

Limit varieties of aperiodic monoids

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Abstract

A *limit variety* is a variety that is minimal with respect to being non-finitely based. We present a new limit variety of aperiodic monoid. We also show that if there exists any other limit variety of aperiodic monoids, then it is contained in the joint of the variety $\mathbb{B}^1$ of all idempotent monoids and certain finitely generated variety $\mathbb{E}^1$ with $\mathbb{B}^1 \wedge \mathbb{E}^1 = \mathbb{L}_2^1$, where $\mathbb{L}_2^1$ is the variety of left-zero monoids. Jackson and Lee proved that $\mathbb{E}^1$ is HFB, that is, its every subvariety is finitely based. We extend this result a step up the classical decomposition $\mathbb{B}^1 = \bigcup_{i \geq 2} \mathbb{L}_i^1$ by showing that $\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{L}_3^1$ is also HFB, where $\overline{\mathbb{E}^1}$ is the variety dual of $\mathbb{E}^1$.

1 Introduction

A variety of algebras is called *finitely based* (abbreviated to FB) if it has a finite basis of its identities, otherwise, the variety is said to be *non-finitely based* (abbreviated to NFB). A variety is *hereditary finitely based* (abbreviated to HFB) if all its subvarieties are FB. A variety is called a *limit variety* if it is NFB but all its proper subvarieties are FB.

According to Zorn's lemma, a variety is HFB if and only if it excludes all limit varieties. This explains the interest in limit varieties. However, constructing explicit examples of limit varieties turns out to be extremely non-trivial. For instance, there are uncountably many limit varieties of periodic groups [13], while no explicit example of a limit variety of groups is known. Locating such an example remains one of the intriguing open problems in the theory of group varieties (see Section 3 in the survey article [6]).

The present article is concerned with the limit varieties of *monoids*, i.e., semi-groups with an identity element. A complete classification of all limit varieties of monoids seems to be highly infeasible since that would include a description of all limit varieties of periodic groups. Therefore, it is logical to focus on the class of monoids with only trivial subgroups. Such monoids are called *aperiodic* monoids.

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The first two examples of limit varieties of aperiodic monoids were presented by Jackson [11] in 2005. Since then, limit varieties of aperiodic monoids have received much attention and several more examples have been found as well as some partial descriptions have been obtained (see the articles [7, 8, 9, 14, 15, 24, 25, 28, 29] and Section 11 of the monograph [16]). We will discuss these results in detail in the next section.

In this paper, based on the previous results, we present a new example of a limit variety of aperiodic monoids. We also show that if there is any other limit variety of aperiodic monoids then it is contained in $\mathbb{Q}^1 \vee \mathbb{B}^1$ and, modulo duality, contains $\mathbb{Q}^1 \vee \mathbb{R}_3^1$, where $\mathbb{B}^1$ is the variety of all idempotent monoids, $\mathbb{R}_3^1$ is the subvariety of $\mathbb{B}^1$ defined by the identity $xyz \approx xyzxzyz$, and $\mathbb{Q}^1$ is the variety generated by the following six-element J -trivial monoid

$$\mathbb{Q}^1 = \langle e, b, c \mid e^2 = e, eb = b, ce = c, ec = be = cb = 0 \rangle \cup \{1\}.$$

The article is structured as follows. In Section 2, we provide a brief overview of the previous results on limit varieties of aperiodic monoids. Several basic facts are collected in Section 3. In Section 4, we remind a description of the lattice of varieties of idempotent monoids and some results on identities of idempotent monoids. Based on the description of the subvariety lattice of $\mathbb{B}^1$, we clarify the structure of the subvariety lattice of the variety $\mathbb{Q}^1 \vee \mathbb{B}^1$ in Section 5. Section 6 provides a new example of a HFB variety of monoids. Section 7 presents a new explicit example of a limit variety of aperiodic monoids. Finally, in Sections 8 and 9, we reduce the further study of limit varieties of aperiodic monoids to the interval $[\mathbb{Q}^1 \vee \mathbb{R}_3^1, \mathbb{Q}^1 \vee \mathbb{B}^1]$.

2 A brief overview of the previous results on limit varieties

We need some definitions and notation. Let $\mathfrak{A}$ be a countably infinite set called an *alphabet*. As usual, let $\mathfrak{A}^+$ and $\mathfrak{A}^*$ denote the free semigroup and the free monoid over the alphabet $\mathfrak{A}$, respectively. Elements of $\mathfrak{A}$ are called *letters* and elements of $\mathfrak{A}^*$ are called *words*. For any set W of words, let $M(W)$ denote the Rees quotient monoid of $\mathfrak{A}^*$ over the ideal consisting of all words that are not subwords of any word in W . We denote by $\mathbb{M}(W)$ the variety generated by $M(W)$.

The varieties $\mathbb{M}(\{xzxyty\})$ and $\mathbb{M}(\{xyzxty, xzytxy\})$ are the two above-mentioned first examples of limit varieties constructed by Jackson [11]. In [14], Lee proved the uniqueness of the limit varieties $\mathbb{M}(\{xzxyty\})$ and $\mathbb{M}(\{xyzxty, xzytxy\})$ in the class of varieties of finitely generated aperiodic monoids with central idempotents. In [15], Lee generalized the result of [14] and established that $\mathbb{M}(\{xzxyty\})$ and $\mathbb{M}(\{xyzxty, xzytxy\})$ are the only limit varieties within the class of varieties of aperiodic monoids with central idempotents. In 2013, Zhang [28] found a NBF variety of monoids that contains neither $\mathbb{M}(\{xzxyty\})$ nor $\mathbb{M}(\{xyzxty, xzytxy\})$ and, therefore, she proved that there exists a limit variety of monoids that differs from $\mathbb{M}(\{xzxyty\})$ and $\mathbb{M}(\{xyzxty, xzytxy\})$.

If S is a semigroup, then the monoid obtained by adjoining a new identity element to S is denoted by S^1 . If M is a monoid, then the variety of monoids generated by M is denoted by $\mathbb{M}$. If $\mathbb{V}$ is a monoid variety, then $\overline{\mathbb{V}}$ denotes the variety *dual to* $\mathbb{V}$, i.e., the variety consisting of monoids anti-isomorphic to monoids from $\mathbb{V}$.

In [29], Zhang and Luo indicated the third explicit example of a limit variety of aperiodic monoids. It is the variety $\mathbb{A}^1 \vee \overline{\mathbb{A}^1}$, where

$$A = \langle a, b, c \mid a^2 = a, b^2 = b, ab = ca = 0, ac = cb = c \rangle.$$

The semigroup A was introduced and shown to generate a FB variety in [18, Section 19].

For any $n \geq 1$, let S_n denote the full symmetric group on the set $\{1, 2, \dots, n\}$. Let $\text{var } \Sigma$ denote the variety defined by a set Σ of identities.

The varieties

$$\mathbb{J} = \text{var} \left\{ \begin{array}{l} x^2 y^2 \approx y^2 x^2, xyx \approx xyx^2, xyzxy \approx yxzxy, xyxztx \approx xyxzztx, \\ xz_{1\pi} z_{2\pi} \cdots z_{n\pi} x t_1 z_1 \cdots t_n z_n \approx x^2 z_{1\pi} z_{2\pi} \cdots z_{n\pi} t_1 z_1 \cdots t_n z_n \end{array} \mid \begin{array}{l} n \geq 1, \\ \pi \in S_n \end{array} \right\},$$

and $\overline{\mathbb{J}}$ are the next pair of limit varieties of monoids [7]. In [8], it was proved that $\mathbb{M}(\{xxxyty\})$, $\mathbb{M}(\{xyzxtxy, xzytxy\})$, $\mathbb{J}$ and $\overline{\mathbb{J}}$ are the only limit varieties of aperiodic monoids with commuting idempotents. In [9], the authors presented one more pair $\mathbb{K}^1$ and $\overline{\mathbb{K}^1}$ of limit varieties, where

$$K = \langle a, b, c \mid a^2 = a, b^2 = b^3, bcb^2 = bcb, ca = c, abc = ac = ba = b^2c = 0 \rangle,$$

and showed that there are exactly seven limit varieties of J -trivial monoids.

Lee and Li [17, Section 14] consider the semigroup

$$E = \langle a, b, c \mid a^2 = ab = 0, ba = ca = a, b^2 = bc = b, c^2 = cb = c \rangle$$

and showed that

$$\mathbb{E}^1 = \text{var}\{xtx \approx xtx^2 \approx x^2tx, xy^2x \approx x^2y^2\}.$$

For each $n \geq 1$, let

$$\mathbb{E}^1\{\sigma_n\} = \text{var}\{xtx \approx xtx^2 \approx x^2tx, xy^2x \approx x^2y^2, \sigma_n\}.$$

where

$$\sigma_n : \mathbf{e}_1 t_1 \mathbf{e}_2 t_2 \cdots \mathbf{e}_n t_n x^2 y^2 \approx \mathbf{e}_1 t_1 \mathbf{e}_2 t_2 \cdots \mathbf{e}_n t_n y^2 x^2,$$

and $\mathbf{e}_{2i-1} = x$ and $\mathbf{e}_{2i} = y$ for all $i \geq 1$. According to Proposition 5.6 in [12], the lattice of subvarieties of $\mathbb{E}^1$ contains an infinite ascending chain $\mathbb{E}^1\{\sigma_1\} \subset \mathbb{E}^1\{\sigma_2\} \subset \cdots \subset \mathbb{E}^1\{\sigma_n\} \subset \cdots$. In [25], the second-named author proved that the varieties $\mathbb{A}^1 \vee \mathbb{E}^1\{\sigma_2\}$ and $\mathbb{A}^1 \vee \overline{\mathbb{E}^1\{\sigma_2\}}$ are also limit.

In the recent paper [10], the first-named author, Li and Zhang proved that the following four varieties are limit:

$$\mathbb{J}_1 = \text{var} \left\{ \begin{array}{l} xyz \approx xyxzx, xyxy \approx x^2yty, \\ (x_1 t_1 x_2 t_2 \cdots x_n t_n) x (z_1 z_2 \cdots z_{2n}) y (t_{n+1} x_{n+1} \cdots t_{2n} x_{2n}) \cdot \\ txy (x_{1\pi} z_{1\tau} x_{2\pi} z_{2\tau} \cdots x_{(2n)\pi} z_{(2n)\tau}) \approx \\ (x_1 t_1 x_2 t_2 \cdots x_n t_n) x (z_1 z_2 \cdots z_{2n}) y (t_{n+1} x_{n+1} \cdots t_{2n} x_{2n}) \cdot \\ tyx (x_{1\pi} z_{1\tau} x_{2\pi} z_{2\tau} \cdots x_{(2n)\pi} z_{(2n)\tau}) \end{array} \middle| \begin{array}{l} n \geq 1, \\ \pi \in S_{2n} \end{array} \right\},$$

$$\mathbb{J}_2 = \text{var} \left\{ \begin{array}{l} xyz \approx xyxzx, xy_1 y s x y x_1 y_1 t x_1 \approx xy_1 y s x y x_1 y_1 t x_1, \\ (x_1 t_1 x_2 t_2 \cdots x_{2n} t_{2n}) x (z_1 s_1 z_2 s_2 \cdots z_n s_n) y (z_{n+1} z_{n+2} \cdots z_{2n}) \cdot \\ xy (x_{1\pi} z_{1\tau} x_{2\pi} z_{2\tau} \cdots x_{(2n)\pi} z_{(2n)\tau}) t (s_1 s_2 \cdots s_n) \approx \\ (x_1 t_1 x_2 t_2 \cdots x_{2n} t_{2n}) x (z_1 s_1 z_2 s_2 \cdots z_n s_n) y (z_{n+1} z_{n+2} \cdots z_{2n}) \cdot \\ yx (x_{1\pi} z_{1\tau} x_{2\pi} z_{2\tau} \cdots x_{(2n)\pi} z_{(2n)\tau}) t (s_1 s_2 \cdots s_n) \end{array} \middle| \begin{array}{l} n \geq 1, \\ \pi \in S_{2n} \end{array} \right\},$$

$\overline{\mathbb{J}_1}$ and $\overline{\mathbb{J}_2}$ are also limit and no other limit variety satisfying $xyz \approx xyxzx$ exists.

Let A_0 be the semigroup given by presentation:

$$A_0 = \langle a, b \mid a^2 = a, b^2 = b, ab = 0 \rangle.$$

The monoid A_0^1 was introduced and shown to be FB in [3]. In the present paper, we show that the variety $\mathbb{A}_0^1 \vee \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$ is a new example of a limit variety mentioned in the introduction.

3 Preliminaries

A letter is called *simple* [*multiple*] in a word $\mathbf{u}$ if it occurs in $\mathbf{u}$ once [at least twice]. The set of all letters in $\mathbf{u}$ is denoted by $\text{con}(\mathbf{u})$. Notice that $\text{con}(\mathbf{u}) = \text{sim}(\mathbf{u}) \cup \text{mul}(\mathbf{u})$, where $\text{sim}(\mathbf{u})$ is the set of all simple letters in $\mathbf{u}$ and $\text{mul}(\mathbf{u})$ is the set of all multiple letters in $\mathbf{u}$.

