equations

We start from the linearized EFE in the Lorenz gauge,

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \quad \square := -\frac{1}{c^2} \partial_t^2 + \Delta. \quad (4)$$

A naive way to “fractionalize” time is

$$\left(-\frac{1}{c^2} C \partial_t^\alpha + \Delta\right) \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \quad 1 < \alpha < 2. \quad (5)$$

Equation (5) is the inhomogeneous Caputo time fractional diffusion–wave equation in 1+3 dimension. The existence and uniqueness of fractional differential equations are discussed (see [5]); the fundamental solutions in the 1+1 dimension are discussed in [31]. However, (5) is dimensionally inconsistent, the time term scales as $[T^{-\alpha}]$ whereas Δ scales as $[L^{-2}]$. One can restore dimensional consistency by introducing a time scale τ ,

$$-\frac{\tau^{\alpha-2}}{c^2} C \partial_t^\alpha \bar{h}_{\mu\nu} + \Delta \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \quad (6)$$

or by using Caputo–type Erdélyi–Kober fractional derivative [32, 33]. Motivated by [30, 34], we also consider a dimensionally consistent sequential fractional operator:

$$\left(-\frac{\Gamma(2-\alpha)}{c t^{1-\alpha}} C \partial_t^\alpha \left[\frac{\Gamma(2-\alpha)}{c t^{1-\alpha}} C \partial_t^\alpha\right] + \Delta\right) \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \quad 0 < \alpha < 1, \quad (7)$$

This sequential Caputo operator is dimensionally consistent and converges to ∂_t^2 as $\alpha \rightarrow 1$. However, $\frac{\Gamma(2-\alpha)}{c t^{1-\alpha}}$ depends explicitly on time, so it breaks time-translation invariance. For analysis and numerical simulations, we take the lower terminal to be $t_a = 0^+$. To understand (7) behavior, we analyze the 1+1 D form and solve it numerically (scheme in the Appendix):

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}} C \partial_t^\alpha \left[\frac{\Gamma(2-\alpha)}{t^{1-\alpha}} C \partial_t^\alpha u(t, x)\right] - \partial_x^2 u(t, x) = s(t, x), \quad 0 < \alpha < 1. \quad (8)$$

We impose Dirichlet boundaries and initial conditions $u(x, 0) = 0$, $u_t(x, 0) = 0$, and take $s(t, x)$ to be localized in space and time:

$$s(t, x) = \exp\left(-\frac{(x-x_0)^2}{2s_x^2}\right) \exp\left(-\frac{(t-t_0)^2}{2s_t^2}\right) \sin(\omega(t-t_0)). \quad (9)$$

We use $\omega \in \{-6\pi, -18\pi, -24\pi\}$, $x_0 = 0$, $t_0 = 0.3$, $s_x = 0.05$, $s_t = 0.06$. Increasing $|\omega|$ results in higher frequency forcing. For each ω we compare several α values at two observation points.

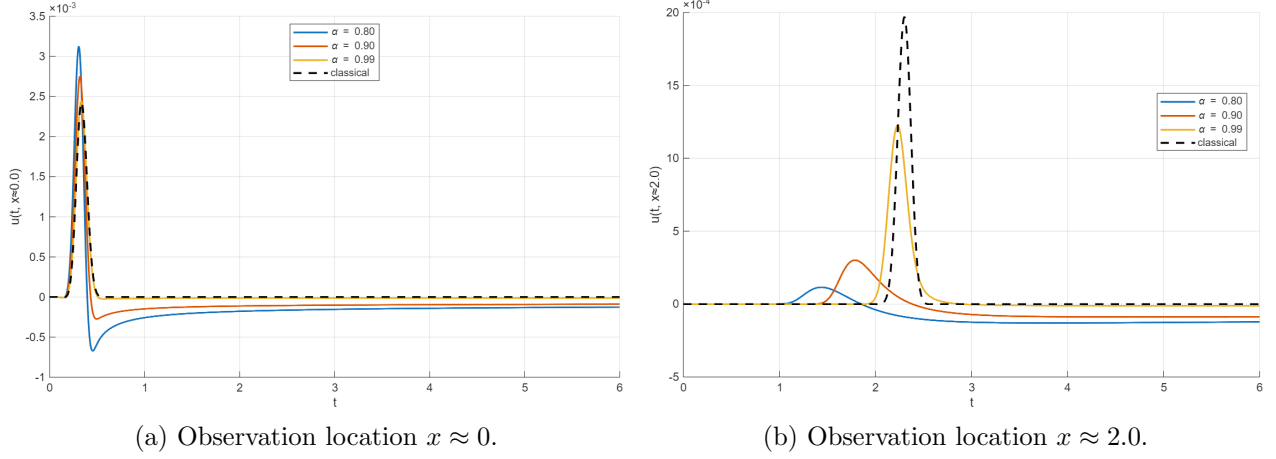


Figure 1: Time evolution of the field $u(t, x)$ at two observation points (a) near the source $x \approx 0$ and (b) at distance $x \approx 2.0$ for the source frequency $\omega = -6\pi$. The curves show the results for several fractional orders α .

