ON KLEE'S PROBLEM OF CONVEX BODIES IN BANACH SPACES

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Abstract. It is well known that every convex body in a finite dimensional normed space can be uniformly approximated by strictly convex and smooth convex bodies. However, in the case of infinite dimensions, little progress has been made since Klee asked how it is in the case of infinite dimensions in 1959. In this paper, we show that for an infinite dimensional Banach space X, (1) every convex body can be uniformly approximated by strictly convex bodies if and only if X admits an equivalent strictly convex norm; (2) every convex body can be uniformly approximated by Gâteaux smooth convex bodies if the dual X^* of X admits an equivalent strictly convex dual norm; in particular, (3) if X is either separable, or reflexive, then every convex body in X can be uniformly approximated by strictly convex and smooth convex bodies. They are done by showing that some correspondences among the sets of all convex bodies endowed with the Hausdorff metric, all continuous coercive Minkowski functionals and Fenchel's transform defined on all quadratic homogenous continuous convex functions equipped with the metric induced by the sup-norm of all bounded continuous functions defined on the closed unit ball B_X are actually locally Lipschitz isomorphisms.

1. Introduction

Let X be a Banach space and X^* its dual. B_X stands for the closed unit ball of X. We use $\mathfrak{C}(X)$, $\mathfrak{K}(X)$, $\mathfrak{C}_{\rm sc}(X)$ and $\mathfrak{C}_{\rm sm}(X)$, to denote successively, the cone of all nonempty bounded closed convex sets, all nonempty compact convex sets, all strictly convex bodies and all Gâteaux smooth convex bodies endowed with the Hausdorff metric (also called Pompeiu-Hausdorff metric) d_H defined for every pair of nonempty bounded convex sets $A, B \subset X$

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by

$$d_H(A, B) = \inf\{r > 0 : A \subset B + rB_X,\ B \subset A + rB_X\}.$$

Since every d dimensional real normed space X is isometric to $(\mathbb{R}^d, \|\cdot\|)$ for some equivalent norm $\|\cdot\|$ on $\mathbb{R}^d$, in the following we often blur the distinction between a d dimensional normed space X and $\mathbb{R}^d$.

In 1959, V. Klee [18] showed that in $\mathbb{R}^d$, the set of convex bodies which are both strictly convex and smooth, i.e. $\mathfrak{C}_{\rm sc}(\mathbb{R}^d) \bigcap \mathfrak{C}_{\rm sm}(\mathbb{R}^d)$ contains a dense G_δ subset of $\mathfrak{C}(\mathbb{R}^d)$. As a consequence of this result, it reveals the following important fact: *Every convex body in $\mathbb{R}^d$ can be uniformly approximated by strictly convex and smooth convex bodies.* Meanwhile, he further pointed out "*we have not succeeded in answering some of the natural questions about infinite-dimensional convex bodies*"[18].

We would like to mention that, in the case of finite dimensions, P. Gruber [15] (1977), among many other things, rediscovered Klee's theorem. In 1987, T. Zamfirescu gave another remarkable result: *The set of strictly convex and smooth convex bodies, i.e. $\mathfrak{C}_{sc}(\mathbb{R}^d) \bigcap \mathfrak{C}_{\rm sm}(\mathbb{R}^d)$ cantains the complement of a σ-porus set of $\mathfrak{C}(\mathbb{R}^d)$ (hence, it contains a dense a dense G_δ subset).* Also, a number of interesting geometrical results have been proven in $\mathbb{R}^d$, so for example, results on most convex bodies (called generic), regarded so far differentiability of the boundary [24, 5, 23], geodesics [25, 27], diameters [6, 4], etc.

However, in the case of infinite dimensions, little progress has been made since 1959. In response to Klee's problem, in this paper, we show the following theorems.

Theorem 1.1. *Every convex body in a Banach space X can be uniformly approximated by strictly convex bodies if and only if there exists an equivalent strictly convex norm on X.*

Theorem 1.2. *Let X be a Banach space.*

i) If there is an equivalent norm on X so that its dual X^* is strictly convex, then every convex body in X can be uniformly approximated by Gâteaux smooth convex bodies.

ii) If X satisfies that every convex body in X can be uniformly approximated by Gâteaux smooth convex bodies, then X admits an equivalent Gâteaux smooth norm.

But we do not know whether the converse of ii) is true.

Theorem 1.3. *Let X be a Banach space. If there is an equivalent strictly convex norm on X so that its dual X^* is strictly convex, then every convex body in X can be uniformly approximated by strictly convex and Gâteaux smooth convex bodies.*

In particular,

Theorem 1.4. *If X is a separable, or, reflexive Banach space, then every convex body in X can be uniformly approximated by strictly convex and Gâteaux smooth convex bodies.*

In order to outline the ideas and methods of our proof, we need more symbols. We use $\mathfrak{C}_{00}(X)$ to denote all convex bodies B of X with the origin $0 \in \mathrm{int}(B)$. $\mathscr{M}_0(X)$ ($\mathscr{H}_c^2(X)$, respectively) stands for the cone of all continuous coercive Minkowski functionals (all positive quadratic homogenous continuous coercive convex functions, respectively) defined on X endowed with the metric d defined for $f, g \in \mathscr{M}_0(X)$ ($\mathscr{H}_c^2(X)$, respectively) by $d(f, g) = \sup_{x \in B_X} |f(x) - g(x)|$, where "$f$ is coercive" means $f(x) \to +\infty$ whenever $\|x\| \to +\infty$. Finally, $\mathscr{H}_c^{*2}(X)$ represents the cone of all continuous and w^*-lower semicontinuous positive quadratic homogenous coercive convex functions on X^*.

To show the theorems mentioned above, we establish a number of correspondences among the cones mentioned previously, and then show the following correspondences are order-reversing and locally Lipschitz isomorphisms.

i) the correspondence $\mathfrak{C}_{00}(X) \to \mathscr{M}_0(X)$ by $C \to p_C$, the Minkowski functional generated by $C \in \mathfrak{C}_{00}(X)$;

ii) the correspondence $\mathscr{H}_c^2(X) \to \mathfrak{C}_{00}(X)$;

iii) Fenchel's transform from $\mathscr{H}_c^2(X)$ to $\mathscr{H}_c^{*2}(X)$.

2. Preliminaries

To begin with, we recall some definitions and basic properties which will be used in the sequel.

2.1. **Convex body.** Let X be a real Banach space. A hyperplane H of X is an affine subspace of codimension one, i.e. there exist a maximal proper subspace N of X and $x_0 \in X$ so that $H = x_0 + N$. Note that for each closed hyperplane H there is a continuous functional $x^*(\neq 0) \in X^*$ so that $H = \{x \in X : \langle x^*, x\rangle = 1\}$ if H is off the origin; and $H = \{x \in X : \langle x^*, x\rangle = 0\}$

if H is a linear subspace. A hyperplane H is said to support a set B at a point $x \in \mathrm{bdr}B$ if $x \in H$ and if H bounds B.

A closed bounded convex set $B \subset X$ is said to be a *convex body* provided its interior $\mathrm{int}B \neq \emptyset$. A convex body B is called *strictly convex* if every boundary point x of B is an extreme point of B, that is, x is not the middle point of any line-segment in B. We say a convex body B is *(Gâteaux) smooth* if it satisfies that every $x \in \mathrm{bdr}(B)$ (the boundary of B) admits a unique supporting hyperplane (closed hyperplane) of B through x.

We use B_X and S_X to denote the closed unit ball and the unit sphere of X, i.e. $B_X = \{x \in X : \|x\| \leq 1\}$, and $S_X = \{x \in X : \|x\| = 1\}$. We say that X is *strictly convex* (resp. *Gâteaux smooth*) if the closed unit ball B_X (as a convex body) is strictly convex (Gâteaux smooth, resp.).

2.2. **Minkowski functional.** Let C be a closed convex set of a Banach space X with $0 \in \mathrm{int}C$. The functional p_C defined for $x \in X$ by $p_C(x) = \inf\{\lambda > 0 : x \in \lambda C\}$ is called the *Minkowski functional* with respect to (or, generated by) C.

Proposition 2.1. *Every Minkowski functional p_C defined on a Banach space X is a nonnegative-valued continuous sublinear functional, that is,*

$$p_C(kx) = kp_C(x), \quad p_C(x+y) \leq p_C(x) + p_C(y), \quad \forall\, x, y \in X \text{ and } k \geq 0.$$

For a convex set C of a Banach space X, let $L_C = \{x \in X : x + C = C\}$, $K_C = \{x \in X : x + C \subset C\}$. Then C is said to be *line-free* (*ray-free,* resp.) provided $L_C = \{0\}$ ($K_C = \{0\}$, resp.).

Proposition 2.2. *Let C be a closed convex set of X with $0 \in \mathrm{int}C$. Then*

i) p_C is strictly positive, i.e. $p_C(x) = 0 \Longrightarrow x = 0$, if and only if C is ray-free.

ii) p_C is a seminorm on X if and only if C is symmetric, i.e. $-C = C$;

iii) p_C is a norm if and only if C is symmetric and line-free.

iv) p_C is an equivalent norm on X if and only if C is a symmetric convex body.