Fact 3.1 ([17, Proposition 4.3]). *An identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{Q}^1$ if and only if $\mathbf{u} = \mathbf{a}_0 \prod_{i=1}^m (t_i \mathbf{a}_i)$ and $\mathbf{v} = \mathbf{b}_0 \prod_{i=1}^m (t_i \mathbf{b}_i)$ for some $m \geq 0$, $\text{sim}(\mathbf{u}) = \text{sim}(\mathbf{v}) = \{t_1, t_2, \dots, t_m\}$, and $\text{con}(\mathbf{a}_i) = \text{con}(\mathbf{b}_i)$ for each $0 \leq i \leq m$. $\square$*

A *block* of a word $\mathbf{u}$ is a maximal subword of $\mathbf{u}$ that does not contain any letters simple in $\mathbf{u}$. If $\mathbf{u} = \mathbf{a}_0 \prod_{i=1}^m (t_i \mathbf{a}_i)$ and $\mathbf{v} = \mathbf{b}_0 \prod_{i=1}^m (t_i \mathbf{b}_i)$ with $\text{sim}(\mathbf{u}) = \text{sim}(\mathbf{v}) = \{t_1, t_2, \dots, t_m\}$, then the blocks $\mathbf{a}_i$ and $\mathbf{b}_i$ are called *corresponding*.

Let

$$\mathbb{L}_2^1 = \text{var}\{x \approx x^2, xy \approx xyx\} \quad \text{and} \quad \mathbb{R}_2^1 = \text{var}\{x \approx x^2, xy \approx yxy\}.$$

Let $\text{ini}(\mathbf{w})$ [respectively, $\text{fin}(\mathbf{w})$] denote the word obtained from $\mathbf{w}$ by retaining the first [respectively, last] occurrence of each letter. The following description of identities of $\mathbb{L}_2^1$ and $\mathbb{R}_2^1$ is well-known and can be easily verified.

Fact 3.2.

- (i) An identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{L}_2^1$ if and only if $\text{ini}(\mathbf{u}) = \text{ini}(\mathbf{v})$.
- (ii) An identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{R}_2^1$ if and only if $\text{fin}(\mathbf{u}) = \text{fin}(\mathbf{v})$. $\square$

Fact 3.3 ([17, Section 14.3]). An identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{E}^1$ if and only if $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{Q}^1$ and the corresponding blocks of $\mathbf{u}$ and $\mathbf{v}$ are equivalent modulo $\mathbb{L}_2^1$. $\square$

Lemma 3.4 ([25, Lemma 4.3]). Let $\mathbb{V}$ be a monoid variety that satisfies $xtx \approx x^2tx \approx xtx^2$ and contains neither $\mathbb{E}^1\{\sigma_2\}$ nor $\overline{\mathbb{A}^1}$. Then $\mathbb{V}$ is FB. $\square$

We say that a set of words $W \subseteq \mathfrak{A}^*$ is *stable with respect to a monoid variety* $\mathbb{V}$ if $\mathbf{v} \in W$ whenever $\mathbf{u} \in W$ and $\mathbb{V}$ satisfies $\mathbf{u} \approx \mathbf{v}$. Recall that a word $\mathbf{u} \in \mathfrak{A}^*$ is an *isoterm* [20] for $\mathbb{V}$ if the set $\{\mathbf{u}\}$ is stable with respect to $\mathbb{V}$.

If τ is an equivalence relation on the free monoid $\mathfrak{A}^*$ and $\mathbb{V}$ is a monoid variety, then a word $\mathbf{u} \in \mathfrak{A}^*$ is said to be a τ -*term* for $\mathbb{V}$ if $\mathbf{u} \tau \mathbf{v}$ whenever $\mathbb{V}$ satisfies $\mathbf{u} \approx \mathbf{v}$. Notice that if $W \subseteq \mathfrak{A}^*$ forms a single τ -class, then W is stable with respect to $\mathbb{V}$ if and only if every word in $\mathbf{u} \in W$ is a τ -term for $\mathbb{V}$. If τ is an equivalence relation on $\mathfrak{A}^*$ and $\mathbf{w} \in \mathfrak{A}^*$, then $[\mathbf{w}]_\tau$ denotes the τ -class of $\mathbf{w}$.

We use regular expressions to describe sets of words, in particular, congruence classes. For brevity, we put $a^+ = \{a\}^+$ and $a^* = \{a\}^*$ for any $a \in \mathfrak{A}^*$. For example, $[atb^2a]_\beta = a^+tbb^+a\{a, b\}^* \cup a^+tb^+a^+b\{a, b\}^*$, where β be the fully invariant congruence of $\mathbb{E}^1$.

Fact 3.5 ([24, Corollary 3.6, Theorem 4.1(iv)]; [25, Example 4.2]). Given a monoid variety $\mathbb{V}$ we have:

- (i) $\mathbb{V}$ contains $\mathbb{A}_0^1$ if and only if the set aa^+bb^+ is stable with respect to $\mathbb{V}$;
- (ii) $\mathbb{V}$ contains $\mathbb{E}^1\{\sigma_2\}$ if and only if β -class $[atb^2a]_\beta$ is stable with respect to $\mathbb{V}$. $\square$

The next statement is a special case of Lemma 2.4 in [25].

Fact 3.6. Suppose that $[atb^2a]_\beta$ is stable with respect to a monoid variety $\mathbb{V}$. Then $\mathbf{u} \in \{a, b, t\}^*$ is β -term for $\mathbb{V}$ in each of the following cases:

- (i) $\text{con}(\mathbf{u}) \subset \{a, b, t\}$;
- (ii) $\mathbf{u} = \mathbf{a}t\mathbf{b}$ such that $\mathbf{a}, \mathbf{b} \in \{a, b\}^*$ and either $\mathbf{a}$ or $\mathbf{b}$ contains no occurrences of a . $\square$

4 Varieties of idempotent monoids

A complete description of the subvariety lattice of the variety of all idempotent monoids $\mathbb{B}^1$ was given by Wismath [27]; see also [1, Section 5.5]. To describe this lattice, for $n \geq 2$, define

$$\begin{aligned}\mathbb{R}_n^1 &= \text{var}\{x \approx x^2, \mathbf{R}_n \approx \mathbf{S}_n\}, \\ \mathbb{L}_n^1 &= \text{var}\{x \approx x^2, \overline{\mathbf{R}_n} \approx \overline{\mathbf{S}_n}\},\end{aligned}$$

where

$$\mathbf{R}_n = \begin{cases} x_2x_1 & \text{for } n = 2, \\ x_1x_2x_3 & \text{for } n = 3, \\ \mathbf{R}_{n-1}x_n & \text{for even } n \geq 4, \\ x_n\mathbf{R}_{n-1} & \text{for odd } n \geq 5 \end{cases}$$

and $\mathbf{S}_n = \begin{cases} x_1x_2x_1 & \text{for } n = 2, \\ x_1x_2x_3x_1x_3x_2x_3 & \text{for } n = 3, \\ \mathbf{S}_{n-1}x_n\mathbf{R}_n & \text{for even } n \geq 4, \\ \mathbf{R}_nx_n\mathbf{S}_{n-1} & \text{for odd } n \geq 5, \end{cases}$

while $\overline{\mathbf{R}}_n$ and $\overline{\mathbf{S}}_n$ are the mirror images of $\mathbf{R}_n$ and $\mathbf{S}_n$, respectively. Let $\mathbb{T}$ denote the variety of trivial monoids.

Proposition 4.1 ([27, Proposition 4.7]). *The subvariety lattice of $\mathbb{B}^1$ is given in Fig. 1.* $\square$

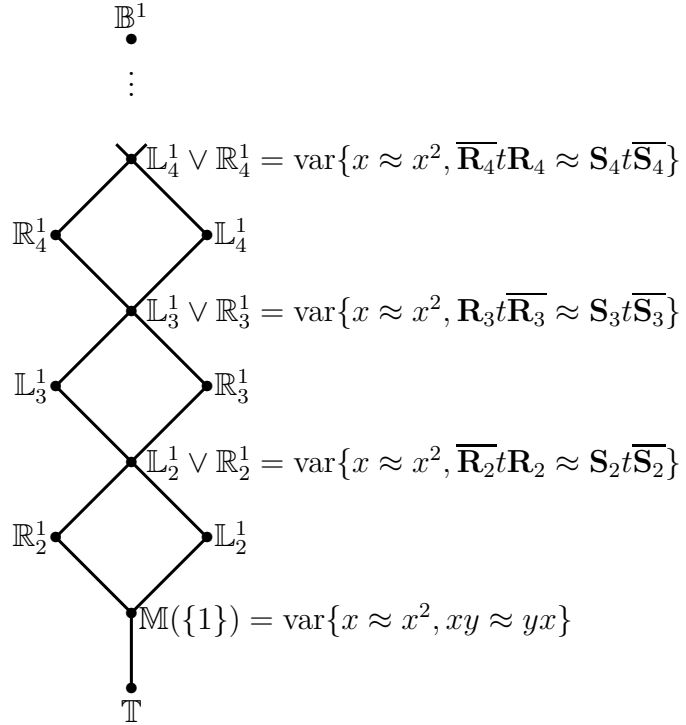


Figure 1: The subvariety lattice of $\mathbb{B}^1$

Proposition 4.1 gives us a freedom in choosing the defining identities for varieties of idempotent monoids as follows.

Corollary 4.2. *Let $k \geq 2$, $\mathbf{u} \approx \mathbf{v}$ be any identity which holds in $\mathbb{R}_k^1$ and fails in $\mathbb{L}_k^1$, and $\mathbf{w} \approx \mathbf{p}$ any identity which holds in $\mathbb{L}_k^1 \vee \mathbb{R}_k^1$ but fails in $\mathbb{L}_{k+1}^1$ and in $\mathbb{R}_{k+1}^1$. Then*

$$\mathbb{R}_k^1 = \text{var}\{x \approx x^2, \mathbf{u} \approx \mathbf{v}\},$$

$$\mathbb{R}_k^1 \vee \mathbb{L}_k^1 = \text{var}\{x \approx x^2, \mathbf{w} \approx \mathbf{p}\} = \mathbb{R}_{k+1}^1 \wedge \mathbb{L}_{k+1}^1. \quad \square$$

We use $\text{Id}(\mathbb{V})$ to denote the set of all identities of the variety $\mathbb{V}$. Let $\ell(\mathbf{u})$ denote the maximal prefix of $\mathbf{u}$ which contains all letters in $\mathbf{u}$ but one. Dually, $r(\mathbf{u})$ denotes the maximal suffix of $\mathbf{u}$ which contains all letters in $\mathbf{u}$ but one. The following result on the variety of all idempotent semigroups $\mathbb{B}$ is useful here because $\text{Id}(\mathbb{B}) = \text{Id}(\mathbb{B}^1)$.

Fact 4.3 ([5]). *An identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{B}^1$ if and only if $\text{con}(\mathbf{u}) = \text{con}(\mathbf{v})$ and the identities $\ell(\mathbf{u}) \approx \ell(\mathbf{v})$ and $r(\mathbf{u}) \approx r(\mathbf{v})$ hold in $\mathbb{B}^1$. $\square$*

The next well-known claim readily follows from Fact 4.3 or from the classical result [2, Exercise 4.2.1] that an idempotent semigroup is a semilattice of rectangular bands.

Corollary 4.4. *If $\text{con}(\mathbf{w}) \subseteq \text{con}(\mathbf{u}) = \text{con}(\mathbf{v})$, then $\mathbb{B}^1$ satisfies $\mathbf{u}\mathbf{v} \approx \mathbf{u}\mathbf{w}\mathbf{v}$. $\square$*

Fact 4.5 ([4, Lemma 3.20]). *An identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{R}_3^1$ if and only if $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{L}_2^1$ and $r(\mathbf{u}) \approx r(\mathbf{v})$ holds in $\mathbb{R}_3^1$. $\square$*

Corollary 4.6. *For any $\mathbf{w} \in \mathfrak{A}^*$, the variety $\mathbb{R}_3^1$ satisfies the identity $\mathbf{w} \approx \text{ini}(\mathbf{w})\mathbf{w}$.*

Proof. Clearly, $\text{ini}(\mathbf{w}) = \text{ini}(\text{ini}(\mathbf{w})\mathbf{w})$ and $r(\mathbf{w}) = r(\text{ini}(\mathbf{w})\mathbf{w})$. Now Facts 3.2(i) and 4.5 apply. $\square$

We use $_{i\mathbf{u}}x$ to refer to the i th from the left occurrence of x in a word $\mathbf{u}$. We use $_{\ell\mathbf{u}}x$ to refer to the last occurrence of x in $\mathbf{u}$. If x is simple in $\mathbf{u}$ then we use $_{\mathbf{u}}x$ to denote the only occurrence of x in $\mathbf{u}$. If the i th occurrence of x precedes the j th occurrence of y in a word $\mathbf{u}$, we write $(_{i\mathbf{u}}x) <_{\mathbf{u}} (_{j\mathbf{u}}y)$.

If $\{z, y\} \subseteq \text{con}(\mathbf{u})$ and $(_{\ell\mathbf{u}}z) <_{\mathbf{u}} (_{\ell\mathbf{u}}y)$, we use $(_{(1z)\mathbf{u}}y)$ to denote the first occurrence of y in $\mathbf{u}$ after $(_{\ell\mathbf{u}}z)$. We say that a triple of pairwise distinct letters $\{x, y, z\} \subseteq \text{con}(\mathbf{u})$ is $\mathbb{R}_3^1$ -stable in $\mathbf{u} \approx \mathbf{v}$ if

$$(_{\ell\mathbf{u}}z) <_{\mathbf{u}} (_{(1z)\mathbf{u}}y) <_{\mathbf{u}} (_{(1z)\mathbf{u}}x) \Leftrightarrow (_{\ell\mathbf{v}}z) <_{\mathbf{v}} (_{(1z)\mathbf{v}}y) <_{\mathbf{v}} (_{(1z)\mathbf{v}}x).$$

Otherwise, we say that $\{x, y, z\}$ is $\mathbb{R}_3^1$ -unstable in $\mathbf{u} \approx \mathbf{v}$.

