

CURLING - II. Improvement on the H_0 Inference from Pixelized Cluster Strong Lens Modeling

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ABSTRACT

Strongly lensed supernovae (glSNe) provide a powerful, independent method to measure the Hubble constant, H_0 , through time delays between their multiple images. The accuracy of this measurement depends critically on both the precision of time delay estimation and the robustness of lens modeling. In many current cluster-scale modeling algorithms, all multiple images used for modeling are simplified as point sources to reduce computational costs. In the first paper of the CURLING program, we demonstrated that such a point-like approximation can introduce significant uncertainties and biases in both magnification reconstruction and cosmological inference. In this study, we explore how such simplifications affect H_0 measurements from glSNe. We simulate a lensed supernova at $z = 1.95$, lensed by a galaxy cluster at $z = 0.336$, assuming time delays are measured from LSST-like light curves. The lens model is constructed using JWST-like imaging data, utilizing both LENSTOOL and a pixelated method developed in CURLING. Under a fiducial cosmology with $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$, the LENSTOOL model yields $H_0 = 69.91^{+6.27}_{-5.50} \text{ km s}^{-1} \text{ Mpc}^{-1}$, whereas the pixelated framework improves the precision by over an order of magnitude, $H_0 = 70.39^{+0.82}_{-0.60} \text{ km s}^{-1} \text{ Mpc}^{-1}$. Our results indicate that in the next-generation observations (e.g., JWST), uncertainties from lens modeling dominate the error budget for H_0 inference, emphasizing the importance of incorporating the extended surface brightness of multiple images to fully leverage the potential of glSNe for cosmology.

Key words: Gravitational Lensing: strong – Cosmology: cosmological parameters – Galaxy: clusters: general

1 INTRODUCTION

The Hubble constant (H_0) is a critical cosmological parameter that describes the expansion rate of the Universe. The value of H_0 , however, is still under debate due to the significant discrepancy between early- and late-Universe measurements. Early-time observations of the cosmic microwave background (CMB, [Planck Collaboration et al. 2020](#)) yield a value of $H_0 = 67.4 \pm 0.5 \text{ km/s/Mpc}$ under the standard Λ Cold Dark Matter (Λ CDM) model. In contrast, type Ia supernovae (SNe Ia) calibrated with distance ladders in the local Universe give $H_0 = 73.0 \pm 1.0 \text{ km/s/Mpc}$ ([Riess et al. 2022](#); [Zhang et al. 2017](#); [Wong et al. 2020](#); [Riess et al. 2024](#)), remaining the problem unresolved as the ‘ H_0 tension’ ([Abdalla et al. 2022](#); [Perivolaropoulos & Skara 2022](#)).

Additional cosmological probes are necessary to resolve this 5σ tension and determine the exact value of H_0 . One promising method involves exploiting the gravitational lensing effect of time-varying sources. In strong lensing, a background source’s light is deflected by the gravitational potential of an intervening mass, producing multiple images with distinct light paths. When the source is variable, these differences lead to measurable time delays between the images. The time delays are related to the so-called ‘time-delay distance’, $D_{\Delta t} \equiv (1 + z_l)D_L D_S / D_{LS}$, which is inversely proportional to H_0 , where z_l is the redshift of the lens, D_L , D_S , and D_{LS} are the angular diameter distances from the observer to the lens, from the observer to the source, and from the lens to the source, respectively.

Time delay cosmology has been extensively used in lensed active galactic nuclei (AGN, e.g., [Kochanek et al. 2006](#); [Suyu et al. 2017](#); [Courbin et al. 2018](#); [Millon et al. 2020b](#); [Ding et al. 2021](#)), albeit the stochastic nature of AGN light curves and their longer

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variability time-scales pose challenges for precise measurement of H_0 . As an alternative, lensed supernovae (gLSNe) offer characteristic light curves and shorter timescales, making them more suitable for accurate time-delay measurements. However, gLSNe are much rarer. The first discovery was SN Refsdal in the galaxy cluster MACS J1149.6+2223 (Kelly et al. 2015), whereas the idea of using the SN time delays to constrain H_0 dates back to 1964 (Refsdal 1964). Among the seven confirmed gLSNe (Kelly et al. 2015; Rodney et al. 2021; Chen et al. 2022b; Frye et al. 2024; Pierel et al. 2024b; Goobar et al. 2017, 2023, 2025) to date, only three have been utilized for H_0 measurement. Grillo et al. (2018) obtained $H_0 = 69.8^{+5.3}_{-4.1}$ km s⁻¹ Mpc⁻¹ from ‘SN Refsdal’, while Pascale et al. (2024) derived $H_0 = 75.4^{+8.1}_{-5.5}$ km s⁻¹ Mpc⁻¹ using the ‘SN H0pe’. A more recent study of ‘SN Encore’ yielded $H_0 = 66.9^{+11.2}_{-8.1}$ km s⁻¹ Mpc⁻¹ (Suyu et al. 2025), leaving significant uncertainties in H_0 inferences with gLSNe.

The low rate of gLSNe has prompted numerous efforts on the searching strategies (Shu et al. 2018; Craig et al. 2021; Huber et al. 2019; Magee et al. 2023; Arendse et al. 2024), while another bottleneck for gLSNe time delay cosmography is the robustness of lens modeling (Grillo et al. 2020; Birrer et al. 2022), which is recognized as the dominant source of uncertainty ($\sim 5.5\%$) over time delay measurement ($\sim 1.5\%$) in H_0 constraints (Kelly et al. 2023). Compared to galaxy-scale lenses, galaxy clusters have definite advantages, such as resolving the intractable mass-sheet degeneracy (Bradač et al. 2004) when more than one family of multiple images are used for mass modeling, and longer time scales between image appearances that are easier to observe. Nevertheless, the complex mass distribution of clusters—featuring multiple dark matter clumps and hundreds of member galaxies—makes lens modeling more difficult, leading to higher uncertainties in cluster-based H_0 estimates (Suyu et al. 2017; Wong et al. 2020; Millon et al. 2020a).