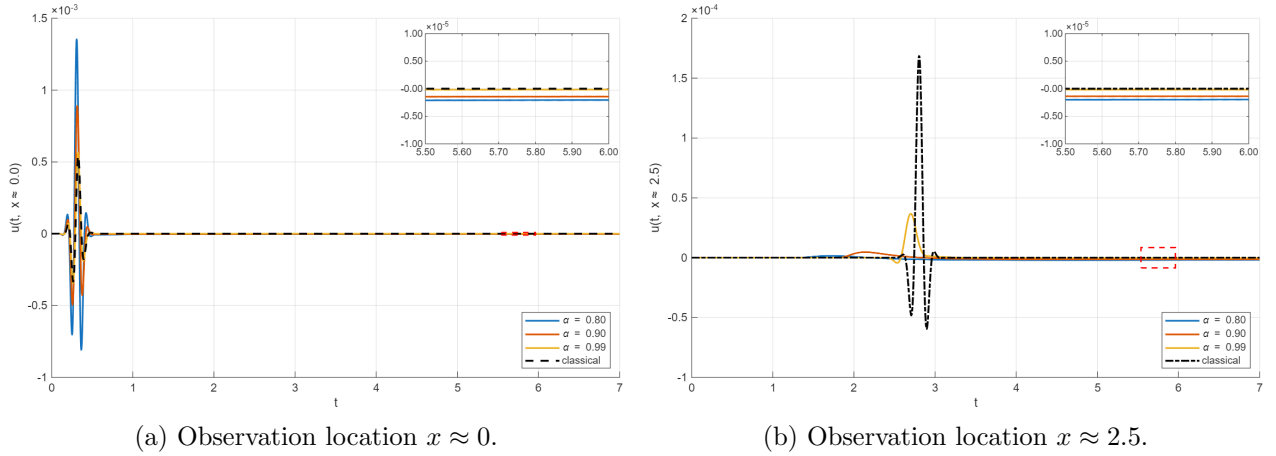


Figure 2: Time evolution of the field $u(t, x)$ at two observation points (a) near the source $x \approx 0$ and (b) at distance $x \approx 2.0$ for the source frequency $\omega = -18\pi$ where the top-right panel shows the zoomed-in view of the region enclosed by the red dashed rectangle. The curves show the results for several fractional orders α .