Fact 4.7. *An identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{R}_3^1$ if and only if $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{L}_2^1 \vee \mathbb{R}_2^1$ and each triple of pairwise distinct letters $\{x, y, z\} \subseteq \text{con}(\mathbf{u})$ is $\mathbb{R}_3^1$ -stable in $\mathbf{u} \approx \mathbf{v}$.*

Proof. Suppose that $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{R}_3^1$. According to Proposition 4.1, $\mathbf{u} \approx \mathbf{v}$ is satisfied by $\mathbb{L}_2^1 \vee \mathbb{R}_2^1$. Take some triple of pairwise distinct letters $\{x, y, z\} \subseteq \text{con}(\mathbf{u})$. Without loss of generality we may assume that $(_{\ell\mathbf{u}}z) <_{\mathbf{u}} (_{(1z)\mathbf{u}}y) <_{\mathbf{u}} (_{(1z)\mathbf{u}}x)$. Since $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{R}_2^1$, Fact 3.2(ii) implies that $(_{\ell\mathbf{v}}z) <_{\mathbf{v}} (_{\ell\mathbf{v}}y)$ and $(_{\ell\mathbf{v}}z) <_{\mathbf{u}} (_{\ell\mathbf{v}}x)$. If $(_{\ell\mathbf{v}}z) <_{\mathbf{v}} (_{(1z)\mathbf{v}}x) <_{\mathbf{v}} (_{(1z)\mathbf{v}}y)$, then $r(\mathbf{u}(x, y, z))$ begins with y , but $r(\mathbf{v}(x, y, z))$

begins with x . Hence $\mathbf{u}(x, y, z) \approx \mathbf{v}(x, y, z)$ fails in $\mathbb{R}_3^1$ by Facts 3.2(i) and 4.5. To avoid the contradiction, the triple $\{x, y, z\} \subseteq \text{con}(\mathbf{u})$ must be $\mathbb{R}_3^1$ -stable in $\mathbf{u} \approx \mathbf{v}$.

Conversely, suppose that $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{L}_2^1 \vee \mathbb{R}_2^1$ and each triple of pairwise distinct letters $\{x, y, z\} \subseteq \text{con}(\mathbf{u})$ is $\mathbb{R}_3^1$ -stable. In view of Fact 3.2(ii), $\text{fin}(\mathbf{u}) = \text{fin}(\mathbf{v}) = x_n x_{n-1} \cdots x_1$ for some $n \geq 1$. We use induction on n . If $n < 3$ then $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{R}_3^1$ by Fact 4.5. Suppose that $n \geq 3$. Clearly, each triple of pairwise distinct letters $\{x, y, z\} \subseteq \text{con}(r(\mathbf{u}))$ is $\mathbb{R}_3^1$ -stable in $r(\mathbf{u}) \approx r(\mathbf{v})$. Since $\text{fin}(r(\mathbf{u})) = \text{fin}(r(\mathbf{v})) = x_{n-1} x_{n-2} \cdots x_1$, Fact 3.2(ii) implies that $r(\mathbf{u}) \approx r(\mathbf{v})$ holds in $\mathbb{R}_2^1$. Further, for each distinct $y, z \in \text{con}(r(\mathbf{u}))$, the triple $\{x_n, y, z\}$ is $\mathbb{R}_3^1$ -stable in $\mathbf{u} \approx \mathbf{v}$. Hence $\text{ini}(r(\mathbf{u})) = \text{ini}(r(\mathbf{v}))$. Then $r(\mathbf{u}) \approx r(\mathbf{v})$ holds in $\mathbb{L}_2^1$ by Fact 3.2(i). By the induction assumption, $r(\mathbf{u}) \approx r(\mathbf{v})$ holds in $\mathbb{R}_3^1$. Now Fact 4.5 applies, yielding that $\mathbf{u} \approx \mathbf{v}$ is satisfied by $\mathbb{R}_3^1$. $\square$

Example 4.8.

- (i) $\mathbb{R}_2^1 \vee \mathbb{L}_2^1 = \text{var}\{x \approx x^2, xszx \approx xsxzx\} = \mathbb{R}_3^1 \wedge \mathbb{L}_3^1$;
- (ii) $\mathbb{R}_3^1 = \text{var}\{x \approx x^2, xytxy \approx xytxxy\}$.

Proof. (i) The identity $xszx \approx xsxzx$ holds in $\mathbb{R}_2^1 \vee \mathbb{L}_2^1$ by Fact 3.2. On the other hand, it fails in $\mathbb{R}_3^1$ and in $\mathbb{L}_3^1$ by Fact 4.7 and its dual. Therefore, $\mathbb{R}_2^1 \vee \mathbb{L}_2^1 = \text{var}\{x \approx x^2, xszx \approx xsxzx\} = \mathbb{R}_3^1 \wedge \mathbb{L}_3^1$ by Corollary 4.2.

(ii) The identity $xytxy \approx xytxxy$ holds in $\mathbb{R}_3^1$ but fails in $\mathbb{L}_3^1$ by Fact 4.7 and its dual. Therefore, $\mathbb{R}_3^1 = \text{var}\{x \approx x^2, xytxy \approx xytxxy\}$ by Corollary 4.2. $\square$

5 The lattice of subvarieties of $\mathbb{Q}^1 \vee \mathbb{B}^1$

Lemma 5.1.

- (i) $\mathbb{Q}^1 \vee \mathbb{B}^1 = \text{var}\{xtx \approx txt^2 \approx x^2tx, (xy)^2 \approx x^2y^2\}$;
- (ii) An identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{Q}^1 \vee \mathbb{B}^1$ if and only if $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{Q}^1$ and the corresponding blocks of $\mathbf{u}$ and $\mathbf{v}$ are equivalent modulo $x \approx x^2$.

Proof. (i) Evidently, $x \approx x^2$ implies $xtx \approx txt^2 \approx x^2tx$ and $(xy)^2 \approx x^2y^2$. It follows from Fact 3.1 that $\mathbb{Q}^1$ satisfies $xtx \approx txt^2 \approx x^2tx$ and $(xy)^2 \approx x^2y^2$ as well. Therefore, $\mathbb{Q}^1 \vee \mathbb{B}^1 \subseteq \text{var}\{xtx \approx txt^2 \approx x^2tx, (xy)^2 \approx x^2y^2\}$.

Conversely, if $\mathbf{u} \approx \mathbf{v}$ is an identity of $\mathbb{Q}^1$, then, by Fact 3.1, $\mathbf{u} = \mathbf{a}_0 \prod_{i=1}^m (t_i \mathbf{a}_i)$ and $\mathbf{v} = \mathbf{b}_0 \prod_{i=1}^m (t_i \mathbf{b}_i)$ for some $m \geq 0$, $\text{sim}(\mathbf{u}) = \text{sim}(\mathbf{v}) = \{t_1, t_2, \dots, t_m\}$, and $\text{con}(\mathbf{a}_i) = \text{con}(\mathbf{b}_i)$ for each $0 \leq i \leq m$. If in addition, $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{B}^1$, then according to Fact 4.3, the corresponding blocks $\mathbf{a}_i$ and $\mathbf{b}_i$ of $\mathbf{u}$ and $\mathbf{v}$, respectively, must be equivalent modulo $x \approx x^2$. Then $\mathbf{u} \approx \mathbf{v}$ can be derived from $\{xtx \approx txt^2 \approx x^2tx, (xy)^2 \approx x^2y^2\}$. Therefore, $\mathbb{Q}^1 \vee \mathbb{B}^1 = \text{var}\{xtx \approx txt^2 \approx x^2tx, (xy)^2 \approx x^2y^2\}$.

Part (ii) readily follows from Part (i) and its proof. $\square$

Observation 5.2. Let $\mathbb{V}$ be a subvariety of $\mathbb{Q}^1 \vee \mathbb{B}^1$ and $\mathbf{u} \approx \mathbf{v}$ an identity of $\mathbb{V} \wedge \mathbb{B}^1$ such that $\text{con}(\mathbf{u}) = \text{mul}(\mathbf{u})$. Then $\mathbf{u} \approx \mathbf{v}$ holds on $\mathbb{V}$.

Proof. Take some monoid $M \in \mathbb{V}$. According to Lemma 5.1, $\{xtx \approx xtx^2 \approx x^2tx, (xy)^2 \approx x^2y^2\}$ holds in M . Since M satisfies $x^2y^2 \approx (xy)^2$, the set of all idempotents $E(M)$ of M forms a submonoid of M . Further, since M satisfies $x^2 \approx x^3$, the square of every element in M is an idempotent. Finally, since $E(M)$ satisfies $\mathbf{u} \approx \mathbf{v}$, the monoid M satisfies the identity obtained from $\mathbf{u} \approx \mathbf{v}$ by replacing every letter $x \in \text{con}(\mathbf{u})$ by x^2 . Since $\text{con}(\mathbf{u}) = \text{mul}(\mathbf{u})$, the resulting identity is equivalent to $\mathbf{u} \approx \mathbf{v}$ modulo $xtx \approx xtx^2 \approx x^2tx$. $\square$

Let $k \geq 2$ and $\mathbf{u} \approx \mathbf{v}$ be any identity which holds in $\mathbb{R}_k^1$ and fails in $\mathbb{L}_k^1$ with $\text{con}(\mathbf{u}) = \text{mul}(\mathbf{u})$. Denote

$$\mathbb{E}_k^1 = \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, \mathbf{u} \approx \mathbf{v}\}.$$

For example,

$$\begin{aligned} \overline{\mathbb{E}^1} &= \mathbb{E}_2^1 = \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, y^2x^2 \approx xy^2x\}, \\ \mathbb{E}^1 &= \overline{\mathbb{E}_2^1} = \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, x^2y^2 \approx xy^2x\}. \end{aligned}$$

See also Example 5.7(ii) below.

The following statement implies that for each $k \geq 2$ the monoid variety $\mathbb{E}_k^1$ is well-defined.

Fact 5.3. *For each $k \geq 2$ the variety $\mathbb{E}_k^1$ is the largest subvariety of $\mathbb{Q}^1 \vee \mathbb{B}^1$ which does not contain $\mathbb{L}_k^1$.*

Proof. Let $\mathbb{V}$ be a subvariety of $\mathbb{Q}^1 \vee \mathbb{B}^1$ which does not contain $\mathbb{L}_k^1$. Then in view of Proposition 4.1, every band subvariety of $\mathbb{V}$ is contained in $\mathbb{R}_k^1 = \text{var}\{x \approx x^2, \sigma\}$, where the identity σ holds in $\mathbb{R}_k^1$ and fails in $\mathbb{L}_k^1$. Then Observation 5.2 implies that $\mathbb{V}$ satisfies every identity $\mathbf{u} \approx \mathbf{v}$ with $\text{con}(\mathbf{u}) = \text{mul}(\mathbf{u})$ that holds in $\mathbb{R}_k^1$ and fails in $\mathbb{L}_k^1$. Therefore, $\mathbb{V}$ is a subvariety of $\mathbb{E}_k^1$. $\square$

Given $k \geq 2$ an identity $\mathbf{u} \approx \mathbf{v}$ is called *block- $\mathbb{R}_k^1$ -balanced* if $\mathbb{Q}^1$ satisfies $\mathbf{u} \approx \mathbf{v}$ and the corresponding blocks in $\mathbf{u}$ and $\mathbf{v}$ form an identity which holds in $\mathbb{R}_k^1$. We say that a property of identities (P) is *derivation-stable* if an identity $\mathbf{u} \approx \mathbf{v}$ satisfies property (P) whenever $\mathbf{u} \approx \mathbf{v}$ follows from Σ such that every identity in Σ satisfies property (P). It is easy to see that the property of being a block- $\mathbb{R}_k^1$ -balanced identity is derivation-stable. The next observation generalizes Fact 3.3 and Lemma 5.1(ii).

Observation 5.4. *Given $k \geq 2$ an identity holds in $\mathbb{E}_k^1$ if and only if it is block- $\mathbb{R}_k^1$ -balanced.*

Proof. Fix some identity $\mathbf{u} \approx \mathbf{v}$ which holds in $\mathbb{R}_k^1$ and fails in $\mathbb{L}_k^1$ with $\text{con}(\mathbf{u}) = \text{mul}(\mathbf{u})$. Since every identity in $\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, \mathbf{u} \approx \mathbf{v}\}$ is block- $\mathbb{R}_k^1$ -balanced and this property of identities is derivation-stable, every identity of $\mathbb{E}_k^1$ is block- $\mathbb{R}_k^1$ -balanced.

Conversely, in view of Proposition 4.1, if $\mathbf{U} \approx \mathbf{V}$ is block- $\mathbb{R}_k^1$ -balanced, then we can derive $\mathbf{U} \approx \mathbf{V}$ from $\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, \mathbf{u} \approx \mathbf{v}\}$ by applying $\mathbf{u} \approx \mathbf{v}$ within the blocks. $\square$

The next proposition shows that the varieties in $\{\mathbb{T}, \mathbb{M}(\{1\}), \mathbb{E}_k^1, \overline{\mathbb{E}_k^1} \mid k \geq 2\}$ generate a lattice isomorphic to the lattice of varieties of band monoids. Compare it with Corollary 4.2.