2.3. **Differentiability.** Let f be a real-valued function defined on a Banach space X, and $x \in X$ be fixed. Then f is said to be *Gâteaux differentiable at x* provided there is a continuous functional $x^* \in X^*$ such that

$$\lim_{t \to 0} \frac{f(x+ty) - f(x)}{t} = \langle x^*, y \rangle, \quad \forall\, y \in X. \tag{2.1}$$

In this case, we denote by $d_G f(x) = x^*$, and call it the *Gâteaux derivative of f at x.*

f is said to be *Fréchet differentiable at x* if there is a continuous functional $x^* \in X^*$ such that

$$\lim_{\|y\|\to 0}\Big(\frac{f(x+y)-f(x)-\langle x^*, y\rangle}{\|y\|}\Big) = 0. \tag{2.2}$$

In this case, we denote by $d_F f(x) = x^*$, and call it the *Fréchet derivative of f at x.*

The norm $\|\cdot\|$ of a Banach space X is said to be *Gâteaux (Fréchet,* resp.*) smooth* provided it is everywhere Gâteaux (Fréchet, resp.) differentiable off the origin.

Theorem 2.3 (Mazur's theorem)**.** *(See, for instance,* [19].*) Every continuous convex function defined on a separable Banach space is everywhere Gâteaux differentiable in a dense G_δ-subset of the space.*

Theorem 2.4 (Asplund's theorem)**.** *(See,*[3]*, also* [19].*) Every continuous convex function defined on a separable Banach space with separable dual is Fréchet differentiable everywhere in a dense G_δ-subset of the space.*

2.4. **Subdifferentiability.** Let f be a convex function defined on a Banach space X. Its *subdifferential mapping $\partial f : X \to 2^{X^*}$* is defined for $x \in X$ by

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle,\ y \in X\}. \tag{2.3}$$

Let $(U, \tau_U), (V, \tau_V)$ be two topological spaces. A set-valued mapping $P : U \to 2^V$ is said to be *upper-semicontinuous* provided that for each $u \in U$ and each open neighborhood $O_{P(u)}$ of $P(u)$, there is an open neighborhood O_u of u such that $P(O_u) \subset O_{P(u)}$. We say that the subdifferential mapping $\partial f : X \to 2^{X^*}$ defined above is *norm-to-w^* upper semicontinuous* if we endow X with the norm-topology and its dual space X^* with its weak-star topology.

The following properties regarding to differentiability and subdifferentiability of convex functions are classical. See, for instance, [19].

Proposition 2.5. *Let f be a continuous convex function defined on a Banach space X. Then*

i) for each $x \in X$, $\partial f(x)$ is a nonempty w^*-compact convex set of X^*;

ii) $\partial f : X \to 2^{X^*}$ is norm-to-w^* upper semicontinuous;

iii) f is Gâteaux differentiable at x if and only if $\partial f(x)$ is a singleton;

iv) f is Fréchet differentiable at x if and only if $\partial f(x)$ is single-valued and norm-to-norm upper semicontinuous at x.

Recall that a selection for ∂f is a single-valued mapping $\varphi : X \to X^*$ satisfying $\varphi(x) \in \partial f(x)$ for all $x \in X$.

Proposition 2.6. *Suppose that f is a continuous convex function defined on a Banach space X. Then the following statements are equivalent.*
i) f is Gâteaux differentiable at x;
ii) ∂f is single-valued at x;
iii) every selection for ∂f is norm-to-w^* continuous at x;
iv) there is a selection for ∂f which is norm-to-w^* continuous at x.

Proposition 2.7. *Suppose that f is a continuous convex function defined on a Banach space X. Then the following statements are equivalent.*
i) f is Fréchet differentiable at x;
ii) ∂f is single-valued at x and norm-to-norm upper semicontinuous at x.;
iii) every selection for ∂f is norm-to-norm continuous at x;
iv) there is a selection for ∂f which is norm-to-norm continuous at x.

Proposition 2.8. *Let C be a convex set of X containing 0 in its interior. Then $x^* \in \partial p_C(x)$ if and only if the following two conditions are satisfied:*
i) $\langle x^*, z\rangle \le p_C(z), \quad \forall\, z \in X$;
ii) $\langle x^*, x\rangle = p_C(x)$.

2.5. **Fenchel's transform.** We say an extended real-valued convex function f is *proper*, if $f(x) > -\infty$ for all $x \in X$ and its essential domain $\mathrm{dom} f \equiv \{x \in X : f(x) < +\infty\} \neq \emptyset$. For a Banach space X, we denote by $\mathscr{C}_{\mathrm{conv}}(X)$ the set of all extended real-valued lower semicontinuous proper convex functions defined on X.

Definition 2.9. Fenchel's transform $\mathscr{F}$ is defined for $f \in \mathscr{C}_{\mathrm{conv}}(X)$ by

$$\mathscr{F}(f)(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}, \quad x^* \in X^*. \tag{2.4}$$

If X is of finite dimension, then X^* is (linearly) isomorphic to X. Therefore, in this case, Fenchel's transform $\mathscr{F}$ defined above is actually from $\mathscr{C}_{\mathrm{conv}}(X)$ to itself, we denote it by $\mathscr{L}$ and call it Legendre's transform.

Proposition 2.10. *Let X be a Banach space. Then for each $f \in \mathscr{C}_{\mathrm{conv}}(X)$, $\mathscr{F}(f)$ is a w^*-lower semicontinuous (proper) convex function.*

Proof. Note that each $f \in \mathscr{C}_{\rm conv}(X)$ is lower semicontinuous and proper. Then by the Brødsted-Rockafellar Theorem (see, for instance, [19]), there exists $x_0 \in \mathrm{dom}(f) \equiv \{x \in X : f(x) < \infty\}$ such that $\partial f(x_0) \neq \emptyset$. Fix any $x_0^* \in \partial f(x_0)$. Since $f(x) - f(x_0) \geq \langle x_0^*, x - x_0\rangle$ for all $x \in X$, we have

$$\mathscr{F}(f)(x_0^*) = \sup\{\langle x_0^*, x\rangle - f(x) : x \in X\} = \langle x_0^*, x_0\rangle - f(x_0) < +\infty.$$

Therefore, $\mathscr{F}(f)$ is a proper convex function on X^*. From (2.4), w^*-lower semicontinuity of $\mathscr{F}(f)$ follows. □

We use $\mathscr{C}^*_{\rm conv}(X^*)$ to denote the set of all extended real-valued and w^*-lower semicontinuous (hence, lower semicontinuous) proper convex functions defined on X^*.

The following result is classical (see, for instance, [11]).

Proposition 2.11. *Let $X = (X, \|\cdot\|)$ be a Banach space, and $f = \frac{1}{2}\|\cdot\|^2$. Then $\mathscr{F}(f) = \frac{1}{2}\|\cdot\|^{*2}$, where $\|\cdot\|^*$ is the dual norm of X^*.*

The notions of order-preserving (order reversing, resp.) and fully order order-preserving (fully order reversing, resp.) were used in [1] [8] and [9]. Let A and B be two partially ordered sets. A mapping $M : A \to B$ is said to be *order preserving* (*order-reversing*, resp.), provided $a \geq b \in A \Longrightarrow M(a) \geq M(b)$ ($a \geq b \in A \Longrightarrow M(a) \leq M(b)$, resp.). If, in addition, M is a bijection from A to B and is order preserving (order-reversing, resp.), then M is called a *fully order-presevering* (*fully order-reversing*, resp.,) mapping.

Proposition 2.12. *Let X be a Banach space. Then*

i) the Fenchel transform $\mathscr{F} : \mathscr{C}_{\rm conv}(X) \to \mathscr{C}_{\rm conv}(X^*)$ is order-reversing;

ii) $\mathscr{F} : \mathscr{C}_{\rm conv}(X) \to \mathscr{C}^*_{\rm conv}(X^*)$ is fully order-reversing;

iii) $\mathscr{F} : \mathscr{C}_{\rm conv}(X) \to \mathscr{C}_{\rm conv}(X^*)$ is fully order-reversing if and only if X is reflexive.

Proof. i) It follows from (2.4) and Proposition 2.10.

ii) By i), $\mathscr{F} : \mathscr{C}_{\rm conv}(X) \to \mathscr{C}^*_{\rm conv}(X^*)$ is order-reversing. Next, we show that $\mathscr{F}$ is surjective. Fix $g \in \mathscr{C}^*_{\rm conv}(X^*)$. Since g is w^*-lower senicontinuous and proper convex on X^*, there is $f \in \mathscr{C}_{\rm conv}(X)$ such that $\mathscr{F}(f) = g$. It remains to show that $\mathscr{F}^{-1} : \mathscr{C}^*_{\rm conv}(X^*) \to \mathscr{C}_{\rm conv}(X)$ is again order-reversing. Let $\mathscr{F}|_*$ be defined for $g \in \mathscr{C}^*_{\rm conv}(X^*)$ by

$$\mathscr{F}|_*(g) = \mathscr{F}(g)|_X,$$

i.e. $\mathscr{F}(g)$ but its domain is restricted to X. Then it is also order-reversing. Note $\mathscr{F}|_* \circ \mathscr{F}$ is just the identity $I = \mathscr{F}^{-1} \circ \mathscr{F}$ on $\mathscr{C}_{\rm conv}(X)$. Therefore, $\mathscr{F}^{-1} = \mathscr{F}|_*$ is order-reversing. Consequently, we finish the proof of ii).

iii) Since $\mathscr{C}^*_{\mathrm{conv}}(X^*) = \mathscr{C}_{\mathrm{conv}}(X^*)$ if and only if X is reflexive, iii) follows from ii). □

2.6. **Inf-convolution.** Let f, g be two continuous positive quadratic homogenous convex functions defined on X. Their inf-convolution $f\Box g$ is defined by

$$f\Box g(x) = \inf\{f(x-y) + g(y) : y \in X\}, \quad x \in X. \tag{2.5}$$

Proposition 2.13. *Let $f, g \geq 0$ be two continuous positive quadratic homogenous convex functions defined on X, and $\mathscr{F}$ be Fenchel's transform. Then*

i) $\mathscr{F}(f+g) = \mathscr{F}(f)\Box\mathscr{F}(g)$;

ii)$\mathscr{F}(f\Box g) = \mathscr{F}(f) + \mathscr{F}(g)$.

