STABILITY OF ASYMPTOTICALLY CONICAL GRADIENT KÄHLER-RICCI EXPANDERS

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Abstract. In this work, we consider a perturbation of an asymptotically conical gradient expanding Kähler-Ricci soliton metric g in the same Kähler class. We demonstrate that, under suitable assumptions, the normalized Kähler-Ricci flow starting from the initial perturbed metric exists for all time and converges uniformly to an asymptotically conical gradient expanding Kähler-Ricci soliton metric g_∞. Moreover, if the perturbed initial metric is asymptotic to g at spatial infinity, then the limiting metric coincides with the original soliton, that is, $g_\infty = g$.

1. Introduction

1.1. **Overview.** In this article, we study the stability of the Kähler–Ricci flow on non compact manifolds admitting a Kähler–Ricci soliton metric. As a natural generalization of Kähler–Einstein metrics, Kähler–Ricci solitons play a central role in complex geometry. An *expanding Kähler-Ricci soliton* is a triple (M, g, X), where M is a complex manifold with complex structure J and a complete Kähler metric g and a complete real-holomorphic vector field X satisfying the equation

$$\frac{1}{2}\mathcal{L}_X g = \mathrm{Ric}(g) + g. \tag{1.1}$$

If $X = \nabla^g f$ for some real-valued smooth function f on M, then we say that (M, g, X) is *gradient*. In this case, the soliton equation (1.1) reduces to

$$\mathrm{Hess}_g(f) = \mathrm{Ric}(g) + g, \tag{1.2}$$

if ω is the Kähler form of g and ρ_ω is the Ricci form of ω, we rewrite (1.2) as:

$$i\partial\bar{\partial} f = \rho_\omega + \omega. \tag{1.3}$$

For a gradient Kähler-Ricci soliton (M, g, X), the vector field X is called the *soliton vector field*. Its completeness is guaranteed by the completeness of g [22]. The smooth real-valued function f satisfying $X = \nabla^g f$ is called the *soliton potential*. It is unique up to a constant.

A Kähler-Ricci expander plays an essential role in the analysis of Kähler-Ricci flow, because each Kähler-Ricci expander provides a *self-similar* solution to Kähler-Ricci flow (see Proposition 2.1).

The first nontrivial gradient expanding Kähler-Ricci soliton is the Gaussian soliton $(\mathbb{C}^n, g_{\mathrm{eucl}}, r\partial_r)$, where the self-similar solution $g(t)$ is static and converges as $t \to 0^+$ to the Kähler cone $(\mathbb{C}^n, g_{\mathrm{eucl}})$. Another example is due to Cao [3], who for any $\lambda > 1$ constructed a complete soliton g_λ with positive bisectional curvature on $\mathbb{C}^n$; its flow $g_\lambda(t)$ converges locally smoothly to the conical metric $\partial\bar{\partial}(|\cdot|^{\frac{2}{\lambda}})$ on $\mathbb{C}^n \setminus \{0\}$.

Feldman–Ilmanen–Knopf [15] extended this picture by constructing, for $k > n \geq 2$, complete gradient expanding solitons on the line bundle $\mathcal{O}(-k)$ over $\mathbb{CP}^{n-1}$. Their associated flow converges as $t \to 0^+$ to the Kähler cone $(\mathbb{C}^n/\mathbb{Z}^k,\ i\partial\bar{\partial}\left(\frac{|\cdot|^{2p}}{p}\right))$, with $p > 0$, away from the zero section. In this case, the soliton carries positive scalar curvature.

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We observe that the above examples describe an important class of Kähler-Ricci solitons modeled by Kähler cones. Inspired by these examples, we define the notion of an asymptotically conical gradient Kähler-Ricci expander, following the work of Conlon-Deruelle-Sun [13].

Definition 1.1 (Asymptotically conical gradient Kähler-Ricci expander)**.** An asymptotically conical gradient Kähler-Ricci expander is a triple (M, g, X) being a complete expanding gradient Kähler-Ricci soliton of complex dimension $n \geq 2$ with complex structure J whose curvature $\mathrm{Rm}(g)$ satisfies

$$\sup_{x \in M} |(\nabla^g)^k \operatorname{Rm}(g)|_g(x) d_g(p,\, x)^{2+k} < \infty \quad \text{for all } k \in \mathbb{N}_0,$$

where $d_g(p,\, \cdot)$ denotes the distance to a fixed point $p \in M$ with respect to g.

It can be verified that the above three examples (Gaussian soliton, Cao's solitons and FIK solitons) are asymptotically conical gradient Kähler-Ricci expanders.

1.2. **Main results.** Let (M, g, X) be an asymptotically conical gradient Kähler-Ricci expander. In this article, we perturb this metric g by adding a term of the form $\partial\bar{\partial}\psi_0$ and investigate the resulting metric's evolution under the Kähler-Ricci flow.

1.2.1. *Statement of main theorems.* Our main results are as follows:

Theorem A (Convergence theorem)**.** *Let (M, g, X) be an asymptotically conical gradient Kähler-Ricci expander with asymptotic Kähler cone (C, g_0). Let f be the normalized soliton potential as in Lemma 2.5.*

*Assume that ψ_0 is a smooth real-valued function on M satisfying **Condition II** (see Definition 1.6). Then there exists a smooth function $\psi \in C^\infty(M \times [0, \infty); \mathbb{R})$ such that $g_\psi(\tau) := g + \partial\bar{\partial}\psi(\tau)$ is a complete immortal solution to the normalized Kähler-Ricci flow (see Definition 2.2) for $\tau \geq 0$, starting from the initial metric $g_{\psi_0} := g + \partial\bar{\partial}\psi_0$.*

Moreover,

(a) *the initial metric g_{ψ_0} is asymptotic conical with a unique asymptotic Kähler cone (C, g_0');*
(b) *as $\tau \to \infty$, the solution $g_\psi(\tau)$ converges smoothly and uniformly to an asymptotically conical gradient Kähler-Ricci expander (M, g_∞, X). More precisely,*
    (i) *for every $k \in \mathbb{N}_0$, there exists a constant $C_k > 0$ such that for all $(x, \tau) \in M \times [0, \infty)$,*

$$\begin{aligned} &\left| \left(\nabla^{g_\psi(\tau)}\right)^k \operatorname{Rm}(g_\psi(\tau)) \right|_{g_\psi(\tau)} (x) \leq C_k f(x)^{-1-\frac{k}{2}}; \\ &\left| (\nabla^{g_\psi(\tau)})^k \left(g_\psi(\tau) - g_\infty\right) \right|_{g_\psi(\tau)} (x) \leq C_k e^{-\tau} f(x)^{-1-\frac{k}{2}}. \end{aligned} \tag{1.4}$$

    (ii) *The expanding soliton (M, g_∞, X) is the unique (up to biholomorphisms) asymptotically conical gradient Kähler-Ricci expander with asymptotic cone (C, g_0').*
(c) *Furthermore, if the initial perturbation satisfies*

$$\left| (\nabla^g)^k (g_{\psi_0} - g) \right|_g = o\left(f^{-\frac{k}{2}}\right) \quad \textit{for all } k \in \mathbb{N}_0, \tag{1.5}$$

then the asymptotic cone of (M, g_{ψ_0}) is (C, g_0) and the limiting metric satisfies $g_\infty = g$.

Remark 1.2. The assumption (1.5) guarantees that the asymptotic cone of (M, g_{ψ_0}) is precisely (C, g_0). If, instead of assuming (1.5), we only require that the asymptotic cone of (M, g_{ψ_0}) is (C, g_0), then by Theorem 2.4 we may conclude that (M, g, X) and (M, g_∞, X) differ by a biholomorphism. However, in the absence of the quantitative condition (1.5), we cannot assert that this biholomorphism is the identity.

If the initial perturbation is not necessarily small, we cannot, in general, expect $g_\infty = g$. A simple counterexample is provided by the Gaussian soliton $(\mathbb{C}^n, g_{\mathrm{eucl}})$. Let $\psi_0 = \alpha\frac{r^2}{2}$ for some $\alpha > -1$, where r denotes the radial function. Then $g_\psi(\tau) \equiv g_{\psi_0} = (1+\alpha)g_{\mathrm{eucl}}$ is a solution to the normalized Kähler–Ricci flow as in Theorem A. Hence, $g_\infty = (1+\alpha)g_{\mathrm{eucl}} \neq g_{\mathrm{eucl}}$ unless $\alpha = 0$.

From a dynamical systems perspective, Theorem A implies that the moduli space of solutions to the normalized Kähler–Ricci flow has infinitely many fixed points. Each fixed point, appearing as an asymptotically conical gradient Kähler–Ricci expander, is uniquely characterized by its asymptotic Kähler cone, as established by Conlon, Deruelle, and Sun [13]. This result drastically differs from Cao's work (Theorem 1.10): on a closed complex manifold M with $c_1(M) < 0$, the classical normalized Kähler-Ricci flow converges to the unique Kähler-Einstein metric on M.

Remark 1.3. This spatial convergence rate (1.4) is optimal. We consider an initial Kähler potential ψ_0 such that $(\nabla^g)^k\psi_0 = O(f^{-\frac{k}{2}})$ for all $k \in \mathbb{N}_0$, then by Theorem A, the solution to the normalized Kähler-Ricci flow $g_\psi(\tau)$ starting from g_{ψ_0} will converge to $g_\infty = g$. Thus when $\tau = 0$, we have for all $k \in \mathbb{N}_0$,

$$\left|(\nabla^g)^k\left(g_{\psi_0} - g\right)\right| = O(f^{-1-\frac{k}{2}}).$$

In 2005, Chau and Schnürer [7] established one of the first stability results for gradient expanding Kähler–Ricci solitons. They proved that Cao's expander with positive bisectional curvature is stable under smooth perturbations of the soliton potential, provided the perturbed metric remains complete with bounded curvature and decays appropriately outside the unit disk in $\mathbb{C}^n$ ($n \geq 2$). In particular, the Kähler–Ricci flow with such initial data, after rescaling, converges back to the soliton as $t \to \infty$. A natural class of admissible perturbations consists of compactly supported smooth functions.

In particular, as a corollary of our main results (Theorem A), the theorem of Chau and Schnürer extends to general asymptotically conical Kähler–Ricci expanders, namely:

Corollary B. *Let (M, g, X) be an asymptotically conical gradient Kähler–Ricci expander with complex structure J. Suppose ψ_0 is a JX-invariant smooth real-valued function on M with compact support, and assume that the perturbed tensor*

$$g_{\psi_0} := g + \partial\bar{\partial}\psi_0$$

defines a Kähler metric. Then there exists a complete immortal solution $g(t)$ to the Kähler–Ricci flow with initial metric $g(0) = g_{\psi_0}$. Moreover, after a suitable rescaling, $g(t)$ converges uniformly to the original soliton metric g as $t \to \infty$.

Estimating the maximal existence time of the Kähler–Ricci flow is often a crucial step in the analysis. By Theorem A, one obtains an immortal solution to the Kähler–Ricci flow (see Proposition 2.3) whenever the initial Kähler potential satisfies Condition II (see Definition 1.6). The next theorem shows that even under weaker assumptions on the initial data, one can still ensure the existence of an immortal solution to the Kähler–Ricci flow:

Theorem C (Long time existence theorem)**.** *Let (M, g, X) be an asymptotically conical gradient Kähler-Ricci expander. Let f be the normalized soliton potential as in Lemma 2.5. Suppose ψ_0 is a smooth real-valued function on M satisfying **Condition I** (see Definition 1.5).*

Then there exists a smooth function $\psi \in C^\infty(M \times [0, \infty); \mathbb{R})$ such that $g_\psi(\tau) := g + \partial\bar{\partial}\psi(\tau)$ is a complete immortal solution to the normalized Kähler-Ricci flow (see Definition 2.2) for $\tau \geq 0$, starting from the initial metric $g_{\psi_0} := g + \partial\bar{\partial}\psi_0$. Moreover, there exists a constant $C > 1$ such that for all $(x, \tau) \in M \times [0, \infty)$

(i) $|\frac{\partial}{\partial\tau}\psi(x,\tau)| + f(x)|\nabla^{g_\psi(x,\tau)}\frac{\partial}{\partial\tau}\psi(x,\tau)|_{g_\psi(x,\tau)} \leq Ce^{-\tau}$;
(ii) $\frac{1}{C}g(x) \leq g_\psi(x,\tau) \leq Cg(x)$;
(iii) $f(x)|\nabla^g g_\psi(x,\tau)|_g^2 \leq C$;
(iv) $|\operatorname{Rm}(g_\psi(x,\tau))|_{g_\psi(x,\tau)} \leq C$.

And for all $m \in \mathbb{N}^*$, for any $\alpha > 0$ there exists a constant $C_m = C(\dim M, m, \psi_0, \alpha) > 0$ such that

$$\sup_M \left|\left(\nabla^{g_\psi(\tau)}\right)^m \operatorname{Rm}(g_\psi(\tau))\right|_{g_\psi(\tau)} \leq C_m, \quad \textit{for all } \tau \geq \alpha.$$

Remark 1.4. Once we have an immortal solution to the normalized Kähler-Ricci flow, the correspondence between the normalized Kähler-Ricci flow and the Kähler-Ricci flow (Proposition 2.3) allows us to obtain an immortal solution to the Kähler-Ricci flow as well.

For closed complex manifolds, Tian–Zhang's criterion (Theorem 1.9) provides a characterization of the maximal existence time. In 2011, Lott and Zhang [17] characterized the first singularity time for a Kähler-Ricci flow solution on a general complex manifold. Building on [17], Chau, Li, and Tam established an estimate for the maximal existence time of the Kähler–Ricci flow in 2016.

Our Theorem C not only recovers Chau–Li–Tam's Theorem [5, Theorem 2.2] in the context of asymptotically conical gradient Kähler–Ricci expanders, but also strengthens it by providing refined quantitative estimates for the Kähler potential on the spacetime $M \times [1, \infty)$. We need to notice that our proof is self-contained and does not rely on their results.

1.2.2. *Initial conditions.* In the context of Kähler geometry, the key idea is to impose appropriate growth conditions on the Kähler potential. To this end, we introduce two growth conditions.

Definition 1.5 (Condition I)**.** Let (M, g, X) be an asymptotically conical gradient Kähler-Ricci expander. Let f denote the normalized soliton potential as in Lemma 2.5. A real-valued smooth function ψ_0 defined on M satisfies Condition I if

- (i) (Metric condition) the function ψ_0 is strictly $g-$psh, i.e. $g_{\psi_0} := g + \partial\bar{\partial}\psi_0 > 0$. And there exists a constant $C_0 > 1$ such that $\frac{1}{C_0} g \leq g_{\psi_0} \leq C_0 g$;
- (ii) (Killing condition) if J denotes the complex structure of M, then $\mathcal{L}_{JX} g_{\psi_0} = 0$;
- (iii) (Asymptotic condition) at spatial infinity $|\psi_0| = O(f)$, and $|(\nabla^g)^i(\frac{X}{2} \cdot \psi_0 - \psi_0)|_g = O(f^{-\frac{i}{2}})$ for $i = 0, 1$, and

$$|\operatorname{Rm}(g_{\psi_0})|_g + f^{\frac{1}{2}} |\nabla^g g_{\psi_0}|_g = O(1).$$

Definition 1.6 (Condition II)**.** Let (M, g, X) be an asymptotically conical gradient Kähler-Ricci expander. Let f denote the normalized soliton potential as in Lemma 2.5. A real-valued smooth function ψ_0 defined on M satisfies Condition II if

- (i) (Metric condition) the function ψ_0 is strictly $g-$psh, i.e. $g_{\psi_0} := g + \partial\bar{\partial}\psi_0 > 0$;
- (ii) (Killing condition) if J denotes the complex structure of M, then $\mathcal{L}_{JX} g_{\psi_0} = 0$;
- (iii) (Asymptotic condition) at spatial infinity $|\psi_0| = O(f)$, and $|(\nabla^g)^i(\frac{X}{2} \cdot \psi_0 - \psi_0)|_g = O(f^{-\frac{i}{2}})$ for $i = 0, 1$, and

$$|(\nabla^g)^k (g_{\psi_0} - g)|_g = O(f^{-\frac{k}{2}}),$$
$$|(\nabla^g)^k (\mathcal{L}_{\frac{X}{2}} g_{\psi_0} - g_{\psi_0})|_g = O(f^{-1-\frac{k}{2}}), \quad \text{for all } k \in \mathbb{N}_0.$$

Remark 1.7. By Lemma 2.6, the function f is asymptotic to $\frac{1}{2} d_g(p, \cdot)^2$ at infinity, where $d_g(p, \cdot)$ denotes the distance function with respect to g from a fixed point $p \in M$. Hence, we use f to measure the asymptotic behavior in place of the distance function $d_g(p, \cdot)$.

In Section 2, we give geometric interpretations for Kähler potentials satisfying Condition I or II. We show that certain asymptotic properties follow directly as corollaries of the metric condition and the Killing condition (see Remark 2.12). We also show that if ψ_0 satisfies Condition II, then there exists a constant $C_0 > 1$ such that $\frac{1}{C_0} g \leq g_{\psi_0} \leq C_0 g$. (see Remark 2.14)

Moreover, the condition

$$\left|(\nabla^g)^k (g_{\psi_0} - g)\right|_g = O\left(f^{-\frac{k}{2}}\right), \quad \text{for all } k \in \mathbb{N}_0,$$

is automatically satisfied, since

$$\left|(\nabla^g)^k \left(\mathcal{L}_{\frac{X}{2}} g_{\psi_0} - g_{\psi_0}\right)\right|_g = O\left(f^{-1-\frac{k}{2}}\right), \quad \text{for all } k \in \mathbb{N}_0.$$

Furthermore, we prove that if ψ_0 satisfies Condition II, then the corresponding initial metric is asymptotically conical with a unique asymptotic Kähler cone.