Frequently investigated systematic errors in cluster-scale lens modeling include the unaccounted line-of-sight (LoS) structures (D’Aloisio & Natarajan 2011; Chirivì et al. 2018), the use of unified scaling relations across all member galaxies (Bergamini et al. 2021), the inaccuracies in the measurement of multiple image redshifts (Acebron et al. 2017). Aside from these, using only the positions of multiple images, instead of their extended surface brightness, can introduce systematic errors in lens modeling (Acebron et al. 2024; Çağan Şengül 2024). This issue has been emphasized in the first paper of our CURLING (CIUster strong Lens modelIng for the Next-Generation observations, Xie et al. 2024, X24) program, where we proposed a ‘pixelized model’ framework. In this approach, lensing constraints are derived from the full surface brightness distribution of the lensed images, rather than just their positions.

We expect to discover and study gLSNe through repeated observations. An ideal survey is the upcoming Large Synoptic Survey Telescope (LSST, LSST Science Collaboration et al. 2009), which is poised to revolutionize time-domain astronomy with its wide field of view and deep imaging capabilities, covering the targeted regions every few nights. Another promising facility is the Multi-Channel Imager (MCI) aboard the China Survey Space Telescope (CSST, Zhan 2018; Zhan 2021), which will offer unique advantages for gLSNe detection. As a space-based instrument, MCI is free from atmospheric seeing, enabling better imaging compared to ground-based surveys. This is particularly valuable for observing the supernovae. MCI features wide wavelength coverage from NUV to NIR, extreme depth up to 30 magnitude, and a total exposure time exceeding 480 000 s (300 s $\times$ 1600 exposures). These capabilities make MCI especially well-suited for monitoring gLSNe with high precision. In this study, we incorporate the observational characteristics of LSST and CSST-

MCI into our simulations to assess their performance in enabling precise time-delay measurements for gLSNe-based H_0 inference.

In this paper, we investigate the improvement of gLSNe time delay cosmography with the pixelated modeling method. The work is organized as follows. In Section 2, we briefly introduce the theoretical framework of time-delay cosmography. We describe the simulated data used in this study, including the lens systems and the lensed supernova, in Section 3. In Section 4, we detail the measurement of time delays and the construction of lens models using both LENSTOOL and our pixelized approach. Section 5 presents the inferred values of H_0 , comparing the results obtained from the two modeling methods. Finally, we summarize the work and discuss the findings in Section 6. Throughout the paper, we assume a flat Λ CDM cosmology ($\Omega_{\Lambda,0} = 0.7$, $\Omega_m = 0.3$, $h = 0.7$) with a constant dark energy equation of state parameter $w = w_0 = -1$.

2 TIME DELAY COSMOGRAPHY

In this section, we briefly introduce the theory of time delay cosmology. For a comprehensive description, we refer the reader to Blandford & Narayan (1992); Schneider et al. (1992); Treu & Marshall (2016); Meneghetti (2022) and references therein.

When light from a distant source is deflected by a gravitational lens, it travels along different paths and through varying gravitational potentials, resulting in a delay in arrival time compared to the unlensed case. The total time delay consists of two components:

$$t = t_{\text{grav}} + t_{\text{geom}}. \quad (1)$$

The first term is the gravitational time delay,

$$t_{\text{grav}} = \int \frac{dz}{c'} - \int \frac{dz}{c} = -\frac{D_L D_S}{D_{LS}} \frac{1}{c} \hat{\Psi}, \quad (2)$$

caused by the intervening gravitational potential $\hat{\Psi}$. The second term denotes the geometrical time delay,

$$t_{\text{geom}} = \frac{\Delta l}{c} = \frac{1}{2c} (\vec{\theta} - \vec{\beta})^2 \frac{D_L D_S}{D_{LS}}, \quad (3)$$

where Δl represents the extra path of the light in the presence of the lens. By substituting Equations 2 and 3 into Equation 1, and accounting for the time dilation by the factor $(1+z_L)$, we obtain the time delay at image position $\vec{\theta}$:

$$t(\vec{\theta}) = \frac{1+z_L}{c} \frac{D_L D_S}{D_{LS}} \left[\frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \hat{\Psi}(\vec{\theta}) \right] = \frac{D_{\Delta t}}{c} \tau(\vec{\theta}). \quad (4)$$

Here, the term $D_{\Delta t} = (1+z_L) \frac{D_L D_S}{D_{LS}}$ is defined as the time-delay distance, which encapsulates three angular diameter distances, D_L from the observer to the lens, D_S from the observer to the source, and D_{LS} from the lens to the source. According to the definition of the angular diameter distance,

$$D(z_1, z_2) = \frac{cH_0^{-1}}{1+z_2} \int_{z_1}^{z_2} dz [\Omega_m(1+z)^3 + \Omega_X(1+z)^{3(w_X+1)}]^{-1/2}, \quad (5)$$

the time-delay distance $D_{\Delta t}$ is inversely proportional to H_0 and weakly dependent on other cosmological parameters. The term $\tau(\vec{\theta})$ in Equation 4 is the Fermat potential, which can be determined through lens modeling.

Though the arrival time of one light ray is not detectable, the time difference between two lensed images i and j is measurable and given by:

$$\Delta t_{ij} = \frac{D_{\Delta t}}{c} [\tau(\vec{\theta}_i) - \tau(\vec{\theta}_j)]. \quad (6)$$

Table 1. Summary of the mock lens system based on MACS J0138.0-2155 at redshift $z = 0.336$. The upper part describes the lensing mass components, which adopts the best-fit mass model of this cluster presented in [Rodney et al. \(2021\)](#) as input. The mock cluster includes a cluster-scale halo, a brightest cluster galaxy (BCG), three individual perturbers, and 24 additional cluster member galaxies. Parameters for the member galaxies follow unified scaling relations. For each component, the table lists the position to the cluster center ($\Delta\alpha$, $\Delta\delta$), ellipticity e , position angle θ , core radius r_{core} , cut radius r_{cut} , and velocity dispersion σ . The lower part presents the lensed sources, each modeled using a Sérsic profile characterized by the position, semi-major and semi-minor axes (a , b), position angle (θ), apparent magnitude, and source redshift z_s . Source 2 corresponds to the strongly lensed supernova.