Proof. i) For every $x^* \in X^*$,

$$\mathscr{F}(f)\Box\mathscr{F}(g)(x^*) = \inf_{y^*\in X^*}\{\mathscr{F}(f)(x^*-y^*) + \mathscr{F}(g)(y^*)\}$$

$$= \inf_{y^*\in X^*}\{\sup\{\langle x^*-y^*, x\rangle - f(x) : x \in X\} + \sup\{\langle y^*, y\rangle - g(y) : y \in X\}\}$$

$$\geq \inf_{y^*\in X^*}\sup_{x\in X}\{\langle x^*, x\rangle - (f(x)+g(x)) : x \in X\} = \mathscr{F}(f+g)(x^*).$$

On the other hand, $x^* \equiv u^* + v^* \in \partial(f+g)(x) = \partial(f)(x) + \partial(g)(x)$ (where $u^* \in \partial f(x), v^* \in \partial g(x)$) implies

$$\mathscr{F}(f+g)(x^*) = \sup_{y\in X}\{\langle x^*, y\rangle - (f(y)+g(y))\} = \langle x^*, x\rangle - (f+g)(x)$$

$$= (\langle u^*, x\rangle - f(x)) + (\langle v^*, x\rangle - g(x))$$

$$= \mathscr{F}(f)(u^*) + \mathscr{F}(g)(x^*-u^*)$$

$$\geq \mathscr{F}(f)\Box\mathscr{F}(f)(x^*).$$

Since f, g are continuous positive quadratic homogenous convex, by the Bishop-Phelps theorem (see, for instance, [19]), the range of $\partial(f+g)$ is dense in X^*. This and continuity of $\mathscr{F}(f+g)$ and $\mathscr{F}(f)\Box\mathscr{F}(f)$ entail that for all $x^* \in X^*$,

$$\mathscr{F}(f+g)(x^*) \geq \mathscr{F}(f)\Box\mathscr{F}(f)(x^*).$$

Therefore, i) is shown.

We can show ii) analogously.

□

3. Strictly convex bodies and strictly convex functions

A convex function f defined on a convex set C of a Banach space X is said to be *strictly convex* provided

$$\lambda f(x) + (1-\lambda)f(y) > f(\lambda x + (1-\lambda)y), \quad \forall\, x \neq y \in C,\ 0 < \lambda < 1. \tag{3.1}$$

If, in addition, f is continuous, then (3.1) is equivalent to

$$\frac{1}{2}\big(f(x) + f(y)\big) > f(\frac{x+y}{2}),\ \forall x \neq y \in X.$$

A function f on a Banach space X is said to be *coercive*, if it satisfies $\lim_{\|x\|\to\infty} f(x) = +\infty$. For example, for every nontrivial Banach space $X = (X, \|\cdot\|)$, the norm $\|\cdot\|$ is a coercive convex function.

Lemma 3.1. *Suppose that f is a nonnegative-valued continuous coercive convex function. Then*

i) for each $r > \inf_{x\in X} f(x)$, the level set $B_r \equiv \{x \in X : f(x) \leq r\}$ is a convex body of X;

ii) if f is strictly convex then for each such r, the convex body B_r is strictly convex.

Proof. i) Clearly, convexity and continuity of f implies that for each $r > \inf_{x\in X} f(x)$, the level set $B_r \equiv \{x \in X : f(x) \leq r\}$ is a closed convex set with nonempty interior. Since f is coercive, B_r is bounded. Therefore, B_r is a convex body.

ii) Suppose that B_r is not strictly convex for some $r > \inf_{x\in X} f(x)$. Note that the boundary bdrB_r of the convex body B_r is $\{x \in X : f(x) = r\}$. Then there exist $x \neq y \in$ bdrB_r and $0 < \lambda < 1$ so that $z \equiv \lambda x + (1-\lambda)y \in$ bdB_r. Therefore,

$$\lambda f(x) + (1-\lambda)f(y) = r = f(z) = f(\lambda x + (1-\lambda)y).$$

This contradicts to that f is strictly convex. $\square$

Remark 3.2. The converse version is not true. For example, assume that $X = (X, \|\cdot\|)$ is a strictly convex Banach space. For each $r > \inf_{x\in X} \|x\| = 0$, B_r is a strictly convex body. But $\|\cdot\|$ is not a strictly convex function. Indeed, let $x \in X$ with $\|x\| = 1$ and $y = 0$. Then $\|\frac{x+y}{2}\| = \frac{1}{2} = \frac{1}{2}\|x\| + \frac{1}{2}\|y\|$. However, the following conclusion holds.

Recall that a convex function f is said to be a positive quadratic homogenous if it satisfies $f(kx) = k^2 f(x),\ \forall x \in X, k \in \mathbb{R}^+$.

Lemma 3.3. *Let f be a positive quadratic homogenous continuous coercive convex function. Then f is strictly convex if and only if for every $r > 0$, the level set $B_r \equiv \{x \in X : f(x) \le r\}$ is a strictly convex body.*

Proof. By Lemma 3.1 ii), it suffices to show the sufficiency. Suppose the contrary that f is not strictly convex. Then there exist $x \neq y \in X$ such that

$$f\Big(\frac{x+y}{2}\Big) = \frac{1}{2}\Big(f(x) + f(y)\Big). \tag{3.2}$$

Note that both $g(t) \equiv f(tx + (1-t)y)$ and $h(t) \equiv tf(x) + (1-t)f(y)$ are convex functions in $t \in [0,1]$ with $h \ge g$ on $[0,1]$ and with $h(\frac{1}{2}) = g(\frac{1}{2})$. Since h is affine, we can see that

$$\text{either } \ g(t) \ge h(t),\ \forall t \in [0, \frac{1}{2}]; \ \text{ or, } g(t) \ge h(t),\ \forall t \in [\frac{1}{2}, 1].$$

This and $h \ge g$ imply

$$\text{either } \ g(t) = h(t),\ \forall t \in [0, \frac{1}{2}]; \ \text{ or, } g(t) = h(t),\ \forall t \in [\frac{1}{2}, 1].$$

This, $h \ge g$ and affine property of h further entail

$$g(t) = h(t),\ \forall t \in [0,1].$$

That is,

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y),\ \forall\ \lambda \in [0,1]. \tag{3.3}$$

Since f is a positive quadratic homogenous continuous coercive convex function, $f(u) > 0$ for all $u \neq 0$. Therefore, x, y are linearly independent and which entails that $x \neq 0 \neq y$. If $f(x) = f(y) \equiv r > 0$, then (3.3) implies that the convex body B_r is not strictly convex, and this is a contradiction.

Assume $f(y) > f(x) > 0$, and let $\alpha = \sqrt{\frac{f(x)}{f(y)}}$, and $z = \alpha y$. Then $0 < \alpha < 1$ and $f(x) = f(z) \equiv r > 0$. Next, let $0 < \lambda < 1$, $\gamma = \frac{\lambda}{\lambda + (1-\lambda)\alpha}$. It follows from (3.3) again that

$$\begin{aligned} f(\lambda x + (1-\lambda)z) = f(\lambda x + (1-\lambda)\alpha y) &= \Big(\lambda + (1-\lambda)\alpha\Big)^2 f(\gamma x + (1-\gamma)y) \\ &= \Big(\lambda + (1-\lambda)\alpha\Big)^2 \Big(\gamma f(x) + (1-\gamma)f(y)\Big) \\ &= \Big(\lambda + (1-\lambda)\alpha\Big)^2 \Big(\gamma f(x) + \frac{(1-\gamma)}{\alpha^2} f(z)\Big) \\ &= r\Big(\lambda + (1-\lambda)\alpha\Big)^2 \Big(\gamma + \frac{(1-\gamma)}{\alpha^2})\Big). \end{aligned}$$

Let $\lambda = \frac{1}{2}$. It follows from the equalities above that

$$f(\lambda x + (1-\lambda)z) = r\Big(\lambda + (1-\lambda)\alpha\Big)^2 \Big(\gamma + \frac{(1-\gamma)}{\alpha^2})\Big)$$

$$= r\Big(\frac{1}{2}+\frac{\alpha}{2}\Big)^2\Big(\frac{1}{1+\alpha}+\frac{1}{\alpha(1+\alpha)}\Big) = r\Big(\frac{1}{2}+\frac{\alpha}{2}\Big)^2\frac{1}{\alpha} > r.$$

This entails that $\alpha = 1$ and consequently, $f(x) = r = f(y)$. This is a contradiction. □

The following lemma directly follows from the definition of strictly convex functions.

Lemma 3.4. *Let f, g be two convex functions defined on a Banach space X. If one of f and g is strictly convex, the $f+g$ is again strictly convex.*

Lemma 3.5. *Let $B \subset X$ be a convex body. Then the following statements are equivalent.*

i) B is strictly convex.

ii) For each fixed $z \in \mathrm{int}B$, $f = \frac{1}{2}p_z^2$ is strictly convex.

iii) There is $z \in \mathrm{int}B$ so that $f = \frac{1}{2}p_z^2$ is strictly convex, where p_z is the Minkowski functional generated by $B - z$.