Remark 1.8. Since (M, g, X) is an expanding gradient Kähler-Ricci soliton, the Reeb vector field JX naturally constitutes a Killing vector field for g (see [9, Lemma 4.1]). To initiate the normalized Kähler-Ricci flow, we consequently require the initial metric g_{ψ_0} to preserve this Killing symmetry - specifically, that JX remains a Killing vector field for g_{ψ_0}. Given that JX is a real-holomorphic vector field, this condition is equivalent to the vanishing of the $(1,1)$-form $\partial\bar{\partial}(JX \cdot \psi_0)$. In Section 3.2, we show that, without altering the initial metric g_{ψ_0}, one may always assume that the Kähler potential is JX-invariant, i.e. $JX \cdot \psi_0 = 0$. This invariance, together with the associated Killing vector field, allows us to reduce the Kähler–Ricci flow equation to a complex Monge–Ampère equation on the non compact complex manifold M.

We can easily see that Condition II is stronger than Condition I. In fact, the class of functions satisfying Condition I or II is considerably larger than it first appears. Let f be the normalized Hamiltonian potential of X with respect to g, as in Lemma 2.5. Let ψ_0 be a $g-$psh function, and we assume that $JX \cdot \psi_0 = 0$ and

$$(\nabla^g)^k \psi_0 = O(f^{-\frac{k}{2}}) \quad \text{for all } k \in \mathbb{N}_0,$$

then ψ_0 automatically satisfies Condition II. This condition implies that our perturbation is *small*. Under this assumption, the initial metric g_{ψ_0} is asymptotic to g at spatial infinity.

Another example is given by $\psi_0 = \alpha f$ for some small constant α. Thus we have $JX \cdot \psi_0 = \alpha JX \cdot f = 0$. By the soliton equation (1.1), we have

$$g_{\psi_0} = g + \alpha\partial\bar{\partial} f = (1+\alpha) g + \alpha \mathrm{Ric}(g).$$

For sufficiently small α, the initial metric g_{ψ_0} is bi-Lipschitz to g.

Moreover, Lemma 2.5 implies

$$\frac{X}{2} \cdot \psi_0 - \psi_0 = \alpha(|\partial f|_g^2 - f) = -\alpha(R_\omega + n) = O(1).$$

For higher-order terms, we verify that

$$(\nabla^g)^k \psi_0 = O(f^{1-\frac{k}{2}});$$
$$(\nabla^g)^k \left(\frac{X}{2} \cdot \psi_0 - \psi_0\right) = O(f^{-\frac{k}{2}}), \quad \text{for all } k \in \mathbb{N}_0.$$

Thus, in this case, $\psi_0 = \alpha f$ satisfies Condition II.

If ψ_0 is a Kähler potential satisfying Condition I or II. Unlike in previous work, where the perturbation of the canonical metric was assumed to be small, the difference $g_{\psi_0} - g$ is not necessarily small in our setting. Nevertheless, we are still able to study the stability of the Kähler–Ricci flow on asymptotically conical gradient Kähler–Ricci expanders.

1.3. Stability of Kähler-Ricci flow on Kähler-Einstein manifolds.

A central problem in Kähler geometry is the construction of *canonical metrics* on Kähler manifolds via geometric flows. Among these, the Kähler–Ricci flow has proven to be an especially powerful method. A smooth family of Kähler metrics $g(t)_{t\in(0,T)}$ on a complex manifold M is a solution to Kähler-Ricci flow if

$$\frac{\partial}{\partial t} g(t) = -\operatorname{Ric}(g(t)), \quad t \in (0, T).$$

If the maximal existence time is finite, meaning that the Kähler–Ricci flow develops singularities in finite time, an important aspect of its analysis is to study the asymptotic behavior of the flow near these singularities. On the other hand, if the maximal existence time is infinite, it becomes crucial to investigate the limiting metric (if it exists) as time tends to infinity. In 2006, Tian and Zhang [20] established an algebraic criterion for determining the maximal existence time of the Kähler–Ricci flow on closed complex manifolds.

Theorem 1.9 (Tian-Zhang [20])**.** *Let M be a closed complex manifold endowed with Kähler metric g_0. Let $c_1(M)$ denote the first Chern class of M and let $[\omega_0]$ be the Kähler class of g_0. If we define*

$$T := \sup\{t > 0 \mid [\omega_0] - tc_1(M) > 0\},$$

then there exists a unique maximal solution $g(t)$ to Kähler-Ricci flow with $g(0) = g_0$ for $t \in [0, T)$.

A direct application arises on manifolds with negative first Chern class: in 1985, Cao [2] used the Kähler–Ricci flow to construct Kähler–Einstein metrics in this setting.

Theorem 1.10 (Cao [2])**.** *Let M be a closed complex manifold with $c_1(M) < 0$. For any Kähler metric g_0 whose Kähler form lies in $-c_1(M)$, there exists a unique solution $g(t)$ to the Kähler–Ricci flow with $g(0) = g_0$ for all $t \geq 0$. Moreover, as $t \to \infty$, the rescaled metrics $\frac{1}{t}g(t)$ converge smoothly to the unique Kähler–Einstein metric $g_{KE} \in -c_1(M)$.*

On non compact complex manifolds, the analysis becomes considerably more difficult, unless one has detailed knowledge of the underlying geometric structure. In 2005, Chau [4] studied the stability of the Kähler–Ricci flow on a non compact complex manifold M admitting a complete negatively curved Kähler–Einstein metric g_{KE} (i.e. $\mathrm{Ric}(g_{\mathrm{KE}}) + g_{\mathrm{KE}} = 0$) with bounded curvature. He proved that if g_{KE} is perturbed by a term of the form $\partial\bar{\partial}u$, where u is a smooth bounded function, then there exists an immortal solution $g(t)$, $t \in [0, \infty)$, to the Kähler–Ricci flow with initial condition $g(0) = g_{\mathrm{KE}} + \partial\bar{\partial}u$. Moreover, as $t \to \infty$, the rescaled metrics $\frac{1}{t}g(t)$ converge smoothly to g_{KE} on every compact subset of M. On $\mathbb{C}^n$, Chau, Li and Tam [6] proved the stability of Kähler-Ricci flow without supposing that the initial metric has bounded curvature. Their main result is the following: for any complete $U(n)-$invariant initial metric with non-negative bisectional curvature on $\mathbb{C}^n$, the Kähler-Ricci flow admits a $U(n)-$invariant long-time solution and converges, after a suitable rescaling at the origin, to the Euclidean metric. Moreover, the curvature becomes immediately bounded and the bisectional curvature stays non-negative.

1.4. **Strategy of proof.** Our approach builds on Shi's fundamental work [19] concerning the Ricci flow on non compact manifolds. The key technical tool is the Killing vector field JX, which allows us to reduce the (normalized) Kähler-Ricci flow to a complex (normalized) Monge-Ampère equation. To establish both theorems, we develop a parabolic maximum principle tailored to asymptotically conical gradient Kähler-Ricci expanders, enabling us to control essential geometric quantities via their initial values.

For long-time existence (Theorem C), by applying the maximum principle on non-compact manifolds, we derive uniform bounds for the Riemann curvature tensor in terms of the initial data. Under Condition I, we prove that the curvature is bounded along the (normalized) Kähler-Ricci flow. This crucial curvature control, combined with Shi's existence criterion, yields the desired long-time existence of solutions.

To prove the convergence Theorem (Theorem A), we systematically study the evolution equations for all higher-order derivatives $(\nabla^{g_\psi})^k \frac{\partial}{\partial\tau} g_\psi$ $(k \in \mathbb{N}_0)$. Under Condition II, careful integration of these evolution equations ultimately establishes the convergence result.

1.5. **Outline of proof.** In Section 2, we establish some fundamental properties of asymptotically conical gradient Kähler-Ricci expanders that will be essential for our analysis. In the remainder of Section 2, we provide a geometric interpretation of Kähler potentials satisfying Conditions I and II.

Section 3 presents the construction of solutions to the (normalized) Kähler-Ricci flow using Shi's existence theorem. Here, we reduce the tensorial flow equation to an equivalent (normalized) complex Monge-Ampère equation, which serves as the foundation for our subsequent arguments.

The proof of Theorem C is developed in Section 4, where we implement a maximum principle adapted to the non-compact manifold M. Finally, in Section 5, we strengthen our initial assumptions to establish the convergence result in Theorem A.

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2. Preliminaries

2.1. **Self-similar solutions and normalized Kähler-Ricci flow.** A Kähler-Ricci expander plays an essential role in the analysis of Kähler-Ricci flow, because each Kähler-Ricci expander provides a *self-similar* solution to Kähler-Ricci flow.

Proposition 2.1 (Self-similar solution)**.** *Given a complete expanding Kähler–Ricci soliton* (M, g, X)*, one can construct a solution* $g(t)$ *to the Kähler–Ricci flow for all* $t > 0$ *such that* $g(1) = g$*.*

Let $\Phi_X^{\cdot} : M \to M$ *denote the flow generated by the vector field* X*, and define* $\Phi_t := \Phi_X^{-\frac{1}{2}\log t}$ *for all* $t > 0$*. Then, it can be verified that*

$$g(t) := t \cdot \Phi_t^* g, \quad t > 0, \tag{2.1}$$

defines a solution to the Kähler–Ricci flow on $M \times (0, \infty)$*. We refer to* $g(t)_{t>0}$ *as the self-similar solution to the Kähler–Ricci flow associated with* (M, g, X)*.*

As $t \to 0^+$, the self-similar solution $g(t)$ may develop a singularity. It is widely believed that singularities of a Kähler–Ricci flow can be modeled by self-similar solutions. Therefore, understanding the singularities of self-similar solutions is of fundamental importance.

To investigate the long-time behavior of solutions to the Kähler–Ricci flow, it is natural to consider the rescaled flow, namely the normalized Kähler–Ricci flow.

Definition 2.2 (Normalized Kähler-Ricci flow)**.** Let (M, g, X) be an asymptotically conical gradient Kähler-Ricci expander. A smooth family $g(\tau)_{\tau\in[0,T)}$ of Kähler metrics on M is a solution to *normalized Kähler-Ricci flow* if

$$\frac{\partial}{\partial\tau} g(\tau) = \mathcal{L}_{\frac{X}{2}} g(\tau) - \mathrm{Ric}(g(\tau)) - g(\tau). \tag{2.2}$$

Notice that the static flow $g(\tau) = g$ for all $\tau \in \mathbb{R}$ is a solution to normalized Kähler-Ricci flow since (M, g, X) is a Kähler-Ricci expander. Moreover, we have a 1-1 correspondence between solutions to normalized Kähler-Ricci flow and solutions to Kähler-Ricci flow.

Proposition 2.3. *Let* $g(\tau)_{\tau\in[0,T)}$ *be a solution to the normalized Kähler-Ricci flow, then*

$$\bar{g}(t) := t\Phi_t^* g(\log t),$$

for $t \in [1, e^T)$ *is a solution to Kähler-Ricci flow.*

2.2. **Asymptotic conical gradient Kähler-Ricci expanders and their structure theorem.** In this section, we summarize the key properties of asymptotically conical gradient Kähler-Ricci expanders that will be used throughout the remainder of the article. Let (M, g, X) be an asymptotically conical gradient Kähler-Ricci expander as in Definition 1.1, let ω be the Kähler form of g and let $f \in C^\infty(M; \mathbb{R})$ be a Hamiltonian potential of X such that $\nabla^g f = X$. In their paper [13], they established a general structure theorem for an asymptotically conical gradient Kähler-Ricci expander.

Theorem 2.4 (Structure theorem [13])**.** *Let* (M, g, X) *be an asymptotically conical gradient Kähler-Ricci expander. Then there exists a Kähler resolution* $\pi : M \to C$ *of a Kähler cone* (C, g_0) *with exceptional set* E*, such that*

(i) *the map* π *is a biholomorphism between* $M \setminus E$ *and* $C \setminus \{o\}$*, where* o *denotes the apex of the cone* C*;*

(ii) *let $g(t)_{t>0}$ denote the self-similar solution to Kähler-Ricci flow associated to g, then when t goes to 0, $\pi_* g(t)$ converges smoothly locally to g_0 on $C \setminus \{o\}$;*
(iii) *let r denote the radial function on the Kähler cone (C, g_0), we have $d\pi(X) = r\partial_r$;*
(iv) *the asymptotic conical gradient Kähler-Ricci expander (M, g, X) is the unique (up to biholomorphisms) asymptotic conical gradient Kähler-Ricci expander with asymptotic cone (C, g_0).*

A fundamental feature of gradient Kähler–Ricci expanders is encapsulated in the celebrated soliton identities, which play a central role in the analysis of their geometry and dynamics.

Lemma 2.5 (Soliton identities)**.**

$$\begin{aligned}
&\Delta_\omega f = n + R_\omega,\\
&\nabla^g R_\omega + \mathrm{Ric}(g)(X) = 0,\\
&|\partial f|_g^2 + R_\omega = f + \textit{constant}.
\end{aligned}$$

Here $n = \dim_{\mathbb{C}} M$, Δ_ω is the Kähler Laplacian, $R_\omega = \frac{1}{2} R_g$ is the Kähler scalar curvature, R_g is the scalar curvature of g.

Proof. The proof is standard (see [11, Section 2 of Chapter 1]) since $\mathrm{Ric}(g) + g = \partial\bar{\partial} f$. ☐

From now on, we normalize f such that $|\partial f|_g^2 + R_\omega + n = f$.

Like the potential of Gauss' soliton in the Euclidean case, our soliton potential f is comparable to $\frac{d_g(p,\cdot)^2}{2}$ for any fixed point $p \in M$ at spatial infinity.

Lemma 2.6. *The normalized potential f is a proper function, and for a fixed point $p \in M$ there exist constants $C, c_1, c_2 > 0$ such that*

$$\frac{(d_g(p,x) - c_1)^2}{2} - C \le f(x) \le \frac{(d_g(p,x) + c_2)^2}{2}, \quad \forall x \in M.$$

Proof. See [13, Proposition 2.19]. ☐

The normalization that we exploit ensures that our soliton potential f is strictly positive.

Proposition 2.7. *There exists an $\varepsilon > 0$ such that*

$$\inf_M f \ge \varepsilon > 0.$$

Proof. This result follows from the fact that $R_\omega + n \ge \varepsilon > 0$ for some positive constant ε, which holds for any asymptotically conical gradient Kähler-Ricci expander (M, g, X). For the proof of this fact, see [9, Lemma 2.2]. ☐

Our Kähler–Ricci expander (M, g, X) is of gradient type. Such expanders enjoy additional symmetries since they admit a Killing vector field JX. This fact is closely related to the following lemma:

Lemma 2.8. *Let (M, g, J) be a Kähler manifold with complex structure J and Kähler metric g. Then a real holomorphic vector field JX is Killing if X is the gradient of a smooth function with respect to g.*

Proof. We suppose that $X = \nabla^g f$ for some smooth function f. Let Y, Z be two vector fields, now we compute,

$$\begin{aligned}
(\mathcal{L}_{JX} g)(Y, Z) &= g(\nabla^g_Y JX, Z) + g(Y, \nabla^g_Z JX)\\
&= g(J\nabla^g_Y X, Z) + g(Y, J\nabla^g_Z X)\\
&= -g(\nabla^g_Y X, JZ) - g(JY, \nabla^g_Z X)\\
&= -(\nabla^g)^2 f(Y, JZ) - (\nabla^g)^2 f(JY, Z).
\end{aligned}$$

Since X is real-holomorphic, the Hessian $(\nabla^g)^2 f$ is Hermitian. Consequently, the last term vanishes, and we conclude that JX is a Killing vector field with respect to g. ☐

2.3. **Geometric interpretation of Condition I and II.** In this section, we provide a geometric interpretations of Kähler potentials satisfying Condition I or II. First, we show that if ψ_0 satisfies Condition I, then certain properties in the asymptotic condition follow as corollaries of the metric condition and the Killing condition.

Proposition 2.9. *Let (M, g, X) be an asymptotically conical gradient Kähler–Ricci expander with normalized soliton potential f, and let ψ_0 be a JX-invariant Kähler potential, i.e. $JX \cdot \psi_0 = 0$. Suppose there exists a constant $C_0 > 0$ such that*

$$\frac{1}{C_0} g \leq \ g_{\psi_0} \leq C_0 g,$$

where $g_{\psi_0} = g + \partial\bar{\partial}\psi_0$. Then there exists a constant $C > 0$ such that on M, $|\psi_0| \leq Cf$ holds.