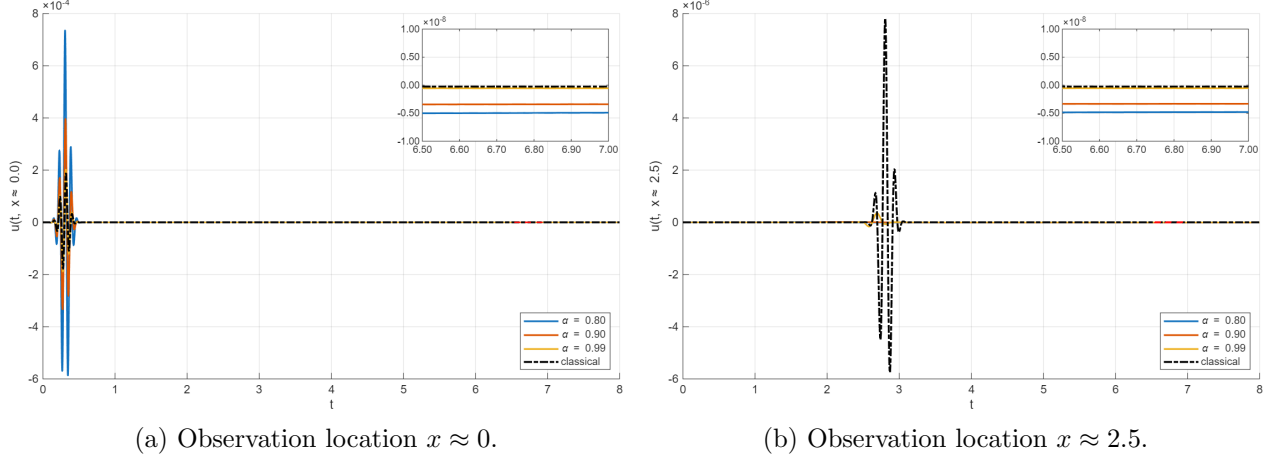


Figure 3: Time evolution of the field $u(t, x)$ at two observation points (a) near the source $x \approx 0$ and (b) at distance $x \approx 2.0$ for the source frequency $\omega = -24\pi$ where the top-right panel shows the zoomed-in view of the region enclosed by the red dashed rectangle. The curves show the results for several fractional orders α .

It's apt to note several observations. (i) As illustrated in Figs. 1–3, the sequential fractional model produces a small memory-like offset in the field $u(t, x)$. The magnitude of this offset increases as the fractional order α decreases. However, the magnitude of this offset decreases as the source frequency $|\omega|$ increases. This trend runs opposite to the gravitational nonlinear memory predicted by GR, which grows with the total radiated energy flux (see (1)). (ii) Comparison of panels (a) and (b) in Figs. 1–3 shows that the apparent offset is slightly reduced at the larger observation distances x , indicating that the effect is localized near the source. (iii) In all cases, the simulations show that the late-time behavior is characterized by the decay of the solution toward zero, regardless of α or ω . This confirms that the operator in (8) behaves like a damped diffusion–wave system rather than producing a permanent displacement in the field. Overall, the fractional model captures some hereditary, memory-like features, but it fails to reproduce the true nonlinear memory effect of GR, which requires a non-vanishing asymptotic offset.

For effective modeling, one may combine the classical and fractionalized operators such that

$$A\Box\bar{h}_{\mu\nu} + B\left(-\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}C\partial_t^\alpha\left[\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}C\partial_t^\alpha\right] + \Delta\right)\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \quad (10)$$

with coefficients A, B satisfying $A + B = 1$; one can allow for more general coefficients, but we have chosen their sum to be 1 below. We use this linear combination approach as a toy model in Section 3.2.

3.2 Fractional quadrupole moment

In Section 3.1, the sequential time–fractional wave model shows a memory-like offset, but its solutions exhibit damped oscillation dynamics. To better preserve the wave nature of the linearized EFE, we suggest a hybrid formulation (10). Nevertheless, for sufficiently large radiated-energy flux, the model still fails to reproduce the qualitative features of the nonlinear memory. This result motivates us to ask whether fractionalizing the stress-energy tensor provides a viable modification to capture the effects of nonlinear memory. To explore this possibility, we begin by introducing a naive fractional modification of the quadrupole moment. In the far zone and the slow motion limit

[35],

$$[h_{ij}^{\text{TT}}(t, \mathbf{x})]_{\text{quad}}' = \frac{2G}{c^4 r} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \ddot{Q}_{kl}\left(t - \frac{r}{c}\right), \quad (11)$$

where $\Lambda_{ij,kl}$ is the projection tensor and $r = |\mathbf{x}|$.

We consider the following fractionalized modification of the second time derivative acting on the source moment:

$$[(\alpha)h_{ij}^{\text{TT}}(t, \mathbf{x})]_{\text{quad}} = \frac{2G}{c^4 r} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \frac{\Gamma(2-\alpha)}{u^{1-\alpha}} C \partial_u^\alpha \left[\frac{\Gamma(2-\alpha)}{u^{1-\alpha}} C \partial_u^\alpha Q_{kl}(u) \right], \quad u := t - \frac{r}{c}, \quad (12)$$

with $0 < \alpha < 1$. As in (7), this sequential Caputo form is dimensionally consistent; however, we do not discuss here whether the modification preserves all tensorial/gauge properties. For effective modeling, we employ a linear combination of (11) and (12),

$$[h_{ij}^{\text{TT}}]_{\text{quad}} = A [h_{ij}^{\text{TT}}]_{\text{quad}}' + B [(\alpha)h_{ij}^{\text{TT}}]_{\text{quad}}, \quad (13)$$

where A, B are (dimensionless) weights chosen after non-dimensionalization; the GR limit corresponds to $A = 1, B = 0$.