Proof. i)$\Longrightarrow$ ii). Since B is strictly convex, every point of $\mathrm{bdr}(B)$ is an extreme point of B. Therefore, for each fixed $z \in X$, every point of $\mathrm{bd}(B-z)$ is an extreme point of $B - z$. In particular, given $x_0 \in \mathrm{int}B$, every point of $\mathrm{bdr}(B - x_0)$ is an extreme point of $B - x_0$. Let p_0 be the Minkowski functional generated by $B - x_0$. Then $\{x \in X : p_0(x) = 1\} = \mathrm{bdr}(B - x_0)$. Strict convexity of B entails that each level set of $f \equiv \frac{1}{2}p_0^2$ is a strictly convex body. By Lemma 3.3, $f \equiv \frac{1}{2}p_0^2$ is strictly convex.

ii) $\Longrightarrow$ iii). Trivial.

iii) $\Longrightarrow$ i). Since $f = \frac{1}{2}p_z^2$ is strictly convex, by Lemma 3.3, $B - z$ is a strictly convex body. Therefore, $B - z$ is again a strictly convex body. □

4. Smooth convex bodies and Gâteaux differentiable convex functions

Lemma 4.1. *Let $B \subset X$ be a convex body. Then the following statements are equivalent.*

i) B is Gâteaux smooth;

ii) For each fixed $z \in \mathrm{bdr}(B)$, p_z is everywhere Gâteaux differentiable off the origin, where p_z is the Minkowski functional generated by B_z;

iii) For each fixed $z \in \mathrm{bdr}(B)$, $f = \frac{1}{2}p_z^2$ is everywhere Gâteaux differentiable;

iv) There is $z \in \mathrm{bdr}(B)$ so that p_z is everywhere Gâteaux differentiable off the origin;

v) There is $z \in \mathrm{bdr}(B)$ so that $f = \frac{1}{2}p_z^2$ is everywhere Gâteaux differentiable.

Proof. Recall that B is Gâteaux smooth if and only if for each $u \in \mathrm{bdr}(B)$ there is a unique closed hyperplane H so that $H \bigcap \mathrm{int}(B) = \emptyset$ and $u \in H \bigcap B$. Since B is Gâteaux smooth if and only if $B - z$ is Gâteaux smooth for each $z \in X$, without loss of generality, we assume that $0 \in \mathrm{int}B$.

i) $\Longrightarrow$ ii). Clearly, $\mathrm{bdr}(B) = \{x \in X : p_B(x) = 1\}$. Fix $u \in \mathrm{bdr}(B)$. Let H_u be the unique supporting hyperplane of B at u. By the separation theorem of convex sets, there exists $x^* \in X^*$ so that

$$\langle x^*, x\rangle < \langle x^*, u\rangle = 1 = \sup_{z\in B}\langle x^*, z\rangle, \forall x \in \mathrm{int}(B).$$

This implies that $H = \{z \in X : \langle x^*, z\rangle = 1 = \langle x^*, u\rangle = p_B(u)\}$. It is easy to check that $x^* \in \partial p_B(u)$. Since every $u^* \in \partial p_B(u)$ satisfies $\langle u^*, u\rangle = p(u) = 1$ and $\langle u^*, x\rangle < 1$ for all $x \in \mathrm{int}(B)$, the uniqueness of H entails $H = \{z \in X : \langle u^*, z\rangle = 1\}$. Consequently, we obtain $u^* = x^*$, that is, $\partial p_B(u)$ is a singleton. Thus, ii) follows from Proposition 2.5.

ii) $\Longrightarrow$ iii). It suffices to note that for each $z \in \mathrm{bdr}(B)$, p_z is everywhere Gâteaux differentiable off the origin entails that p_z^2 is everywhere Gâteaux differentiable in X.

iii) $\Longrightarrow$ iv). It suffices to note that for each $z \in \mathrm{bdr}(B)$, p_z^2 is everywhere Gâteaux differentiable entails that p_z is everywhere Gâteaux differentiable off the origin.

iv) $\Longrightarrow$ v). It suffices to note that p_z is everywhere Gâteaux differentiable off the origin implies that p_z^2 is everywhere Gâteaux differentiable.

v) $\Longrightarrow$ i). Since $f = \frac{1}{2}p_z^2$ is everywhere Gâteaux differentiable implies that p_z is everywhere Gâteaux differentiable off the origin, we obtain that for each $u \in \mathrm{bd}(B - z)$, there is a unique $x^*(= \partial p_z(u)) \in X^*$ so that $\langle x^*, u\rangle = 1 > \langle x^*, x\rangle$ for all $x \in \mathrm{int}(B)$. Therefore, $H \equiv \{v \in X : \langle x^*, v\rangle = 1\}$ is the unique closed hyperplane satisfying $H \bigcap \mathrm{int}(B) = \emptyset$ and $u \in H$. Thus, $B - z$ is a Gâteaux smooth convex body.

□

5. Strictly convex bodies in $\mathfrak{C}(X)$

In this section, we will show that for a Banach space X, strictly convex bodies are dense in $\mathfrak{C}(X)$ if and only if X admits an equivalent strictly convex norm, i.e. Theorem 1.1 stated in the first section.

Let $\mathfrak{C}(X)$ be the cone of all convex bodies of X with the following two "linear" operations

$$\alpha B = \{\alpha b : b \in B\}, \ \alpha \in \mathbb{R}, B \in \mathfrak{C}(X),$$

$$A \oplus B = \overline{\{a+b : a \in A,\ b \in B\}},\ A, B \in \mathfrak{C}(X)$$

and endowed with the Hausdorff metric d_H, where $\overline{D}$ denotes the norm closure of $D \subset X$. The following property is classical and easy to prove.

Proposition 5.1. *For every Banach space X, $\mathfrak{C}(X) = (\mathfrak{C}(X), d_H)$ is a complete metric cone.*

Let $\mathfrak{C}_0(X) = \{B \in \mathfrak{C}(X) : 0 \in B\}$ and $\mathfrak{C}_{00}(X) = \{B \in \mathfrak{C}(X) : 0 \in \mathrm{int}B\}$.

Lemma 5.2. *Assume that X is a Banach space. Then*
i) $\mathfrak{C}_0(X)$ is again a complete cone;
ii) $\mathfrak{C}_{00}(X)$ is a dense open cone of $\mathfrak{C}_0(X)$;
iii) $\mathfrak{C}_{00}(X)$ is an open cone of $\mathfrak{C}(X)$.

Proof. i) Clearly, $\mathfrak{C}_0(X)$ is again a subcone of $\mathfrak{C}(X)$ and it is complete in the Hausdorff metric d_H.

ii) It is not difficult to observe that $\mathfrak{C}_{00}(X)$ is dense in $\mathfrak{C}_0(X)$. Indeed, for every $\varepsilon > 0$, we have $C + \varepsilon B_X \in \mathfrak{C}_{00}(X)$ with $d_H(C, C + \varepsilon B_X) = \varepsilon$. Thus, $\mathfrak{C}_{00}(X)$ is dense in $\mathfrak{C}_0(X)$. Since iii) implies $\mathfrak{C}_{00}(X)$ is an open set of $\mathfrak{C}_0(X)$, it remains to show iii) that $\mathfrak{C}_{00}(X)$ is open in $\mathfrak{C}(X)$. Let $C \in \mathfrak{C}_{00}(X)$. Since $0 \in \mathrm{int}C$, there is $\delta > 0$ so that $\delta B_X \subset C$. Therefore, $\{D \in \mathfrak{C}(X) : d_H(C, D) < \delta\} \subset \mathfrak{C}_{00}(X)$. Consequently, $\mathfrak{C}_{00}(X)$ is open in $\mathfrak{C}(X)$. □

Recall that $\mathscr{H}^2(X)$ denotes the cone of all positive quadratic homogenous continuous convex functions defined on X endowed with the metric d defined for $p, q \in \mathscr{H}(X)$ by $d(p, q) = \sup_{x \in B_X} |p(x) - q(x)|$, and that $\mathscr{H}^2_c(X)$ stands for the subcone of $\mathscr{H}^2(X)$ consisting of all positive quadratic homogenous continuous coercive convex functions defined on X.

Lemma 5.3. *Let X be a Banach space. Then*
i) $\mathscr{H}^2(X)$ is a complete metric cone;
ii) given $f \in \mathscr{H}^2(X)$ and $r > 0$, $B_r(\equiv \{x \in X : f(x) \leq r\})$ is a convex body if and only if f is coercive;
iii) $\mathscr{H}^2_c(X)$ is a dense open subset of $\mathscr{H}^2(X)$;
iv) given $f \in \mathscr{H}^2(X)$ and $r > 0$, B_r is a strictly convex body if and only if f is strictly convex and coercive;
v) for each $f \in \mathscr{H}^2(X)$, there is a closed convex set $C \subset X$ with $0 \in \mathrm{int}C$ so that $f = \frac{1}{2}p^2_C$;
vi) the convex set C defined in v) is a convex body if and only if f is coercive.