Proof. Since X is a real holomorphic vector field, let U denote the holomorphic part of X, so that $X = U + \overline{U}$. Notice that $JX \cdot \psi_0 = 0$, we obtain that

$$X \cdot \psi_0 = 2U \cdot \psi_0 = 2\overline{U} \cdot \psi_0.$$

Locally, the vector field X can be written as

$$X = U + \overline{U} = X^i \partial_i + X^{\bar{j}} \partial_{\bar{j}},$$

with X^i holomorphic and $X^{\bar{j}}$ anti-holomorphic. Then, for any vector field Y of the form

$$Y = Y^i \partial_i + Y^{\bar{j}} \partial_{\bar{j}},$$

we have

$$\begin{aligned}
\partial\bar{\partial}\psi_0(X, Y) &= X^i Y^{\bar{j}} \partial_i \bar{\partial}_{\bar{j}} \psi_0 + Y^i X^{\bar{j}} \partial_i \bar{\partial}_{\bar{j}} \psi_0 \\
&= Y^{\bar{j}} \partial_{\bar{j}} \left(X^i \partial_i \psi_0 \right) + Y^i \partial_i \left(X^{\bar{j}} \partial_{\bar{j}} \psi_0 \right) \\
&= Y^{\bar{j}} \partial_{\bar{j}} \left(\frac{1}{2} X \cdot \psi_0 \right) + Y^i \partial_i \left(\frac{1}{2} X \cdot \psi_0 \right) \\
&= Y \cdot \left(\frac{1}{2} X \cdot \psi_0 \right)
\end{aligned}$$

Therefore, we conclude that

$$g_{\psi_0}(X, \cdot) = g(X, \cdot) + \frac{1}{2} d(X \cdot \psi_0).$$

Since $\frac{1}{C_0} g \leq g_{\psi_0} \leq C_0 g$, then there exists a constant $C_1 > 0$ such that

$$|X \cdot (X \cdot \psi_0)| = | < X, d(X \cdot \psi_0) > | \leq C_1 |X|_g^2 = C_1 X \cdot f.$$

From Lemma 2.6, we know that f is a proper function. Let us define

$$C_2 = \max_{f(x) \leq 1} X \cdot \psi_0(x) - C_1 f(x).$$

Since $X \cdot (X \cdot \psi_0 - C_1 f) \leq 0$, it follows that on M

$$X \cdot \psi_0 - C_1 f \leq C_2.$$

By Proposition 2.7, there exists an $\varepsilon > 0$ such that $f \geq \varepsilon$ on M. Hence

$$X \cdot \psi_0 \leq \left(\frac{C_2}{\varepsilon} + C_1 \right) f := C_3 f.$$

Recall the soliton identity, $f = n + R_\omega + |\partial f|_g^2$, we define $C_4 = n + \sup_M |R_\omega|$, hence $f \leq C_4 + |\partial f|_g^2$. It follows that

$$X \cdot \psi_0 \leq \frac{C_3}{2} X \cdot f + C_3 C_4. \tag{2.3}$$

Let Φ^s denote the flow of X for all $s \in \mathbb{R}$, then (2.3) implies that for all $x \in M$, $s < 0$, we have

$$\left(\psi_0 - \frac{C_3}{2} f\right)(x) - \left(\psi_0 - \frac{C_3}{2} f\right)(\Phi^s(x)) \le -C_3 C_4 s. \tag{2.4}$$

We define $C_5 = \max_{f(y)\le 1}\left(\psi_0 - \frac{C_3}{2} f\right)(y)$. Now we fix $x \in M$ such that $f(x) \ge 1$ and let $s_0 \le 0$ such that $f(\Phi^s(x)) = 1$. It follows that, by (2.4),

$$\left(\psi_0 - \frac{C_3}{2} f\right)(x) \le -C_3 C_4 s_0 + C_5.$$

Now we estimate s_0. Let us define $h(s) = f(\Phi^s(x))$ for $s \le 0$, we compute

$$\frac{d}{ds} h(s) = X \cdot f(\Phi^s(x)) \le 2f(\Phi^s(x)).$$

By integration, we have for $s_0 \le 0$ such that $h(s_0) = f(\Phi^{s_0}(x)) = 1$

$$e^{-2s_0} = e^{-2s_0} h(s_0) \le h(0) = f(x).$$

Thus $-s_0 \le \frac{1}{2}\log f(x) \le f(x)$. Therefore, we have that for $x \in M$ such that $f(x) \ge 1$,

$$\left(\psi_0 - \frac{C_3}{2} f\right)(x) \le C_3 C_4 f(x) + C_5 \le \left(C_3 C_4 + \frac{C_5}{\varepsilon}\right) f(x).$$

For $x \in M$ such that $f(x) \le 1$, by the definition of C_5, we have,

$$\left(\psi_0 - \frac{C_3}{2} f\right)(x) \le C_5 \le \frac{C_5}{\varepsilon} f(x).$$

We conclude that there exists a constant $C > 0$ such that on M, $\psi_0 \le Cf$ holds. Similarly, we can prove $\psi_0 \ge -Cf$. □

Remark 2.10. The bi-Lipschitz condition $\frac{1}{C_0} g \le\ g_{\psi_0} \le C_0 g$ could not control the full covariant derivatives of ψ_0.

Let (C, g_0) be the Kähler cone appearing in Theorem 2.4. It is clear that (C, g_0) is the unique asymptotic cone of (M, g). Let (S, g_S) denote the closed Sasaki manifold such that (C, g_0) is the Kähler cone with link (S, g_S). Suppose ψ_0 is a Kähler potential satisfying Condition I. In fact, Condition I implies that ψ_0 is asymptotic to some function on C at spatial infinity.

In the remainder of this article, we identify $M \setminus E$ with its $C \setminus \{o\}$ via the biholomorphism π as in Theorem 2.4.

Proposition 2.11. *Let r denote the radial function of (C, g_0), let ψ_0 be a Kähler potential satisfying Condition I. Then there exists a function $\psi_0^S \in C^1(S)$ such that at spatial infinity, we have*

$$\begin{aligned} &\left|\frac{2\psi_0}{r^2} - \psi_0^S\right| = O(r^{-2}); \\ &\left|\nabla^{g_S}\left(\frac{2\psi_0}{r^2} - \psi_0^S\right)\right|_{g_S} = O(r^{-2}). \end{aligned} \tag{2.5}$$

Proof. First we recall [9, Lemma 3.4] and [9, Corollary 5.6]: there exist constants $\lambda, C > 0$ such that on $\{r^2 \ge \lambda\}$, we have

$$\begin{aligned} &\frac{1}{C} g \le g_0 \le Cg; \\ &|\nabla^g g_0|_g \le \frac{C}{r}; \\ &\left|f - \frac{r^2}{2}\right| \le C. \end{aligned} \tag{2.6}$$

Since on C, $X = r\partial_r$ thanks to Theorem 2.4, thus $\frac{X}{2}\psi_0 - \psi_0 = O(1), |\nabla^g\left(\frac{X}{2}\cdot\psi_0 - \psi_0\right)|_g = O(f^{-\frac{1}{2}})$ in Condition I as in Definition 1.6 becomes

$$|r\frac{\partial_r\psi_0}{2} - \psi_0| = O(1);$$
$$\left|\nabla^{g_0}\left(r\frac{\partial_r\psi_0}{2} - \psi_0\right)\right|_{g_0} = O(r^{-1}).$$

Define $\kappa := \frac{2\psi_0}{r^2}$, thus at spatial infinity of C, we have,

$$|\partial_r\kappa| = O(r^{-3});$$
$$|\nabla^{g_0}\partial_r\kappa|_{g_0} = O(r^{-4}).$$

Then κ converges uniformly to a continuous function ψ_0^S which is a function defined on S such that $\left|\frac{2\psi_0}{r^2} - \psi_0^S\right| = O(r^{-2})$.

Moreover, $|\nabla^{g_0}\partial_r\kappa|_{g_0} = O(r^{-4})$ implies that $|\nabla^{g_S}\partial_r\kappa|_{g_S} = O(r^{-3})$, thus ψ_0^S is a C^1 function and $\left|\nabla^{g_S}\left(\frac{2\psi_0}{r^2} - \psi_0^S\right)\right|_{g_S} = O(r^{-2})$. □

Remark 2.12. In the proof of Proposition 2.11, we only require the conditions

$$\left|\tfrac{X}{2}\cdot\psi_0 - \psi_0\right| = O(1), \qquad \left|\nabla^g\left(\tfrac{X}{2}\cdot\psi_0 - \psi_0\right)\right|_g = O(r^{-1}).$$

Since [(2.5), Proposition 2.11] holds, and because ψ_0^S is a C^1 function defined on S, it follows that at spatial infinity

$$|\psi_0| = O(r^2), \qquad |\partial\psi_0|_{g_0} = O(r).$$

Moreover, by [(2.6), Proposition 2.11], we conclude that on M

$$|\psi_0| = O(f), \qquad |\partial\psi_0|_g = O(f^{\frac{1}{2}}).$$

Here we reprove $|\psi_0| = O(f)$ by asymptotic condition instead of metric condition. We need to pay attention the fact that from the metric condition, we cannot conclude that $|\frac{X}{2}\cdot\psi_0 - \psi_0| = O(1)$.

Proposition 2.13. *Let ψ_0 be a Kähler potential satisfying Condition II. Then the associated metric g_{ψ_0} is asymptotically conical.*

More precisely, there exists a Sasaki metric g'_S on S such that the corresponding metric space (M, g_{ψ_0}) admits a unique asymptotic Kähler cone (C, g'_0) with link (S, g'_S). Moreover, (C, g'_0) is bi-Lipschitz to (C, g), and there exists a smooth function ψ_0^S, defined on the link S, such that

$$g'_0 \;=\; \partial\bar{\partial}\Big(\tfrac{1+\psi_0^S}{2}\,r^2\Big).$$

Proof. Since the soliton metric g is asymptotically conical (see [13, Theorem A]), indeed, at spatial infinity, we have that

$$|(\nabla^{g_0})^j(g - g_0)|_{g_0} = O(r^{-2-j}), \quad \text{for all } j \in \mathbb{N}_0.$$

Then the asymptotic condition in Definition 1.6 becomes

$$\left|(\nabla^{g_0})^j\left(\mathcal{L}_{\frac{X}{2}}g_{\psi_0} - g_{\psi_0}\right)\right|_{g_0} = O(r^{-2-j}).$$

From Proposition 2.11, we know that ψ_0 is asymptotic to $\psi_0^S\frac{r^2}{2}$ for some C^1 function ψ_0^S. Now we prove the function ψ_0^S has more regularities if ψ_0 satisfies Condition II.

Now we consider $t\Phi_t^*\psi_0$ for all $t > 0$ where Φ_t is the flow of $-\frac{1}{2t}X$ and we compute

$$\frac{\partial}{\partial t}(t\Phi_t^*\psi_0) = \Phi_t^*\left(\psi_0 - \frac{X}{2}\cdot\psi_0\right) = O(1).$$

It follows that for any $x \in C$ such that $r(x) > 0$, $t\Phi_t^*\psi_0(x)$ has a limit. Similarly, by the asymptotic condition in Definition 1.6, for any $\lambda > 0$, on $\{r(x)^2 > \lambda\}$, for all $k \in \mathbb{N}_0$, $(\nabla^{g_0})^k\left(\partial\bar{\partial}(t\Phi_t^*\psi_0)\right)$ converges uniformly when t tends to 0.

Let $(\rho, s) \in (0,\infty)\times S = C$, then $t\Phi_t^*\psi_0(\rho,s) = t\psi_0(\frac{\rho}{\sqrt{t}}, s)$. By Proposition 2.11, we have

$$\left|t\Phi_t^*\psi_0(\rho,s) - \frac{\rho^2}{2}\psi_0^S(s)\right| = \left|t\psi_0(\frac{\rho}{\sqrt{t}}, s) - \frac{\rho^2}{2}\psi_0^S(s)\right| = tO(\frac{t}{\rho^2}).$$

This implies when t goes to 0, the function $t\Phi_t^*\psi_0$ converges to $\frac{r^2}{2}\psi_0^S$. Therefore for any $\lambda > 0$, on $\{r(x)^2 > \lambda\}$, for any $k \in \mathbb{N}_0$ $(\nabla^{g_0})^k\left(\partial\bar\partial(t\Phi_t^*\psi_0)\right)$ converges uniformly to $(\nabla^{g_0})^k\left(\partial\bar\partial\left(\frac{r^2}{2}\psi_0^S\right)\right)$.

For any $\lambda > 0$, on $\{r(x)^2 > \lambda\}$, by Theorem 2.4, the self-similar solution $g(t) = t\Phi_t^* g$ converges smoothly uniformly to $g_0 = \partial\bar\partial\left(\frac{r^2}{2}\right)$. It turns out that for any $\lambda > 0$, on $\{r(x)^2 > \lambda\}$, the metric $t\Phi_t^* g_{\psi_0} = g(t) + \partial\bar\partial(t\Phi_t^*\psi_0)$ converges smoothly uniformly to the metric $g_0' = \partial\bar\partial\left(\frac{1+\psi_0^S}{2}r^2\right)$.

We define $G = \frac{1+\psi_0^S}{2}$, now we prove there exists a Sasaki metric g_S' on S such that (C, g_0') is a Kähler cone with link (S, g_S').

Since JX is a Killing vector field of g_{ψ_0}, in Proposition 3.7, we can assume that $JX\cdot\psi_0 = 0$, that is, $JX\cdot G = 0$ without changing the metric g_{ψ_0}.

First, we compute $\partial\bar\partial G$. Since $JX\cdot G = X\cdot G = 0$, by the same reason as in Proposition 2.9, we conclude that for any vector field Y, $\partial\bar\partial G(X,Y) = Y\cdot(\frac{1}{2}X\cdot G) = 0$. Furthermore $\mathcal{L}_X\partial\bar\partial G = \partial\bar\partial(X\cdot G) = 0$, these facts imply that $\partial\bar\partial G$ is a symmetric 2-tensor which is only defined on S.

Now we are going to find g_S' and the corresponding radial function. Notice that

$$\begin{aligned} g_0' &= G\partial\bar\partial\left(\frac{r^2}{2}\right) + \partial\left(\frac{r^2}{2}\right)\otimes\bar\partial G + \partial G\otimes\bar\partial\left(\frac{r^2}{2}\right) + \frac{r^2}{2}\partial\bar\partial G \\ &= Gdr^2 + Gr^2\left(g_S + \frac{1}{2G}\partial\bar\partial G\right) + r\partial r\otimes\bar\partial G + \partial G\otimes r\bar\partial r. \end{aligned}$$

The metric g_0' is a conical metric, thus we conclude that $G > 0$ and $\left(g_S + \frac{1}{2G}\partial\bar\partial G\right)$ is a Riemannian metric on S. Define $R := \sqrt{G}r$ and $g_S' := g_S + \frac{1}{2G}\partial\bar\partial G$, now we prove

$$g_0' = dR^2 + R^2 g_S'.$$

It suffices to prove that $dr\otimes dG = 2\partial r\otimes\bar\partial G$. Consider Y being a vector field defined on S, then we have

$$dr\otimes dG(X,Y) = rY\cdot G.$$

On the one hand

$$g_0'(X,Y) = r\partial r\otimes\bar\partial G(X,Y) = \frac{r^2}{2} < Y, \bar\partial G > .$$

One the other hand since $JX\cdot R = 0$ and X is real-holomorphic,

$$g_0'(X,Y) = \partial\bar\partial\left(\frac{R^2}{2}\right)(X,Y) = Y\cdot\left(\frac{1}{2}X\cdot R^2\right) = \frac{r^2}{2}Y\cdot G.$$

hence we have $Y\cdot G = < Y, \bar\partial G >$, therefore $dr\otimes dG = 2\partial r\otimes\bar\partial G$.

Observe that $g_0' = dR^2 + R^2 g_S'$, thus $g_0' = (\nabla^{g_0'})^2\frac{R^2}{2}$. At the same time g_0' is Kähler, this implies that the radial vector field $R\partial_R$ of g_0' is real-holomorphic. Now we prove that $R\partial_R = r\partial_r = X$. It suffices to prove the Hamiltonian potential of X with respect to g_0' is $\frac{R^2}{2}$. Suppose that locally we have $X = X^i dz_i + X^{\bar j}dz_{\bar j}$, we compute

$$g_{0,i\bar j}' X^i = X^i\partial_i\partial_{\bar j}\frac{R^2}{2} = \partial_{\bar j}\left(\frac{X}{2}\cdot\frac{R^2}{2}\right) = \partial_{\bar j}\frac{R^2}{2}.$$

Hence we get $r\partial_r = X = R\partial_R$.

Since G is a function which is defined on S, then g_0' is bi-Lipschitz equivalent to g_0, i.e. there exists a constant $C_0 > 0$ such that $\frac{1}{C_0}g_0 \le g_0' \le C_0 g_0$.

We now prove that (M, g_{ψ_0}) is asymptotically conical. In fact, we prove that for all $j \in \mathbb{N}_0$, on $\{r(x)^2 \geq 1\}$,

$$|(\nabla^{g_0'})^j \left(g_{\psi_0} - g_0'\right)|_{g_0'} = O(R^{-2-j}).$$

For any $j \in \mathbb{N}_0$, we have

$$\frac{\partial}{\partial t}\left((\nabla^{g_0})^j \partial\bar{\partial}(t\Phi_t^* g_{\psi_0})\right) = \Phi_t^* \left((\nabla^{g_0})^j \left(\mathcal{L}_{\frac{X}{2}} g_{\psi_0} - g_{\psi_0}\right)\right).$$

By the asymptotic condition in Definition 1.6, we have

$$\left|(\nabla^{g_0})^j \left(\mathcal{L}_{\frac{X}{2}} g_{\psi_0} - g_{\psi_0}\right)\right|_{g_0} = O(r^{-2-j}).$$

We conclude that $\left|(\nabla^{g_0})^j (g_{\psi_0} - g_0')\right|_{g_0} = O(r^{-2-j})$ for all $j \in \mathbb{N}_0$. Thus it is sufficient to show that for any $j \in \mathbb{N}_0$,

$$|(\nabla^{g_0})^j g_0'|_{g_0} = O(r^{-j}).$$

Notice that $g_0' - Gg_0 = r^2 R$, where $R = Gg_S' - Gg_S$ is a symmetric 2-tensor defined on S, now we compute for all $j \in \mathbb{N}_0$, on the one hand,

$$(\nabla^{g_0})^j (g_0' - Gg_0) = (\nabla^{g_0})^j g_0' - g_0(\nabla^{g_0})^j G = (\nabla^{g_0})^j g_0' - (\nabla^{g_0})^j G \otimes g_0.$$

On the other hand,

$$(\nabla^{g_0})^j (g_0' - Gg_0) = (\nabla^{g_0})^j (r^2 R) = \sum_{i \leq j} (\nabla^{g_0})^i r^2 * (\nabla^{g_0})^{j-i} R.$$

Hence we conclude that

$$(\nabla^{g_0})^j g_0' = \sum_{i \leq j} (\nabla^{g_0})^i r^2 * (\nabla^{g_0})^{j-i} R + (\nabla^{g_0})^j G \otimes g_0,$$

and there exists a dimensional constant $C(n) > 0$ such that

$$\left|(\nabla^{g_0})^j g_0'\right|_{g_0} \leq C(n) \sum_{i \leq j} |(\nabla^{g_0})^i r^2|_{g_0} |(\nabla^{g_0})^{j-i} R|_{g_0} + |g_0|_{g_0} |(\nabla^{g_0})^j G|_{g_0} = O(r^{-j}).$$

$\square$

Remark 2.14. In the proof of Proposition 2.13, we relied solely on the assumption

$$\left|(\nabla^g)^j \left(\mathcal{L}_{\frac{X}{2}} g_{\psi_0} - g_{\psi_0}\right)\right|_g = O\left(f^{-1-\frac{j}{2}}\right), \quad j \in \mathbb{N}_0.$$

We now show that if this condition holds, then it follows that

$$\left|(\nabla^g)^j (g_{\psi_0} - g)\right|_g = O\left(f^{-\frac{j}{2}}\right), \quad j \in \mathbb{N}_0.$$

Since for any $j \in \mathbb{N}_0$, $|(\nabla^{g_0})^j \left(g_{\psi_0} - g_0'\right)|_{g_0} = O(r^{-2-j})$, it suffices to show that for any $j \in \mathbb{N}_0$,

$$|(\nabla^{g_0})^j g_0'|_{g_0} = O(r^{-j}).$$

And this is proved at the end of the proof of Proposition 2.13.