MACS0138-like, $z = 0.336$							
Component	$\Delta\alpha$ [']	$\Delta\delta$ [']	e	θ [deg]	r_{core} [kpc]	r_{cut} [kpc]	σ [km/s]
Cluster Halo	-0.7	-1.2	0.81	114.9	31.0	1000.0	446.0
BCG	0.1	-0.1	0.52	-41.1	0.15	136.0	700.0
Perturber 1	20.04	-15.11	0.49	86.2	0.15	25.5	152.0
Perturber 2	-5.0	6.9	0.06	4.4	0.15	12.0	23.0
Perturber 3	0.12	-18.25	0.24	-63.1	0.15	6.0	110.0
scaling relations	$N(\text{gal}) = 24$	$m^{\text{ref}} = 19.5$	$r_{\text{core}}^{\text{ref}} = 0.15 \text{ kpc}$	$r_{\text{cut}}^{\text{ref}} = 45.0 \text{ kpc}$	$\sigma^{\text{ref}} = 158.0 \text{ km/s}$		
Sources							
Component	$\Delta\alpha$ [']	$\Delta\delta$ [']	a [kpc]	b [kpc]	θ [deg]	mag	z_s
Source 1	3.05	-1.47	2.17	3.47	0.0	15.5	1.95
Source 2 (SN)	3.1	-1.73	/	/	0.0	20.8	1.95
Source 3	-2.0	4.01	1.88	1.13	0.0	18.5	0.7336

Therefore, if the time delay Δt_{ij} is measured from observations and the Fermat potential is accurately inferred from lens modeling, the value of H_0 can be well constrained.

3 SIMULATIONS

The primary aim of this study is to compare the use of point-like versus extended multiple images as lensing constraints, in order to assess how extended surface brightness information can improve the inference of the Hubble constant H_0 from strongly lensed supernovae in galaxy clusters. To focus on this comparison, we use mock data instead of real observations, which allows us to avoid additional challenges, such as light contamination on the extraction of arcs, deviations between analytical models and real mass distributions, that would complicate the modeling of extended sources.

In this section, we describe the construction of the simulated data used in this work. Simulating a lensed supernova involves two main components: defining the mass distribution of the lensing galaxy cluster and generating multiple images of the lensed sources which will serve as ‘observational’ constraints for lens modeling; specifying the properties of the background supernova.

3.1 The strong lensing system: MACS J0138.0-2155

To create a realistic mock lensing cluster, we simulate the field of MACS J0138.0-2155, a galaxy cluster of particular interest due to the discoveries of two lensed supernovae, SN Requiem and SN Encore, located in the same host galaxy ([Rodney et al. 2021](#); [Pierel et al. 2024b](#)). Making the mock image for this cluster field involves several key steps.

(i) We generate the galaxies in the field, which includes not only cluster member galaxies but also foreground and background galaxies along the line of sight. To ensure a reliable spatial and photometric distribution, we perform source detection and photometry using PHOTUTILS ([Bradley et al. 2016](#)) on the JWST NIRC2 F277W image (PI: Justin Pierel, Proposal ID: 6549) of MACS J0138.0-2155. From this, we extract the positions, shapes, and fluxes of all detected galaxies. Each galaxy is then simulated using a Sérsic profile, adopting the measured morphological parameters and flux.

(ii) We define the mass distribution of the cluster lens system by adopting the best-fit lens model from [Rodney et al. \(2021\)](#) as the input cluster mass, as detailed in Table 1. The model includes a cluster-scale dark matter halo, a brightest cluster galaxy (BCG), three perturbers close to the multiple images, and member galaxies. While [Rodney et al. \(2021\)](#) modeled 32 members, our simulation includes only 24, as we restrict the field-of-view to $60'' \times 60''$. This clipping reduces computational cost while preserving the strong-lensing region at the core of the cluster, ensuring that the critical information for lens modeling are retained. Each component is described using a pseudo-isothermal elliptical mass distribution (PIEMD, [Elíasdóttir et al. 2007](#)),

$$\rho(r) = \frac{\rho_0}{(1 + \frac{r^2}{r_{\text{core}}^2})(1 + \frac{r^2}{r_{\text{cut}}^2})}, \quad (7)$$

where r_{core} , r_{cut} are the core radius and cut radius, respectively. A PIEMD profile is thereby described by the parameter set $\{x, y, e, \theta, r_{\text{core}}, r_{\text{cut}}, \sigma\}$, where x, y describe the center of the potential, e and θ are the ellipticity and position angle, respectively, and σ represents the velocity dispersion of the potential, which is related with ρ_0 via $\sigma^2 = \frac{4}{3} G \pi \rho_0 r_{\text{core}}^2 r_{\text{cut}}^3 / [(r_{\text{cut}} - r_{\text{core}})(r_{\text{cut}} + r_{\text{core}})^2]$. Core radius, cut radius, and velocity dispersion of each member galaxy follow scaling relations based on its magnitude ([Granata et al. 2022](#)):

$$\begin{cases} \sigma_i^{\text{gal}} = \sigma^{\text{ref}} 10^{0.4 \frac{m_i^{\text{ref}} - m_i}{\alpha}} \\ r_{\text{core},i}^{\text{gal}} = r_{\text{core}}^{\text{ref}} 10^{0.4 \frac{m_i^{\text{ref}} - m_i}{\beta}} \\ r_{\text{cut},i}^{\text{gal}} = r_{\text{cut}}^{\text{ref}} 10^{0.4 \frac{m_i^{\text{ref}} - m_i}{\beta}}, \end{cases} \quad (8)$$

where the typical values σ^{ref} , $r_{\text{core}}^{\text{ref}}$, $r_{\text{cut}}^{\text{ref}}$, and m^{ref} are 158 km s^{-1} , 0.15 kpc , 45 kpc , and 19.5 mag .

(iii) We generate the lensed arcs based on the mass distributions obtained in step (ii). To do so, we first estimate the properties of the sources, including their positions, shapes, and fluxes, assuming a Sérsic profile for each. Using these sources, we perform ray tracing through the cluster’s gravitational lensing potential to create mock arcs, and adjust the source properties to find the best arcs that resemble those observed in the real cluster. The properties of the two sources corresponding to the arcs and the supernova are summarized in Table 1.