Proposition 5.5. *For each $k \geq 2$ we have*

$$\mathbb{E}_k^1 \vee \overline{\mathbb{E}_k^1} = \text{var}\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2, \mathbf{u} \approx \mathbf{v}\} = \mathbb{E}_{k+1}^1 \wedge \overline{\mathbb{E}_{k+1}^1},$$

where $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{L}_k^1 \vee \mathbb{R}_k^1$, fails in $\mathbb{L}_{k+1}^1$ and in $\mathbb{R}_{k+1}^1$, and $\text{con}(\mathbf{u}) = \text{mul}(\mathbf{u})$.

Proof. Observation 5.4 and its dual imply that

$$\mathbb{E}_k^1 \vee \overline{\mathbb{E}_k^1} \subseteq \text{var}\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2, \mathbf{u} \approx \mathbf{v}\}.$$

Conversely, let $\mathbf{U} \approx \mathbf{V}$ be an identity of $\mathbb{E}_k^1 \vee \overline{\mathbb{E}_k^1}$. Then it holds in $\mathbb{Q}^1$ and its corresponding blocks form the identities which hold both in $\mathbb{R}_k^1$ and in $\mathbb{L}_k^1$ by Observation 5.4 and its dual. So, we can derive $\mathbf{U} \approx \mathbf{V}$ from $\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2, \mathbf{u} \approx \mathbf{v}\}$ by applying $\mathbf{u} \approx \mathbf{v}$ within the blocks. Therefore,

$$\mathbb{E}_k^1 \vee \overline{\mathbb{E}_k^1} = \text{var}\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2, \mathbf{u} \approx \mathbf{v}\}.$$

Since

$$\mathbb{E}_{k+1}^1 \wedge \overline{\mathbb{E}_{k+1}^1} = \text{var}\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2, \mathbf{u}_1 \approx \mathbf{v}_1, \mathbf{u}_2 \approx \mathbf{v}_2\},$$

where $\mathbf{u}_1 \approx \mathbf{v}_1$ holds in $\mathbb{R}_{k+1}^1$ and fails in $\mathbb{L}_{k+1}^1$ with $\text{con}(\mathbf{u}_1) = \text{mul}(\mathbf{u}_1)$, $\mathbf{u}_2 \approx \mathbf{v}_2$ holds in $\mathbb{L}_{k+1}^1$ and fails in $\mathbb{R}_{k+1}^1$ with $\text{con}(\mathbf{u}_2) = \text{mul}(\mathbf{u}_2)$, Corollary 4.2 implies that the sets of identities $\{\mathbf{u}_1 \approx \mathbf{v}_1, \mathbf{u}_2 \approx \mathbf{v}_2\}$ and $\{\mathbf{u} \approx \mathbf{v}\}$ are equivalent modulo $x \approx x^2$. Therefore,

$$\mathbb{E}_{k+1}^1 \wedge \overline{\mathbb{E}_{k+1}^1} = \text{var}\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2, \mathbf{u} \approx \mathbf{v}\},$$

as required. $\square$

Corollary 5.6. *For each $k \geq 2$, $\mathbb{E}_k^1 \vee \overline{\mathbb{E}_k^1}$ is the largest subvariety of $\mathbb{Q}^1 \vee \mathbb{B}^1$ which contains neither $\mathbb{L}_{k+1}^1$ nor $\mathbb{R}_{k+1}^1$.*

Proof. Recall that, by Lemma 5.1(i), $\mathbb{Q}^1 \vee \mathbb{B}^1 = \text{var}\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2\}$. Let $\mathbb{V}$ be a subvariety of $\mathbb{Q}^1 \vee \mathbb{B}^1$ which contains neither $\mathbb{L}_{k+1}^1$ nor $\mathbb{R}_{k+1}^1$. Then $\mathbb{V}$ is contained in $\mathbb{E}_{k+1}^1 \wedge \overline{\mathbb{E}_{k+1}^1}$ by Fact 5.3 and its dual. Since $\mathbb{E}_k^1 \vee \overline{\mathbb{E}_k^1} = \mathbb{E}_{k+1}^1 \wedge \overline{\mathbb{E}_{k+1}^1}$ by Proposition 5.5, $\mathbb{V}$ is contained in $\mathbb{E}_k^1 \vee \overline{\mathbb{E}_k^1}$. $\square$

Compare the following with Example 4.8.

Example 5.7.

- (i) $\mathbb{E}^1 \vee \overline{\mathbb{E}^1} = \text{var}\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xs^2z^2x \approx xs^2xz^2x\} = \mathbb{E}_3^1 \wedge \overline{\mathbb{E}_3^1}.$
- (ii) $\mathbb{E}_3^1 = \text{var}\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xyt^2xy \approx xyxt^2xy\}.$

Proof. (i) The identity $xs^2z^2x \approx xs^2xz^2x$ holds in $\mathbb{R}_2^1 \vee \mathbb{L}_2^1$ by Fact 3.2. On the other hand, it fails in $\mathbb{R}_3^1$ and in $\mathbb{L}_3^1$ by Fact 4.7 and its dual. Therefore, $\mathbb{E}^1 \vee \overline{\mathbb{E}^1} = \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xs^2z^2x \approx xs^2xz^2x\} = \mathbb{E}_3^1 \wedge \overline{\mathbb{E}_3^1}$ by Proposition 5.5.

Part (ii) is by the definition above. $\square$

Observation 5.8. *The following pairs of sets of identities are equivalent within the variety $\mathbb{Q}^1 \vee \mathbb{B}^1$:*

- (i) $\{xytyx \approx xytxyx\}$ and $\{xytxy \approx xytyxy\}$;
- (ii) $\{xytyx \approx xyxtxyx\}$ and $\{xytyx \approx xytxyx, xytyx \approx xyxtxyx\}$;
- (iii) $\{x^2y^2 \cdot \mathbf{a}_1x\mathbf{a}_2y \cdots \mathbf{a}_nx \approx xy^2x \cdot \mathbf{a}_1x\mathbf{a}_2y \cdots \mathbf{a}_nx\}$ and $\{x^2y^2 \cdot \mathbf{a}_1y\mathbf{a}_2x \cdots \mathbf{a}_ny \approx xy^2x \cdot \mathbf{a}_1y\mathbf{a}_2x \cdots \mathbf{a}_ny\}$, where $n \geq 0$ and $\{x, y\} \cap \text{con}(\mathbf{a}_1 \dots \mathbf{a}_n) = \emptyset$.
- (iv) $\{xyt^2ysx \approx xyxt^2ysx\}$ and $\{xyt^2xsy \approx xyxt^2xsy\}$;
- (v) $\{x^2y^2ty \approx xy^2xty\}$ and $\{x^2y^2tx \approx xy^2xtx\}$.

Proof. (i) If we replace t by xt and multiply $xytyx \approx xytxyx$ by y on the left, then we obtain an identity which is equivalent within $\mathbb{Q}^1 \vee \mathbb{B}^1$ to $yxyx \approx yxxyx$. Swapping x and y we obtain $xytxy \approx xytyxy$.

Conversely, if we replace t by xt and multiply $xytxy \approx xytyxy$ by y on the left, then we obtain an identity which is equivalent within $\mathbb{Q}^1 \vee \mathbb{B}^1$ to $yxyx \approx yxxyx$. Swapping x and y we obtain $xytyx \approx xytxyx$.

(ii) First, $xytyx \approx xyxtxyx$ implies $xytyx \approx xytxyx$ within $\mathbb{Q}^1 \vee \mathbb{B}^1$ because of the following deduction:

$$xytyx \approx (xy)xyt(yx) \stackrel{\mathbb{Q}^1 \vee \mathbb{B}^1}{\approx} (xyx)xyt(xyx) \approx xytxyx.$$

Similarly, $xytxy \approx xyxtxyx$ implies $xytxy \approx xytyxy$ within $\mathbb{Q}^1 \vee \mathbb{B}^1$.

Reversely, $\{xytyx \approx xytxyx, xytyx \approx xyxtxyx\}$ implies $xytyx \approx xyxtxyx$ within $\mathbb{Q}^1 \vee \mathbb{B}^1$ as follows:

$$xytyx \approx xytxyx \approx xy(xy)tx(yx) \approx xy(xy)xtx(yx) \approx xyxtxyx,$$

and we are done.

(iii) Multiplying both sides of $x^2y^2 \cdot \mathbf{a}_1x\mathbf{a}_2y \cdots \mathbf{a}_nx \approx xy^2x \cdot \mathbf{a}_1x\mathbf{a}_2y \cdots \mathbf{a}_nx$ by y on the left and using $\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2\}$ gives an identity which is equivalent to $x^2y^2 \cdot \mathbf{a}_1y\mathbf{a}_2x \cdots \mathbf{a}_ny \approx xy^2x \cdot \mathbf{a}_1y\mathbf{a}_2x \cdots \mathbf{a}_ny$ modulo swapping x and y .

Parts (iv) and (v) readily follow from Part (iii). $\square$

6 The variety $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$ is HFB

Lemma 6.1. *Let $\mathbb{V}$, $\mathbb{S}$ and $\mathbb{P}$ be three varieties such that $\mathbb{P} \subseteq \mathbb{V}$ and $\mathbb{P} \subseteq \mathbb{S}$. Let Σ be a set of identities such that $\mathbb{V}$ satisfies Σ .*

Let $\text{Dist}(\mathbb{P} \rightarrow \mathbb{S})$ be a function which associates with each identity $\mathbf{u} \approx \mathbf{v}$ of $\mathbb{P}$ a set $\text{Dist}(\mathbb{P} \rightarrow \mathbb{S})(\mathbf{u} \approx \mathbf{v})$ so that the set $\text{Dist}(\mathbb{P} \rightarrow \mathbb{S})(\mathbf{u} \approx \mathbf{v})$ is empty if and only if $\mathbf{u} \approx \mathbf{v}$ holds on $\mathbb{S}$.

Suppose that for every identity $\mathbf{u} \approx \mathbf{v}$ of $\mathbb{V}$ which fails on $\mathbb{S}$, one can find a word $\mathbf{u}_1$ such that Σ implies $\mathbf{u} \approx \mathbf{u}_1$ and $|\text{Dist}(\mathbb{P} \rightarrow \mathbb{S})(\mathbf{u}_1 \approx \mathbf{v})| < |\text{Dist}(\mathbb{P} \rightarrow \mathbb{S})(\mathbf{u} \approx \mathbf{v})|$.

Then every identity of $\mathbb{V}$ can be derived from Σ and an identity of $\mathbb{V} \vee \mathbb{S}$.

Proof. Let $\text{Dist}(\mathbb{V} \rightarrow \mathbb{S})$ be the restriction of $\text{Dist}(\mathbb{P} \rightarrow \mathbb{S})$ to the identities of $\mathbb{V}$. Then by Lemma 3.1 in [21] and its proof, for every identity $\mathbf{u} \approx \mathbf{v}$ of $\mathbb{V}$ there is a derivation

$$\mathbf{u} = \mathbf{u}_0 \approx \mathbf{u}_1 \approx \mathbf{u}_2 \approx \cdots \approx \mathbf{u}_k \approx \mathbf{v},$$

such that for each $i \geq 0$, the identity $\mathbf{u}_i \approx \mathbf{u}_{i+1}$ follows from Σ and $\mathbf{u}_k \approx \mathbf{v}$ holds on $\mathbb{S}$. Since $\mathbb{V}$ satisfies Σ , the identity $\mathbf{u}_k \approx \mathbf{v}$ holds on $\mathbb{V} \vee \mathbb{S}$. $\square$

Let $\mathbf{u} \approx \mathbf{v}$ be an identity of $\mathbb{Q}^1$. Define $\text{Dist}(\mathbb{Q}^1 \rightarrow \mathbb{E}^1)$ as the set of all unordered pairs of occurrences $\{_{1\mathbf{a}}x, _{1\mathbf{a}}y\}$ such that $x, y \in \text{con}(\mathbf{u}) = \text{con}(\mathbf{v})$, $\mathbf{a}$ is a block of $\mathbf{u}$ and one of the following holds:

- $(_{1\mathbf{a}}x) <_{\mathbf{a}} (_{1\mathbf{a}}y)$ but $(_{1\mathbf{b}}y) <_{\mathbf{b}} (_{1\mathbf{b}}x)$;
- $(_{1\mathbf{a}}y) <_{\mathbf{a}} (_{1\mathbf{a}}x)$ but $(_{1\mathbf{b}}x) <_{\mathbf{b}} (_{1\mathbf{b}}y)$,

where $\mathbf{b}$ is the block of $\mathbf{v}$ corresponding to $\mathbf{a}$. Notice that the dual to Observation 5.4 implies that $\text{Dist}(\mathbb{Q}^1 \rightarrow \mathbb{E}^1)(\mathbf{u} \approx \mathbf{v})$ is empty if and only if $\mathbf{u} \approx \mathbf{v}$ holds on $\mathbb{E}^1$.