Now we show that the metric g_{ψ_0} is bi-Lipschitz to g. Since g_{ψ_0} (resp. g) is asymptotically conical to conical metric g_0' (resp. g_0), and g_0' is bi-Lipschitz to g_0, we conclude that g_{ψ_0} is bi-Lipschitz to g, that is, there exists a constant $C_0 > 1$ such that $\frac{1}{C_0} g \leq g_{\psi_0} \leq C_0 g$.

3. Shi's solution

In 1989, Shi [19] constructed a solution to the Ricci flow on non compact manifolds. Given the correspondence between the Kähler-Ricci flow and the normalized Kähler-Ricci flow (Proposition 2.3), the Ricci flow evolving from the initial data g_{ψ_0} arises as a natural candidate for the establishment of Theorem C.

Let (M, g, X) be an asymptotically conical gradient Kähler-Ricci expander and let ψ_0 be a smooth function defined on M satisfying Condition I, as in Definition 1.5.

3.1. **Shi's solution to Ricci flow.** Since the initial metric g_{ψ_0} is a complete Kähler metric with bounded Riemann curvature, Shi's existence theorem [19] ensures the existence of a complete solution to the Kähler-Ricci flow.

Theorem 3.1 (Shi's solution)**.** *There exists an $1 < \tilde{T}_{\max} \leq \infty$ such that there is a complete smooth solution to Ricci flow $(g_\varphi(t))_{t\in[1,\tilde{T}_{\max})}$ starting from g_{ψ_0} such that*

(i) $\frac{\partial}{\partial t} g_\varphi(t) = -\operatorname{Ric}(g_\varphi(t))$*, for all* $t \in [1, \tilde{T}_{\max})$*;*

(ii) *for all* $1 < T < \tilde{T}_{\max}$*, for all* $m \in \mathbb{N}_0$*, there exists a constant* $C_m = C(n, m, T, \psi_0)$ *such that*

$$\sup_M \left| \left(\nabla^{g_\varphi(t)}\right)^m \operatorname{Rm}(g_\varphi(t)) \right|_{g_\varphi(t)} \leq \frac{C_m}{(t-1)^{\frac{m}{2}}}, \quad \text{for all } t \in (1, T];$$

(iii) *if* $\tilde{T}_{\max} \neq +\infty$*, then* $\limsup_{t\to\tilde{T}_{\max}} \sup_M |\operatorname{Rm}(g_\varphi(t)|_{g_\varphi(t)} = \infty$*.*

Proof. We apply Shi's existence theorem [19, Theorem 1.1]; for additional details on Shi's estimates, see [12, Theorem 6.9]. □

Remark 3.2. Here we slightly abuse notation by writing g_φ, since a priori the metric $g_\varphi(t)$ is not necessarily a solution to the Kähler–Ricci flow.

Since Shi's solution possesses uniformly bounded Riemann curvature on the entire space time $M \times [1, T]$ for all $T < \tilde{T}_{\max}$, several geometric properties are preserved along the Ricci flow. In particular, both the Kähler structure and the Killing vector field JX are preserved throughout the evolution.

Lemma 3.3. *The solution $(g_\varphi(t))_{t\in[1,\tilde{T}_{\max})}$ from Theorem 3.1 is a solution to the Kähler-Ricci flow; that is, $g_\varphi(t)$ remains a Kähler metric for all $t \in [1, \tilde{T}_{\max})$.*

Proof. On each compact time interval, the Riemann curvature of $g_\varphi(t)$ remains bounded. Moreover, since the initial metric g_{ψ_0} is Kähler, it follows from Huang and Tam's theorem [16, Theorem 1.1] that $g_\varphi(t)$ remains a Kähler metric for all $t \in [1, \tilde{T}_{\max})$. □

Lemma 3.4. *The Reeb vector field JX is a Killing vector field for the solution $g_\varphi(t)$ from Theorem 3.1 for all $t \in [1, \tilde{T}_{\max})$.*

Proof. On each compact time interval, the Riemann curvature of $g_\varphi(t)$ is bounded and the Reeb vector field JX is a Killing vector field for g_{ψ_0}, due to the uniqueness theorem of Chen-Zhu [8, Theorem 1.1], JX is Killing for all $g_\varphi(t)$ with $t \in [1, \tilde{T}_{\max})$. □

Thanks to the correspondence (Proposition 2.3) between the Kähler-Ricci flow and the normalized Kähler-Ricci flow, we can obtain a solution to the normalized Kähler-Ricci flow based on Shi's solution in Theorem 3.1.

Lemma 3.5. *The flow $g_\psi(\tau)$ defined by $g_\psi(\tau) := e^{-\tau}\Phi^*_{e^{-\tau}} g_\varphi(e^\tau)$ for $\tau \in [0, \log \tilde{T}_{\max})$ is a solution to the normalized Kähler-Ricci flow, where $\Phi_t = \Phi_X^{-\frac{1}{2}\log t}$ for $t > 0$ and $\Phi_X^\cdot : M \mapsto M$ denotes the flow of X.*

Proof. The metric $g_\psi(\tau)$ is a Kähler metric due to the Lemma 3.3 and the fact that X is real-holomorphic. Now we compute

$$\begin{aligned}
\frac{\partial}{\partial \tau} g_\psi(\tau) &= -e^{-\tau}\Phi^*_{e^{-\tau}} g_\varphi(e^\tau) + e^{-\tau}\frac{\partial}{\partial \tau}\left(\Phi^*_{e^{-\tau}} g_\varphi(e^\tau)\right) \\
&= -g_\psi(\tau) + e^{-\tau}\mathcal{L}_{\frac{X}{2}}\left(\Phi^*_{e^{-\tau}} g_\varphi(e^\tau)\right) - \Phi^*_{e^{-\tau}} \operatorname{Ric}(g_\varphi(e^\tau)) \\
&= \mathcal{L}_{\frac{X}{2}} g_\psi(\tau) - \operatorname{Ric}(g_\psi(\tau)) - g_\psi(\tau).
\end{aligned}$$

□

By combining Theorem 3.1 and Lemma 3.5, we obtain the following theorem:

Theorem 3.6. *There exists an $0 < T_{\max} \leq \infty$ such that there is a complete smooth solution to normalized Kähler-Ricci flow $g_\psi(\tau)_{\tau \in [0,T_{\max})}$ starting from initial data g_{ψ_0}. Moreover,*

(i) *for all $T < T_{\max}$, for all $m \in \mathbb{N}_0$ there exists a constant $C_m = C(n,m,\psi_0,T) > 0$ such that*

$$\sup_M \left|\left(\nabla^{g_\psi(\tau)}\right)^m \mathrm{Rm}(g_\psi(\tau))\right|_{g_\psi(\tau)} \leq \frac{C_m}{\tau^m}, \quad \text{for all } \tau \in (0,T].$$

(ii) *If $T_{\max} \neq +\infty$, then* $\limsup_{\tau \to T_{\max}} \sup_M |\mathrm{Rm}(g_\psi(\tau))|_{g_\psi(\tau)} = \infty$.

3.2. **Reduction to complex Monge-Ampère equation.** In Kähler geometry, a common strategy is to reduce the Kähler-Ricci flow equation to a complex Monge-Ampère equation for the Kähler potential.

Our first reduction concerns the initial Kähler potential ψ_0 which satisfies Condition I. We show that there exists another Kähler potential, inducing the same initial metric, but enjoying additional symmetries.

Proposition 3.7. *There exists a smooth function $\bar\psi_0 \in C^\infty(M)$ satisfying Condition I such that $JX \cdot \psi_0 = 0$ and $\partial\bar\partial\psi_0 = \partial\bar\partial\bar\psi_0$.*

Moreover, if ψ_0 satisfies Condition II, then $\bar\psi_0$ also satisfies Condition II.

Proof. We consider the isometry group of (M,g) that fixes E as in Theorem 2.4 endowed with the topology induced by uniform convergence on compact subsets of M. By the Arzelà-Ascoli theorem, this is a compact Lie group. And it has been proved that the flows of X and JX preserve E (see [13, Lemma 2.6]). Thus in particular, the flow of JX lies in such isometry group, taking the closure of the flow of JX in this group therefore yields the holomorphic isometric action of a torus T on (M,g). Let μ_T be the normalized Haar measure of T, and we define $\bar\psi_0 := \int_T a^*\psi_0 d\mu_T(a)$, since f and $\partial\bar\partial\psi_0$ is $T-$invariant, hence $\bar\psi_0$ satisfies the same condition as ψ_0. Moreover, since $\bar\psi_0$ is the average of ψ_0 by the action of T, hence $JX \cdot \bar\psi_0 = 0$. □

Since $\bar\psi_0$ and ψ_0 determine the same initial metric and $\bar\psi_0$ satisfies Condition I (resp. II) as in Definition 1.5 (resp. Definition 1.6) if ψ_0 satisfies Condition I (resp. II), from now on we use $\bar\psi_0$ instead of ψ_0, for convenience, we still denote it by ψ_0.

Proposition 3.8. *Let $g_\varphi(t)_{t\in[1,\tilde T_{\max})}$ be Shi's solution to Kähler-Ricci flow as in Theorem 3.1. Let $g(t)_{t\geq 1} := t\Phi_t^* g$ be the self-similar solution associated with (M,g,X) as in Proposition 2.1. Then there exists a smooth real-valued function $\varphi \in C^\infty(M \times [1,\tilde T_{\max}))$ such that*

$$\begin{aligned} &g_\varphi(t) = g(t) + \partial\bar\partial\varphi(t),\\ &JX \cdot \varphi(t) = 0, \quad \forall t \in [1,\tilde T_{\max}),\\ &\varphi(1) = \psi_0. \end{aligned}$$

Proof. Recall the Kähler-Ricci flow equation

$$\begin{aligned} &\frac{\partial}{\partial t} g_\varphi(t) = -\mathrm{Ric}(g_\varphi(t));\\ &\frac{\partial}{\partial t} g(t) = -\mathrm{Ric}(g(t)). \end{aligned}$$

We compute the difference, if $\omega_\varphi(t)$ (resp. $\omega(t)$) denotes the Kähler form of $g_\varphi(t)$(resp. $g(t)$), we get

$$\frac{\partial}{\partial t}\left(g_\varphi(t) - g(t)\right) = \mathrm{Ric}(g(t)) - \mathrm{Ric}(g_\varphi(t)) = \partial\bar\partial \log\frac{\omega_\varphi(t)^n}{\omega(t)^n}.$$

By integrating, we get

$$\begin{aligned} g_\varphi(t) - g(t) &= g_{\psi_0} - g + \partial\bar\partial\int_1^t \log\frac{\omega_\varphi(s)^n}{\omega(s)^n}ds\\ &= \partial\bar\partial\left(\psi_0 + \int_1^t \log\frac{\omega_\varphi(s)^n}{\omega(s)^n}ds\right). \end{aligned}$$

Now we keep the notations as in Proposition 3.7 and we define

$$\varphi(t) := \int_T a^* \left(\psi_0 + \int_1^t \log \frac{\omega_\varphi(s)^n}{\omega(s)^n} ds \right) d\mu_T(a).$$

We observe that $\varphi(1) = \psi_0$. Since $\varphi(t)$ is the average of a function over the torus T, we get $JX \cdot \varphi(t) = 0$. Moreover, since $\mathcal{L}_{JX} g_\varphi(t) = 0$, we have

$$\mathcal{L}_{JX} \left(\partial\bar{\partial} \left(\psi_0 + \int_1^t \log \frac{\omega_\varphi(s)^n}{\omega(s)^n} ds \right) \right) = 0,$$

which implies that $\partial\bar{\partial} \left(\psi_0 + \int_1^t \log \frac{\omega_\varphi(s)^n}{\omega(s)^n} ds \right)$ is invariant under the action of T. It turns out that

$$\partial\bar{\partial}\varphi(t) = \int_T a^* \left(\partial\bar{\partial} \left(\psi_0 + \int_1^t \log \frac{\omega_\varphi(s)^n}{\omega(s)^n} ds \right) \right) d\mu_T(a) = \partial\bar{\partial} \left(\psi_0 + \int_1^t \log \frac{\omega_\varphi(s)^n}{\omega(s)^n} ds \right).$$

Hence $g_\varphi(t) - g(t) = \partial\bar{\partial}\varphi(t)$. □

We now establish a Liouville-type lemma, which allows us to reduce the Kähler–Ricci flow equation to a complex Monge–Ampère equation.

Lemma 3.9. *If κ is a smooth real-valued function such that $JX \cdot \kappa = 0$ and $\partial\bar{\partial}\kappa = 0$, then κ is a constant function.*

Proof. We can find a detailed proof in [9, Proposition 4.4].

Let π be the Kähler resolution as in Theorem 2.4 form M to (C, g_0) with exceptional set E. If $X^{1,0}$ denotes the holomorphic part of X, then $JX \cdot \kappa$ is equivalent to saying that $X \cdot \kappa = 2X^{1,0} \cdot \kappa$.

Since $\partial\bar{\partial}\kappa = 0$, hence $\bar{\partial}(X \cdot \kappa) = \bar{\partial}\left(2X^{1,0} \cdot \kappa\right) = 0$. This implies that $X \cdot \kappa$ is a real-valued holomorphic function, hence a constant function. Therefore, there exists a constant c such that $X \cdot \kappa = c$.

Now we consider $c = \pi_*(X \cdot \kappa) = d\pi(X) \cdot \pi_*\kappa$, since $d\pi(X) = r\partial_r$, where r denotes the radial function, we deduce that $\pi_*\kappa = c \log r + \alpha$ with α a smooth function that only defines on the link S of (C, g_0). We compute the Laplacian of $\pi_*\kappa$ with respect to g_0. On the one hand, it should be 0. On the other hand

$$0 = \Delta_{g_0} \pi_*\kappa(t) = cr^{-2}(2n-2) + r^{-2}\Delta_{g_S}\alpha.$$

Since $n \geq 2$, this implies that $c = 0$ and α is a constant. As a consequence, κ is constant on $M \setminus E$. Because $M \setminus E$ is dense, we deduce that κ is constant. □

Proposition 3.10 (Complex Monge-Ampère equation)**.** *Let $g_\varphi(t)_{t\in[1,\tilde{T}_{\max})}$ be Shi's solution to Kähler-Ricci flow as in Theorem 3.1. Let φ be the Kähler potential as in Proposition 3.8.*

There exists a smooth real-valued function $\bar{\varphi} \in C^\infty(M \times [1, \tilde{T}_{\max}))$ such that

$$\begin{aligned} &\frac{\partial}{\partial t}\bar{\varphi}(t) = \log \frac{\omega_\varphi(t)^n}{\omega(t)^n}, \\ &JX \cdot \bar{\varphi}(t) = 0, \quad \forall t \in [1, \tilde{T}_{\max}), \\ &\bar{\varphi}(1) - \varphi(1) = \textit{constant}. \end{aligned}$$

Proof. Recall in Proposition 3.8, we have that:

$$\partial\bar{\partial}\left(\frac{\partial}{\partial t}\varphi(t)\right) = \partial\bar{\partial} \log \frac{\omega_\varphi(t)^n}{\omega(t)^n}.$$

Thanks to Lemma 3.9, there exists a smooth function $c(t)$ which only depends on time such that

$$\frac{\partial}{\partial t}\varphi(t) = \log \frac{\omega_\varphi(t)^n}{\omega(t)^n} + c(t).$$

Let $C(t)$ be the primitive function of $c(t)$ and let $\bar{\varphi}(t) = \varphi(t) - C(t)$, we get

$$\frac{\partial}{\partial t}\bar{\varphi}(t) = \log \frac{\omega_\varphi(t)^n}{\omega(t)^n}.$$

In particular $\bar{\varphi}(1) = \varphi(1) - C(1)$, thus $\bar{\varphi}(1)$ satisfies Condition I (resp. II) as in Definition 1.5 (resp. Definition 1.6) if ψ_0 satisfies Condition I (resp. II). □

From now on, we still use φ to denote the Kähler potential $\bar{\varphi}$, and we still use ψ_0 to denote $\bar{\varphi}(1)$ which satisfies Condition I (resp. II) as in Definition 1.5 (resp. Definition 1.6) if ψ_0 satisfies Condition I (resp. II)..