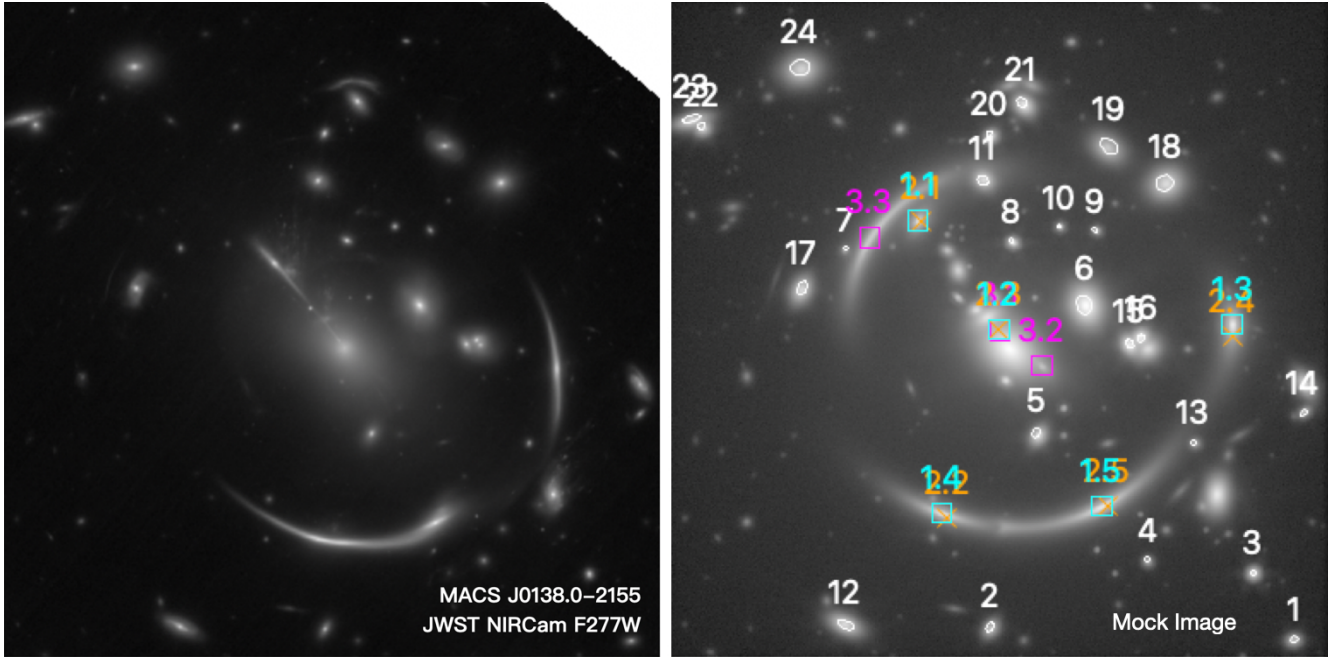


Figure 1. Left: JWST NIRCам (F277W) Observation on the lensing cluster MACS J0138.0-2155. Right: Mock image of the simulated cluster used in this work. Member galaxies, multiple images as lensing constraints, and lensed supernova images are marked with white ellipses, cyan/magenta boxes, and orange crosses, respectively. The field-of-view (FoV) is $60'' \times 60''$.

(iv) We model the intra-cluster light (ICL, Zwicky 1937; DeMaio et al. 2018; Chen et al. 2022a; Contini et al. 2023), which accounts for the diffuse light of individual stars that are not associated with any galaxy. The ICL is represented by a smooth component following a Navarro-Frenk-White (NFW, Navarro et al. 1997) profile, of which its mass is determined according to the total cluster stellar mass $M_* = 9.26 \times 10^{12} M_\odot$, and a concentration parameter $c = 3.88$, based on the concentration-mass relation of Child et al. (2018). More details for the generation of mock ICL distribution can be found in Chen et al. (2022a).

(v) Finally, we simulate observational effects to create the mock image. The mock image is convolved with a JWST NIRCам Point Spread Function (PSF) generated using WEBBPSF (Perrin et al. 2015), and we add noise to reflect observing conditions. This includes both Poisson noise from the sources and sky background, as well as Gaussian noise from detector readout. The total noise level in each pixel is computed (Stark et al. 2024; Cao et al. 2018) using:

$$\sigma = \frac{\sqrt{N_{\text{pix}} \times (S_{\text{img}} + S_{\text{sky}}) \times t_{\text{expo}} + N_{\text{expo}} \times R^2}}{t_{\text{expo}}}, \quad (9)$$

where S_{sky} is the sky brightness in unit of $e^- \text{ s}^{-1} \text{ pixel}^{-1}$, converted from $m_{\text{sky}} = 29.5 \text{ mag arcsec}^{-2}$, given a zero magnitude of 27.295 in JWST F277W filter, N_{pix} is the number of pixels of the object, the readout noise is $R = 15.88 e^-$, the total exposure time is $t_{\text{expo}} = 7537 \text{ s}$, and the number of exposures is $N_{\text{expo}} = 9$.

The mock image, generated through the process detailed above, is presented in Figure 1 alongside the real JWST/NIRCам F277W observation for visual comparison.

3.2 The supernova: Requiem

Two aspects need to be taken into account for simulating the lensed supernova: the intrinsic properties of the supernova and the effects of gravitational lensing.

For the supernova itself, we model a Type Ia event inspired by SN Requiem in MACS J0138.0-2155 (Rodney et al. 2021), located at a source redshift of $z_s = 1.95$. The SN light curve is generated using the `sal2` model (version T23, Taylor et al. 2023) provided by SNCosmo, adopting typical parameters from the literature (Kenthworthy et al. 2021), $x_1 = 0.4$ for the light curve shape, $c = -0.05$ for the color, and $t_0 = 0$ days as the observer-frame time corresponding to the source's phase = 0. The SN is simulated with a peak absolute magnitude of -19.2 AB mag in B band, which is a typical value for Type Ia supernova (Phillips 1993; Riess et al. 1998).