Proof. i) Trivial.

ii) Let $f \in \mathscr{H}^2(X)$ be coercive and $r > 0$. Since $f(x) \ge 0$ with $f(0) = 0$, and since f is continuous and convex, $B_r \equiv \{x \in X : f(x) \le r\}$ is a closed convex set with $\mathrm{int}B_r \ne \emptyset$. Coerciveness of f entails that B_r is bounded. Conversely, if B_r is a convex body for some $r > 0$, then coerciveness of f follows from the homogeneity of f and boundedness of B_r.

iii) For each $f \in \mathscr{H}^2(X)$ and $\varepsilon > 0$, let $g_\varepsilon = \varepsilon\|\cdot\|^2$. Then $f + g_\varepsilon \in \mathscr{H}_c^2(X)$ with $d(f, g_\varepsilon) \le \varepsilon$. Therefore, $\mathscr{H}_c^2(X)$ is dense in $\mathscr{H}^2(X)$. Conversely, for each $f \in \mathscr{H}_c^2(X)$, there is $\delta > 0$ such that $f \ge \delta\|\cdot\|^2$. This implies that the open neighborhood $\{g \in \mathscr{H}^2(X) : d(f, g)(\equiv \sup_{x\in B_X} |f(x) - g(x)|) < \delta\}$ of f is contained in $\mathscr{H}_c^2(X)$.

iv) This is just Lemma 3.3.

v) Let $f \in \mathscr{H}^2(X)$. Since f is a positive quadratic homogenous continuous convex function, $p_C \equiv \sqrt{2f}$ is a continuous positively homogenous function, i.e. a continuous coercive Minkowski functional. Clearly, p_C is generated by $C \equiv \{x \in X : f(x) \le \frac{1}{2}\}$. Therefore, $f = \frac{1}{2}p_C^2$.

vi) It suffices to note that C is bounded with $0 \in \mathrm{int}C$ if and only if p_C is continuous and coercive, which is equivalent to f is continuous coercive. □

Theorem 5.4. *Suppose that X is a Banach space. Then $T : \mathfrak{C}_{00}(X) \to \mathscr{H}_c^2(X)$ defined for $B \in \mathfrak{C}_{00}(X)$ by $T(B) = \frac{1}{2}p_B^2$ is a fully order-reversing locally Lipschitz isomorphism.*

Proof. We first show that $T : \mathfrak{C}_{00}(X) \to \mathscr{H}_c^2(X)$ defined for $B \in \mathfrak{C}_{00}(X)$ by $T(B) = \frac{1}{2}p_B^2$ is fully order-reversing locally Lipschitz.

Since every $B \in \mathfrak{C}_{00}(X)$ is a convex body with $0 \in \mathrm{int}(B)$, T is order-reversing. Conversely, for each $f \in \mathscr{H}_c^2(X)$, let $p = \sqrt{2f}$. Since f is positive quadratic homogenous continuous convex, p is a continuous coercive Minkowski functional. Consequently, $B \equiv \{x \in X : p(x) \le 1\}$ is a convex body with $0 \in \mathrm{int}(B)$ so that $T(B) = f$. Therefore, $T : \mathfrak{C}_{00}(X) \to \mathscr{H}_c^2(X)$ is fully order-reversing.

Next, we show that T is locally Lipschitz. For any fixed $A \in \mathfrak{C}_{00}(X)$, there is $\beta > 0$ such that $2\beta B_X \subset A$. Note that for each $\beta > \delta > 0$, and for any $B \in \mathfrak{C}(X)$, $B \in \mathfrak{A}_\delta \equiv \{C \in \mathfrak{C}(X) : d_H(A, C) < \delta\}$ entails that $\delta B_X \subset B \in \mathfrak{C}_{00}(X)$, and that $B \subset A \oplus \delta B_X$ and $A \subset B \oplus \delta B_X$. We claim that T is Lipschitz on the neighborhood $\mathfrak{A}_\delta$ of A.

For any $B, C \in \mathfrak{A}_\delta$, let $\alpha = d_H(B, C)$. Then $B \subset C \oplus \alpha B_X$ and $C \subset B \oplus \alpha B_X$. Consequently, $p_{B\oplus\alpha B_X} \le p_C$ and $p_{C\oplus\alpha B_X} \le p_B$. Set $K_1 = \{x \in B_X : p_B(x) \ge$

$p_C(x)\}$ and $K_2 = \{x \in B_X : p_B(x) \le p_C(x)\}$. Then

$$\sup_{x\in B_X} |p_B(x) - p_C(x)| = \max\Big\{\sup_{x\in K_1}(p_B(x) - p_C(x)), \sup_{x\in K_2}(p_C(x) - p_B(x))\Big\}$$

$$\le \max\Big\{\sup_{x\in B_X}(p_B(x) - p_{B\oplus\alpha B_X}(x)), \sup_{x\in B_X}(p_C(x) - p_{C\oplus\alpha B_X}(x))\Big\}.$$

Note that for any $x(\neq 0) \in X$, $\frac{p_B(x)}{\|x\|} \le \frac{1}{\delta}$, $\frac{x}{p_B(x)} \in \mathrm{bdr}B$, and that $\frac{x}{p_B(x)} + \alpha\frac{x}{\|x\|} \in \mathrm{bdr}(B \oplus \alpha B_X)$. Then

$$p_{B\oplus\alpha B_X}(x) = \frac{p_B(x)\|x\|}{\|x\| + \alpha p_B(x)} = \frac{p_B(x)}{1 + \alpha\frac{p_B(x)}{\|x\|}} \ge \frac{p_B(x)}{1 + \frac{\alpha}{\delta}}.$$

Therefore, $x \in K_1$ entails

$$0 \le p_B(x) - p_{B\oplus\alpha B_X}(x) \le p_B(x)\Big(1 - \frac{1}{1 + \frac{\alpha}{\delta}}\Big) \le p_B(x)\Big(\frac{\alpha}{\delta}\Big) = p_B(x)\Big(\frac{d_H(B, C)}{\delta}\Big).$$

We can show the following inequality in the same way: For every $x(\neq 0) \in K_2$,

$$0 \le p_C(x) - p_{C\oplus\alpha B_X}(x) \le p_C(x)\Big(\frac{d_H(B, C)}{\delta}\Big).$$

Consequently,

$$d(T(B), T(C)) = \sup_{x\in B_X} |T(B)(x) - T(C)(x)|$$

$$= \frac{1}{2}\sup_{x\in B_X} |p_B^2(x) - p_C^2(x)| = \frac{1}{2}\sup_{x\in B_X}\Big(p_B(x) + p_C(x)\Big) \cdot |p_B(x) - p_C(x)|$$

$$\le M d_H(B, C),$$

where

$$M = \frac{1}{2\delta}\max\Big\{\sup_{x\in B_X} p_B(x), \sup_{x\in B_X} p_C(x)\Big\}\Big(\sup_{x\in B_X} p_B(x) + \sup_{x\in B_X} p_C(x)\Big).$$

Finally, we show that $T^{-1} : \mathscr{H}_c^2(X) \to \mathfrak{C}_{00}(X)$ is again fully order-reversing locally Lipschitz. By definition, T^{-1} is fully order-reversing if and only if T is fully order-reversing. It remains to show that T^{-1} is locally Lipschitz on $\mathscr{H}_c^2(X)$.

Let $f \in \mathscr{H}_c^2(X)$. By Lemma 5.3, there is $C \in \mathfrak{C}_{00}(X)$ such that $f = \frac{1}{2}p_C^2$. Let $\mathscr{M}_c(X)$ be the cone of all continuous coercive Minkowski functionals on X endowed with the metric $d_{\mathscr{M}}$ which is defined for $p, q \in \mathscr{M}_c(X)$ by $d_{\mathscr{M}}(p, q) = \sup_{x\in B_X} |p(x) - q(x)|$. Then $S : \mathscr{H}_c^2(X) \to \mathscr{M}_c(X)$ defined for $f \in \mathscr{H}_c^2(X)$ by $S(f) = \sqrt{2f}$ is locally Lipschitz. To show T^{-1} is locally Lipschitz, it suffices to prove that the mapping $U : \mathscr{M}_c(X) \to \mathfrak{C}_{00}(X)$ defined for $p_C \in \mathscr{M}_c(X)$ by $U(p_C) = C$ is locally Lipschitz.

Note that the metric $d_{\mathscr{M}}$ on $\mathscr{M}_c(X)$ is defined for $g, h \in \mathscr{M}_c(X)$ by $d_{\mathscr{M}}(g,h) = \sup_{x\in B_X} |g(x) - h(x)|$. Let $g = p_C$, $h = p_D$ with $d_{\mathscr{M}}(g,h) = r > 0$. We can assume that C and D are bounded by $\alpha > 0$.

Positive homogeneity of g and h entails

$$g \le h + r\|\cdot\|, \text{ and } \ h \le g + r\|\cdot\|,$$

where $\|\cdot\|$ is the norm of X. This in turn deduces that

$$C_r \equiv \{x \in X : h(x) + r\|x\| \le 1\} = \{x \in X : p_D(x) + r\|x\| \le 1\} \subset C \cap D \tag{5.1}$$

and

$$D_r \equiv \{x \in X : g(x) + r\|x\| \le 1\} = \{x \in X : p_C(x) + r\|x\| \le 1\} \subset D \cap C. \tag{5.2}$$

On the other hand, $x \in X$, $h(x) = p_D(x) = 1$ implies

$$1 = p_{C_r}(\frac{x}{p_{C_r}(x)}) = p_{C_r}(\frac{x}{p_D(x) + r\|x\|}) = \frac{1}{1 + r\|x\|} p_{C_r}(x) \ge \frac{1}{1 + r\beta} p_{C_r}(x). \tag{5.3}$$

where $\beta = \sup_{x\in \mathrm{bdr}(D)} \|x\|$ (> 0). Therefore,

$$D \subset (1 + r\beta)C_r \subset (1 + r\beta)(C \cap D) \subset (1 + r\beta)C. \tag{5.4}$$