As a result of Theorem 3.6 and Proposition 3.10, we can drive an evolution equation to the Kähler potential along the normalized Kähler-Ricci flow:

Proposition 3.11 (Normalized complex Monge-Ampère equation)**.** *Let $g_\psi(\tau)_{\tau\in[0,T_{\max})}$ be Shi's solution to the normalized Kähler-Ricci flow as in Theorem 3.6. Let ω_ψ be the Kähler form of g_ψ. Then there exists a real-valued function $\psi \in C^\infty(M \times [0, T_{\max}))$ such that for $\tau \in [0, T_{\max})$ we have $g_\psi(\tau) = g + \partial\bar{\partial}\psi(\tau)$, and*

$$\begin{aligned} &\frac{\partial}{\partial \tau}\psi(\tau) = \log \frac{\omega_\psi(\tau)^n}{\omega^n} + \frac{X}{2}\cdot\psi(\tau) - \psi(\tau), \\ &JX \cdot \psi(\tau) = 0, \quad \forall \tau \in [0, T_{\max}), \\ &\psi(0) = \psi_0. \end{aligned} \tag{3.1}$$

Proof. Let φ be the solution to the complex Monge-Ampère equation as in Proposition 3.10. Let $\Phi_t = \Phi_X^{-\frac{1}{2}\log t}$ for $t > 0$ and $\Phi_X^{\cdot}\colon M \mapsto M$ denotes the flow of X.

For all $\tau \in [0, T_{\max})$, we define $\psi(\tau) = e^{-\tau}\Phi^*_{e^{-\tau}}\varphi(e^\tau)$. Therefore $\psi(0) = \varphi(1) = \psi_0$. Moreover, we can verify that $g_\psi(\tau) = g + \partial\bar{\partial}\psi(\tau)$ and

$$\begin{aligned} &\frac{\partial}{\partial \tau}\psi(\tau) = \log \frac{\omega_\psi(\tau)^n}{\omega^n} + \frac{X}{2}\cdot\psi(\tau) - \psi(\tau), \\ &JX \cdot \psi(\tau) = 0. \end{aligned}$$

□

4. Proof of long time existence theorem C

In this section, we give a detailed proof of Theorem C. Let $g(t) = t\Phi_t^* g$ be the self-similar solution associated with asymptotically conical gradient Kähler-Ricci expander (M, g, X) as in Proposition 2.1. Let ψ_0 be a smooth function defined on M satisfying Condition I.

Let $(g_\psi(\tau))_{\tau\in[0,T_{\max})}$ be the solution to normalized Kähler-Ricci flow as in Theorem 3.6 and let $(g_\varphi(t))_{t\in[1,e^{T_{\max}})}$ be Shi's solution as in Theorem 3.1.

4.1. **Rough estimates along normalized Kähler-Ricci flow.**

In this section, we show some rough estimates results along this normalized Kähler-Ricci flow.

Proposition 4.1. *For all $0 \le T < T_{\max}$, there exists a constant $B > 1$ such that for all $(x, \tau) \in M \times [0, T]$,*

$$\frac{1}{B} g \le g_\psi(x, \tau) \le Bg.$$

Proof. According to Theorem 3.6, there exists a constant $B(T) > 0$ such that

$$\sup_{M\times[1,e^T]} |\operatorname{Rm}(g_\varphi(\cdot))|_{g_\varphi(\cdot)} \le B(T).$$

Hence by [12, Corollary 6.11], there exists a constant $B_1 > 1$ such that

$$\frac{1}{B_1} g_\varphi(x, 1) \le g_\varphi(x, t) \le B_1 g_\varphi(x, 1), \quad \text{for all } (x, t) \in M \times [1, e^T].$$

Moreover, we observe that for all $t \in [1, e^T]$

$$\sup_M |\operatorname{Rm}(g(t))|_{g(t)} = \frac{1}{t}\sup_M |\operatorname{Rm}(g)|_g \le \sup_M |\operatorname{Rm}(g)|_g < \infty.$$

We deduce that there exists a constant $B_2 > 1$ such that for all $t \in [1, e^T]$

$$\frac{1}{B_2} g(0) \leq g(t) \leq B_2 g(0).$$

At time $t = 1$, $g_\varphi(1) = g_{\psi_0}, g(1) = g$, the metric assumption in condition I, as in Definition 1.5 implies that there exists a constant $B_3 > 1$ such that

$$\frac{1}{B_3} g(t) \leq g_\varphi(t) \leq B_3 g(t), \quad \text{for all } t \in [1, e^T].$$

Since $g(t) = t\Phi_t^* g$ and $g_\psi(\tau) = e^{-\tau}\Phi_{e^{-\tau}}^* g_\varphi(e^\tau)$ for $t = e^\tau$, there exists a constant $B = B(T) > 1$ such that

$$\frac{1}{B} g \leq g_\psi(\tau) \leq Bg, \quad \text{for all } \tau \in [0, T].$$

□

Along the normalized Kähler-Ricci flow, the Christoffel symbols are also roughly bounded on every compact time interval.

Proposition 4.2. *For all $0 \leq T < T_{\max}$, there exists a constant $B = B(T) > 0$ such that on $M \times [0, T]$,*

$$|\nabla^g g_\psi|_g \leq B.$$

Proof. Recall the correspondence $\nabla^g g_\psi(\tau) = e^{-\tau}\Phi_{e^{-\tau}}^* \left(\nabla^{g(e^\tau)} g_\varphi(e^\tau)\right)$ and the fact $\nabla^{g(e^\tau)} g_\varphi(e^\tau) = \nabla^g g(e^\tau) * g_\varphi(e^\tau) + \nabla^g g_\varphi(e^\tau)$. Hence we only need to estimate $\nabla^g g_\varphi(e^\tau)$, and the control $\nabla^g g(e^\tau)$ comes from the same proof.

Now we estimate $\nabla^g g_\varphi(e^\tau)$. Take $t = e^\tau$ and define $y(s) = |\nabla^g g_\varphi(x, s)|_g^2$ for $1 \leq s \leq t$ and fixed $x \in M$. Now we compute

$$\frac{d}{ds} y(s) \leq 2|\nabla^g g_\varphi(s)|_g |\nabla^g \operatorname{Ric}(g_\varphi(s))|_g.$$

Since $\nabla^g \operatorname{Ric}(g_\varphi(s)) = \nabla^g g_\varphi(s) * \operatorname{Ric}(g_\varphi(s)) + \nabla^{g_\varphi(s)} \operatorname{Ric}(g_\varphi(s))$, we conclude that there is a dimensional constant $C(n)$ such that

$$\frac{d}{ds} y(s) \leq C(n)\sqrt{y(s)} \left(\sqrt{y(s)} |\operatorname{Ric}(g_\varphi(s))|_g + |\nabla^{g_\varphi(s)} \operatorname{Ric}(g_\varphi(s))|_g\right)$$

By the curvature estimates of Shi's solution in Lemma 3.1 and the metric equivalence indicated in Proposition 4.1, there is a constant $C_1 > 1$ such that for all $1 \leq s \leq e^T$

$$|\operatorname{Ric}(g_\varphi(s))|_g \leq C_1,$$

and

$$|\nabla^{g_\varphi(s)} \operatorname{Ric}(g_\varphi(s))|_g \leq \frac{C_1}{\sqrt{s-1}}.$$

Hence we get

$$\frac{d}{ds} y(s) \leq C(n) C_1 \left(y(s) + \frac{\sqrt{y(s)}}{\sqrt{s-1}}\right),$$

which implies that

$$\frac{d}{ds}\sqrt{y(s)} \leq \frac{1}{2} C(n) C_1 \left(\sqrt{y(s)} + \frac{1}{\sqrt{s-1}}\right).$$

When $s = 1$, $y(1) = |\nabla^g g_{\psi_0}|_g^2$ is bounded globally due to Definition 1.5. Since $\frac{1}{\sqrt{s-1}}$ is integrable at 1 thus by integration, we conclude that there is a constant $B_0 > 0$ such that $y(t) \leq B_0$ for all $t \in [1, e^T]$. Hence, there is a constant $B > 0$ such that $|\nabla^g g_\psi|_g \leq B$ holds on $M \times [0, T]$ as required. □

A key observation is that the vector field X remains a gradient vector field along the normalized Kähler–Ricci flow.

Definition & Proposition 4.3. Let f be the normalized Hamiltonian potential of X with respect to g, as in Lemma 2.5. Let $g_\psi(\tau)_{\tau\in[0,T_{\max})}$ denote the solution to the normalized Kähler–Ricci flow given by Theorem 3.6. Then, for each $\tau\in[0,T_{\max})$, the vector field X is gradient with respect to $g_\psi(\tau)$, with Hamiltonian potential

$$f+\tfrac{1}{2}\,X\cdot\psi(\tau).$$

Hence we define $f_\psi(\tau):=f+\frac{1}{2}\,X\cdot\psi(\tau)$ for each $\tau\in[0,T_{\max})$.

Proof. Take local holomorphic coordinates of M, since X is real holomorphic and $JX\cdot\psi=0$, let $X^{1,0}$ denote the holomorphic part of X, then we have $X^{1,0}\cdot\psi=\frac{1}{2}X\cdot\psi$. This allows us to compute

$$\begin{aligned}\bar\partial_{\bar j}(f+\frac{1}{2}X\cdot\psi)&=\bar\partial_{\bar j}f+\bar\partial_{\bar j}(X^{1,0}\cdot\psi)\\&=g_{i\bar j}X^i+(X^{1,0})^i\partial_i\bar\partial_{\bar j}\psi\\&=g_{i\bar j}X^i+X^i\partial_i\bar\partial_{\bar j}\psi\\&=g_{\psi\,i\bar j}X^i.\end{aligned}$$

Hence $f+\frac{1}{2}X\cdot\psi$ is a Hamiltonian potential of X with respect to g_ψ. □

These two Hamiltonian potentials are roughly comparable along the flow.

Proposition 4.4. *For all $0\le T<T_{\max}$, there exists a constant $B>1$ such that for all $(x,\tau)\in M\times[0,T]$,*

$$\frac{1}{B}f(x)\le f_\psi(x,\tau)\le Bf(x). \tag{4.1}$$

Moreover, there exists an $\varepsilon>0$ such that for each $\tau\in[0,T_{\max})$ $\inf_M f_\psi(\tau)\ge\inf_M f\ge\varepsilon>0$.

Proof. By Proposition 4.1, there exists a constant $B=B(T)>1$ such that

$$\frac{1}{B}g\le g_\psi(\tau)\le Bg,\quad\text{for all }\tau\in[0,T].$$

We observe that

$$X\cdot(f_\psi-Bf)=g_\psi(X,X)-Bg(X,X)\le 0.$$

Let $\Phi_X^{\cdot}$ denotes the flow of X then for all $x\in M,\rho<0$, $(f_\psi-Bf)(x)\le(f_\psi-Bf)(\Phi_X^\rho(x))$.

For all $\rho<0$, $f(\Phi_X^\rho(x))\le f(x)$, thus $\{\Phi_X^\rho(x)\}_{\rho<0}\subset\{f\le f(x)\}$, where $\{f\le f(x)\}$ is a compact set. The set $\{\Phi_X^\rho(x)\}_{\rho<0}$ is a pre-compact set. We suppose that $x_0\in M$ is a limit point of $\Phi_X^{\rho_i}(x)$ with $\lim_{i\to\infty}\rho_i=-\infty$. Hence, we have $(f_\psi-Bf)(x)\le(f_\psi-Bf)(x_0)$. Since $f(\Phi_X^\rho(x))$ is increasing with respect to ρ, thus

$$\lim_{\rho\to-\infty}f(\Phi_X^\rho(x))=\lim_{i\to\infty}f(\Phi_X^{\rho_i}(x))=f(x_0).$$

For all $\eta\in\mathbb{R}$, on the one hand,

$$\lim_{i\to\infty}f\circ\Phi_X^\eta(\Phi_X^{\rho_i}(x))=f(\Phi_X^\eta(x_0)).$$

On the other hand

$$\lim_{i\to\infty}f\circ\Phi_X^\eta(\Phi_X^{\rho_i}(x))=\lim_{i\to\infty}f(\Phi_X^{\rho_i+\eta}(x))=\lim_{\rho\to-\infty}f(\Phi_X^\rho(x))=f(x_0).$$

It turns out that $f(x_0)=f(\Phi_X^\eta(x_0))$ for all $\eta\in\mathbb{R}$, which implies that $X(x_0)=0$. In particular, $(f_\psi-Bf)(x_0)=f(x_0)-Bf(x_0)=(1-B)f(x_0)\le 0$. Therefore, $(f_\psi-Bf)(x)\le 0$ for all $x\in M$.

For the same reason, we prove that

$$f_\psi\ge\frac{1}{B}f.$$

From (4.1) and the fact that f is a proper function, we deduce that $f_\psi(\tau)$ is a proper function for all $\tau\in[0,T_{\max})$. Suppose that for fixed τ, $x_0\in M$ is a minimum point of $f_\psi(\tau)$, i.e. $f_\psi(x_0,\tau)=\inf_M f_\psi(\tau)$, then $X(x_0)=\left(\nabla^{g_\psi(\tau)}f_\psi(\tau)\right)(x_0)=0$, hence $f_\psi(x_0,\tau)=f(x_0)\ge\inf_M f$. Thanks to

Proposition 2.7 there exists an $\varepsilon > 0$ such that on $M \times [0, T_{\max})$,

$$\inf_M f_\psi \geq \inf_M f \geq \varepsilon > 0. \tag{4.2}$$

□

The following Proposition gives us rough estimates on the Kähler potential ψ and its radial derivative $X \cdot \psi$.

Proposition 4.5. *For all $0 \leq T < T_{\max}$, there exists a constant $B > 1$ such that for all $(x, \tau) \in M \times [0, T]$,*

$$\left(|X \cdot \psi| + |\psi|\right)(x, \tau) \leq Bf.$$

Proof. First we prove that there exists a constant $B_1 = B_1(T)$ such that on $M \times [0, T]$

$$|X \cdot \psi| \leq Bf.$$

Notice that on the one hand by [(4.1), Proposition 4.1], there exists a constant $B > 1$ such that

$$X \cdot \psi = 2f_\psi - 2f \leq 2f_\psi \leq 2Bf.$$

On the other hand,

$$X \cdot \psi = 2f_\psi - 2f \geq -2f.$$

Thus, we can take $B_1 = 2B$ so that $|X \cdot \psi| \leq B_1 f$ on $M \times [0, T]$.

Now we consider φ defined in Proposition 3.10. Since $\dot{\varphi}(t) = \log \frac{\omega_\varphi(t)^n}{\omega(t)^n}$, Lemma 4.1 ensures that there exists a constant $B_2 = B_2(T) > 0$ such that for $t \in [1, e^T]$

$$|\varphi(t) - \varphi(1)| \leq B_2.$$

Since $e^{-\tau}\Phi^*_{e^{-\tau}}\varphi(e^\tau) = \psi(\tau)$ and $\varphi(1) = \psi_0$, we have for $\tau \in [0, T]$, $x \in M$,

$$|e^\tau \psi(\Phi_{e^\tau}(x), \tau) - \psi_0(x)| \leq B_2.$$

Since $\psi_0 = O(f)$ by Condition I (see Definition 1.5), there exists a constant $B_3 > 0$ such that $\psi_0(x) \leq B_3 f(x)$, hence

$$|e^\tau \psi(\Phi_{e^\tau}(x), \tau)| \leq B_2 + B_3 f(x).$$

This yields that for all $(x, \tau) \in M \times [0, T]$

$$|\psi(x, \tau)| \leq B_2 e^{-\tau} + B_3 e^{-\tau} f(\Phi_{e^{-\tau}}(x)). \tag{4.3}$$

Let us consider the function $e^{-\rho} f(\Phi_{e^{-\rho}}(x))$ for all $\rho \geq 0$. We compute as follows:

$$\frac{d}{d\rho}\left(e^{-\rho} f(\Phi_{e^{-\rho}}(x))\right) = e^{-\rho}(-f + |\partial f|^2_g)(\Phi_{e^{-\rho}}(x)) = e^{-\rho}(-n - R_\omega)(\Phi_{e^{-\rho}}(x)) \leq 0. \tag{4.4}$$

This implies that $e^{-\tau} f(\Phi_{e^{-\tau}}(x)) \leq f(x)$ for $(x, \tau) \in M \times [0, T]$, the combination of (4.3) and (4.4) leads to

$$|\psi(x, \tau)| \leq B_2 e^{-\tau} + B_3 f(x) \leq B_2 + B_3 f(x) \leq \left(\frac{B_2}{\varepsilon} + B_3\right) f(x).$$

We can take $B = \left(\frac{B_2}{\varepsilon} + B_3\right)$ such that on $M \times [0, T]$,

$$|\psi| \leq Bf.$$

□

4.2. **Uniform estimates on space-time along normalized Kähler-Ricci flow.** In this section, we omit the time dependence in $g_\psi(\tau)$ for ease of reading. Let ω_ψ be the Kähler form of g_ψ. We denote the drift Laplacian $\Delta_{\omega_\psi} + \frac{X}{2}$ by $\Delta_{\omega_\psi, X}$, and define $\dot{\psi} := \frac{\partial \psi}{\partial \tau}$. This first observation is that the Hamiltonian potential f_ψ satisfies a *good* evolution equation.