For lensing, the SN is placed with an offset of $\Delta x = 0.43 \text{ kpc}$ and $\Delta y = 2.24 \text{ kpc}$ from the host galaxy's center. Computations through the cluster potential described in Section 3.1 yield five lensed images of the SN (orange crosses in Figure 1) with respective magnifications and arrival times of (1.33, 3.58, 3.88×10^{-4} , 3.23, 6.45) and (5968.85, 421.97, 9318.08, 0, 781.98) days after the first image appears. Given the low magnifications and late arrival times of SN image 2.1 and 2.3, we focus on the other three for time delay measurements, which we label as SN image 2.2, 2.4, and 2.5. These images exhibit time delays relative to the earliest image 2.4 of $\Delta t_{42} = 421.97$ days and $\Delta t_{45} = 781.98$ days.

Based on the original SN light curve, we compute the theoretical light curves for these three lensed images and show them as the dashed lines in Figure 2. To simulate observational conditions, we assume that the supernova images are monitored by LSST with a Gaussian-distributed cadence (a mean of 10 days and a standard deviation of 0.5 day), a seeing of $0.68''$ (Arendse et al. 2024), and noise as described in Equation 9 with sky magni-

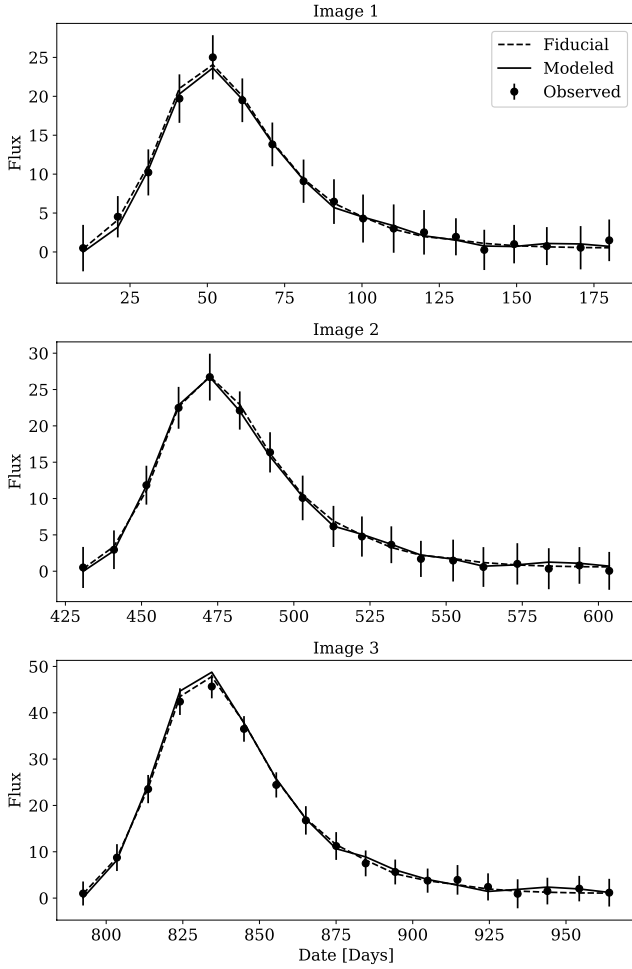


Figure 2. Light curves of the three multiple images of the simulated supernova under the LSST observational configuration. Each panel corresponds to one of the SN images. The dashed lines represent the theoretical light curves computed from the intrinsic supernova model and the associated lensing magnification and arrival time. Data points with error bars show the measured fluxes, incorporating observational effects from LSST. Solid lines indicate the best-fit light curves obtained using the `SNTP` package.

tude $m_{\text{sky}} = 22.61 \text{ mag arcsec}^{-2}$, zero point magnitude in g-band $m_{\text{zp}} = 26.68$, $t_{\text{expo}} = 30 \text{ s}$, $N_{\text{expo}} = 2$, and $R = 10 \text{ e}^-$.

After generating the simulations, we replicate observational procedures to extract the light curves. At each epoch, we subtract a reference frame without SN flux from a science frame containing the corresponding flux, with both sets of images produced under identical observing conditions. Force photometry is then performed at the known SN image positions on the difference images, measuring fluxes within an aperture matched to the PSF size. These measurements yield the data points shown in Figure 2, which form the basis for deriving the time delays of the lensed SN images, as depicted in Section 4.2.

4 METHODS

4.1 Lens modeling

For the purpose of assessing whether the pixel-based modeling approach can improve the H_0 inference, we model the simulated galaxy

cluster introduced in the previous section by using two techniques: a traditional one that uses only the positions of multiple images as observational constraints, and a new one developed in our earlier work (Xie et al. 2024), which fits the extended surface brightness distribution of lensed arcs. Both methods are applied under a simulated JWST/NIRCam observation setup and make use of the same two families of multiple images as constraints.

Due to computational limitations, only the parameters of the primary cluster-scale halo are allowed to vary, while the remaining model parameters are fixed to their inputs. Systematics such as uncertainties in redshift measurements, the presence of line-of-sight structures are not included.

Cluster-scale lens modeling has historically treated multiple images of background sources as point-like constraints, fitting lens models by minimizing the discrepancies between the observed and predicted positions of lensed images, while ignoring morphological and photometric information. Here, we use `LENSTOOL` (Kneib et al. 1996; Jullo et al. 2007) as a representative of this method. The optimization is performed by minimizing a χ^2 defined on the image plane:

$$\chi_i^2 = \sum_{j=1}^{n_i} \frac{[\vec{\theta}_{\text{obs}}^j - \vec{\theta}^j(p)]^2}{\sigma_{ij}^2}, \quad (10)$$

where $\vec{\theta}_{\text{obs}}^j$ and $\vec{\theta}^j(p)$ are the observed and modeled positions of image j , respectively. The positional uncertainty σ_{ij} is uniformly assumed to be $0.1''$, consistent with the value used for JWST strong lensing cluster modeling (e.g., Bergamini et al. 2023). The posterior distribution of the lens parameters p is sampled using the `BAYESys` MCMC sampler (Skilling 2005) implemented within `LENSTOOL`.