Analogously,

$$C \subset (1 + r\gamma)D_r \subset (1 + r\gamma)(C \cap D) \subset (1 + r\gamma)D, \tag{5.5}$$

where $\gamma = \sup_{x\in \mathrm{bdr}(C)} \|x\|$. Since C, D are bounded by α, it follows from (5.4) and (5.5)

$$\begin{aligned} d_H(C, D) &\le d_H(C, (1 + r\beta)C) + d_H((1 + r\beta)C, D) \\ &\le d_H(C, (1 + r\beta)C) + d_H((1 + r\gamma)(1 + r\beta)D, D) \\ &\le r\beta\alpha + ((1 + r\gamma)(1 + r\beta) - 1))\alpha \\ &= r(\beta + \gamma + r\beta\gamma)\alpha = L \cdot d_{\mathscr{M}}(g, h), \end{aligned}$$

where $L = (\beta + \gamma + r\beta\gamma)\alpha$. Thus, we have shown that the mapping $U : \mathscr{M}_c(X) \to \mathfrak{C}_{00}(X)$ is locally Lipschitz. □

Lemma 5.5. *Assume that X is a Banach space admitting an equivalent strictly convex norm. Then the subset $\mathfrak{C}_{sc,00}(X)$ consisting of all strictly convex bodies in $\mathfrak{C}_{00}(X)$ contains a dense F_σ-subset of $\mathfrak{C}_0(X)$.*

Proof. Since X has an equivalent strictly convex norm, without loss of generality, we can assume that $X = (X, \|\cdot\|)$ is itself strictly convex. For each $n \in \mathbb{N}$, let $f_n = \frac{1}{2n}\|\cdot\|^2$. Then each of $\{f_n : n \in \mathbb{N}\}$ is continuous, strictly convex and coercive. By Lemma 3.4, for each $f \in \mathscr{H}^2(X)$, $g_n \equiv f + f_n, n = 1, 2, \cdots$ are strictly convex and coercive. Note that for each $n \in \mathbb{N}$, $\mathscr{H}_n^2(X) \equiv f_n + \mathscr{H}^2(X)$ is a closed cone of $\mathscr{H}^2(X)$. Clearly, $\bigcup_{n=1}^{\infty} \mathscr{H}_n^2(X)$ is a dense F_σ-subset of $\mathscr{H}^2(X)$. Since $T : \mathfrak{C}_{00}(X) \to \mathscr{H}_c^2(X)$ is a locally Lipschitz isomorphism (Theorem 5.4), $T^{-1}(\mathscr{H}_n^2(X))$ is a closed subset of $\mathfrak{C}_{00}(X)$. Consequently, $\mathfrak{F} \equiv T^{-1}(\bigcup_{n=1}^{\infty} \mathscr{H}_n^2(X)) = (\bigcup_{n=1}^{\infty} T^{-1}\mathscr{H}_n^2(X))$ (which consists of strictly convex bodies) is a dense F_σ-subset of $\mathfrak{C}_{00}(X)$. We finish the proof by noting that $\mathfrak{C}_{00}(X)$ is a dense open set of $\mathfrak{C}_0(X)$ (Proposition 5.1 iii). □

Now, we restate and prove Theorem 1.1 as follows.

Theorem 5.6. *In a Banach space X, the set $\mathfrak{C}_{sc}(X)$ consisting of all strictly convex bodies of X is dense in $\mathfrak{C}(X)$ if and only if X admits an equivalent strictly convex norm.*

Proof. Necessity. Assume that $\mathfrak{C}_{sc}(X)$ is dense in $\mathfrak{C}(X)$. Take any $B \in \mathfrak{C}_{sc}(X)$ with $0 \in \mathrm{int}B$, and let $C = B \bigcap (-B)$. Since B is strictly convex, C is again strictly convex. Let $|||\cdot||| = p_C$, the Minkowski functional generated by C. Then it is easy to see that $|||\cdot|||$ is an equivalent strictly convex norm of X.

Sufficiency. Suppose that X admits an equivalent strictly convex norm. Then by Lemma 5.5, the set $\mathfrak{C}_{sc,00}(X)$ of all strictly convex bodies in $\mathfrak{C}_{00}(X)$ contains a dense F_σ set of $\mathfrak{C}_0(X)$. We finish the proof by noting $\mathfrak{C}(X) = X + \mathfrak{C}_{00}(X)$, and $X + \mathfrak{C}_{sc,00}(X) = \mathfrak{C}_{sc}$.

□

Recall that a Banach space X is said to be weakly compactly generated (WCG) if there is a weakly compact subset W of X such that span$W$ is dense in X. Since every WCG space admits an equivalent strictly convex norm, and every separable Banach space is a WCG space (see, for instance, [11]), we have the following result.

Corollary 5.7. *If X is a Banach space in one of the following classes,*

$$\mathfrak{F} = \{\text{finite dimensional spaces}\},$$

$$\mathfrak{S} = \{\text{separable Banach spaces}\},$$

$$\mathfrak{S}_{WCG} = \{\text{weakly compactly generated spaces}\},$$

then the set consisting of all strictly convex bodies of X is dense in $\mathfrak{C}(X)$.

Corollary 5.8. *If X is a Banach space such that X^* is w^*-separable, then the set consisting of all strictly convex bodies of X is dense in $\mathfrak{C}(X)$.*

Proof. By Theorem 5.6, it suffices to note that every Banach space with w^*-separable dual admits an equivalent strictly convex norm. □

Corollary 5.9. *If X is one of the following Banach spaces: $c_0(\Gamma)$ for an arbitrary set Γ, ℓ_p $(1 \leq p \leq \infty)$, $L_p(\mu)$ $(1 \leq p \leq \infty, \mu$ is σ- finite$)$ and $\ell_1[0,1]$, then the set consisting of all strictly convex bodies of X is dense in $\mathfrak{C}(X)$.*

Proof. By Theorem 5.6, it is true for $X = c_0(\Gamma)$ because $c_0(\Gamma)$ is WCG for an arbitrary set Γ. Again by Theorem 5.6, the conclusion holds for $X = \ell_p$ $(1 \leq p < \infty)$, $L_p(\mu)$ $(1 \leq p < \infty, \mu$ is σ- finite$)$ because they are separable. According to Corollary 5.7, it is true when $X = \ell_\infty, L_\infty(\mu)$ because $\ell_\infty^* = \ell_1 \oplus c_0^\perp$ is w^*-separable, and $L_\infty(\mu)$ is isomorphic to ℓ_∞. Since the dual $\ell_1^*[0,1]$ of $\ell_1[0,1]$ is w^*-separable, the conclusion for $X = \ell_1[0,1]$ follows from Corollary 5.7. □

6. Gâteaux smooth convex bodies in $\mathfrak{C}(X)$

In this section, we will show that if a Banach space X admits an equivalent norm so that its dual X^* is strictly convex with respect to the new dual norm, then Gâteaux smooth convex bodies are dense in $\mathfrak{C}(X)$, i.e. Theorem 1.2 stated in Section 1.

For a Banach space X, we again use $\mathscr{C}_{\mathrm{conv}}(X)$ to denote the set of all extended real-valued lower semicontinuous proper convex functions on X, and $\mathscr{C}^*_{\mathrm{conv}}(X^*)$ the set of all extended real-valued w^*-lower semicontinuou proper convex functions on X^*. If X is nontrivial, then both $\mathscr{C}_{\mathrm{conv}}(X)$ and $\mathscr{C}^*_{\mathrm{conv}}(X^*)$ are not cones. For example, given any $u \neq v \in X$, let δ_u, and δ_v be the indicators of the singletons $\{u\}$ and $\{v\}$, i.e. $\delta_u(x) = 0$, if $x = u$; $= +\infty$, otherwise. Then δ_u, and δ_v are extended real-valued lower semicontinuous proper convex functions, but $\delta_u + \delta_v = +\infty$. However, the following subsets of $\mathscr{C}_{\mathrm{conv}}(X)$ form cones. $\mathscr{C}_{\mathrm{cconv}}(X)$ denotes the cone of all lower continuous coercive convex functions on X, and $\mathscr{C}^*_{\mathrm{cconv}}(X^*)$, the cone of all lower continuous and w^*-lower semicontinuous coercive convex functions on X^*. Finally, $\mathscr{H}_c^2(X)$ $(\mathscr{H}_c^{*2}(X^*)$, resp.) stands for the cone of all real-valued continuous (continuous and w^*-lower semicontinuous, resp.) positive quadratic homogenous coercive convex functions on X (X^*, resp.).

A convex body B in X^* is said to be *w^*-convex* if B is w^*-closed. The following property is easy to follow.

Proposition 6.1. *Let C be a closed convex set of X^* with $0 \in \mathrm{int}(C)$. Then*

i) p_C is w^*-lower semicontiuous if and only if C is w^*-closed;

ii) C is a w^*-convex body if and only if p_C is w^*-lower semicontinuous coercive.

Let X be a Banach space. We use $\mathfrak{C}^*(X^*)$ to denote the cone of all w^*-compact convex sets of X^* endowed with the Hausdorff metric d_H. $\mathfrak{C}_0^*(X^*)$ ($\mathfrak{C}_{00}^*(X^*)$, resp.) is the cone of all w^*-compact convex sets containing the origin (w^*-compact convex sets containing the origin in their interiors, resp.).

Let $\mathscr{H}^{*2}(X^*)$ be the cone of all continuous and w^*-lower semicontinuous positive quadratic homogenous convex functions endowed with the metric d defined by

$$d(f,g) = \sup_{x^* \in X^*, \|x^*\| \leq 1} |f(x^*) - g(x^*)|, \quad f, g \in \mathscr{H}^{*2}(X^*),$$

and $\mathscr{H}_c^{*2}(X^*)$ ($\mathscr{H}_{\mathrm{cs}}^{*2}(X^*)$, resp.) be the subcone consisting of all continuous positive quadratic homogenous coercive convex (strictly convex, resp.) functions of $\mathscr{H}^{*2}(X^*)$.