Proposition 4.6. *The Hamiltonian potential f_ψ satisfies the following evolution equation:*

$$\frac{\partial}{\partial \tau} f_\psi = \Delta_{\omega_\psi, X} f_\psi - f_\psi. \tag{4.5}$$

Proof. First we compute

$$\frac{\partial}{\partial \tau} f_\psi = \frac{X}{2} \cdot \dot{\psi} = \frac{X}{2} \cdot \left(\log \frac{\omega_\psi^n}{\omega^n} + \frac{X}{2} \cdot \psi - \psi \right).$$

Since

$$\frac{X}{2} \cdot \log \frac{\omega_\psi^n}{\omega^n} = \mathrm{tr}_{\omega_\psi} \mathcal{L}_{\frac{X}{2}} \omega_\psi - \mathrm{tr}_\omega \mathcal{L}_{\frac{X}{2}} \omega = \Delta_{\omega_\psi} f_\psi - \Delta_\omega f,$$

and

$$\frac{X}{2} \cdot \left(\frac{X}{2} \cdot \psi - \psi \right) = \frac{X}{2} \cdot (f_\psi - f) - f_\psi + f = \frac{X}{2} \cdot f_\psi - f_\psi + f - |\partial f|_g^2.$$

The soliton identities (Lemma 2.5) tell us that $f = \Delta_\omega f + |\partial f|_g^2$, hence

$$\frac{\partial}{\partial \tau} f_\psi = \Delta_{\omega_\psi, X} f_\psi - f_\psi,$$

holds. □

The Hamiltonian potential f_ψ plays a role analogous to $\frac{r^2}{2}$ in the case of the Gaussian soliton. On Euclidean space, a maximum principle holds for subsolutions to the heat equation with exponential growth (see [14, Theorem 6 of Chapter 2.3]). Therefore, it is natural to expect a similar maximum principle to hold on our asymptotically conical gradient Kähler-Ricci expander (M, g, X).

Proposition 4.7 (Maximum principle)**.** *If u is a real-valued smooth function defined on $M \times [0, T_{\max})$ such that*

(i) *the function u is a subsolution to the drift heat equation along normalized Kähler-Ricci flow, i.e. $\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) u \leq 0$;*
(ii) *for all $0 < T < T_{\max}$, there exists a constant $B > 0$ and integer $k > 0$ such that $u \leq B f^k$;*
(iii) $\sup_M u|_{\tau=0} \leq 0$.

Then $\sup_{M \times [0, T_{\max})} u \leq 0$.

Proof. We fix $0 < T < T_{\max}$ and consider a weight function $h(x, \tau) := \frac{e^\tau f_\psi(x,\tau)}{T - \beta\tau}$ with $\beta > 0$ to be determined for $(x, \tau) \in M \times [0, \frac{T}{2\beta}]$. By Proposition 4.1 and Proposition 4.4, there exists a constant $B > 0$ such that

$$2|\partial f_\psi|_{g_\psi}^2 = |X|_{g_\psi}^2 \leq B|X|_g^2 = 2B|\partial f|_g^2 \leq 2Bf \leq 2B^2 f_\psi.$$

Now we compute

$$\begin{aligned}
\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X} \right) e^h(x, \tau) &= \left(\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X} \right) h - |\partial h|_{g_\psi}^2 \right) e^h \\
&= \left(\frac{\beta e^\tau f_\psi - e^{2\tau} |\partial f_\psi|_{g_\psi}^2}{(T - \beta\tau)^2} + \frac{\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(e^\tau f_\psi(x, \tau))}{T - \beta\tau} \right) e^h \\
&= \frac{\beta e^\tau f_\psi - e^{2\tau} |\partial f_\psi|_{g_\psi}^2}{(T - \beta\tau)^2} e^h \\
&\geq \frac{\beta e^\tau f_\psi - e^{2\tau} B^2 f_\psi}{(T - \beta\tau)^2} e^h.
\end{aligned}$$

Here in line 3, we have used the (4.5) in Proposition 4.6. Then we take $\beta > e^T B^2 + 1$ so that

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X} \right) e^h > 0,$$

on $M \times [0, \frac{T}{2\beta}]$.

Now we consider auxiliary function $u_\eta = u - \eta e^h$ for $\eta > 0$. Since the growth of u at infinity is polynomial according to assumption (ii), the auxiliary function u_η tends to $-\infty$ uniformly at spatial infinity. Then u_η admits a maximum point. Now we apply the weak maximum principle, we conclude that

$$\sup_{M\times[0,\frac{T}{2\beta}]} u_\eta \leq \sup_M u_\eta|_{\tau=0} \leq 0.$$

Letting $\eta \to 0$, we obtain $\sup_{M\times[0,\frac{T}{2\beta}]} u \leq 0$. We then repeat the above argument inductively on the intervals $[\frac{T}{2\beta}, \frac{T}{\beta}], [\frac{T}{\beta}, \frac{3T}{2\beta}], \ldots$, and ultimately conclude that $\sup_{M\times[0,T]} u \leq 0$. Therefore, we have

$$\sup_{M\times[0,T_{\max})} u \leq 0,$$

as desired. □

The first application of the above maximum principle is the following: we can give a precise bound for $\dot{\psi}$ on the entire space-time $M \times [0, T_{\max})$.

Corollary 4.8. *There exists a constant $C > 0$ such that for all $(x, \tau) \in M \times [0, T_{\max})$*

$$|\dot{\psi}(x, \tau)| \leq Ce^{-\tau}.$$

Proof. Recall the complex Monge-Ampère equation

$$\dot{\psi} = \log \frac{\omega_\psi^n}{\omega^n} + \frac{1}{2} X \cdot \psi - \psi.$$

Now we compute the evolution equation of $\dot{\psi}$:

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \dot{\psi} = -\dot{\psi}. \tag{4.6}$$

It turns out that $e^\tau \dot{\psi}$ is a solution to the drift heat equation along the normalized Kähler-Ricci flow, i.e.

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) e^\tau \dot{\psi} = 0.$$

At time $\tau = 0$, we have

$$\dot{\psi}(0) = \log\left(\frac{\omega_{\psi_0}^n}{\omega^n}\right) + \frac{X}{2} \cdot \psi_0 - \psi_0.$$

By the asymptotic condition in Definition 1.5, there exists a constant $C > 0$ such that $|\dot{\psi}(0)| \leq C$. For all $0 < T < T_{\max}$, Proposition 4.5 guarantees the existence of a constant $B > 0$ such that $|\dot{\psi}| \leq Bf$. Applying Proposition 4.7, we conclude that

$$e^\tau |\dot{\psi}| \leq C,$$

holds on $M \times [0, T_{\max})$. □

Remark 4.9. From the above estimates, the Kähler potential ψ converges uniformly to a continuous function ψ_∞ on M. In general, however, one cannot conclude that ψ_∞ is constant, unlike in the compact case.

When M is a closed complex manifold with $c_1(M) < 0$, we have $X \equiv 0$ and ω is a Kähler–Einstein metric. By [1, Chapter 3.4.1], the Kähler potential then converges uniformly to a smooth function, which must be constant by the uniqueness of Kähler–Einstein metrics on closed manifolds.

In contrast, for non compact manifolds the general uniqueness of asymptotically conical gradient expanders is not known, and hence no such conclusion can be drawn.

Another application of the maximum principle above is to control $|X|^2_{g_\psi}$. Before doing so, we require Bochner-type formulas to derive the evolution equation of the gradient of a solution to the heat equation along the normalized Kähler-Ricci flow.

Lemma 4.10. *Let u be a smooth function defined on $M \times [0, T_{\max})$ such that*

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) u = -u.$$

Then the evolution equation of $|\nabla^{g_\psi} u|^2_{g_\psi}$ along the normalized Kähler-Ricci flow is given by:

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) |\nabla^{g_\psi} u|^2_{g_\psi} = -|\nabla^{g_\psi} u|^2_{g_\psi} - |(\nabla^{g_\psi})^2 u|^2_{g_\psi}. \tag{4.7}$$

Proof. First we compute

$$\begin{aligned}
\frac{\partial}{\partial \tau}|\nabla^{g_\psi} u|^2_{g_\psi} &= -(\mathcal{L}_{\frac{X}{2}} g_\psi - \mathrm{Ric}(g_\psi) - g_\psi)(\nabla^{g_\psi} u, \nabla^{g_\psi} u) + 2 g_\psi(\nabla^{g_\psi} \Delta_{\omega_\psi, X} u, \nabla^{g_\psi} u) \\
&\quad - 2|\nabla^{g_\psi} u|^2_{g_\psi} \\
&= -|\nabla^{g_\psi} u|^2_{g_\psi} - (\mathcal{L}_{\frac{X}{2}} g_\psi - \mathrm{Ric}(g_\psi))(\nabla^{g_\psi} u, \nabla^{g_\psi} u) + g_\psi(\nabla^{g_\psi} \Delta_{g_\psi} u, \nabla^{g_\psi} u) \\
&\quad + g_\psi(\nabla^{g_\psi}(X \cdot u), \nabla^{g_\psi} u) \\
&= -|\nabla^{g_\psi} u|^2_{g_\psi} + \mathrm{Ric}(g_\psi)(\nabla^{g_\psi} u, \nabla^{g_\psi} u) + g_\psi(\nabla^{g_\psi} \Delta_{g_\psi} u, \nabla^{g_\psi} u) \\
&\quad + \nabla^{g_\psi} u \cdot g_\psi(\nabla^{g_\psi} f_\psi, \nabla^{g_\psi} u) - \mathcal{L}_{\frac{X}{2}} g_\psi(\nabla^{g_\psi} u, \nabla^{g_\psi} u) \\
&= -|\nabla^{g_\psi} u|^2_{g_\psi} + \mathrm{Ric}(g_\psi)(\nabla^{g_\psi} u, \nabla^{g_\psi} u) + g_\psi(\nabla^{g_\psi} \Delta_{g_\psi} u, \nabla^{g_\psi} u) \\
&\quad + (\nabla^{g_\psi})^2 u(\nabla^{g_\psi} f_\psi, \nabla^{g_\psi} u).
\end{aligned} \tag{4.8}$$

By Bochner's formula,

$$\begin{aligned}
\Delta_{\omega_\psi} |\nabla^{g_\psi} u|^2_{g_\psi} &= \frac{1}{2} \Delta_{g_\psi} |\nabla^{g_\psi} u|^2_{g_\psi} \\
&= |(\nabla^{g_\psi})^2 u|^2 + \mathrm{Ric}(g_\psi)(\nabla^{g_\psi} u, \nabla^{g_\psi} u) + g_\psi(\nabla^{g_\psi} \Delta_{g_\psi} u, \nabla^{g_\psi} u).
\end{aligned} \tag{4.9}$$

Moreover,

$$\frac{X}{2} \cdot |\nabla^{g_\psi} u|^2_{g_\psi} = g_\psi(\nabla^{g_\psi}_X \nabla^{g_\psi} u, \nabla^{g_\psi} u) = (\nabla^{g_\psi})^2 u(\nabla^{g_\psi} f_\psi, \nabla^{g_\psi} u). \tag{4.10}$$

Combining (4.8) (4.9) and (4.10), we have (4.7) as desired. □

Corollary 4.11. *There exists a constant $A > 0$ such that*

$$|X|^2_{g_\psi} \leq A f_\psi,$$

holds on $M \times [0, T_{\max})$.

Proof. Since $X = \nabla^{g_\psi} f_\psi$ and the evolution equation of f_ψ from Proposition 4.6:

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) f_\psi = -f_\psi.$$

Thus by [(4.7), Lemma 4.10], we deduce that

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) e^\tau |X|^2_{g_\psi} \leq 0.$$

By the initial condition (metric condition) in Definition 1.5, there exists a constant $A > 0$ such that $|X|^2_{g_{\psi_0}} \leq A f_\psi(0)$. Hence by Proposition 4.7, $e^\tau |X|^2_{g_\psi} - A e^\tau f_\psi \leq 0$ on $M \times [0, T_{\max})$. Indeed, the assumption (ii) of Proposition 4.7 is justified as follows, for any $0 < T < T_{\max}$, by Proposition 4.1, there exists a constant $B > 1$ such that $|X|^2_{g_\psi} \leq B|X|^2_g \leq Bf$.

Therefore,

$$|X|^2_{g_\psi} \leq A f_\psi,$$

holds on $M \times [0, T_{\max})$. □

With control over $|X|^2_{g_\psi}$, we construct a useful barrier function in the remainder of this article, which will allow us to effectively and uniformly eliminate negligible terms.

Lemma 4.12 (Barrier function)**.** *There exist constants $K, \delta > 0$ such that on $M \times [0, T_{\max})$*

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \frac{1}{f_\psi + K} \geq \delta \frac{1}{f_\psi}.$$

Proof. Let $A > 0$ be the constant stated in Corollary 4.11, take $K \geq A$, now we compute

$$\begin{aligned}\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \frac{1}{f_\psi + K} &= -\frac{(\partial_\tau - \Delta_{\omega_\psi, X}) f_\psi}{(f_\psi + K)^2} - 2\frac{|\partial f_\psi|^2_{g_\psi}}{(f_\psi + K)^3} \\ &= \frac{f_\psi}{(f_\psi + K)^2} - \frac{|X|^2_{g_\psi}}{(f_\psi + K)^3} \\ &\geq \frac{f_\psi (f_\psi + K) - A f_\psi}{(f_\psi + K)^3} \geq \frac{f_\psi^2}{(f_\psi + K)^3}.\end{aligned}$$

Due to [(4.2), Proposition 4.1], $\frac{f_\psi^2}{(f_\psi + K)^3} \geq (\frac{\varepsilon}{\varepsilon + K})^3 \frac{1}{f_\psi}$. Taking $\delta = (\frac{\varepsilon}{\varepsilon + K})^3$, we have that

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \frac{1}{f_\psi + K} \geq \delta \frac{1}{f_\psi}.$$

□

The following lemma, which can be viewed as a variation of Proposition 4.7, will be fundamental in the proofs of Theorem C and Theorem A.

Lemma 4.13. *Assume that $u \in C^\infty(M \times [0, T_{\max}))$ satisfies*

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) u \leq D f_\psi^{-1},$$

for some positive constant $D > 0$. Moreover, assume that for all $0 < T < T_{\max}$, there exists $k \in \mathbb{N}_0$ and $B > 0$ such that $u \leq B f^k$ on $M \times [0, T]$. Then there exists a constant $C > 0$ which only depends on D, the lower bound of f as in Proposition 2.7 and δ, K as in Lemma 4.12 such that

$$\sup_{M \times [0, T_{\max})} u \leq \sup_M u(0) + C.$$

Proof. Let δ, K be the constants as in Lemma 4.12, we have

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \frac{1}{f_\psi + K} \geq \delta \frac{1}{f_\psi}.$$

Hence

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \left(u - \frac{D}{\delta} \frac{1}{f_\psi + K}\right) \leq 0.$$

Since $u - \frac{D}{\delta} \frac{1}{f_\psi + K}$ is bounded by $B f^k$ for some positive constant B and integer k on $M \times [0, T]$ for all $0 < T < T_{\max}$, the maximum principle (Proposition 4.7) implies that

$$\sup_{M \times [0, T_{\max})} u - \frac{D}{\delta} \frac{1}{f_\psi + K} \leq \sup_M u(0) - \frac{D}{\delta} \frac{1}{f_\psi + K}(0) \leq \sup_M u(0).$$

By Proposition 4.1, there exists a constant $\varepsilon > 0$ such that $\inf_M f_\psi \geq \varepsilon > 0$. Take $C = \frac{D}{\delta} \frac{1}{\varepsilon + K}$ to get

$$\sup_{M \times [0, T_{\max})} u \leq \sup_M u(0) + C,$$

as desired. □

With all the tools developed above, we now present the second uniform estimate for the Kähler potential ψ:

Corollary 4.14. *There exists a constant $C > 0$ such that on $M \times [0, T_{\max})$ we have:*

$$|\nabla^{g_\psi} \dot{\psi}|^2_{g_\psi} \leq C e^{-2\tau} \frac{1}{f_\psi}.$$

Proof. First we compute

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(e^{2\tau}f_\psi|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}) &= e^{2\tau}f_\psi\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}\\ &\quad+|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(e^{2\tau}f_\psi)\\ &\quad-2e^{2\tau}\operatorname{Re}\left(<\partial f_\psi,\bar{\partial}|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}>_{g_\psi}\right)\\ &=-e^{2\tau}f_\psi|(\nabla^{g_\psi})^2\dot{\psi}|^2_{g_\psi}-e^{2\tau}f_\psi|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}\\ &\quad+|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}e^{2\tau}f_\psi-e^{2\tau}X\cdot|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}\\ &=-e^{2\tau}f_\psi|(\nabla^{g_\psi})^2\dot{\psi}|^2_{g_\psi}-e^{2\tau}X\cdot|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}.\end{aligned}\tag{4.11}$$

Let $A>1$ be the constant as in Corollary 4.11. We notice that by the Cauchy-Schwarz inequality

$$X\cdot|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}\leq 2|(\nabla^{g_\psi})^2\dot{\psi}|_{g_\psi}|X|_{g_\psi}|\nabla^{g_\psi}\dot{\psi}|_{g_\psi}\leq\frac{1}{A}|X|^2_{g_\psi}|(\nabla^{g_\psi})^2\dot{\psi}|^2_{g_\psi}+A|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}.\tag{4.12}$$

Hence we conclude that

$$\begin{aligned}-e^{2\tau}X\cdot|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}&\leq e^{2\tau}\left(\frac{1}{A}|X|^2_{g_\psi}|(\nabla^{g_\psi})^2\dot{\psi}|^2_{g_\psi}+A|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}\right)\\ &\leq Ae^{2\tau}|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}+e^{2\tau}f_\psi|(\nabla^{g_\psi})^2\dot{\psi}|^2_{g_\psi}\end{aligned}\tag{4.13}$$

Thus we get

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(e^{2\tau}f_\psi|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi})\leq Ae^{2\tau}|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}.\tag{4.14}$$

Now we consider the evolution equation of $e^{2\tau}\dot{\psi}^2$:

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)e^{2\tau}\dot{\psi}^2=2e^{2\tau}\dot{\psi}^2+e^{2\tau}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)\dot{\psi}^2=-e^{2\tau}|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}.$$

As a consequence, on $M\times[0,T_{\max})$, we have

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(e^{2\tau}f_\psi|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}+Ae^{2\tau}\dot{\psi}^2)\leq 0.$$

Observe that $Ae^{2\tau}\dot{\psi}^2$ is uniformly bounded on $M\times[0,T_{\max})$ by Corollary 4.8. To use the maximum principle stated in Proposition 4.7, we need a rough estimate on $|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}$. In the remainder of this proof, we use B to denote a constant that may vary from line to line.