Toy binary source. To investigate the offset response vs. source frequency, consider an equal-mass circular binary with $m_1 = m_2 = 1.4 M_\odot$ in the xy -plane and center of mass (CM) at the origin. Let the relative separation be

$$\mathbf{r}(t) = a(t) [\cos \phi(t), \sin \phi(t), 0], \quad a(t) = a_0 \exp\left[-\frac{(t-t_0)^2}{2s_t^2}\right], \quad (14)$$

with $a_0 = 200$ (length units), $t_0 = 20$ s, $s_t = 7.15$ s, and $\phi(t) = \omega_0 t$ where $\omega_0 = 2\pi f_0$, $f_0 \in \{0.2, 0.4, 0.6\}$ Hz. The component positions (CM frame) are

$$\mathbf{x}_1(t) = \frac{m_2}{M} \mathbf{r}(t), \quad \mathbf{x}_2(t) = -\frac{m_1}{M} \mathbf{r}(t), \quad M = m_1 + m_2, \quad (15)$$

and we observe along the $\hat{\mathbf{z}}$ axis at a distance $r = 0.0065$ pc.

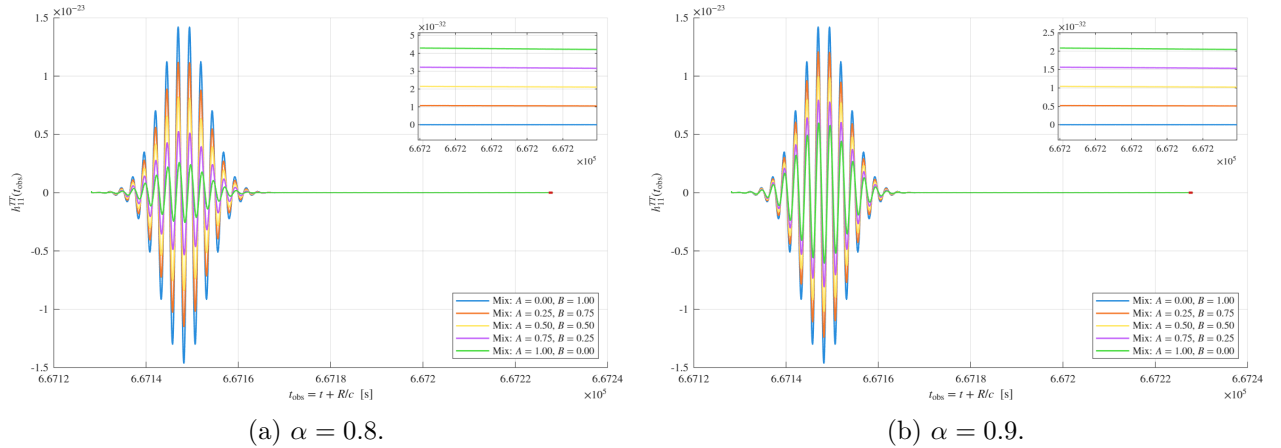


Figure 4: Time evolution of the transverse-traceless gravitational waveform component h_{11}^{TT} for an equal mass circular binary with orbital frequency $f_0 = 0.2$ Hz, observed along the z -axis. The two panels compare the fractional model for (a) $\alpha = 0.8$ and (b) $\alpha = 0.9$. In figures (a) and (b), bottom-left panel shows the chosen A and B values in (13), and top-right panel shows the zoomed-in view of the region enclosed by the red point.