Let $Z(x, y)$ denote the set of all words $\mathbf{u}$ with $\text{mul}(\mathbf{u}) = \{x, y\}$ such that each block of $\mathbf{u}$ is either x or y , and no two blocks containing the same letter are adjacent. Denote:

- $\Phi = \{\mathbf{c}tx^2y^2 \approx \mathbf{c}tyx^2y \mid \mathbf{c} \in Z(x, y)\}$;
- $\overline{\Phi} = \{x^2y^2t\mathbf{d} \approx xy^2xt\mathbf{d} \mid \mathbf{d} \in Z(x, y)\}$;
- $\phi : xytxy \approx xytyxy$;
- $\overline{\phi} : xytxy \approx xyxxy$.

When $_{1\mathbf{u}}x$ and $_{1\mathbf{u}}y$ are adjacent in $\mathbf{u}$ and $(_{1\mathbf{u}}x) <_{\mathbf{u}} (_{1\mathbf{u}}y)$, we write $(_{1\mathbf{u}}x) \ll_{\mathbf{u}} (_{1\mathbf{u}}y)$.

Fact 6.2. *Let $\mathbb{V}$ be a variety such that $(\mathbb{Q}^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1) \subseteq \mathbb{V} \subseteq \mathbb{E}^1 \vee \overline{\mathbb{E}^1}$. Then every identity of $\mathbb{V}$ follows within $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$ from $(\Phi \cup \{\phi\}) \cap \text{Id}(\mathbb{V})$ and an identity of $\mathbb{V} \vee \mathbb{E}^1$.*

Proof. Take an arbitrary identity $\mathbf{u} \approx \mathbf{v}$ of $\mathbb{V}$ which fails in $\mathbb{E}^1$. Lemma 3.2 and the dual to Observation 5.4 imply that $\text{Dist}(\mathbb{Q}^1 \rightarrow \mathbb{E}^1)(\mathbf{u} \approx \mathbf{v})$ is not empty. This means that for some block $\mathbf{a}$ of $\mathbf{u}$ and $x, y \in \text{con}(\mathbf{u})$ we have $(1_{\mathbf{a}}x) <_{\mathbf{u}} (1_{\mathbf{a}}y)$ but $(1_{\mathbf{b}}y) <_{\mathbf{v}} (1_{\mathbf{b}}x)$, where $\mathbf{b}$ is the block of $\mathbf{v}$ corresponding to $\mathbf{a}$. Since $\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \subset \mathbb{E}_3^1$ by Proposition 5.5, Corollary 4.6 allows us to assume that $\mathbf{a}$ begins with the sequence of first occurrences of all variables in $\text{con}(\mathbf{a})$. We can further assume that $(1_{\mathbf{a}}x) \ll_{\mathbf{u}} (1_{\mathbf{a}}y)$. Indeed, if the initially chosen pair $\{1_{\mathbf{a}}x, 1_{\mathbf{a}}y\} \in \text{Dist}(\mathbb{Q}^1 \rightarrow \mathbb{E}^1)(\mathbf{u} \approx \mathbf{v})$ is not adjacent in $\mathbf{a}$, then either $\{1_{\mathbf{a}}x, 1_{\mathbf{a}}z\} \in \text{Dist}(\mathbb{Q}^1 \rightarrow \mathbb{E}^1)(\mathbf{u} \approx \mathbf{v})$ or $\{1_{\mathbf{a}}z, 1_{\mathbf{a}}y\} \in \text{Dist}(\mathbb{Q}^1 \rightarrow \mathbb{E}^1)(\mathbf{u} \approx \mathbf{v})$ for some $z \in \text{con}(\mathbf{a})$ with $(1_{\mathbf{a}}x) <_{\mathbf{u}} (1_{\mathbf{a}}z) <_{\mathbf{u}} (1_{\mathbf{a}}y)$. Iterating this argument we find a pair which is adjacent in $\mathbf{a}$.

Let t denote the simple letter to the left of $\mathbf{a}$ in $\mathbf{u}$ (we may assume without any loss that such a letter exists). We multiply both sides of $\mathbf{u} \approx \mathbf{v}$ by y on the right, erase all multiple letters in $\mathbf{u} \approx \mathbf{v}$ other than x and y , erase all simple letters on the right of block $\mathbf{a}$ and some simple letters on the left of $\mathbf{a}$ depending on which of the two cases takes place.

Case 1: some block $\mathbf{q}$ on the left of $\mathbf{a}$ contains both x and y .

In this case, the identity $xy\mathbf{u}y \approx xy\mathbf{v}y$ implies some identity which is equivalent modulo $\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2\}$ to ϕ (see Observation 5.8).

Since $\mathbf{q}$ contains both x and y , Corollary 4.4 implies that using $\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2\}$ the block $\mathbf{q}$ can be modified to some block which contains xy as a subword. So, we may assume that $\mathbf{q}$ contains xy as a subword. Then, since $(1_{\mathbf{a}}x) \ll_{\mathbf{u}} (1_{\mathbf{a}}y)$, we can apply ϕ to $\mathbf{u}$ so that the image of x is a power of x and the image of y is a power of y . As a result, we obtain $\mathbf{u}_1$ such that $(1_{\mathbf{a}_1}y) \ll_{\mathbf{u}_1} (1_{\mathbf{a}_1}x)$, where $\mathbf{a}_1$ is the block of $\mathbf{u}_1$ corresponding to $\mathbf{a}$.

Case 2: no block on the left of $\mathbf{a}$ contains both x and y .

In this case, the identity $\mathbf{u}y \approx \mathbf{v}y$ implies some identity which is equivalent modulo $\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2\}$ to an identity $(\mathbf{r} \approx \mathbf{s}) \in \Phi$.

Since $(1_{\mathbf{a}}x) \ll_{\mathbf{u}} (1_{\mathbf{a}}y)$, we can apply $\mathbf{r} \approx \mathbf{s}$ to $\mathbf{u}$ so that the image of x is a power of x and the image of y is a power of y . As a result, we obtain $\mathbf{u}_1$ such that $(1_{\mathbf{a}_1}y) \ll_{\mathbf{u}_1} (1_{\mathbf{a}_1}x)$, where $\mathbf{a}_1$ is the block of $\mathbf{u}_1$ corresponding to $\mathbf{a}$.

Since in every case we have $|\text{Dist}(\mathbb{Q}^1 \rightarrow \mathbb{E}^1)(\mathbf{u}_1 \approx \mathbf{v})| < |\text{Dist}(\mathbb{Q}^1 \rightarrow \mathbb{E}^1)(\mathbf{u} \approx \mathbf{v})|$, Lemma 6.1 implies that every identity of $\mathbb{V}$ can be derived from $(\Phi \cup \{\phi\}) \cap \text{Id}(\mathbb{V})$ and an identity of $\mathbb{V} \vee \mathbb{E}^1$. $\square$

The following theorem generalizes Proposition 5.11(i) in [12] which says that the variety $\mathbb{E}^1$ is HFB.

Theorem 6.3. *Every subvariety of $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$ is FB.*

Proof. Let $\mathbb{V}$ be a subvariety of $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$. If $\mathbb{V}$ does not contain $\mathbb{Q}^1$, then $\mathbb{V}$ contains neither $\mathbb{E}^1\{\sigma_2\}$ nor $\overline{\mathbb{A}^1}$. Then $\mathbb{V}$ is FB by Lemma 3.4. Hence we may assume that $\mathbb{V}$ contains $\mathbb{Q}^1$. If either $\mathbb{L}_2^1$ or $\mathbb{R}_2^1$ is not contained in $\mathbb{V}$, then $\mathbb{V}$ is a subvariety of $\mathbb{E}^1$ or $\overline{\mathbb{E}^1}$ by Fact 5.3 or the dual to it. Since $\mathbb{E}^1$ is HFB by Proposition 5.11(i) in [12], we may assume that $\mathbb{V}$ contains $\mathbb{L}_2^1 \vee \mathbb{R}_2^1$.

By Fact 6.2, every identity of $\mathbb{V}$ follows within $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$ from $(\Phi \cup \{\phi\}) \cap \text{Id}(\mathbb{V})$ and an identity of $\mathbb{V} \vee \mathbb{E}^1$. The dual of Fact 6.2 implies that every identity of $\mathbb{V} \vee \mathbb{E}^1$ follows within $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$ from $(\overline{\Phi} \cup \{\overline{\phi}\}) \cap \text{Id}(\mathbb{V} \vee \mathbb{E}^1) = (\overline{\Phi} \cup \{\overline{\phi}\}) \cap \text{Id}(\mathbb{V})$ and an identity of $\mathbb{V} \vee \mathbb{E}^1 \vee \overline{\mathbb{E}^1} = \mathbb{E}^1 \vee \overline{\mathbb{E}^1}$. Hence every identity of $\mathbb{V}$ follows within $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$ from $((\Phi \cup \{\phi\}) \cap \text{Id}(\mathbb{V})) \cup ((\overline{\Phi} \cup \{\overline{\phi}\}) \cap \text{Id}(\mathbb{V})) = (\Phi \cup \overline{\Phi} \cup \{\phi, \overline{\phi}\}) \cap \text{Id}(\mathbb{V})$. Clearly, any subset of $\Phi \cup \overline{\Phi} \cup \{\phi, \overline{\phi}\}$ is equivalent to some its finite subset. Therefore, $\mathbb{V}$ is FB. $\square$

Since $\mathbb{Q}^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$ satisfies $xy^2tx \approx xy^2xtx$ and $xtxy^2 \approx xtyxy$ by Facts 3.1 and 3.2, the proof of Theorem 6.3 and Observation 5.8 give us the following.

Example 6.4. *The variety*

$$\begin{aligned} \mathbb{Q}^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1 &= \\ &= \text{var}\{xtx \approx xtx^2 \approx x^2tx, (xy)^2 \approx x^2y^2, x^2y^2tx \approx xy^2xtx, xtx^2y^2 \approx xtyx^2y\} \\ &= \text{var}\{xtx \approx xtx^2 \approx x^2tx, (xy)^2 \approx x^2y^2, x^2y^2ty \approx xy^2xty, ytx^2y^2 \approx ytyx^2y\} \end{aligned}$$

is HFB. $\square$

7 New limit variety of monoids

For a set of identities Σ and $k > 0$, we use Σ_k to denote the set of all identities from Σ which involve at most k letters.

Fact 7.1 ([22, Fact 2.1]; see also [26, Section 4.2]). *Let $\mathbb{V} = \text{var } \Sigma$ be a semigroup variety and Σ a set of identities. Suppose that for infinitely many n , the variety $\mathbb{V}$ satisfies an identity $\mathbf{U}_n \approx \mathbf{V}_n$ in at least n letters such that $\mathbf{U}_n$ has some Property (P_n) but $\mathbf{V}_n$ does not. Suppose that for every word $\mathbf{U}$ such that $\mathbb{V}$ satisfies $\mathbf{U} \approx \mathbf{U}_n$ and $\mathbf{U}$ has Property (P_n) , for every substitution $\Theta: \mathfrak{A} \rightarrow \mathfrak{A}^+$ and every identity $\mathbf{u} \approx \mathbf{v} \in \Sigma_k$ such that $\Theta(\mathbf{u}) = \mathbf{U}$ and $k < n/2$, the word $\Theta(\mathbf{v})$ also has property (P_n) . Then $\mathbb{V}$ is NFB.*

If $\mathbf{U} = \Theta(\mathbf{u})$ for some substitution $\Theta: \mathfrak{A} \rightarrow \mathfrak{A}^+$ and ${}_i\mathbf{U}x$ is an occurrence of a letter x in $\mathbf{U}$, then $\Theta_{\mathbf{u}}^{-1}({}_i\mathbf{U}x)$ denotes an occurrence ${}_j\mathbf{u}z$ of a letter z in $\mathbf{u}$ such that $\Theta({}_j\mathbf{U}z)$ regarded as a subword of $\mathbf{U}$ contains ${}_i\mathbf{U}x$.

Recall that β is the fully invariant congruence of $\mathbb{E}^1$. Let $\overline{\beta}$ denote the congruence dual to β .

Sufficient Condition 7.2. *Let $\mathbb{V}$ be a monoid variety that satisfies the identity*

$$\mathbf{U}_n = xy_1^2y_2^2 \cdots y_{n-1}^2y_n^2x \approx xy_1^2xy_2^2 \cdots y_{n-1}^2xy_n^2x = \mathbf{V}_n \quad (1)$$

for any $n \geq 1$. If the sets $[ab^2ta]_{\overline{\beta}}$, $[atb^2a]_{\beta}$ and aa^+bb^+ are stable with respect to $\mathbb{V}$, then $\mathbb{V}$ is NFB.

Proof. Consider the following property of a word $\mathbf{U}$ with $\text{con}(\mathbf{U}) = \{x, y_1, \dots, y_n\}$:

(P) there is no x in $\mathbf{U}$ between the last occurrence of y_1 and the first occurrence of y_n .

Notice that $\mathbf{U}_n$ satisfies Property (P) but $\mathbf{V}_n$ does not.

Let $\mathbf{U}$ be such that $\mathbb{V}$ satisfies $\mathbf{U}_n \approx \mathbf{U}$. Since aa^+bb^+ is stable with respect to $\mathbb{V}$, we have:

$$({}_1\mathbf{U}x) <_{\mathbf{U}} ({}_{\ell}\mathbf{U}y_1) <_{\mathbf{u}} ({}_1\mathbf{U}y_2) <_{\mathbf{U}} ({}_{\ell}\mathbf{U}y_2) <_{\mathbf{U}} \cdots <_{\mathbf{U}} ({}_{\ell}\mathbf{U}y_{n-1}) <_{\mathbf{U}} ({}_1\mathbf{U}y_n) <_{\mathbf{U}} ({}_{\ell}\mathbf{U}x). \quad (2)$$

Let $\mathbf{u} \approx \mathbf{v}$ be an identity of $\mathbb{V}$ in less than $n/2$ letters and let $\Theta : \mathfrak{A} \rightarrow \mathfrak{A}^+$ be a substitution such that $\Theta(\mathbf{u}) = \mathbf{U}$. In view of (2), the following holds:

(*) for $t \in \text{con}(\mathbf{u})$, if $\Theta(t)$ contains both y_i and y_j for some $1 \leq i < j \leq n$, then the letter t is simple in $\mathbf{u}$.