Since $|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}=2|\partial\dot{\psi}|^2_{g_\psi}$, by Proposition 4.2 we only need a rough estimate of $|\partial\dot{\psi}|^2_g$. Recall that

$$\partial\dot{\psi}=\partial\log\frac{\omega_\psi^n}{\omega^n}+\partial\left(\frac{X}{2}\cdot\psi\right)-\partial\psi.$$

Since X is real holomorphic and $JX\cdot\psi=0$, $\partial\left(\frac{X}{2}\cdot\psi\right)=\partial(X^{\bar{i}}\bar{\partial}_i\psi)=X^{\bar{i}}\partial\bar{\partial}_i\psi$. Hence for all $0<T<T_{\max}$, by Proposition 4.1 there exists a constant $B>0$ such that on $M\times[0,T]$,

$$\left|\partial\left(\frac{X}{2}\cdot\psi\right)\right|_g\leq Bf.\tag{4.15}$$

Moreover, $|\nabla^g\log\frac{\omega_\psi^n}{\omega^n}|_g\leq C(n)|\nabla^g g_\psi|_g$ by some dimensional constant $C(n)$. By Proposition 4.2, for all $0<T<T_{\max}$, there is a constant $B>0$ such that on $M\times[0,T]$,

$$\left|\partial\log\frac{\omega_\psi^n}{\omega^n}\right|_g\leq B.\tag{4.16}$$

Combining (4.15) and (4.16), we get for $\tau\in[0,T]$, $x\in M$

$$|\partial\dot{\psi}|_g(x,\tau)\leq B\left(f(x)+|\partial\psi|_g(x,\tau)\right).\tag{4.17}$$

When $\tau = 0$, Remark 2.12 implies that $|\partial\psi_0|_g = O(f^{\frac{1}{2}})$. By integration, there are constants $B, D > 0$ such that $|\partial\psi|_g(x,\tau) \le Bf(x)$ and hence $|\partial\dot{\psi}|_g(x,\tau) \le Bf(x)$ for all $(x,\tau) \in M \times [0,T]$.

With this rough estimate, by the maximum principle as in Proposition 4.7,

$$\sup_{M\times[0,T_{\max})} e^{2\tau} f_\psi |\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi} + Ae^{2\tau}\dot{\psi}^2 \le \sup_M f_\psi(0)|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}(0) + A\dot{\psi}^2(0).$$

Since ψ_0 satisfies Condition I, we have

$$\sup_M f_\psi(0)|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}(0) = \sup_M f_\psi(0)\left|\partial\left(\log\frac{\omega^n_{\psi_0}}{\omega^n} + \frac{X}{2}\cdot\psi_0 - \psi_0\right)\right|^2_{g_{\psi_0}} < \infty.$$

Take $C = \sup_M f_\psi(0)|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi}(0) + A\dot{\psi}^2(0) < \infty$, we have that

$$|\nabla^{g_\psi}\dot{\psi}|^2_{g_\psi} \le e^{-2\tau}\frac{C}{f_\psi},$$

holds on $M \times [0, T_{\max})$. □

Corollary 4.15. *There exists a constant $C > 1$ such that for all $(x,\tau) \in M \times [0,T_{\max})$,*

(i) $|\dot{\psi}| + |X\cdot\dot{\psi}|(x,\tau) \le Ce^{-\tau}$;
(ii) $|\frac{X}{2}\cdot\psi - \psi|(x,\tau) \le C$;
(iii) $\frac{1}{C}\omega^n \le \omega_\psi(\tau)^n \le C\omega^n$;
(iv) $|\psi|(x,\tau) \le Cf(x)$, $\psi(x,\tau) \ge -Cf_\psi(x,\tau)$;
(v) $\frac{1}{C}f(x) \le f_\psi(x,\tau) \le Cf(x)$.

Proof. We notice that (i) comes by Corollary 4.8, Corollary 4.11 and Corollary 4.14. Since $\frac{X}{2}\cdot\psi_0 - \psi_0 = O(1)$, hence by integration, we have $|\frac{X}{2}\cdot\psi - \psi| \le C$. Together with the bound of $\dot{\psi}$, we get (iii).

For (iv), by integration, we get $|\psi - \psi_0| \le C$, since $\psi_0 = O(f)$ and f is bounded form below by $\varepsilon > 0$, hence there exists a constant $C > 0$ such that $|\psi| \le Cf$. Moreover, recall that ψ is a supersolution to the drift heat equation along the normalized Kähler-Ricci flow, i.e.

$$\dot{\psi} \ge \Delta_{\omega_\psi,X}\psi - \psi.$$

At time $\tau = 0$, there exists a constant $C > 0$ such that $\psi_0 \ge -Cf_\psi(0)$, therefore by the maximum principle stated in Proposition]4.7, we have $\psi \ge -Cf_\psi$. (Recall the evolution equation of f_ψ stated in Proposition 4.6: $(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X})f_\psi = -f_\psi$.)

For (v), since $|\frac{X}{2}\cdot\psi - \psi| \le C$, on the one hand, $\psi \le Cf$, we have $\frac{X}{2}\cdot\psi \le C + Cf$, and hence $f_\psi \le C + Cf + f \le (C + 1 + \frac{C}{\varepsilon})f$. On the other hand, $\psi \ge -Cf_\psi$, we have that

$$\frac{X}{2}\cdot\psi \ge -C - Cf_\psi.$$

Hence $f_\psi \ge -C - Cf_\psi + f$, since f_ψ is also bounded from below by $\varepsilon > 0$, there exists a constant $C > 0$ such that $f_\psi > \frac{1}{C}f$. □

4.3. **The second order estimates along normalized Kähler-Ricci flow.** In this section, we adopt Yau's C^2-estimate to prove the uniform equivalence of metrics evolving under the normalized Kähler-Ricci flow.

Proposition 4.16. *There exists a constant $C > 1$ such that on $M \times [0, T_{\max})$,*

$$\frac{1}{C} \le g_\psi \le Cg$$

Before proving Proposition 4.16, we will need the evolution equation of $\operatorname{tr}_\omega\omega_\psi$ along the normalized Kähler-Ricci flow.

Lemma 4.17. *Along the normalized Kähler-Ricci flow, we have,*

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)\operatorname{tr}_\omega\omega_\psi &= \operatorname{Ric}(g)^{i\bar{j}}g_{\psi i\bar{j}}-g_\psi^{p\bar{q}}g_{\psi i\bar{j}}g^{i\bar{s}}g^{m\bar{j}}\operatorname{Rm}(g)_{p\bar{q}m\bar{s}}\\ &\quad -g_\psi^{p\bar{q}}g^{i\bar{j}}g_\psi^{a\bar{b}}\nabla_p^g g_{\psi i\bar{b}}\nabla_{\bar{q}}^g g_{\psi a\bar{j}}.\end{aligned}\tag{4.18}$$

Proof. In the normal holomorphic coordinate of g, let's compute the Laplacian of $\operatorname{tr}_\omega\omega_\psi$ with respect to metric g_ψ,

$$\begin{aligned}\Delta_{\omega_\psi}\operatorname{tr}_\omega\omega_\psi &= g_\psi^{p\bar{q}}\partial_p\partial_{\bar{q}}(g^{i\bar{j}}g_{\psi i\bar{j}})\\ &= g_\psi^{p\bar{q}}g^{i\bar{j}}\partial_p\partial_{\bar{q}}g_{\psi i\bar{j}}+g_\psi^{p\bar{q}}g_{\psi i\bar{j}}\partial_p\partial_{\bar{q}}g^{i\bar{j}}\\ &= -g_\psi^{p\bar{q}}g^{i\bar{j}}\operatorname{Rm}(g_\psi)_{p\bar{q}i\bar{j}}+g_\psi^{p\bar{q}}g^{i\bar{j}}g_\psi^{a\bar{b}}\partial_p g_{\psi i\bar{b}}\partial_{\bar{q}}g_{\psi a\bar{j}}+g_\psi^{p\bar{q}}g_{\psi i\bar{j}}g^{i\bar{s}}g^{m\bar{j}}\operatorname{Rm}(g)_{p\bar{q}m\bar{s}}\\ &= -g^{i\bar{j}}\operatorname{Ric}(g_\psi)_{i\bar{j}}+g_\psi^{p\bar{q}}g^{i\bar{j}}g_\psi^{a\bar{b}}\nabla_p^g g_{\psi i\bar{b}}\nabla_{\bar{q}}^g g_{\psi a\bar{j}}+g_\psi^{p\bar{q}}g_{\psi i\bar{j}}g^{i\bar{s}}g^{m\bar{j}}\operatorname{Rm}(g)_{p\bar{q}m\bar{s}}.\end{aligned}\tag{4.19}$$

Now we compute $\frac{X}{2}\cdot\operatorname{tr}_\omega\omega_\psi=\mathcal{L}_{\frac{X}{2}}\operatorname{tr}_\omega\omega_\psi$:

$$\begin{aligned}\frac{X}{2}\cdot\operatorname{tr}_\omega\omega_\psi &= \mathcal{L}_{\frac{X}{2}}\operatorname{tr}_\omega\omega_\psi\\ &= \mathcal{L}_{\frac{X}{2}}(g^{i\bar{j}}g_{\psi i\bar{j}})\\ &= -(\mathcal{L}_{\frac{X}{2}}g)^{i\bar{j}}g_{\psi i\bar{j}}+g^{i\bar{j}}\mathcal{L}_{\frac{X}{2}}g_{\psi i\bar{j}}\\ &= -\operatorname{Ric}(g)^{i\bar{j}}g_{\psi_{i\bar{j}}}-\operatorname{tr}_\omega\omega_\psi+g^{i\bar{j}}\mathcal{L}_{\frac{X}{2}}g_{\psi i\bar{j}}.\end{aligned}\tag{4.20}$$

Here the last line is ensured by the soliton equation (1.1). The time derivative of $\operatorname{tr}_\omega\omega_\psi$ is given by:

$$\frac{\partial}{\partial\tau}\operatorname{tr}_\omega\omega_\psi=\operatorname{tr}_\omega(\mathcal{L}_{\frac{X}{2}}\omega_\psi-\operatorname{Ric}(\omega_\psi)-\omega_\psi).\tag{4.21}$$

Combine these three equations (4.19),(4.20) and (4.21), to get

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)\operatorname{tr}_\omega\omega_\psi &= \operatorname{Ric}(g)^{i\bar{j}}g_{\psi i\bar{j}}-g_\psi^{p\bar{q}}g_{\psi i\bar{j}}g^{i\bar{s}}g^{m\bar{j}}\operatorname{Rm}(g)_{p\bar{q}m\bar{s}}\\ &\quad -g_\psi^{p\bar{q}}g^{i\bar{j}}g_\psi^{a\bar{b}}\nabla_p^g g_{\psi i\bar{b}}\nabla_{\bar{q}}^g g_{\psi a\bar{j}}.\end{aligned}$$

□

To get a universal metric equivalence, we perform the Yau's C^2 estimate in the parabolic setting.

Lemma 4.18. *There exists a constant $C>0$ independent of time such that*

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)\log\operatorname{tr}_\omega\omega_\psi\le\frac{C}{f_\psi}(\operatorname{tr}_{\omega_\psi}\omega+1).\tag{4.22}$$

Proof. Firstly recall that

$$\Delta_{\omega_\psi,X}\log\operatorname{tr}_\omega\omega_\psi=\frac{\Delta_{\omega_\psi,X}\operatorname{tr}_\omega\omega_\psi}{\operatorname{tr}_\omega\omega_\psi}-\frac{|\partial\operatorname{tr}_\omega\omega_\psi|^2_{g_\psi}}{\operatorname{tr}_\omega\omega_\psi^2}.$$

By [(4.18), Lemma 4.17], we have

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)\log\operatorname{tr}_\omega\omega_\psi &= \frac{\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)\operatorname{tr}_\omega\omega_\psi}{\operatorname{tr}_\omega\omega_\psi}+\frac{|\partial\operatorname{tr}_\omega\omega_\psi|^2_{g_\psi}}{\operatorname{tr}_\omega\omega_\psi^2}\\ &= \frac{1}{\operatorname{tr}_\omega\omega_\psi}(\operatorname{Ric}(g)^{i\bar{j}}g_{\psi i\bar{j}}-g_\psi^{p\bar{q}}g_{\psi i\bar{j}}g^{i\bar{s}}g^{m\bar{j}}\operatorname{Rm}(g)_{p\bar{q}m\bar{s}})\\ &\quad+\frac{1}{\operatorname{tr}_\omega\omega_\psi}\left(\frac{|\partial\operatorname{tr}_\omega\omega_\psi|^2_{g_\psi}}{\operatorname{tr}_\omega\omega_\psi}-g_\psi^{p\bar{q}}g^{i\bar{j}}g_\psi^{a\bar{b}}\nabla_p^g g_{\psi i\bar{b}}\nabla_{\bar{q}}^g g_{\psi a\bar{j}}\right).\end{aligned}$$

We claim that $\frac{|\partial \operatorname{tr}_\omega \omega_\psi|^2_{g_\psi}}{\operatorname{tr}_\omega \omega_\psi} - g_\psi^{p\bar{q}} g^{i\bar{j}} g_\psi^{a\bar{b}} \nabla^g_p g_{\psi i\bar{b}} \nabla^g_{\bar{q}} g_{\psi a\bar{j}} \leq 0$. This is a celebrated inequality in Yau's C^2 estimate. For detailed proof, see [1, Proposition 3.2.5].

The Kähler-Ricci expander (M, g, X) being asymptotically conical, there exists a constant $C(g) > 0$ such that $f|\operatorname{Rm}(g)|_g \leq C(g)$. In normal holomorphic coordinates,

$$-g_\psi^{p\bar{q}} g_{\psi i\bar{j}} g^{i\bar{s}} g^{m\bar{j}} \operatorname{Rm}(g)_{p\bar{q}m\bar{s}} = \sum_{p,i} -g_\psi^{p\bar{p}} g_{\psi i\bar{i}} \operatorname{Rm}(g)_{p\bar{p}i\bar{i}} \leq \frac{C(g)}{f} \operatorname{tr}_\omega \omega_\psi \operatorname{tr}_{\omega_\psi} \omega$$

Finally we get

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \log \operatorname{tr}_\omega \omega_\psi \leq \frac{C(g)}{f} (\operatorname{tr}_{\omega_\psi} \omega + 1).$$

Since f and f_ψ are comparable by Corollary 4.15, (4.22) holds as required. $\square$

Remark 4.19. If (M, g, X) admits positive holomorphic bisectional curvature, then we can reduce (4.22) to

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \log \operatorname{tr}_\omega \omega_\psi \leq \frac{C(g)}{f},$$

by some constant that only depends on g.

In 2016, Chodosh and Fong [10] have proved that an asymptotically conical gradient Kähler-Ricci expander with positive holomorphic bisectional curvature must be isometric to one of the $U(n)-$rotationally symmetric expanding gradient solitons on C^n, as constructed by Cao [3].

With this key lemma, we can now prove Proposition 4.16.

Proof of Proposition 4.16. We choose to use $\frac{\psi}{f_\psi}$ as a barrier function to control $\log \operatorname{tr}_\omega \omega_\psi$.

First we compute the evolution equation of $\frac{\psi}{f_\psi}$

$$\begin{aligned}\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \frac{\psi}{f_\psi} &= -\frac{\Delta_{\omega_\psi}\psi}{f_\psi} + \frac{\dot{\psi}}{f_\psi} - \frac{\frac{X}{2}\cdot\psi}{f_\psi} + \psi \left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \frac{1}{f_\psi} \\ &\quad - 2\operatorname{Re}\left(< \partial\psi, \bar{\partial}\frac{1}{f_\psi} >_{g_\psi}\right) \\ &= -\frac{\Delta_{\omega_\psi}\psi}{f_\psi} + \frac{\dot{\psi}}{f_\psi} - \frac{\frac{X}{2}\cdot\psi}{f_\psi} + \psi\left(\frac{1}{f_\psi} - \frac{|X|^2_{g_\psi}}{f_\psi^3}\right) + \frac{X\cdot\psi}{f_\psi^2} \\ &= -\frac{\Delta_{\omega_\psi}\psi}{f_\psi} + \frac{\dot{\psi}}{f_\psi} + \frac{\psi - \frac{X}{2}\cdot\psi}{f_\psi} - \psi\frac{|X|^2_{g_\psi}}{f_\psi^3} + \frac{X\cdot\psi}{f_\psi^2}.\end{aligned}$$

By Corollary 4.15, there exists a constant $C_1 > 0$ independent of time such that

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \frac{\psi}{f_\psi} \geq -\frac{\Delta_{\omega_\psi}\psi}{f_\psi} - C_1\frac{1}{f_\psi} = \frac{\operatorname{tr}_{\omega_\psi}\omega}{f_\psi} - \frac{C_1 + n}{f_\psi}.$$

Due to Lemma 4.18, there exists a constant $C > 0$ such that

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \log \operatorname{tr}_\omega \omega_\psi \leq \frac{C}{f_\psi} (\operatorname{tr}_{\omega_\psi} \omega + 1).$$

Hence we compute the evolution equation of $\log \operatorname{tr}_\omega \omega_\psi - C\frac{\psi}{f_\psi}$ as follows:

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right) \left(\log \operatorname{tr}_\omega \omega_\psi - C\frac{\psi}{f_\psi}\right) \leq \frac{C}{f_\psi} + C\frac{C_1 + n}{f_\psi} := \frac{C_3}{f_\psi}.$$

Thanks to Proposition 4.1, $\log \operatorname{tr}_\omega \omega_\psi$ is bounded on $M \times [0, T]$ for any $0 < T < T_{\max}$. By applying Lemma 4.13, there exists a constant $C_4 > 0$ such that for all $\tau \in [0, T_{\max})$,

$$\sup_M \left(\log \operatorname{tr}_\omega \omega_\psi - C\frac{\psi}{f_\psi}\right)(\tau) \leq \sup_M \left(\log \operatorname{tr}_\omega \omega_{\psi_0} - C\frac{\psi_0}{f_\psi(0)}\right) + C_4 < \infty.$$

Since $\frac{\psi}{f_\psi}$ and $\frac{1}{f_\psi+K}$ are bounded due to Corollary 4.15, we thus find a constant $C'>0$ such that on $M\times[0,T_{\max})$,

$$\operatorname{tr}_\omega \omega_\psi \le C'.$$

Moreover, $\frac{\omega_\psi^n}{\omega^n}$ being globally bounded from below on $M\times[0,T_{\max})$ due to Corollary 4.15. Thanks to the elementary inequality:

$$\frac{\omega_\psi^n}{\omega^n}\operatorname{tr}_{\omega_\psi}\omega \le (\operatorname{tr}_\omega \omega_\psi)^{n-1},$$

we can find another constant $C''>0$ such that on $M\times[0,T_{\max})$

$$\operatorname{tr}_{\omega_\psi}\omega \le C''.$$

Take $C=\max\{C',C''\}$, we get the desired result. $\square$

4.4. **The higher order estimates along normalized Kähler-Ricci flow.** In this section we omit all the time dependence of $g_\psi(\tau)$, we denote it simply by g_ψ. Moreover, we use ∇(resp. $\overline{\nabla}$) to denote the holomorphic (resp. anti-holomorphic) part of ∇^{g_ψ}, we also use $|\cdot|$ to denote $|\cdot|_{g_\psi}$ and $<>$ to denote the scalar product of g_ψ.