Parallel to Lemma 5.2, we have

Lemma 6.2. *i) Both $\mathfrak{C}(X^*)$ and $\mathfrak{C}_0^*(X^*)$ are complete cones;*

ii) $\mathfrak{C}_{00}^*(X^*)$ is a dense open set of $\mathfrak{C}_0^*(X^*)$.

Recall the definition of Fenchel's transform $\mathscr{F} : \mathscr{C}_{\mathrm{conv}}(X) \to \mathscr{C}_{\mathrm{conv}}(X^*)$ (Definition 2.9) is defined for $f \in \mathscr{C}_{\mathrm{conv}}(X)$ by

$$\mathscr{F}(f)(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}, \quad x^* \in X^*.$$

The following property follows from definitions of subdifferential mapping of convex functions and of Fenchel's transform.

Lemma 6.3. *Let f be a lower semicontinuous proper convex function defined on a Banach space X, $x_0 \in X$ and $x_0^* \in X^*$. Then*

$$x_0^* \in \partial f(x_0) \iff f(x_0) + \mathscr{F}(f)(x_0^*) = \langle x_0^*, x_0 \rangle. \tag{6.1}$$

Consequently,

$$x_0^* \in \partial f(x_0) \iff x_0 \in \partial \mathscr{F}(f)(x_0^*). \tag{6.2}$$

Proof. By the definition of the subdifferential mapping,

$$x_0^* \in \partial f(x_0) \iff f(x) - f(x_0) \geq \langle x_0^*, x - x_0 \rangle, \ \forall \ x \in X$$

$$\iff \langle x_0^*, x_0 \rangle - f(x_0) \geq \langle x_0^*, x \rangle - f(x), \ \forall \ x \in X$$

$$\iff \mathscr{F}(f)(x_0^*) = \langle x_0^*, x_0\rangle - f(x_0)$$
$$\iff \mathscr{F}(f)(x_0^*) + f(x_0) = \langle x_0^*, x_0\rangle.$$

□

Recall that for a subset $A \subset X$, its polar A° is defined by

$$A^\circ = \{x^* \in X^* : \langle x^*, x\rangle \le 1,\ \forall x \in A\};$$

and the support function σ_A of A is defined for $x^* \in X^*$ by

$$\sigma_A(x^*) = \sup\{\langle x^*.x\rangle : x \in A\}.$$

The following property is easy to obtain.

Proposition 6.4. *Let $B \subset \mathfrak{C}_{00}(X)$. Then its polar $B^\circ \in \mathfrak{C}_{00}(X^*)$, and the Minkowski functional p_{B° satisfies $p_{B^\circ} = \sigma_B$.*

Lemma 6.5. *Suppose that X is a Banach space. Then*

i) for each $f \in \mathscr{H}_c^2(X)$ there is a convex body $B \in \mathfrak{C}_{00}(X)$ such that $f = \frac{1}{2}p_B^2$, where p_B is the Minkowski functional generated by B;

ii) for each $f \in \mathscr{H}_c^2(X)$, we have $\mathscr{F}(f) = \frac{1}{2}p_{B^\circ}^2 = \frac{1}{2}\sigma_B^2$;

iii) $\mathscr{F} : \mathscr{H}_c^2(X) \to \mathscr{H}_c^{2*}(X)$ is a fully order-reversing isomorphism.

Proof. i) is just Lemma 5.3 v).

ii) By i), for each $f \in \mathscr{H}_c^2(X)$ there is a convex body $B \in \mathfrak{C}_{00}(X)$ such that $f = \frac{1}{2}p_B^2$. Fix any $x_0 \in X$ and $x_0^* \in \partial f(x_0) = p_B(x_0)\partial p_B(x_0)$. Then $x_0^* = p_B(x_0)z_0^*$ for some $z_0^* \in \partial p_B(x_0)$. Definition of Fenchel's transform entails

$$\mathscr{F}(f)(x_0^*) = \sup\{\langle x_0^*, x\rangle - f(x) : x \in X\} = \sup\{p_B(x_0)\langle z_0^*, x\rangle - \frac{1}{2}p_B^2(x) : x \in X\}.$$

Since $g(x) \equiv \frac{1}{2}p_B^2(x) - p_B(x_0)\langle z_0^*, x\rangle$ is a continuous convex function with $0 \in \partial g(x_0)$,

$$\min_{x\in X} g(x) = g(x_0) = \frac{1}{2}p_B^2(x_0) - p_B(x_0)\langle z_0^*, x_0\rangle. \tag{6.3}$$

Note that $z_0^* \in \partial p_B(x_0)$ implies $\langle z_0^*, x_0\rangle = p_B(x_0)$. This and (6.3) entail

$$\mathscr{F}(f)(x_0^*) = \sup\{-g(x) : x \in X\} = -\min_{x\in X} g(x) = -g(x_0) = \frac{1}{2}p_B^2(x_0). \tag{6.4}$$

On the other hand, note that there is $z_0^* \in \partial p_B(x_0)$ such that $x_0^* = p_B(x_0)z_0^*$. It follows from (6.4)

$$\frac{1}{2}p_B^2(x_0) = \mathscr{F}(f)(x_0^*) = \sup\{\langle x_0^*, x\rangle - f(x) : x \in X\}$$
$$= \sup\{p_B(x_0)\langle z_0^*, x\rangle - \frac{1}{2}p_B^2(x) : x \in X\}.$$

This and $\langle z_0^*, x\rangle = p_B(x_0)$ imply

$$\frac{1}{2}p_B^2(x_0) = \sup\{p_B(x_0)\langle z_0^*, x\rangle - \frac{1}{2}p_B^2(x) : x \in X, p_B(x) \le p_B(x_0)\}$$

$$\frac{1}{2}p_B^2(x_0) = \sup\{p_B(x_0)\langle z_0^*, x\rangle - \frac{1}{2}p_B^2(x) : x \in X, p_B(x) = p_B(x_0)\}$$

$$= \sup\{p_B(x_0)\langle x_0^*, x\rangle : x \in X, p_B(x) \le 1\} - \frac{1}{2}p_B^2(x_0)$$

$$= p_B(x_0)\sigma_B(x_0^*) - \frac{1}{2}p_B^2(x_0),$$

where $\sigma_B(x_0^*) = \sup\{\langle x_0^*, x\rangle : x \in B\}$. This entails that

$$p_B(x_0) = \sigma_B(x_0^*), \quad \forall\, x_0 \in X,\ x_0^* \in \partial f(x_0). \tag{6.5}$$

This combined with (6.4) derive

$$\mathscr{F}(f)(x_0^*) = \frac{1}{2}\sigma_B^2(x_0^*),\ \forall x_0^* \in \partial f(X). \tag{6.6}$$

Since f is continuous coercive convex, by an argument of the Bishop-Phelps theorem (see, for instance, [19]), we see the range $\partial f(X)$ of the subdifferential mapping ∂f is dense in X^*. This, local Lipschitz continuity of $\mathscr{F}(f)$ on X^* and (6.6) together imply

$$\mathscr{F}(f)(x^*) = \frac{1}{2}\sigma_B^2(x^*),\ \forall x^* \in X^*.$$

To finish the proof of of ii), it suffices to note $p_{B^*} = \sigma_B$ on X^*.

iii). By ii), it is easy to check that $\mathscr{F} : \mathscr{H}_c^2(X) \to \mathscr{H}_c^{2*}(X)$ is a fully order-reversing isomorphism. □

Remark 6.6. Indeed, we can show that $\mathscr{F} : \mathscr{H}_c^2(X) \to \mathscr{H}_c^{2*}(X)$ is also a locally Lipschitz isomorphism.

Lemma 6.7. *Let X be a Banach space.*

i) If $g \in \mathscr{H}_c^{*2}(X^*)$ *is strictly convex, then* $f \equiv \mathscr{F}^{-1}(g) \in \mathscr{H}_c^2(X)$ *is everywhere Gâteaux differentiable.*

ii) If $g \in \mathscr{H}_c^{*2}(X^*)$ *is everywhere Gâteaux differentiable, then* $f \equiv \mathscr{F}^{-1}(g) \in \mathscr{H}_c^2(X)$ *is strictly convex.*

Proof. Let $g \in \mathscr{H}_c^{*2}(X^*)$. By Lemma 6.5 ii), there is a convex body $B \in \mathfrak{C}_{00}(X)$ with $B^\circ \in \mathfrak{C}_{00}^*(X^*)$ such that $g = \frac{1}{2}\sigma_B^2 = \frac{1}{2}p_{B^\circ}^2$ and $f = \mathscr{F}^{-1}(g) = \frac{1}{2}p_B^2$.

i) It follows from (6.5) and (6.6) that for each $x_0 \in X$, we have

$$p_B(x_0) = \sigma_B(x_0^*),\ x_0^* \in \partial f(x_0),$$

and

$$g(x_0^*) = \frac{1}{2}\sigma_B^2(x_0^*) = \frac{1}{2}p_B^2(x_0),\ x_0^* \in \partial f(x_0).$$

Therefore, g is constant on $\partial f(x_0)$. Convexity of the set $\partial f(x_0)$ and strict convexity of g entail that $\partial f(x_0)$ is a singleton, which is equivalent to that f is Gâteaux differentiable at x_0.