Suppose that $\mathbf{U}$ has Property (P). Let us verify that $\mathbf{V} = \Theta(\mathbf{v})$ also has Property (P). To obtain a contradiction, assume that there is an occurrence of x in $\mathbf{V}$ between ${}_{\ell}\mathbf{v}y_1$ and ${}_1\mathbf{v}y_n$. By symmetry, we may assume without any loss that there is an occurrence ${}_k\mathbf{v}x$ of x in $\mathbf{V}$ such that $({}_1\mathbf{v}y_{n/2}) <_{\mathbf{v}} ({}_k\mathbf{v}x) <_{\mathbf{v}} ({}_1\mathbf{v}y_n)$.

Since $\mathbf{u}$ has less than $n/2$ letters, for some $t \in \text{con}(\mathbf{u})$ the word $\Theta(t)$ contains both y_i and y_j for some $1 \leq i < j \leq n/2$. In view of (*), the letter t is simple in $\mathbf{u}$. By Fact 3.6(i), t is an isoterm for $\mathbb{V}$. Hence the letter t is simple in $\mathbf{v}$ as well. Clearly, $\Theta_{\mathbf{v}}^{-1}({}_k\mathbf{v}x) = {}_{p\mathbf{v}}z$ is an occurrence of some letter z in $\mathbf{v}$ such that $\Theta(z)$ contains x . Since the empty word 1 is an isoterm for $\mathbb{V}$ by Fact 3.6(i), the letter z occurs in $\mathbf{u}$ as well.

In view of Fact 2.6 in [22], $\Theta_{\mathbf{u}}^{-1}({}_1\mathbf{u}y_n) = {}_{1\mathbf{u}}y$ and $\Theta_{\mathbf{v}}^{-1}({}_1\mathbf{v}y_n) = {}_{1\mathbf{v}}y'$ for some $y, y' \in \text{con}(\mathbf{u}) = \text{con}(\mathbf{v})$. If $y \neq y'$ then $({}_1\mathbf{u}y) <_{\mathbf{u}} ({}_1\mathbf{u}y')$ but $({}_1\mathbf{v}y') <_{\mathbf{v}} ({}_1\mathbf{v}y)$. This is impossible, because $\mathbf{u}(y, y')$ is β -term for $\mathbb{V}$ by Fact 3.6(i). Thus $y = y'$.

Further, in view of (2) and the fact that aa^+bb^+ is stable with respect to $\mathbb{V}$, we have:

$$({}_1\mathbf{v}x) <_{\mathbf{v}} ({}_{\ell}\mathbf{v}y_1) <_{\mathbf{u}} ({}_1\mathbf{v}y_2) <_{\mathbf{v}} ({}_{\ell}\mathbf{v}y_2) <_{\mathbf{v}} \cdots <_{\mathbf{v}} ({}_{\ell}\mathbf{v}y_{n-1}) <_{\mathbf{v}} ({}_1\mathbf{v}y_n) <_{\mathbf{v}} ({}_{\ell}\mathbf{v}x).$$

Hence if $\Theta(z)$ is not a power of x , then $\Theta(z)$ contains either xy_i for some $n/2 \leq i \leq n$ or y_jx for some $n/2 \leq j < n$. This is impossible, because $\Theta(z)$ is a subword of $\mathbf{U}$ and $\mathbf{U}$ has Property (P) and (2). Therefore, $\Theta(z)$ is a power of x . Then $z \neq y$ and $z \neq t$. Since $({}_1\mathbf{v}y_{n/2}) <_{\mathbf{v}} ({}_k\mathbf{v}x) <_{\mathbf{v}} ({}_1\mathbf{v}y_n)$, we have $({}_{\mathbf{v}}t) <_{\mathbf{v}} ({}_{p\mathbf{v}}z) <_{\mathbf{v}} ({}_1\mathbf{v}y)$. In particular, $y \neq t$.

Since $\mathbf{U}$ has Property (P), no z occurs between t and ${}_{1\mathbf{u}}y$ in $\mathbf{u}$. Hence $\mathbf{u}(z, y, t) \in z^*ty\{y, z\}^*$. On the other hand, $\mathbf{v}(z, y, t) \in z^*tz\{y, z\}^*$. This is impossible, because $\mathbf{u}(z, y, t)$ is β -term for $\mathbb{V}$ by Fact 3.6(ii). We conclude that $\mathbf{V}$ must also satisfy Property (P). Therefore, $\mathbb{V}$ is NFB by Fact 7.1. $\square$

Corollary 7.3. *Every monoid variety $\mathbb{V}$ that contains $\mathbb{A}_0^1 \vee \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$ and is contained in $\mathbb{A}_0^1 \vee \mathbb{E}^1 \vee \overline{\mathbb{E}^1}$ is NFB.*

Proof. Facts 3.2 and 3.3 and the dual to Fact 3.3 imply that $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$ satisfies $\mathbf{U}_n \approx \mathbf{V}_n$ for each $n \geq 1$. The variety $\mathbb{A}_0^1$ also satisfies $\mathbf{U}_n \approx \mathbf{V}_n$ by Proposition 4.2 in [21].

Since $\mathbb{V}$ contains $\overline{\mathbb{E}^1\{\sigma_2\}}$, $\mathbb{E}^1\{\sigma_2\}$ and $\mathbb{A}_0^1$, the sets $[ab^2ta]_{\beta}$, $[atb^2a]_{\beta}$ and aa^+bb^+ are stable with respect to $\mathbb{V}$ by Fact 3.5 and the dual of Fact 3.5(ii). Hence $\mathbb{V}$ is NFB by the Sufficient Condition 7.2. $\square$

Fact 7.4. *Let $\mathbb{V}$ be a variety satisfying $xtx \approx xtx^2 \approx x^2tx$. If $\mathbb{V}$ does not contain $\mathbb{A}_0^1$, then $\mathbb{V}$ satisfies $x^2y^2 \approx (xy)^2$.*

Proof. Since $\mathbb{V}$ does not contain $\mathbb{A}_0^1$, Fact 3.5(i) implies that $\mathbb{V}$ satisfies an identity $x^2y^2 \approx \mathbf{u}$ such that $\mathbf{u}$ contains yx as a subword. Together with $xtx \approx xtx^2 \approx x^2tx$ this implies $x^2y^2 \approx (xy)^2$. $\square$

Theorem 7.5. *The variety $\mathbb{A}_0^1 \vee \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$ is limit and is different from all the previously found limit varieties.*

Proof. The variety $\mathbb{A}_0^1 \vee \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$ is NFB by Corollary 7.3.

Since $\mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$ is a subvariety of

$$\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \stackrel{\text{Example 5.7}}{=} \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xs^2z^2x \approx xs^2xz^2x\},$$

it satisfies $xs^2z^2x \approx xs^2xz^2x$. Since $\mathbb{A}_0^1$ satisfies the identity $xszx \approx xsxzx$ (see [3, Proposition 3.2] or [21, Proposition 4.2]), the variety $\mathbb{A}_0^1 \vee \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$ satisfies $\{xtx \approx xtx^2 \approx x^2tx, xs^2z^2x \approx xs^2xz^2x\}$.

Let $\mathbb{V}$ be a proper subvariety of $\mathbb{A}_0^1 \vee \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$. If $\mathbb{V}$ does not contain either $\mathbb{E}^1\{\sigma_2\}$ or $\overline{\mathbb{E}^1\{\sigma_2\}}$, then $\mathbb{V}$ is FB by Lemma 3.4. Hence we may assume that $\mathbb{V}$ does not contain $\mathbb{A}_0^1$. Then $\mathbb{V}$ satisfies $x^2y^2 \approx (xy)^2$ by Fact 7.4. We see that $\mathbb{V}$ is a subvariety of $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$. Now Theorem 6.3 applies, yielding that the variety $\mathbb{V}$ is HFB.

Therefore, $\mathbb{A}_0^1 \vee \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$ is a limit variety. It is different from $\mathbb{A}^1 \vee \overline{\mathbb{A}^1}$, $\mathbb{A}^1 \vee \mathbb{E}^1\{\sigma_2\}$ and $\overline{\mathbb{A}^1} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$ because it contains neither $\mathbb{A}^1$ nor $\overline{\mathbb{A}^1}$. No other limit variety mentioned Section 2 contains $\mathbb{A}_0^1$ as a subvariety. $\square$

8 New Sorting Lemma

The following lemma is a combination of Sorting lemma 2 in [9] and Theorem 5.11 in [10].

Sorting Lemma 1. *Let $\mathbb{V}$ be a variety of aperiodic monoids. Then either $\mathbb{V}$ is HFB or one of the following holds:*

- (i) $\mathbb{V}$ contains one of the 11 varieties: $\mathbb{A}^1 \vee \overline{\mathbb{A}^1}$, $\mathbb{J}$, $\overline{\mathbb{J}}$, $\mathbb{J}_1$, $\overline{\mathbb{J}_1}$, $\mathbb{J}_2$, $\overline{\mathbb{J}_2}$, $\mathbb{K}$, $\overline{\mathbb{K}}$, $\mathbb{M}(\{xzyty\})$, $\mathbb{M}(\{xyzxy, xzytx\})$.
- (ii) $\mathbb{V}$ satisfies either $\{xtx \approx xtx^2, xy^2tx \approx (xy)^2tx\}$ or dually, $\{xtx \approx x^2tx, xty^2x \approx xt(yx)^2\}$. $\square$

Fact 8.1. $\mathbb{Q}^1 \vee \mathbb{R}_3^1 = \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}} \vee \mathbb{R}_3^1$.

Proof. Since by Figure 4 in [12], the variety $\mathbb{E}^1\{\sigma_2\}$ contains $\mathbb{Q}^1$, we have $\mathbb{Q}^1 \vee \mathbb{R}_3^1 \subseteq \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}} \vee \mathbb{R}_3^1$.

If $[atb^2a]_\beta$ is not stable with respect to $\mathbb{Q}^1 \vee \mathbb{R}_3^1$, then in view of Facts 3.1, 3.2 and 3.3, $\mathbb{Q}^1 \vee \mathbb{R}_3^1$ satisfies an identity $\mathbf{u} \approx \mathbf{v}$ with $\mathbf{u} \in x^+ty^+x^+\{x, y\}^*$ and $\mathbf{v} \in x^+tx^+y^+\{x, y\}^*$ for some $x \neq y \in \text{mul}(\mathbf{u}) = \text{mul}(\mathbf{v})$. Then $r(\mathbf{u}) \in y^+x^+\{x, y\}^*$ but $r(\mathbf{v}) \in x^+y^+\{x, y\}^*$. Since $r(\mathbf{u}) \approx r(\mathbf{v})$ does not hold in $\mathbb{L}_2^1$ by Fact 3.2, the identity $\mathbf{u} \approx \mathbf{v}$ fails in $\mathbb{R}_3^1$ by Fact 4.5. To avoid a contradiction, the β -class $[atb^2a]_\beta$ must be stable with respect to $\mathbb{Q}^1 \vee \mathbb{R}_3^1$.

If $[ab^2ta]_{\bar{\beta}}$ is not stable with respect to $\mathbb{Q}^1 \vee \mathbb{R}_3^1$, then in view of Facts 3.1, 3.2 and 3.3, $\mathbb{Q}^1 \vee \mathbb{R}_3^1$ satisfies an identity $\mathbf{u} \approx \mathbf{v}$ with $\mathbf{u} \in \{x, y\}^*x^+y^+tx^+$ and $\mathbf{v} \in \{x, y\}^*y^+x^+tx^+$ for some $x \neq y \in \text{mul}(\mathbf{u}) = \text{mul}(\mathbf{v})$. Then $r(\mathbf{u}) \in tx^+$ but $r(\mathbf{v}) \in x^+tx^+$. Since $r(\mathbf{u}) \approx r(\mathbf{v})$ does not hold in $\mathbb{L}_2^1$ by Fact 3.2, the identity $\mathbf{u} \approx \mathbf{v}$ fails in $\mathbb{R}_3^1$ by Fact 4.5. To avoid a contradiction, the $\bar{\beta}$ -class $[ab^2ta]_{\bar{\beta}}$ must be stable with respect to $\mathbb{Q}^1 \vee \mathbb{R}_3^1$.