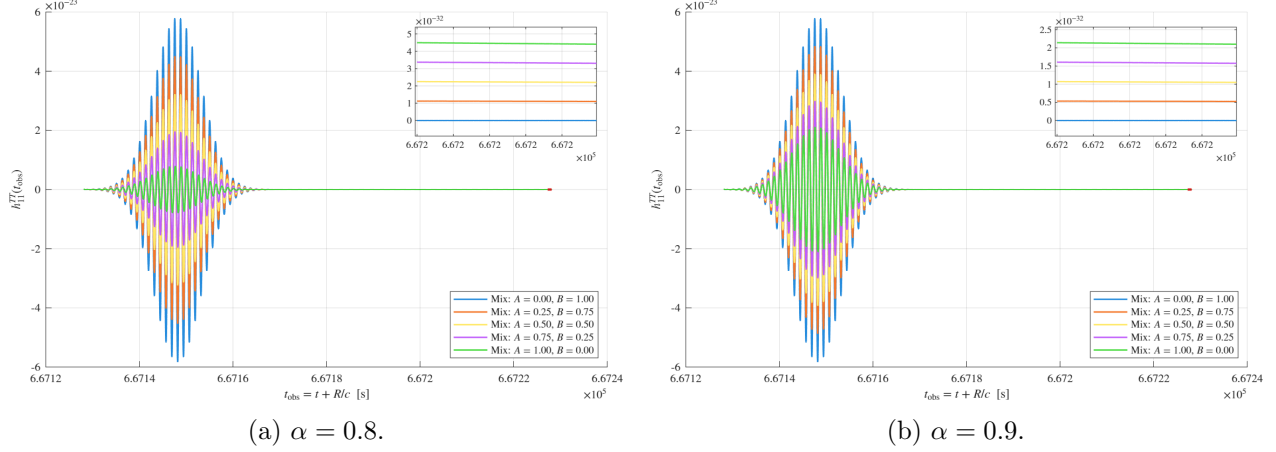


Figure 5: Time evolution of the transverse-traceless gravitational waveform component h_{11}^{TT} for an equal mass circular binary with orbital frequency $f_0 = 0.4$ Hz, observed along the z -axis. The two panels compare the fractional model for (a) $\alpha = 0.8$ and (b) $\alpha = 0.9$. In figures (a) and (b), bottom-left panel shows the chosen A and B values in (13), and top-right panel shows the zoomed-in view of the region enclosed by the red point.

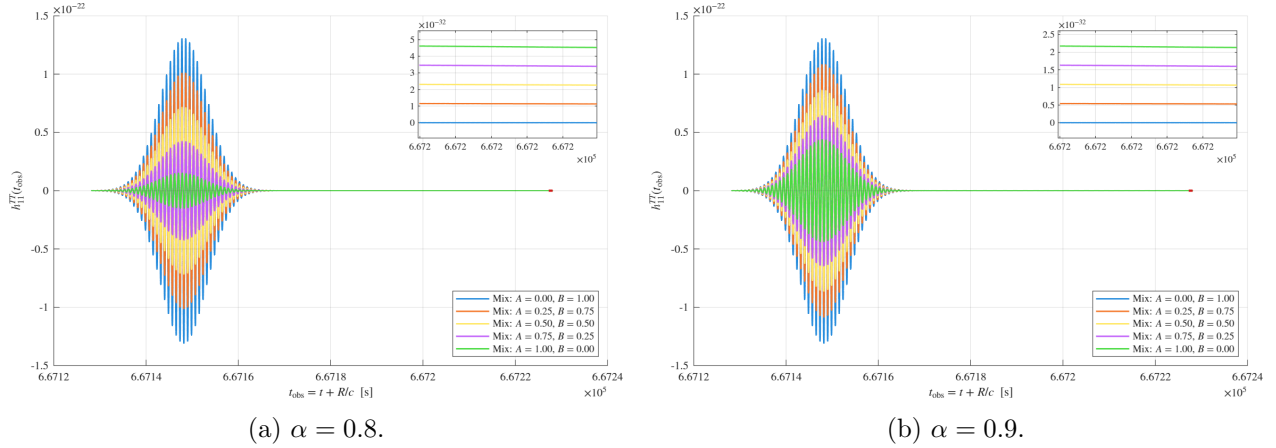


Figure 6: Time evolution of the transverse-traceless gravitational waveform component h_{11}^{TT} for an equal mass circular binary with orbital frequency $f_0 = 0.6$ Hz, observed along the z -axis. The two panels compare the fractional model for (a) $\alpha = 0.8$ and (b) $\alpha = 0.9$. In figures (a) and (b), bottom-left panel shows the chosen A and B values in (13), and top-right panel shows the zoomed-in view of the region enclosed by the red point.

Observations. As shown in Figs. 4–6, the apparent memory, measured by the late-time offset of h_{11}^{TT} , increases slightly as the binary orbital frequency f_0 increases and as the fractional order α decreases. This trend is visible by comparing different simulations: for example, in Fig. 4, the offset is larger in (a) ($\alpha = 0.8$) than in (b) ($\alpha = 0.9$), and the same behavior persists at higher frequencies in Figs. 5 and 6. These results suggest that the contribution of the fractional operator depends sensitively on both α and the source frequency. Nevertheless, in all numerical simulations the waveform incrementally decays toward zero as $t \rightarrow \infty$, indicating that fractionalizing the quadrupole moment cannot produce a permanent displacement in the detectors. Thus, it still fails to capture the true nonlinear memory predicted by GR, which requires a permanent asymptotic offset related

to the radiated energy flux from GWs.