ii) Suppose, to the contrary, that f is not strictly convex. Then by Lemma 3.3, there is $r > 0$ such that $B_r \equiv \{x \in X : f(x) \leq r\}$ is not strictly convex. Since f is positively quadric homogenous, we can assume that $r = \frac{1}{2}$, that is, $B_r = B$. Therefore, there exist $u \neq v \in \mathrm{bdr}(B)$ such that the segment $[u, v] \equiv \{\lambda u + (1 - \lambda)v : 0 \leq \lambda \leq 1\} \subset B$. By Zorn's lemma, there is a maximal convex set K of $\mathrm{bdr}(B)$ such that $[u, v] \subset K$. It is trivial to get that K is closed with $K \cap \mathrm{int}B = \emptyset$. By the Hahn-Banach separation theorem of convex sets, there exists $x_0^* \in X^*$ such that

$$\langle x_0^*, x\rangle = 1, \ \forall\ x \in K \ \text{ and } \ \langle x_0^*, x\rangle < 1, \ \forall\ x \in B \setminus K.$$

This implies that $K \subset \partial\sigma_B(x_0^*)$. It follows from $\sigma_B(x_0^*) = 1$ that

$$K \subset \sigma_B(x_0^*)\partial\sigma_B(x_0^*) = \partial(\frac{1}{2}\sigma_B^2(x_0^*)) = \partial g(x_0^*).$$

This is a contradiction to that g is Gâteaux differentiable at x_0^*. □

Lemma 6.8. *Let X be a Banach space admitting an equivalent norm so that X^* is strictly convex with respect to the new norm. Then the cone $\mathfrak{C}_{00,\mathrm{gsm}}(X)$ consisting of all Gâteaux smooth convex bodies B with $0 \in \mathrm{int}(B)$ contains a dense open subset of $\mathfrak{C}_0(X)$.*

Proof. Without loss of generality, we can assume that X^* is itself strictly convex. Let $f_n = \frac{1}{n}(\|\cdot\|^*)^2, \ n = 1, 2, \cdots$, where $\|\cdot\|^*$ is the dual norm of X^*. Then $f_n \in \mathscr{H}_{cs}^{*2}(X^*)$ for all $n \in \mathbb{N}$. Therefore, $\mathscr{H}_n^{*2}(X^*) \equiv f_n + \mathscr{H}^{*2}(X^*) \subset \mathscr{H}_{cs}^{*2}(X^*)$. Clearly, $\mathfrak{D} \equiv \bigcup_{n\in\mathbb{N}} \left(f_n + \mathscr{H}^2(X^*)\right)$ is a dense F_σ-set of $\mathscr{H}^2(X^*)$. By Lemma 6.5 iii), $\mathscr{F}^{-1} : \mathscr{H}_c^{*2}(X^*) \to \mathscr{H}_c^2(X)$ is a continuous fully order-reversing isomorphism. Consequently, $\mathfrak{S} \equiv \mathscr{F}^{-1}(\mathfrak{D})$ is a dense subset of $\mathscr{H}_c^2(X)$. Since each $g \in \mathfrak{D}$ is strictly convex, by Lemma 6.7 ii), $f \equiv \mathscr{F}^{-1}(g)$ is everywhere Gâteaux differentiable. Let p be the Minkowski functional so that $f = \frac{1}{2}p^2$. Then the convex body

$$B = \{x \in X : p(x) \leq 1\} = \{x \in X : f(x) \leq \frac{1}{2}\} \tag{6.7}$$

is Gâteaux smooth. Since the correspondence (from $f \in \mathscr{H}_c^2(X)$ to $B \in \mathfrak{C}_{00}(X)$) defined by (6.7)) is a continuous isomorphism from $\mathscr{H}_c^2(X)$ to $\mathfrak{C}_{00}(X)$, Gâteaux smooth convex bodies are dense in $\mathfrak{C}_{00}(X)$. Consequently, $\mathfrak{C}_{00,\mathrm{gsm}}(X)$ (the cone of Gâteaux smooth convex bodies) is dense in $\mathfrak{C}_0(X)$.

□

Now, we are ready to prove Theorem 1.2. First, we restate it as follows.

Theorem 6.9. *Let X be a Banach space admitting an equivalent norm so that X^* is strictly convex with respect to the new norm. Then the cone $\mathfrak{C}_{\rm sm}(X)$ consisting of all Gâteaux smooth convex bodies contains a dense subset of $\mathfrak{C}(X)$.*

Proof. By Lemma 6.7, the cone $\mathfrak{C}_{00,\rm sm}(X)$ consisting of all Gâteaux smooth convex bodies B with $0 \in \mathrm{int}(B)$ contains a dense subset $\mathfrak{S}$ of $\mathfrak{C}_0(X)$. This entails that $X + \mathfrak{S}$ is dense in $X + \mathfrak{C}_0(X) = \mathfrak{C}(X)$.

□

7. Strictly convex and Gâteaux smooth convex bodies

In this section, we will show that if a Banach space X admits an equivalent strictly convex norm so that its dual norm is also strictly convex on X^*, then every convex body of X can be uniformly approximated by strictly convex and Gâteaux smooth convex bodies, i.e. Theorem 1.3 stated in Section 1. In particular, every convex body in a separable Banach space, or, a reflexive Banach space is uniformly approximated by strictly convex and Gâteaux smooth convex ones, i.e. Theorem 1.4.

Again as before, we restate Theorem 1.3 as follows.

Theorem 7.1. *Let X be a Banach space admitting an equivalent strictly convex norm so that its dual is also strictly convex. Then every convex body of X can be uniformly approximated by strictly convex and Gâteaux smooth convex bodies.*

Proof. Our proof was motivated by Asplund's averaging technique [2]. Clearly, it suffices to show that it is true for every $B \in \mathfrak{C}_{00}(X)$. We can assume that the original norm $\|\cdot\|$ of X itself is strictly convex and its dual norm $\|\cdot\|^*$ on X^* is also strictly convex.

Fix any $B \in \mathfrak{C}_{00}(X)$. We can claim that for every $\varepsilon > 0$, there exist a strictly convex body $A \in \mathfrak{C}_{00}(X)$ contained in B and a convex body $C \in \mathfrak{C}_{00}(X)$ containing B so that its polar $C^\circ = \{x^* \in X^* : \langle x^*, x\rangle \leq 1;\ \forall x \in C\}$ is a w^*-closed strictly convex body of X^* such that $d_H(A, C) < \varepsilon$. Indeed, let p_A, p_B and p_C be successively, the Minkowski functionals generated by A, B and C. Then for any $\delta > 0$, both $f_\delta \equiv \frac{1}{2}(\delta\|\cdot\|^2 + p_A^2)$ and $g_\delta^* \equiv \frac{1}{2}(\delta\|\cdot\|^{*2} + p_{C^*}^2)$ are strictly convex, and g^*_δ is w^*-lower semicontinuous on X^*. Put $A_\delta = \{x \in X : f_\delta(x) \leq 1\}$ and $C_\delta = (C^*_\delta)^\circ$. Clearly, $A_\delta \subset B \subset C_\delta$ and $d_H(A_\delta, C_\delta) < \varepsilon$ for all sufficiently small $\delta > 0$.

Starting with $f_0 = \frac{1}{2}p_A^2, g_0 = \frac{1}{2}p_C^2$, let

$$f_n = \frac{1}{2}(f_{n-1} + g_{n-1}),\ g_n = f_{n-1}\Box g_{n-1},\ n = 1, 2, \cdots,$$

where $f\Box g$ is the inf-convolution of f and g defined in Section 2. Then for all $n \in \mathbb{N}$,

$$f_0 \geq f_1 \geq \cdots \geq f_{n-1} \geq f_n \geq g_n \geq g_{n-1} \geq \cdots \geq g_0,$$

and

$$\mathscr{F}(f_0) \leq \mathscr{F}(f_1) \leq \cdots \leq \mathscr{F}(f_{n-1}) \leq \mathscr{F}(f_n) \leq \mathscr{F}(g_n) \leq \mathscr{F}(g_{n-1}) \leq \cdots \leq \mathscr{F}(g_0).$$

Note that f_0 is strictly convex on X and $\mathscr{F}(g_0) = \frac{1}{2}p_{C^*}^2$ is strictly convex on X^*, and that $\mathscr{F}(g_n) = \frac{1}{2}(\mathscr{F}(f_{n-1}) + \mathscr{F}(g_{n-1}))$. Then for each $n \in \mathbb{N}$, f_n is strictly convex on X and $\mathscr{F}(g_n)$ is strictly convex on X^*. Monotonicity of both the sequences $\{f_n\}$ and $\{g_n\}$ and $\sup_{x\in B_X} |f_n(x) - g_n(x)| \to 0$ entail that that there is a convex body $D \in \mathfrak{C}_{00}$ such that

$$\lim_n f_n = \frac{1}{2}p_D^2 = \lim_n g_n, \quad \text{uniformly on } B_X.$$

Clearly, $A \subset D \subset C$. Since $A \subset B \subset C$ with $d_H(A, C) < \varepsilon$, we get $d(B, D) < \varepsilon$. By an argument as the same as the proof of Theorem 1 in [2], we see that $\frac{1}{2}p_D^2$ is strictly convex on X, and $\mathscr{F}(\frac{1}{2}p_D^2)$ is strictly convex on X^*. It follows that D is both strictly convex and Gâteaux smooth. □

The following consequence is just Theorem 1.4.

Corollary 7.2. *Suppose that X is either separable, or, reflexive. Then every convex body can be uniformly approximated by strictly convex and Gâteaux smooth convex bodies.*

Proof. By Theorem 7.1, it suffices to note that every separable (reflexive, respectively) Banach space admits an equivalent strictly convex norm so that it dual norm is also strictly convex on X^*. □

Remark 7.3. We do not know whether Gâteaux smooth convex bodies are dense in $\mathfrak{C}(X)$ whenever X is a Gâteaux smooth Banach space; and whether it can be deduced that strictly convex and Gâteaux smooth convex bodies are dense in $\mathfrak{C}(X)$ if X is both strictly convex and Gâteaux smooth.

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