Pixel-based lens modeling becomes increasingly significant with the availability of high-resolution imaging (Acebron et al. 2024; Urcelay et al. 2025; Cao et al. 2025; Ding et al. 2025), since the traditional lens modeling approach discards the rich information contained in the surrounding pixels. As a result, much of the morphological detail and flux distribution within the lensed arcs is ignored, leading to a substantial loss of constraining power. Furthermore, this method relies on the assumption that positional uncertainties follow a Gaussian distribution, while the complex and arc-like morphology of these sources violates this assumption, potentially introducing biases into the inferred lens model parameters.

To overcome the limitations in traditional lens modeling techniques, we implement a more sophisticated modeling framework developed in X24, which fits the observed surface brightness distributions of the lensed images on a pixel-by-pixel basis, enabling leveraging full information of the arcs and more precise parameter inference. The modeling procedure consists of the following steps. First is the arc pixel selection, we identify high-signal-to-noise regions of the arcs by selecting pixels centered around the flux peaks of each lensed image. Only pixels with fluxes greater than 60 times the mean value of the error map are included, ensuring reliable photometric constraints. 23,000 pixels are selected as the arcs in this simulated cluster. The next step is forward modeling, for each proposed set of model parameters p , we generate the corresponding image distributions via lens equation. Bayesian inference is finally applied, where the χ^2 over all selected image pixels is defined as:

$$\chi^2 = \sum_{i=1}^{n_{\text{pix}}} \frac{[S_{\text{obs}}^i - S^i(p)]^2}{\sigma_i^2}, \quad (11)$$

where S_{obs}^i is the observed surface brightness at pixel i , $S^i(p)$ is the model-predicted surface brightness given model parameters p , and σ_i

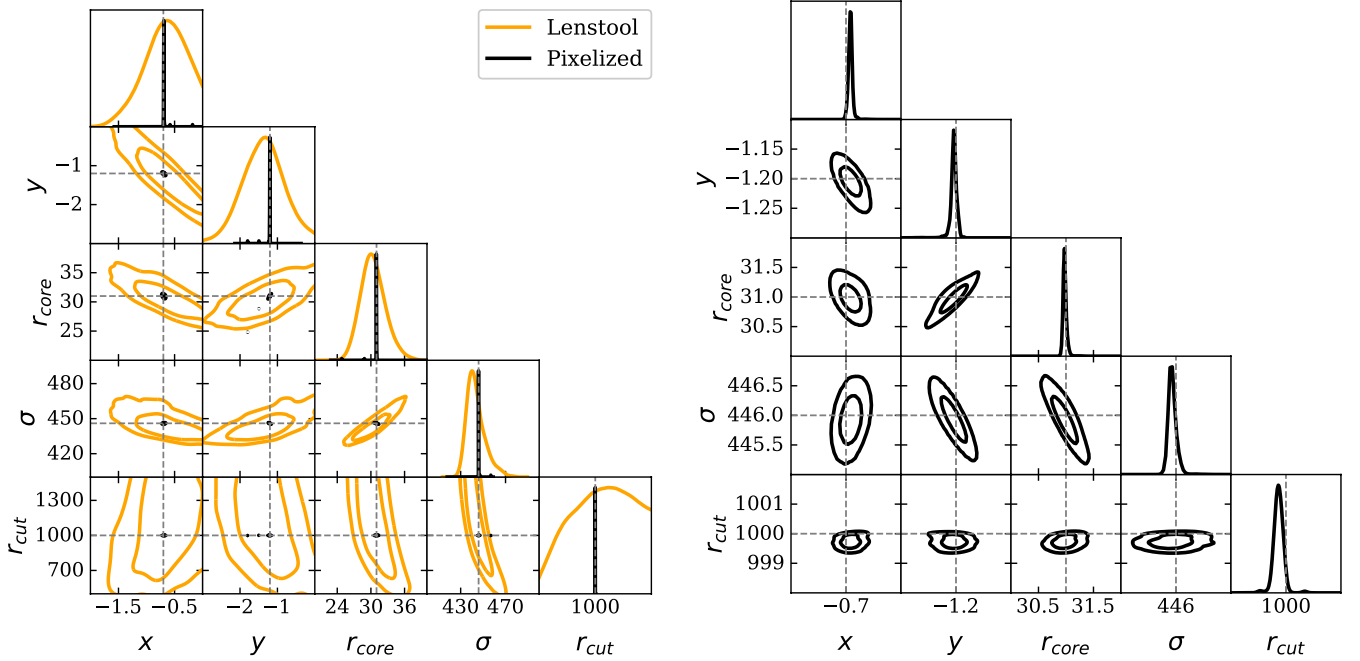


Figure 3. Posterior distributions ($1\text{-}\sigma$ and $2\text{-}\sigma$ confidence levels) of the lens mass parameters for the main halo. The left panel compares the results obtained using LEnSTOOL (orange) and the pixel-based modeling approach (black). The right panel provides a zoomed-in view of the pixel-based modeling posteriors.

is the associated error. The noise model is based on a JWST/NIRCam configuration as mentioned in Section 3.1.

Figure 3 presents the posterior distributions of the primary halo parameters obtained from the two modeling approaches. The left panel compares the results from LEnSTOOL (shown in orange) with those from the pixel-based modeling (shown in black). A zoomed-in version of the pixel-based modeling posteriors is presented in the right panel. The true parameter values used in the simulation are indicated by dashed lines. As previously emphasized in X24, the pixel-based modeling yields significantly improved constraints, the posterior contours are narrower, demonstrating its precision in parameter estimation.

4.2 Time delay measurement

We employ the open-source Python package SNTD (SuperNova Time Delays, Pierel & Rodney 2019a,b) to measure the time delays between the multiple images of the simulated lensed supernova (see Section 3.2). Built on top of the well-established SNCOSMO framework (Barbary et al. 2016), SNTD enables time delay inference by fitting the light curves of the multiple images. We adopt the ‘parallel’ fitting approach, which allows for fitting the light curve of each image in parallel, provided that the light curves are well sampled before and after the brightness peaks. Given the fact that the various light curves for the multiple images are generated from the same SN explosion, the parameters for fitting the light curves are divided into two categories, θ_{SN} , including parameters that describe the SN properties (e.g., redshift, type, light curve shape, etc), and θ_{L} , containing parameters affected by lensing (i.e., the magnification effect and changes in arrival times). The parameter space is explored using nested sampling, SNTD first constrains θ_{SN} using all light curves jointly, and then derives θ_{L} for each image individually.