4 Conclusions and Further Work

In this work, we investigated whether fractional calculus, with its characteristic long-tail and the nonlocal features, can serve as a framework for modeling the nonlinear gravitational-wave memory effect. We examined two toy constructions: (i) a sequential Caputo fractional modification of the linearized Einstein field equations and (ii) a fractionalized quadrupole formula in which the source moment is acted upon by the same operator. In both cases, the resulting waveforms displayed history-dependent, memory-like offsets. However, in every numerical experiment, the signal decayed to zero at late times, and therefore failed to reproduce the permanent displacement that GR robustly predicts.

This outcome demonstrates a key negative result: naive fractionalization of differential operators, while it can mimic hereditary response functions, is not sufficient to capture the true nonlinear memory effect. Importantly, such a null result is not merely technical—it provides a valuable boundary condition on the search for generalized models of gravity. It shows that only models which preserve the flux-balance structure of the BMS symmetries, or an equivalent conservation law tied to the radiated energy flux, can correctly encode permanent gravitational-wave memory. Thus, our study can be viewed as a no-go result for simple fractional generalizations.

The implications are twofold. First, if future theoretical work identifies a fractional framework that does preserve flux balance and yields permanent memory, this would represent a genuine new mathematical structure for radiation in strong gravity. Second, if all such fractional generalizations fail, this reinforces the uniqueness of GR’s prediction of nonlinear memory and highlights the deep role of asymptotic symmetries in governing radiative phenomena.

Looking forward, there is a strong observational motivation for refining theoretical models of memory. Forecasts suggest that the detection of the nonlinear memory may already be possible with large event catalogs in the advanced LIGO/Virgo era (e.g. [18, 19]), though it is more likely to be achieved with next-generation detectors such as the Einstein Telescope and Cosmic Explorer, or with the space-based LISA mission [36–38]. A confirmed detection would constitute the first direct observational proof of the nonlinear structure of Einstein’s equations in the radiative sector, providing a novel window into strong-field, nonlinear gravity. Moreover, recent work shows that including nonlinear memory in waveform models can even improve parameter estimation by breaking degeneracies (e.g. [39]).

Several directions emerge from this work:

1. Developing hybrid models in which the hereditary flux integral of GR is replaced or augmented by fractional kernels.
2. Investigating whether fractional calculus can serve as an effective description of modified gravity including massive gravity theories [15] that incorporate intrinsic nonlocality.
3. Exploring the contrast between standard GR, which predicts no memory in higher even-dimensional spacetimes $D > 4$ [16], and fractional frameworks, which may alter this conclusion.
4. Establishing the explicit criteria—dimensional consistency, causal structure, and compatibility with flux balance—that any fractional generalization must meet to be physically viable.

In summary, our findings underscore that fractional operators alone cannot reproduce gravitational-wave memory, but the attempt clarifies what structures are required. As the era of precision gravitational wave astronomy unfolds, both positive and negative results of this kind will be crucial in guiding how we test and extend our understanding of gravity in its most nonlinear regime.

Acknowledgments

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5 Appendix

Here, we provide a pedagogical derivation of the finite-difference scheme used for the time-fractional PDE in (8). The purpose is to show each step in a way that highlights the physical meaning of the discretization. We begin with (8)

$$\frac{\Gamma(2-\alpha)}{t^{1-\alpha}} \partial_t^\alpha \left(\frac{\Gamma(2-\alpha)}{t^{1-\alpha}} \partial_t^\alpha u(t, x) \right) - \partial_x^2 u(t, x) = s(t, x). \quad (16)$$

with Dirichlet boundaries and initial conditions as $u(x, 0) = 0$, $u_t(x, 0) = 0$. To simplify the notation for the governing relation (8), define the additional function $S(t)$ as

$$S(t) := \frac{\Gamma(2-\alpha)}{t^{1-\alpha}}, \quad (17)$$

so that the relation (8) becomes

$$S(t) \partial_t^\alpha [S(t) \partial_t^\alpha u(t, x)] - \partial_x^2 u(t, x) = s(t, x). \quad (18)$$

We divide the time interval $[0, T]$ into N_t subintervals of equal size $k = T/N_t$ with nodes $t_n = nk$, $n = 0, 1, \dots, N_t$. Similarly, we divide the spatial interval $[0, L]$ into N_x subintervals of size $h = L/N_x$ with nodes $x_i = ih$, $i = 0, 1, \dots, N_x$. To avoid the singularity of $S(t) \propto t^{\alpha-1}$ at $t = 0$, we start the time marching at the first positive level $t_1 = k$ (i.e., $n = 1$) and interpret all Caputo integrals with lower terminal 0^+ . In particular, $S(t_n)$ is evaluated only for $n \geq 1$, and the $n = 1$ history sums (e.g., K_i^1 , H_i^1) vanish by definition.