Overall, both $[atb^2a]_\beta$ and $[ab^2ta]_{\bar{\beta}}$ are stable with respect to $\mathbb{Q}^1 \vee \mathbb{R}_3^1$. In view of Fact 3.5(ii) and its dual, we have $\mathbb{Q}^1 \vee \mathbb{R}_3^1 = \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}} \vee \mathbb{R}_3^1$. $\square$

Sorting Lemma 2. *Let $\mathbb{V}$ be a variety of aperiodic monoids. Then either $\mathbb{V}$ is HFB or one of the following holds:*

- (i) $\mathbb{V}$ contains one of the 14 varieties: $\mathbb{A}^1 \vee \overline{\mathbb{A}^1}$, $\mathbb{J}$, $\overline{\mathbb{J}}$, $\mathbb{J}_1$, $\overline{\mathbb{J}_1}$, $\mathbb{J}_2$, $\overline{\mathbb{J}_2}$, $\mathbb{K}$, $\overline{\mathbb{K}}$, $\mathbb{M}(\{xzyty\})$, $\mathbb{M}(\{xyzxy, xzytxy\})$, $\mathbb{A}^1 \vee \mathbb{E}^1\{\sigma_2\}$, $\overline{\mathbb{A}^1} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$, $\mathbb{A}_0^1 \vee \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$.
- (ii) $\mathbb{V}$ satisfies $\{xtx \approx xtx^2 \approx x^2tx, (xy)^2 \approx x^2y^2\}$ and contains either $\mathbb{Q}^1 \vee \mathbb{L}_3^1$ or $\mathbb{Q}^1 \vee \mathbb{R}_3^1$.

Proof. Suppose that $\mathbb{V}$ is not HFB and does not contain any of the fourteen varieties. Then in view of the Sorting Lemma 1, we may assume without any loss that $\mathbb{V}$ satisfies $\{xtx \approx xtx^2, xy^2tx \approx (xy)^2tx\}$. According to Theorem 4.3 in [24], $\mathbb{V}$ does not contain $\mathbb{A}^1$. Then $\mathbb{V}$ contains $\overline{\mathbb{E}^1\{\sigma_2\}}$ by the dual to Lemma 3.4. Since $\mathbb{V}$ does not contain $\overline{\mathbb{A}^1} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$, the variety $\mathbb{V}$ does not contain $\overline{\mathbb{A}^1}$. Now Lemma 3.4 applies, yielding that $\mathbb{V}$ contains $\mathbb{E}^1\{\sigma_2\}$. Hence $\mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$ is a subvariety of $\mathbb{V}$. Since $\mathbb{V}$ does not contain $\mathbb{A}_0^1 \vee \mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$, the variety $\mathbb{V}$ does not contain $\mathbb{A}_0^1$. Therefore, $\mathbb{V}$ satisfies $x^2y^2 \approx (xy)^2$ by Fact 7.4.

Overall, $\mathbb{V}$ satisfies $\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2\}$ and contains $\mathbb{E}^1\{\sigma_2\} \vee \overline{\mathbb{E}^1\{\sigma_2\}}$. If $\mathbb{V}$ contains neither $\mathbb{L}_3^1$ nor $\mathbb{R}_3^1$, then $\mathbb{V}$ is a subvariety of $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$ by Corollary 5.6. Since $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$ is HFB by Theorem 6.3, the variety $\mathbb{V}$ must contain either $\mathbb{L}_3^1$ or $\mathbb{R}_3^1$. In view of Fact 8.1 and its dual, $\mathbb{V}$ contains either $\mathbb{Q}^1 \vee \mathbb{L}_3^1$ or $\mathbb{Q}^1 \vee \mathbb{R}_3^1$. $\square$

Sorting Lemma 2 leaves only two options for the variety $\mathbb{Q}^1 \vee \mathbb{R}_3^1$: to be either limit or HFB. Theorem 9.3 below shows that $\mathbb{Q}^1 \vee \mathbb{R}_3^1$ is HFB.

9 The variety $\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1$ is HFB

Let $\mathbf{u} \approx \mathbf{v}$ be an identity of $\mathbb{E}^1 \vee \overline{\mathbb{E}^1}$. Define $\text{Dist}(\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \rightarrow \mathbb{E}_3^1)$ as the set of all unordered triples $\{\ell_{\mathbf{a}}z, (1z)_{\mathbf{a}}y, (1z)_{\mathbf{a}}x\}$ such that $x, y, z \in \text{con}(\mathbf{u}) = \text{con}(\mathbf{v})$, $\mathbf{a}$ is a block of $\mathbf{u}$ and one of the following holds:

- $(\ell_{\mathbf{a}}z) <_{\mathbf{a}} ((1z)_{\mathbf{a}}y) <_{\mathbf{a}} ((1z)_{\mathbf{a}}x)$ but $(\ell_{\mathbf{b}}z) <_{\mathbf{b}} ((1z)_{\mathbf{b}}x) <_{\mathbf{b}} ((1z)_{\mathbf{b}}y)$;
- $(\ell_{\mathbf{a}}z) <_{\mathbf{a}} ((1z)_{\mathbf{a}}x) <_{\mathbf{a}} ((1z)_{\mathbf{a}}y)$ but $(\ell_{\mathbf{b}}z) <_{\mathbf{b}} ((1z)_{\mathbf{b}}y) <_{\mathbf{b}} ((1z)_{\mathbf{b}}x)$,

where $\mathbf{b}$ is the block of $\mathbf{v}$ corresponding to $\mathbf{a}$.

Example 9.1.

$$\begin{aligned} \mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1 &= \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xyt^2xsy \approx xyxt^2xsy\} \\ &\stackrel{\text{Observation 5.8}}{=} \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xyt^2ysx \approx xyxt^2ysx\}. \end{aligned}$$

Proof. First, notice that $\mathbb{V} = \mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1 \subseteq \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xyt^2xsy \approx xyxt^2xsy\}$ by Facts 3.2, 3.3, 4.7 and the dual results.

Now let $\mathbf{u} \approx \mathbf{v}$ be an identity of $\mathbb{V}$ which fails in $\mathbb{E}_3^1$. Observation 5.4 and Fact 4.7 imply that $\text{Dist}(\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \rightarrow \mathbb{E}_3^1)(\mathbf{u} \approx \mathbf{v})$ is not empty. This means that for some block $\mathbf{a}$ of $\mathbf{u}$ and $x, y, z \in \text{con}(\mathbf{u})$ we have $(\ell_{\mathbf{a}}z) <_{\mathbf{a}} ((1z)_{\mathbf{a}}x) <_{\mathbf{a}} ((1z)_{\mathbf{a}}y)$ but $(\ell_{\mathbf{b}}z) <_{\mathbf{b}} ((1z)_{\mathbf{b}}y) <_{\mathbf{b}} ((1z)_{\mathbf{b}}x)$, where $\mathbf{b}$ is the block of $\mathbf{v}$ corresponding to $\mathbf{a}$. Since $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{R}_3^1$, Fact 4.7 implies that $\ell_{\mathbf{a}}z$ is not the last occurrence of z in $\mathbf{u}$. Let t denote the simple letter adjacent to the left of $\mathbf{a}$ in $\mathbf{u}$ (we may assume without any loss that such t exists). Let

$$(\mathbf{u}t) = (\ell_{\mathbf{u}}z_1) <_{\mathbf{u}} (\ell_{\mathbf{a}}z_2) <_{\mathbf{u}} \cdots <_{\mathbf{u}} (\ell_{\mathbf{a}}z_r) <_{\mathbf{u}} (\ell_{\mathbf{a}}z)$$

be the sequence of all last occurrences of letters between $\mathbf{u}t$ and $\ell_{\mathbf{a}}z$ for some $z_2, \dots, z_r \in \text{con}(\mathbf{a}) = \text{con}(\mathbf{b})$. Since $\mathbf{u} \approx \mathbf{v}$ holds in $\overline{\mathbb{E}^1}$, Fact 3.2(ii) and the dual of Fact 3.3 imply that

$$(\mathbf{v}t) = (\ell_{\mathbf{v}}z_1) <_{\mathbf{v}} (\ell_{\mathbf{b}}z_2) <_{\mathbf{v}} \cdots <_{\mathbf{v}} (\ell_{\mathbf{b}}z_r) <_{\mathbf{v}} (\ell_{\mathbf{b}}z).$$

Choose $1 \leq m \leq r$ to be maximal such that the discussed occurrence of z_m is the last occurrence of z_m in $\mathbf{u}$ and rename $p = z_m$ (it is possible that $p = t = z_1$). Since $\mathbf{u} \approx \mathbf{v}$ holds on $\mathbb{Q}^1$, the corresponding occurrence of p in $\mathbf{v}$ is the last occurrence of p in $\mathbf{v}$. By construction, no letter q between $\ell_{\mathbf{u}}p$ and $\ell_{\mathbf{a}}z$ is the last occurrence of q in $\mathbf{u}$ and no letter q between $\ell_{\mathbf{v}}p$ and $\ell_{\mathbf{b}}z$ is the last occurrence of q in $\mathbf{v}$.

Since $\mathbf{u}(p, x, y) \approx \mathbf{v}(p, x, y)$ holds on $\mathbb{R}_3^1$, Fact 4.7 implies that there is an occurrence of y between $\ell_{\mathbf{u}}p$ and $\ell_{\mathbf{a}}z$ or there is an occurrence of x between $\ell_{\mathbf{v}}p$ and $\ell_{\mathbf{b}}z$. Modulo symmetry we have:

$$(\ell_{\mathbf{v}}p) <_{\mathbf{v}} (\ell_{\mathbf{b}}x) <_{\mathbf{v}} (\ell_{\mathbf{b}}z) <_{\mathbf{v}} ((1z)_{\mathbf{b}}y) <_{\mathbf{v}} ((1z)_{\mathbf{b}}x).$$

Since no letter q between $\ell_{\mathbf{v}}p$ and $\ell_{\mathbf{b}}z$ is the last occurrence of q in $\mathbf{v}$, we can insert x multiple times using $xqt^2xsq \approx xqxt^2xsq$ and $xqt^2qsx \approx xqxt^2qsx$ together with $\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2\}$ until we obtain a word $\mathbf{w}$ with $(\ell_{\mathbf{c}}z) \ll_{\mathbf{w}} (\ell_{1z}\mathbf{c}x)$, where $\mathbf{c}$ is the block in $\mathbf{w}$ corresponding to $\mathbf{b}$.

Since $|\text{Dist}(\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \rightarrow \mathbb{E}_3^1)(\mathbf{w} \approx \mathbf{v})| < |\text{Dist}(\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \rightarrow \mathbb{E}_3^1)(\mathbf{u} \approx \mathbf{v})|$, Lemma 6.1 implies that every identity of $\mathbb{V}$ can be derived from $\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xyt^2xsy \approx xyxt^2xsy\}$ and an identity of $\mathbb{V} \vee \mathbb{E}_3^1 \stackrel{\text{Fact 5.3}}{=} \mathbb{E}_3^1$. In view of Example 5.7(ii), we have $\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1 = \text{var}\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xyt^2xsy \approx xyxt^2xsy\}$. $\square$

Let $\mathbf{u} \approx \mathbf{v}$ be an identity of $\mathbb{Q}^1$. Define $\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})$ as the set of all unordered pairs of occurrences $\{\ell_{\mathbf{a}}x, \ell_{\mathbf{a}}y\}$ such that $x, y \in \text{con}(\mathbf{u}) = \text{con}(\mathbf{v})$, $\mathbf{a}$ is a block of $\mathbf{u}$ and one of the following holds:

- $(\ell_{\mathbf{a}}x) <_{\mathbf{a}} (\ell_{\mathbf{a}}y)$ but $(\ell_{\mathbf{b}}y) <_{\mathbf{b}} (\ell_{\mathbf{b}}x)$;
- $(\ell_{\mathbf{a}}y) <_{\mathbf{a}} (\ell_{\mathbf{a}}x)$ but $(\ell_{\mathbf{b}}x) <_{\mathbf{b}} (\ell_{\mathbf{b}}y)$,

where $\mathbf{b}$ is the block of $\mathbf{v}$ corresponding to $\mathbf{a}$. Notice that Observation 5.4 implies that $\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u} \approx \mathbf{v})$ is empty if and only if $\mathbf{u} \approx \mathbf{v}$ holds on $\overline{\mathbb{E}^1}$.

Lemma 9.2. *Let $\mathbf{u} \approx \mathbf{v}$ be an identity of $\mathbb{Q}^1 \vee \mathbb{R}_3^1$. If $\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u} \approx \mathbf{v})$ is not empty, then one can find a word $\mathbf{w}$ such that $\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1$ satisfies $\mathbf{u} \approx \mathbf{w}$, $\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{w} \approx \mathbf{v}) = \text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u} \approx \mathbf{v})$, and for some block $\mathbf{c}$ of $\mathbf{w}$ and $x, y \in \text{con}(\mathbf{c})$, we have $(\ell_{\mathbf{c}}x) \ll_{\mathbf{w}} (\ell_{\mathbf{c}}y)$ but $(\ell_{\mathbf{b}}y) <_{\mathbf{v}} (\ell_{\mathbf{b}}x)$, where $\mathbf{b}$ is the block of $\mathbf{v}$ corresponding to $\mathbf{c}$.*

Proof. Since $\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u} \approx \mathbf{v})$ is not empty, for some block $\mathbf{a}$ of $\mathbf{u}$ and $x, y \in \text{con}(\mathbf{a})$ we have $(\ell_{\mathbf{a}}x) <_{\mathbf{u}} (\ell_{\mathbf{a}}y)$ but $(\ell_{\mathbf{b}}y) <_{\mathbf{v}} (\ell_{\mathbf{b}}x)$, where $\mathbf{b}$ is the block of $\mathbf{v}$ corresponding to $\mathbf{a}$. We may assume that no letter z between $\ell_{\mathbf{a}}x$ and $\ell_{\mathbf{a}}y$ in $\mathbf{a}$ is the last occurrence of z in $\mathbf{a}$. (Indeed, if $(\ell_{\mathbf{a}}x) <_{\mathbf{u}} (\ell_{\mathbf{a}}z) <_{\mathbf{u}} (\ell_{\mathbf{a}}y)$, then we choose either $\{\ell_{\mathbf{a}}x, \ell_{\mathbf{a}}z\} \in \text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u} \approx \mathbf{v})$ or $\{\ell_{\mathbf{a}}z, \ell_{\mathbf{a}}y\} \in \text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u} \approx \mathbf{v})$ instead.)