We fit the simulated light curves using the `salto` model, consistent with that used in the simulation in Section 3.2. The free parameters

include the light curve shape parameters x_0 and x_1 , the color parameter c , and the time of maximum brightness t_0 in the observer frame. The supernova redshift is assumed to be known, and microlensing effects are not included in the simulations or fitting.

The solid lines in Figure 2 present the results of the light curve fitting, for the three images 1, 2, and 3 from top to bottom. SNTD fits the ‘LSST-observed’ fluxes (indicated as error bars) with high accuracy, as illustrated by the solid lines in Figure 2. Although these observed fluxes deviate from the theoretical light curves (dashed lines) due to the effects of seeing and observational noise, we find that SNTD still delivers robust and precise time delay measurements: $\Delta t_{42, \text{obs}} = 421.18^{+2.16}_{-2.21}$ days, $\Delta t_{45, \text{obs}} = 781.51^{+1.53}_{-1.73}$ days, in excellent agreement with the theoretical time delays of $\Delta t_{42} = 421.97$ days and $\Delta t_{45} = 781.98$ days. The deviations in measured fluxes due to seeing do not significantly affect the inferred delays, highlighting the robustness of light curve shape information in time delay measurements.

5 RESULTS

Based on the derived lens models and the time delays for the simulated lensed supernova in our analyses, we are able to constrain the value of the Hubble constant H_0 . The lens models are constructed assuming a fiducial value of $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$, yielding fiducial time delays $\Delta t_{i,j}^{\text{fid}}$ for each image pair (i, j). Given that the time delay between lensed images is inversely proportional to H_0 (Equation 6), we can rescale the fiducial model-predicted time delays to infer the actual value of H_0 by comparing them to the observed time delays $\Delta t_{i,j}^{\text{meas}}$. The measured time delay at its corresponding H_0 is given by:

$$\Delta t_{i,j}^{\text{meas}}(H_0) = \Delta t_{i,j}^{\text{fid}} \times \frac{70 \text{ km s}^{-1} \text{ Mpc}^{-1}}{H_0}. \quad (12)$$

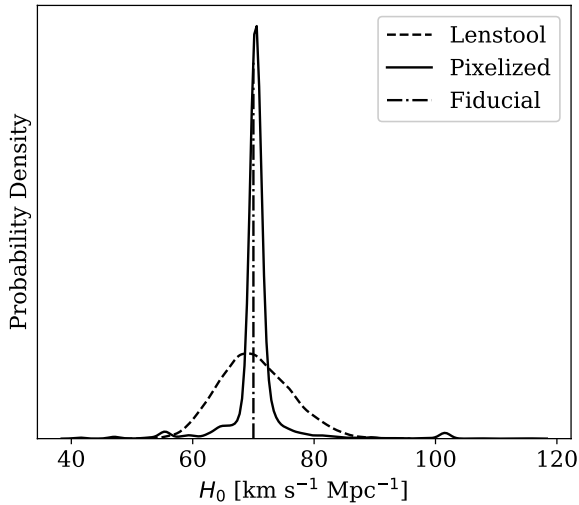


Figure 4. Inferred H_0 from lens model obtained with LENSTOOL (dashed) and pixelized method (solid curve). The fiducial value of $H_{0, \text{fid}}$ is marked as the vertical dash-dotted line.

By comparing these model-predicted time delays to the measured values for all image pairs, constraint on the value of H_0 is obtained assuming that the lens model accurately captures the underlying mass distribution (Pascale et al. 2024; Napier et al. 2023).

All other cosmological parameters are fixed at their true values, as our goal is to isolate and examine the impact of lens modeling accuracy on the inferred value of H_0 . The constraints on H_0 derived from the lens models using both methods are shown in Figure 4. We obtain $H_0 = 69.91^{+6.27}_{-5.50} \text{ km s}^{-1} \text{ Mpc}^{-1}$ (LENSTOOL, dashed curve) and a significantly tighter constraint $H_0 = 70.39^{+0.82}_{-0.60} \text{ km s}^{-1} \text{ Mpc}^{-1}$ (pixel-based modeling, solid curve). Both methods produce robust results under ideal conditions, while pixel-based approach offers an order of magnitude enhancement in precision.

This enhancement is particularly important. In current studies using cluster-lensed supernovae, uncertainties in H_0 are dominated by lens modeling errors ($\sim 5.5\%$, whereas uncertainties from time-delay measurements are much smaller ($\sim 1.5\%$) (Kelly et al. 2023). Our findings suggest that adopting pixel-based modeling can substantially improve cosmological precision by delivering more accurate and constrained lens models.

6 DISCUSSION AND CONCLUSION

The gISNe are regarded as a promising one-step probe for measuring the distance scale of the Universe, potentially resolving the 5σ H_0 tension between the early- and late-time observations. Nevertheless, the robustness of lens modeling is a critical factor in achieving reliable constraints on H_0 . As the second paper in the CURLING program, this work investigates the impact of incorporating pixel-level information from multiple images into lens modeling on the H_0 constraints, leveraging time delays between gISN multiple images.

We simulate a system analogous to the multiply imaged supernova ‘Requiem’, discovered in the galaxy cluster MACS J0138.0-2155, and perform lens modeling using two approaches: the traditional parametric method implemented in LENSTOOL, which relies on the positions of multiple images, and a pixel-based method that incor-