Following [40], we use the first-order discretization for the Caputo derivative as

$$D_t^\alpha f(t_n) \approx \sigma \sum_{j=1}^n w_j (f^{n-j+1} - f^{n-j}), \quad (19)$$

with the weights and the coefficient

$$w_j = j^{1-\alpha} - (j-1)^{1-\alpha}, \quad \sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)(1-\alpha)k^\alpha}. \quad (20)$$

This formula explicitly shows the “memory” effect: the derivative at step n depends on *all previous time steps*, with power-law weights w_j that implement the fractional kernel. The schematic description of weights w_j for several α values is given in [34]. To simplify the notation, we denote $\sigma \equiv \sigma_{\alpha,k}$, since the α and k values are predetermined before the simulation.

We use (19) to evaluate partial fractional time derivatives on (18) as follows

$$\partial_t^\alpha u(t_n, x_i) \approx \sigma \sum_{j=1}^n w_j (u_i^{n-j+1} - u_i^{n-j}), \quad (21)$$

where the upper indices represent time grid points and lower indices represent space grid points.

Our purpose is to discretize relation (18) at time t_n , so we define every summation compactly and start from the interior part. Thus, let $q_i^n := \partial_t^\alpha u(t_n, x_i)$, then use the first-order approximation (19)

$$q_i^n = \sigma[(u_i^n - u_i^{n-1}) + \sum_{j=2}^n w_j (u_i^{n-j+1} - u_i^{n-j})]. \quad (22)$$

Observe that $j = 1$ corresponds to a higher term for u on (22). Define K_i^n for the summation on (22) as

$$K_i^n := \sum_{j=2}^n w_j (u_i^{n-j+1} - u_i^{n-j}). \quad (23)$$

Then, (22) becomes

$$q_i^n = \sigma[(u_i^n - u_i^{n-1}) + K_i^n]. \quad (24)$$

We define g_i^n for the interior part of the time derivative part on (18) for the discrete case as follows

$$g_i^n := S(t_n) \partial_t^\alpha u(t_n, x_i) \Rightarrow g_i^n = S^n q_i^n, \quad (25)$$

where q_i^n was defined in (22). Use the first-order approximation in (19) to discretize the $\partial_t^\alpha g(t_n, x_i)$,

$$\partial_t^\alpha g(t_n, x_i) \approx \sigma[(g_i^n - g_i^{n-1}) + \sum_{j=2}^n w_j (g_i^{n-j+1} - g_i^{n-j})]. \quad (26)$$

Similar to K_i^n , we define H_i^n for the summation on (26) as

$$H_i^n := \sum_{j=2}^n w_j (g_i^{n-j+1} - g_i^{n-j}). \quad (27)$$

Then, (26) becomes

$$\partial_t^\alpha g(t_n, x_i) \approx \sigma[(g_i^n - g_i^{n-1}) + H_i^n]. \quad (28)$$

We are able to discretize the interior part of (18) in a simple notational manner as

$$S^n(\partial_t^\alpha g_i^n) - \partial_x^2 u_i^n = s_i^n. \quad (29)$$

The partial fractional derivative $\partial_t^\alpha g_i^n$ must be written by (28) and (29) becomes

$$S^n \sigma[(g_i^n - g_i^{n-1}) + H_i^n] - \partial_x^2 u_i^n = s_i^n. \quad (30)$$

We insert the definition of g_i^n in (26) to (30) and expand

$$\sigma(S^n)^2 q_i^n - \sigma S^n S^{n-1} q_i^{n-1} + \sigma S^n H_i^n - \partial_x^2 u_i^n = s_i^n. \quad (31)$$

Similarly, we insert the definition of q_i^n in (22) to (31) as

$$\sigma(S^n)^2 \sigma[(u_i^n - u_i^{n-1}) + K_i^n] - \sigma S^n S^{n-1} \sigma[(u_i^{n-1} - u_i^{n-2}) + K_i^{n-1}] + \sigma S^n H_i^n - \partial_x^2 u_i^n = s_i^n. \quad (32)$$