Since $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{R}_3^1$, Fact 4.7 implies that $\ell_{\mathbf{a}}x \neq \ell_{\mathbf{u}}x$ and $\ell_{\mathbf{a}}y \neq \ell_{\mathbf{u}}y$. Taking into account Example 9.1, one can apply $xyt^2ysx \approx xyxt^2ysx$ together with $\{xtx \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2\}$ to $\mathbf{u}$ as many times as needed until we obtain a word $\mathbf{w}$ with $(\ell_{\mathbf{c}}x) \ll_{\mathbf{w}} (\ell_{\mathbf{c}}y)$, where $\mathbf{c}$ is the block in $\mathbf{w}$ corresponding to $\mathbf{a}$. Since no two last occurrences of variables in the same block are being switched in the process, we have $\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{w} \approx \mathbf{v}) = \text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u} \approx \mathbf{v})$. $\square$

Proposition 5.6 in [12] implies that

$$\mathbb{Q}^1 \vee \mathbb{R}_2^1 = \text{var}\{xtx \approx txt^2 \approx x^2tx, xy^2tx \approx y^2xtx\}.$$

Compare it to the following.

Theorem 9.3. *The variety*

$$\begin{aligned} \mathbb{Q}^1 \vee \mathbb{R}_3^1 &= \mathbb{E}^1 \vee \mathbb{R}_3^1 = \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xytxsy \approx xytxsy\} \\ &\stackrel{\text{Observation 5.8}}{=} \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xytyxsx \approx xytyxsx\} \end{aligned}$$

is HFB.

Proof. First, let us verify that $\mathbb{Q}^1 \vee \mathbb{R}_3^1 \supseteq \mathbb{E}^1 \vee \mathbb{R}_3^1$. Indeed, let $\mathbf{u} \approx \mathbf{v}$ be an identity of $\mathbb{Q}^1 \vee \mathbb{R}_3^1$. Let $\mathbf{a}$ and $\mathbf{b}$ be the corresponding blocks of $\mathbf{u} \approx \mathbf{v}$ and $t \in \text{sim}(\mathbf{u})$ be such that $\mathbf{u}$ contains $t\mathbf{a}$ as a subword and $\mathbf{v}$ contains $t\mathbf{b}$ as a subword (we may assume without any loss that such t exists). Take any $x, y \in \text{con}(\mathbf{a}) = \text{con}(\mathbf{b})$. Since $\mathbf{u}(t, x, y) \approx \mathbf{v}(t, x, y)$ holds on $\mathbb{R}_3^1$, Fact 4.7 implies that $(1_{\mathbf{a}}x) <_{\mathbf{u}} (1_{\mathbf{a}}y)$ if and only if $(1_{\mathbf{b}}x) <_{\mathbf{v}} (1_{\mathbf{b}}y)$. Consequently $\mathbf{u} \approx \mathbf{v}$ holds on $\mathbb{E}^1$ by Facts 3.2 and 3.3.

Now we verify that

$$\mathbb{V} = \mathbb{E}^1 \vee \mathbb{R}_3^1 \supseteq \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xytxsy \approx xytxsy\}.$$

Indeed, let $\mathbf{u} \approx \mathbf{v}$ be an identity of $\mathbb{V}$ which fails on $\overline{\mathbb{E}^1}$. Observation 5.4 implies that $\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u} \approx \mathbf{v})$ is not empty. By Lemma 9.2 we may assume that for some block $\mathbf{a}$ of $\mathbf{u}$ and $x, y \in \text{con}(\mathbf{u})$ we have $(\ell_{\mathbf{a}}x) \ll_{\mathbf{u}} (\ell_{\mathbf{a}}y)$ but $(\ell_{\mathbf{b}}y) <_{\mathbf{v}} (\ell_{\mathbf{b}}x)$, where $\mathbf{b}$ is the block of $\mathbf{v}$ corresponding to $\mathbf{a}$. Since $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{R}_3^1$, Fact 4.7 implies that $\ell_{\mathbf{a}}x \neq \ell_{\mathbf{u}}x$ and $\ell_{\mathbf{a}}y \neq \ell_{\mathbf{u}}y$.

Hence we can apply either $xytxsy \approx xytxsy$ or $xytyxsx \approx xytyxsx$ (see Observation 5.8) to $\mathbf{u}$ to obtain $\mathbf{u}_1$ such that $\ell_{\mathbf{n}}y \ll \ell_{\mathbf{n}}x$, where $\mathbf{n}$ is the block of $\mathbf{u}_1$ corresponding to $\mathbf{a}$. Since $|\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u}_1 \approx \mathbf{v})| < |\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u} \approx \mathbf{v})|$, Lemma 6.1 implies that every identity of $\mathbb{V}$ can be derived from $\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xytyxsx \approx xytyxsx\}$ and an identity of $\mathbb{V} \vee \overline{\mathbb{E}^1} = \mathbb{E}^1 \vee \mathbb{R}_3^1 \vee \overline{\mathbb{E}^1}$. Since $xytyxsx \approx xytyxsx$ implies $xyt^2ysx \approx xyxt^2ysx$, in view of Example 9.1, we have $\text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xytyxsx \approx xytyxsx\} \subseteq \mathbb{Q}^1 \vee \mathbb{R}_3^1$.

To complete the circle of inclusions, notice that $\mathbb{Q}^1 \vee \mathbb{R}_3^1 \subseteq \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, xytxsy \approx xytxsy\}$ by Facts 3.1 and 4.7.

Since $\mathbb{Q}^1 \vee \mathbb{R}_3^1$ is finitely based, Sorting Lemma 2 implies that it is HFB. $\square$

The next theorem generalizes Theorems 6.3 and 9.3.

Theorem 9.4. *Every subvariety $\mathbb{V}$ of $\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1$ is FB. If $\mathbb{V}$ contains $(\mathbb{Q}^1 \vee \mathbb{R}_3^1)$, then $\mathbb{V}$ coincides with one of the following:*

$$(\mathbb{E}^1 \vee \mathbb{R}_3^1) \subset \mathbb{V}_1 \subset \mathbb{V}_2 \subset \mathbb{V}_3 \subset \cdots \subset \mathbb{V}_{\infty} \subset (\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1),$$

where

$$\begin{aligned} (\mathbb{E}^1 \vee \mathbb{R}_3^1) &\stackrel{\text{Theorem 9.3}}{=} (\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1) \wedge \text{var}\{xyt_1xt_2y \approx xyxt_1xt_2y\}, \\ \mathbb{V}_1 &= (\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1) \wedge \text{var}\{xyt_1xt_2yt_3x \approx xyxt_1xt_2yt_3x\}, \\ \mathbb{V}_2 &= (\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1) \wedge \text{var}\{xyt_1xt_2yt_3xt_4y \approx xyxt_1xt_2yt_3xt_4y\}, \\ &\dots\dots\dots \\ \mathbb{V}_{\infty} &= (\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1) \wedge \text{var}\{xytyx \approx xyxyx\}. \end{aligned}$$

Proof. In view of Sorting Lemma 2, Example 9.1 and Theorem 9.3, in order to show that $\mathbb{V}$ is FB it is enough to assume that

$$(\mathbb{E}^1 \vee \mathbb{R}_3^1) \subset \mathbb{V} \subset (\mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{R}_3^1).$$

Let $\mathbf{u} \approx \mathbf{v}$ be an identity of $\mathbb{V}$ which fails on $\overline{\mathbb{E}^1}$. Observation 5.4 implies that $\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u} \approx \mathbf{v})$ is not empty. By Lemma 9.2 we may assume that for some block $\mathbf{a}$ of $\mathbf{u}$ and $x, y \in \text{con}(\mathbf{u})$ we have $(\ell_{\mathbf{a}}x) \ll_{\mathbf{u}} (\ell_{\mathbf{a}}y)$ but $(\ell_{\mathbf{b}}y) <_{\mathbf{v}} (\ell_{\mathbf{b}}x)$, where $\mathbf{b}$ is the block of $\mathbf{v}$ corresponding to $\mathbf{a}$. Since $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbb{R}_3^1$, Fact 4.7 implies that $\ell_{\mathbf{a}}x \neq \ell_{\mathbf{u}}x$ and $\ell_{\mathbf{a}}y \neq \ell_{\mathbf{u}}y$.

Let t denote the simple letter immediately to the right of $\mathbf{a}$ (and of $\mathbf{b}$). We multiply both sides of $\mathbf{u} \approx \mathbf{v}$ by x on the left, erase all multiple letters in $\mathbf{u} \approx \mathbf{v}$ other than x and y , erase all linear letters on the left of block $\mathbf{a}$ and some linear letters on the right of $\mathbf{a}$ depending on which of the two cases takes place.

Case 1: some block $\mathbf{q}$ on the right of $\mathbf{a}$ contains both x and y .

In this case, we multiply both sides of $\mathbf{u} \approx \mathbf{v}$ by x or y on the right if necessary, and obtain an identity which is equivalent within $\mathbb{Q}^1 \vee \mathbb{B}^1$ to either $xyxxy \approx xytxy$ or $xytxy \approx xytyx$. In view of Observation 5.8, each of these identities together with $\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2\}$ implies $\overline{\phi} : \mathbf{r} = xytxy \approx yxxtxy = \mathbf{s}$.

Since $\mathbf{q}$ contains both x and y , Corollary 4.4 implies that modulo $\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2\}$, the block $\mathbf{q}$ contains xy as a subword. Since $1_{\mathbf{a}}x \ll 1_{\mathbf{a}}y$, we can apply $\overline{\phi}$ to $\mathbf{u}$ so that the image of x is a power of x and the image of y is a power of y . As a result, we obtain $\mathbf{u}_1$ such that $1_{\mathbf{n}}y \ll 1_{\mathbf{n}}x$, where $\mathbf{n}$ is the block of $\mathbf{u}_1$ corresponding to $\mathbf{a}$.

Case 2: no block on the right of $\mathbf{a}$ contains both x and y .

In this case, we obtain an identity which is equivalent within $\mathbb{Q}^1 \vee \mathbb{B}^1$ to some identity $\mathbf{r} \approx \mathbf{s} \in \overline{\Phi}$. Since $1_{\mathbf{a}}x \ll 1_{\mathbf{a}}y$, we can apply $\mathbf{r} \approx \mathbf{s}$ to $\mathbf{u}$ so that the image of x is a power of x and the image of y is a power of y . As a result, we obtain $\mathbf{u}_1$ such that $1_{\mathbf{n}}y \ll 1_{\mathbf{n}}x$, where $\mathbf{n}$ is the block of $\mathbf{u}_1$ corresponding to $\mathbf{a}$.

Since in every case we have $|\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u}_1 \approx \mathbf{v})| < |\text{Dist}(\mathbb{Q}^1 \rightarrow \overline{\mathbb{E}^1})(\mathbf{u} \approx \mathbf{v})|$, Lemma 6.1 implies that every identity of $\mathbb{V}$ can be derived from $\mathbf{r} \approx \mathbf{s}$ and an identity of $\mathbb{V} \vee \overline{\mathbb{E}^1} = \mathbb{E}^1 \vee \mathbb{R}_3^1 \vee \overline{\mathbb{E}^1}$. In view of Example 9.1, we have $\mathbb{V} = \text{var}\{xtx \approx xtx^2 \approx x^2tx, x^2y^2 \approx (xy)^2, x^2y^2 \approx (xy)^2, xytx^2xy \approx xytx^2xy, \mathbf{r} \approx \mathbf{s}\}$. $\square$

In view of Sorting Lemma 2 and Theorem 9.3, if there exists any other limit variety of aperiodic monoids, then it satisfies $\{xtx \approx xtx^2 \approx x^2tx, (xy)^2 \approx x^2y^2\}$ and properly contains either $\mathbb{Q}^1 \vee \mathbb{L}_3^1$ or $\mathbb{Q}^1 \vee \mathbb{R}_3^1$.

Question 1. Is every monoid satisfying the identities $xtx \approx xtx^2 \approx x^2tx$ and $x^2y^2 \approx (xy)^2$ finitely based?

Question 1 is equivalent to asking whether for each $k > 3$ every subvariety of $\mathbb{Q}^1 \vee \mathbb{L}_k^1 \vee \mathbb{R}_k^1$ is finitely based. This motivates the following conjecture. Compare it with Example 6.4.

Conjecture 9.5. *The variety*

$$\mathbb{Q}^1 \vee \mathbb{L}_3^1 \vee \mathbb{R}_3^1 \stackrel{\text{Theorem 9.3}}{=} \mathbb{E}^1 \vee \overline{\mathbb{E}^1} \vee \mathbb{L}_3^1 \vee \mathbb{R}_3^1 = \text{var} \left\{ \begin{array}{l} txt \approx txt^2 \approx x^2tx, x^2y^2 \approx (xy)^2, \\ xyzxytz \approx xzyxytz, \\ ztxyzxy \approx ztxyzxy \end{array} \right\}$$

is HFB.

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