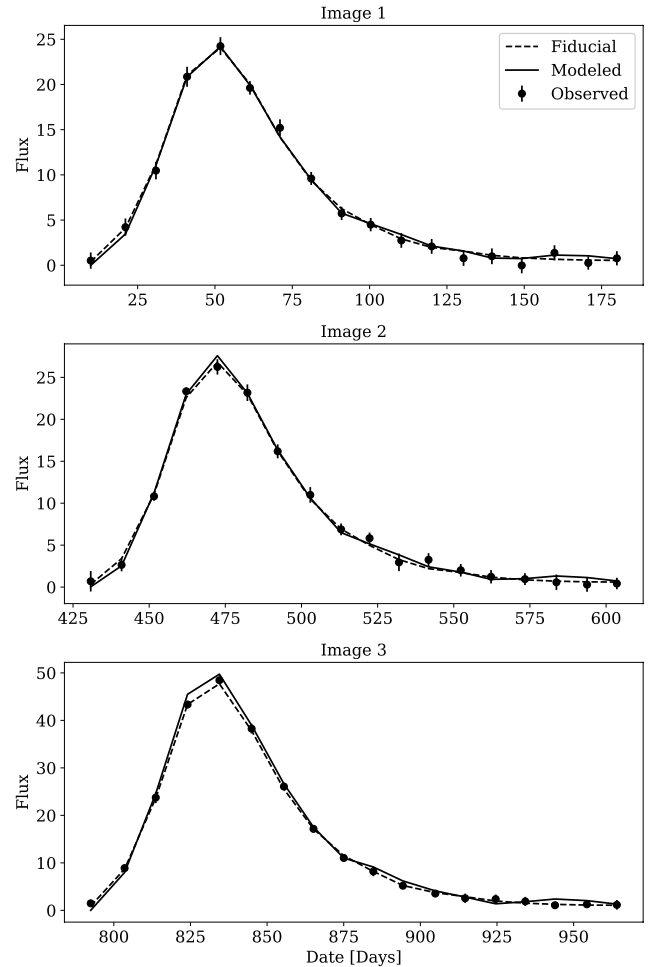


Figure 5. Light curves of the three multiple images of the simulated supernova under the CSST-MCI observational configuration. The line styles are consistent with those in Figure 2.

porates the surface brightness distribution across image pixels. Our results show that the pixel-based approach yields a more robust and precise reconstruction of the lensing mass distribution. On the other hand, we measure the time delays between the SN images using the public package `sntd`, which fits the simulated light curves. Using the inferred time delays and lens models, we then derive H_0 . While traditional modeling provides reasonable constraints, the pixel-based approach significantly improves precision, highlighting the value of using full surface brightness information in lens modeling.

To investigate the contribution of lens modeling uncertainties to H_0 inference, we compare the two modeling methods while fixing time delay uncertainties based on the LSST observing setup. Nonetheless, uncertainties in H_0 arise from both time delay measurements and lens modeling. To assess the impact of improved time delay precision, we consider an alternative observational configuration: the CSST-MCI, a space-based instrument which can avoid the influence from atmospheric seeing effects and offer high quality imaging data (observational parameters: sky magnitude $m_{\text{sky}} = 21.68 \text{ mag arcsec}^{-2}$, zero point magnitude in CBU band: $m_{\text{sky}} = 26.373 \text{ mag}$, readout noise $R = 10 \text{ e}^-$, total exposure time $t_{\text{expo}} = 300 \text{ s}$, number of exposures $N_{\text{expo}} = 2$, and a pixel scale of $0.05''$). The fitted light curves for the three supernova images observed by MCI are shown in Figure 5. Using `sntd`, we obtain time delays of

$\Delta t_{42} = 421.74^{+0.63}_{-0.64}$ days, and $\Delta t_{45} = 781.74^{+0.62}_{-0.70}$ days. These measurements yield $H_0 = 69.88^{+6.25}_{-5.49}$ km s⁻¹ Mpc⁻¹ with LENSTOOL, and $H_0 = 69.84^{+0.11}_{-0.10}$ km s⁻¹ Mpc⁻¹ using the pixel-based modeling. Despite the significantly improved time delay precision with MCI compared to LSST, the constraint on H_0 with LENSTOOL remains similar, further confirming that lens modeling with this method dominates the error budget in H_0 inference. However, once lens modeling uncertainties are reduced, the high-precision time delay measurements from MCI, compared to those from LSST, can be fully leveraged to achieve much more stringent constraints on H_0 .

Micro lensing from passing stars or other small objects can perturb the time-delay measurements for gISNe, albeit it is largely mitigated comparing with strongly lensed AGN (Bonvin et al. 2019; Tie & Kochanek 2018) due to the compact sizes of SNe and their characteristic light curves. Microlensing introduces chromatic flux variations (Dobler & Keeton 2006) that can distort the intrinsic SN brightness changes, making it challenging to accurately match corresponding features in the light curves of different multiple images. In order to avoid such variations, Pierel et al. (2024a) proposed to use colored light curves. Additionally, if the flux ratio of two images is taken into account in the time delay measurement, microlensing can introduce perturbation to the magnification, leading to a few tenths of magnitude bias, further complicating the time delay analysis. While we have not accounted for these effects in this work, we note the importance and will incorporate microlensing into our framework in future work to improve the accuracy and reliability.

We have made several simplifications for modeling the lensing cluster to ensure the time-consuming pixelized algorithm working: 1) we use the same form of analytical potentials for the simulation and fitting, incorporating no other systematic errors but instrumental and Poisson noises in the image; 2) we treat the parameters of the main halo as free parameters in the lens modeling, while assuming that the remaining potentials are well constrained already; 3) we ignore the complex structures of sources and masses on the line of sight, which can worsen the modeling results (Li et al. 2021); 4) we use only two families of multiple images as the observables, whereas more than tens of lensed sources could be identified thanks to the capabilities of deep imaging and spectroscopic follow-up observations. Since our aim here is to investigate the enhancement of our new framework with the pixelized images for lens modeling, a careful consideration for the realistic simulation of the lensing cluster is beyond the scope, while we believe that making the pixelized algorithm applicable to real observations is vital. With the aid of the promising GPU-based JAX framework (Bradbury et al. 2018) combined with the high-performance samplers, e.g., NUMPYRO (Phan et al. 2019), our method is anticipated to be accelerated by orders of magnitude, thus making it feasible for real cases.

This work is carried out under the flat Λ CDM scenario, with the Hubble constant as the only free cosmological parameter. However, we note that the time delays and lens model are also weakly dependent on the other cosmological parameters, such as the matter density of the Universe Ω_m , the dark energy equation-of-state parameter w , etc. Therefore, gISNe are expected to put constraints on different cosmological parameters simultaneously, works can be found in e.g., Grillo et al. (2024); Birrer et al. (2024).

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DATA AVAILABILITY

The data underlying this article will be shared on a reasonable request to the authors.

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