Expand the relation (32)

$$\begin{aligned} s_i^n = & \sigma^2(S^n)^2 u_i^n - \sigma^2(S^n)^2 u_i^{n-1} + \sigma^2(S^n)^2 K_i^n - \sigma^2 S^n S^{n-1} u_i^{n-1} \\ & + \sigma^2 S^n S^{n-1} u_i^{n-2} - \sigma^2 S^n S^{n-1} K_i^{n-1} - \sigma S^n H_i^n - \partial_x^2 u_i^n. \end{aligned} \quad (33)$$

For the approximation of the spatial derivative, the well-known central difference approximation for the second-order partial derivative is used:

$$\partial_x^2 u(t, x) \approx \frac{u(t, x+h) - 2u(t, x) + u(t, x-h)}{h^2}. \quad (34)$$

Then, (33) becomes

$$\begin{aligned} s_i^n = & \sigma^2(S^n)^2 u_i^n - \sigma^2(S^n)^2 u_i^{n-1} + \sigma^2(S^n)^2 K_i^n - \sigma^2 S^n S^{n-1} u_i^{n-1} \\ & + \sigma^2 S^n S^{n-1} u_i^{n-2} - \sigma^2 S^n S^{n-1} K_i^{n-1} - \sigma S^n H_i^n - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}. \end{aligned} \quad (35)$$

To solve (35), we need to rearrange the equation such that the unknown variables are collected on the left-hand side (LHS) and the known variables are collected on the right-hand side (RHS). Consider that iteration case $b = i$, we know every value of u_b^a for all $a < n$ and every value of u_b^a for all $a > n$, we do not know. After the rearrangement, we obtain

$$\begin{aligned} \sigma^2(S^n)^2 u_i^n - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} = & s_i^n + \sigma^2(S^n)^2 u_i^{n-1} - \sigma^2(S^n)^2 K_i^n + \sigma^2 S^n S^{n-1} u_i^{n-1} \\ & - \sigma^2 S^n S^{n-1} u_i^{n-2} + \sigma^2 S^n S^{n-1} K_i^{n-1} + \sigma S^n H_i^n. \end{aligned} \quad (36)$$

The definitions of K_i^n and H_i^n contain only the values of u_b^a where $a < n$. We define the parameter $C^n = \sigma^2(S^n)^2$, and rearrange the LHS to obtain a linear equation.

$$\begin{aligned} -\frac{u_{i+1}^n}{h^2} + (C^n + \frac{2}{h^2})u_i^n - \frac{u_{i-1}^n}{h^2} = & s_i^n + C^n(u_i^{n-1} - K_i^n) + \sigma^2 S^n S^{n-1} u_i^{n-1} - \sigma^2 S^n S^{n-1} u_i^{n-2} \\ & + \sigma^2 S^n S^{n-1} K_i^{n-1} + \sigma S^n H_i^n. \end{aligned} \quad (37)$$

Define everything in the RHS as RHS_i^n

$$RHS_i^n := s_i^n + C^n(u_i^{n-1} - K_i^n) + \sigma^2 S^n S^{n-1} u_i^{n-1} - \sigma^2 S^n S^{n-1} u_i^{n-2} + \sigma^2 S^n S^{n-1} K_i^{n-1} + \sigma S^n H_i^n \quad (38)$$

By combining the time-fractional approximation with the spatial finite difference, we obtain a linear system of the form $Au^n = b^n$, where A is a tridiagonal matrix encoding the spatial coupling, and b^n collects known terms from earlier time steps (the memory terms). At each time step, this linear system is solved to update the field u^n .

$$\begin{bmatrix} C^n + \frac{2}{h^2} & -\frac{1}{h^2} & 0 & \cdots & 0 \\ -\frac{1}{h^2} & C^n + \frac{2}{h^2} & -\frac{1}{h^2} & \cdots & 0 \\ 0 & -\frac{1}{h^2} & C^n + \frac{2}{h^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{1}{h^2} & C^n + \frac{2}{h^2} \end{bmatrix} \begin{bmatrix} u_2^n \\ u_3^n \\ \vdots \\ u_{L_n-1}^n \end{bmatrix} = \begin{bmatrix} RHS_2^n \\ RHS_3^n \\ \vdots \\ RHS_{L_n-1}^n \end{bmatrix}$$

Physically, the iterative scheme demonstrates how the fractional operator induces long-term memory in the evolution: each new step is influenced by the entire history of the source and the field. This is consistent with the hereditary nature of the gravitational memory effect.

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