Define a tensor Ψ by

$$\Psi_{ij}^k=\Gamma(g_\psi)_{ij}^k-\Gamma(g)_{ij}^k=g_\psi^{k\bar{l}}\nabla_i^g g_{\psi j\bar{l}}.$$

Define a smooth function S by

$$S:=|\Psi|^2=g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l}.$$

Proposition 4.20 (Modified Phong-Šešum-Sturm equality [18])**.** *Along the normalized Kähler-Ricci flow, S evolves by*

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)S=&-|\overline{\nabla}\Psi|^2-|\nabla\Psi|^2+S\\&+2\operatorname{Re}\left(g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}\left(\nabla^g\operatorname{Ric}(g)_{ip}^k-\nabla^{\bar{b}}\operatorname{Rm}(g)_{i\bar{b}p}^k\right)\overline{\Psi_{jq}^l}\right),\end{aligned}\tag{4.23}$$

where $\nabla^{\bar{b}}=g_\psi^{a\bar{b}}\nabla_a$, $\operatorname{Rm}(g)_{i\bar{b}p}^k:=g^{m\bar{s}}\operatorname{Rm}(g)_{i\bar{b}p\bar{s}}$ and $\nabla^g\operatorname{Ric}(g)_{ip}^k:=g^{k\bar{s}}\nabla_{\bar{s}}^g\operatorname{Ric}(g)_{ip}$.

Proof. Compute

$$\Delta_{\omega_\psi}S=g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}\left(\Delta_{\omega_\psi}\Psi_{ip}^k\overline{\Psi_{jq}^l}+\Psi_{ip}^l\overline{\overline{\Delta}_{\omega_\psi}\Psi_{jq}^l}\right)+|\overline{\nabla}\Psi|^2+|\nabla\Psi|^2,\tag{4.24}$$

where we are writing $\Delta_{\omega_\psi}=g_\psi^{a\bar{b}}\nabla_a\nabla_{\bar{b}}$ for the *rough* Laplacian and $\overline{\Delta}_{\omega_\psi}=g_\psi^{\bar{b}a}\nabla_{\bar{b}}\nabla_a$ for its conjugate. In particular, using the commutation formulae

$$\overline{\Delta}_{\omega_\psi}\Psi_{jq}^l=\Delta_{\omega_\psi}\Psi_{jq}^l+\operatorname{Ric}(g_\psi)_j^b\Psi_{bq}^l+\operatorname{Ric}(g(\psi))_q^b\Psi_{jb}^l-\operatorname{Ric}(g_\psi)_b^l\Psi_{jq}^b.\tag{4.25}$$

Combining (4.24) and (4.25),

$$\begin{aligned}\Delta_{\omega_\psi}S=&\,2\operatorname{Re}(g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}\Delta_{\omega_\psi}\Psi_{ip}^k\overline{\Psi_{jq}^l})+|\overline{\nabla}\Psi|^2+|\nabla\Psi|^2\\&+\operatorname{Ric}(g_\psi)^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l}+g_\psi^{i\bar{j}}\operatorname{Ric}(g_\psi)^{p\bar{q}}g_{\psi k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l}-g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}\operatorname{Ric}(g_\psi)_{k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l}.\end{aligned}\tag{4.26}$$

Now we compute

$$\mathcal{L}_{\frac{X}{2}}S=\frac{X}{2}\cdot S=2\operatorname{Re}(g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}\nabla_{\frac{X}{2}}\Psi_{ip}^k\overline{\Psi_{jq}^l}).\tag{4.27}$$

Notice that

$$\nabla_{\frac{X}{2}}\Psi_{ip}^k=\mathcal{L}_{\frac{X}{2}}\Psi_{ip}^k-\frac{1}{2}(\mathcal{L}_{\frac{X}{2}}g_\psi)_i^a\Psi_{ap}^k-\frac{1}{2}(\mathcal{L}_{\frac{X}{2}}g_\psi)_p^a\Psi_{ia}^k+\frac{1}{2}(\mathcal{L}_{\frac{X}{2}}g_\psi)_a^k\Psi_{ip}^a.\tag{4.28}$$

Combining (4.27) and (4.28),

$$\begin{aligned}\frac{X}{2}\cdot S=&\,2\operatorname{Re}(g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}\mathcal{L}_{\frac{X}{2}}\Psi_{ip}^k\overline{\Psi_{jq}^l})\\&-(\mathcal{L}_{\frac{X}{2}}g_\psi)^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l}-g_\psi^{i\bar{j}}(\mathcal{L}_{\frac{X}{2}}g_\psi)^{p\bar{q}}g_{\psi k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l}+g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}(\mathcal{L}_{\frac{X}{2}}g_\psi)_{k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l}.\end{aligned}\tag{4.29}$$

We now compute the time derivative of S. We claim that

$$\frac{\partial}{\partial\tau}\Psi_{ip}^k = \Delta_{\omega_\psi}\Psi_{ip}^k + \mathcal{L}_{\frac{X}{2}}\Psi_{ip}^k - \nabla^{\bar{b}}\operatorname{Rm}(g)_{i\bar{b}p}^k + g^{k\bar{s}}\nabla_{\bar{s}}^g\operatorname{Ric}(g)_{ip}. \tag{4.30}$$

Given this, together with

$$\frac{\partial}{\partial\tau}g_\psi = \mathcal{L}_{\frac{X}{2}}g_\psi - \operatorname{Ric}(g_\psi) - g_\psi,$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial\tau}S =& \operatorname{Ric}(g_\psi)^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l} + g_\psi^{i\bar{j}}\operatorname{Ric}(g_\psi)^{p\bar{q}}g_{\psi k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l} - g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}\operatorname{Ric}(g_\psi)_{k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l}\\ &- (\mathcal{L}_{\frac{X}{2}}g_\psi)^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l} - g_\psi^{i\bar{j}}(\mathcal{L}_{\frac{X}{2}}g_\psi)^{p\bar{q}}g_{\psi k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l} + g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}(\mathcal{L}_{\frac{X}{2}}g_\psi)_{k\bar{l}}\Psi_{ip}^k\overline{\Psi_{jq}^l}\\ &+ S + 2\operatorname{Re}\left(g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}(\Delta_{\omega_\psi}\Psi_{ip}^k + \mathcal{L}_{\frac{X}{2}}\Psi_{ip}^k)\overline{\Psi_{jq}^l}\right)\\ &+ 2\operatorname{Re}\left(g_\psi^{i\bar{j}}g_\psi^{p\bar{q}}g_{\psi k\bar{l}}\left(\nabla^g\operatorname{Ric}(g)_{ip}^k - \nabla^{\bar{b}}\operatorname{Rm}(g)_{i\bar{b}p}^k\right)\overline{\Psi_{jq}^l}\right).\end{aligned} \tag{4.31}$$

Then (4.23) follows from (4.26), (4.29) and (4.31).

To establish (4.30), compute

$$\frac{\partial}{\partial\tau}\Psi_{ip}^k = \frac{\partial}{\partial\tau}\Gamma(g_\psi)_{ip}^k = \nabla_i(\mathcal{L}_{\frac{X}{2}}g_\psi - \operatorname{Ric}(g_\psi))_p^k. \tag{4.32}$$

On the other hand

$$\begin{aligned}\nabla_{\bar{b}}\Psi_{ip}^k &= \partial_{\bar{b}}(\Gamma(g_\psi)_{ip}^k - \Gamma(g)_{ip}^k) = \operatorname{Rm}(g)_{i\bar{b}p}^k - \operatorname{Rm}(g_\psi)_{i\bar{b}p}^k,\\ \mathcal{L}_{\frac{X}{2}}\Psi_{ip}^k &= \nabla_i(\mathcal{L}_{\frac{X}{2}}g_\psi)_p^k - \nabla_i^g(\mathcal{L}_{\frac{X}{2}}g)_p^k = \nabla_i(\mathcal{L}_{\frac{X}{2}}g_\psi)_p^k - \nabla^g\operatorname{Ric}(g)_{ip}^k,\end{aligned} \tag{4.33}$$

and hence

$$\Delta_{\omega_\psi}\Psi_{ip}^k = g_\psi^{a\bar{b}}\nabla_a\nabla_{\bar{b}}\Psi_{ip}^k = \nabla^{\bar{b}}\operatorname{Rm}(g)_{i\bar{b}p}^k - \nabla_i\operatorname{Ric}(g_\psi)_p^k, \tag{4.34}$$

where for the last equality we have used the second Bianchi identity. Then (4.30) follows from (4.32), (4.33) and (4.34). □

With this evolution equation of S, we can now give a uniform estimate of S along the normalized Kähler-Ricci flow.

Proposition 4.21. *There exists a constant $C > 0$ such that on $M \times [0, T_{\max})$, we have*

$$S \leq C\frac{1}{f_\psi}.$$

Before proving Proposition 4.21, we need to prove the following lemma.

Lemma 4.22. *There exists a constant $C_1 > 0$ such that on $M \times [0, T_{\max})$, we have,*

$$\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)(f_\psi S) \leq C_1 S - \frac{1}{2}f_\psi(|\nabla\Psi|^2 + |\overline{\nabla}\Psi|^2) + \frac{C_1}{f_\psi}. \tag{4.35}$$

Proof. We compute

$$\begin{aligned}\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)(f_\psi S) &= f_\psi\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)S + S\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)f_\psi\\ &\quad - 2\operatorname{Re}(< \partial f_\psi, \bar{\partial}S >)\\ &= f_\psi\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)S - f_\psi S - X\cdot S.\end{aligned}$$

Notice that $\nabla\operatorname{Rm}(g) = \nabla g * \operatorname{Rm}(g) + \nabla^g\operatorname{Rm}(g)$, then by [(4.23), Proposition 4.20], there exits a constant $C_2 > 0$ such that

$$\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)S \leq -|\overline{\nabla}\Psi|^2 - |\nabla\Psi|^2 + S + C_2\left(\frac{S}{f_\psi} + \frac{\sqrt{S}}{f_\psi^{\frac{3}{2}}}\right).$$

Thus,

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(f_\psi S)\leq -f_\psi|\overline{\nabla}\Psi|^2-f_\psi|\nabla\Psi|^2+C_2(S+\frac{\sqrt{S}}{f_\psi^{\frac{1}{2}}})-X\cdot S. \tag{4.36}$$

We notice that by Cauchy-Schwarz inequality, for any $\sigma>0$

$$|X\cdot S|\leq 2|X||\nabla^{g_\psi}\Psi||\Psi|\leq \sigma|X|^2|\nabla^{g_\psi}\Psi|^2+\frac{1}{\sigma}S.$$

Take $\sigma<1$ such that $\sigma|X|^2_{g_\psi}\leq\frac{1}{2}f_\psi$ thanks to Corollary 4.11, hence we get

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(f_\psi S)\leq -\frac{1}{2}f_\psi|\overline{\nabla}\Psi|^2-\frac{1}{2}f_\psi|\nabla\Psi|^2+C_2(S+\frac{\sqrt{S}}{f_\psi^{\frac{1}{2}}})+\frac{1}{\sigma}S.$$

Using Cauchy-Schwarz inequality $\frac{\sqrt{S}}{f_\psi^{\frac{1}{2}}}\leq S+\frac{1}{f_\psi}$, there exists a constant $C_1>0$ such that

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(f_\psi S)\leq -\frac{1}{2}f_\psi|\overline{\nabla}\Psi|^2-\frac{1}{2}f_\psi|\nabla\Psi|^2+C_1(S+\frac{1}{f_\psi}),$$

as required. □

Proof of Proposition 4.21. By [(4.18), Lemma 4.17], there exist constants $C_2,\alpha>0$ such that

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)\operatorname{tr}_\omega\omega_\psi\leq\frac{C_2}{f_\psi}-\alpha S.$$

Together with [(4.35), Lemma 4.22], we get

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)\left(f_\psi S+\frac{C_1}{\alpha}\operatorname{tr}_\omega\omega_\psi\right)\leq C_1(\frac{C_2}{\alpha}+1)\frac{1}{f_\psi}.$$

Thanks to Proposition 4.2, we have a rough estimate that allows us to apply Lemma 4.13: there exists a constant $C_3>0$ such that for all $\tau\in[0,T_{\max})$,

$$\sup_M\left(f_\psi S+\frac{C_1}{\alpha}\operatorname{tr}_\omega\omega_\psi\right)\leq\sup_M\left(f_\psi S+\frac{C_1}{\alpha}\operatorname{tr}_\omega\omega_\psi\right)\Big|_{\tau=0}+C_3.$$

Since $\psi(0)$ satisfies Condition I, $\sup_M\left(f_\psi S+\frac{C_1}{\alpha}\operatorname{tr}_\omega\omega_\psi\right)|_{\tau=0}$ is finite, and because $\operatorname{tr}_\omega\omega_\psi$ is positive, we can find a constant $C>0$ such that on $M\times[0,T_{\max})$, $S\leq\frac{C}{f_\psi}$ holds. □

Building upon the metric equivalence established in Proposition 4.16 and the C^3-estimates obtained in Proposition 4.21, we now derive estimates for the Riemann curvature tensor under the normalized Kähler-Ricci flow.

Proposition 4.23. *Along the normalized Kähler-Ricci flow, the curvature tensor evolves by*

$$\frac{\partial}{\partial\tau}\operatorname{Rm}(g_\psi)=\frac{1}{2}\Delta_{g_\psi}\operatorname{Rm}(g_\psi)+\mathcal{L}_{\frac{X}{2}}\operatorname{Rm}(g_\psi)-\operatorname{Rm}(g_\psi)+\operatorname{Rm}(g_\psi)*\operatorname{Rm}(g_\psi). \tag{4.37}$$

Proof. Recall that

$$\begin{aligned}\frac{\partial}{\partial\tau}\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}}&=\frac{\partial}{\partial\tau}(-g_{\psi p\bar{j}}\partial_{\bar{l}}\Gamma(g_\psi)^p_{ik})\\&=-(\mathcal{L}_{\frac{X}{2}}g_\psi-\operatorname{Ric}(g_\psi)-g_\psi)_{p\bar{j}}\partial_{\bar{l}}\Gamma(g_\psi)^p_{ik}+\nabla_{\bar{l}}\nabla_k\operatorname{Ric}(g_\psi)_{i\bar{j}}-\nabla_{\bar{l}}\nabla_k(\mathcal{L}_{\frac{X}{2}}g_\psi)_{i\bar{j}}\\&=\mathcal{L}_{\frac{X}{2}}\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}}-\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}}-\operatorname{Ric}(g_\psi)^{\bar{a}}_{\bar{j}}\operatorname{Rm}(g_\psi)_{i\bar{a}k\bar{l}}+\nabla_{\bar{l}}\nabla_k\operatorname{Ric}(g_\psi)_{i\bar{j}}\end{aligned}$$

Using the Bianchi identity and the commutation formulae, we obtain

$$\begin{aligned}\Delta_{\omega_\psi}\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}} &= g_\psi^{a\bar{b}}\nabla_a\nabla_{\bar{b}}\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}}\\ &= g_\psi^{a\bar{b}}\nabla_a\nabla_{\bar{l}}\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{b}}\\ &= g_\psi^{a\bar{b}}\nabla_{\bar{l}}\nabla_a\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{b}} + g_\psi^{a\bar{b}}[\nabla_a,\nabla_{\bar{l}}]\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{b}}\\ &= \nabla_{\bar{l}}\nabla_k\operatorname{Ric}(g_\psi)_{i\bar{j}} + \operatorname{Rm}(g_\psi)*\operatorname{Rm}(g_\psi).\end{aligned}$$

By conjugate formulae of $\overline{\Delta}_{\omega_\psi}$ and the fact $\Delta_{g_\psi} = \Delta_{\omega_\psi} + \overline{\Delta}_{\omega_\psi}$, we conclude that

$$\frac{\partial}{\partial\tau}\operatorname{Rm}(g_\psi) = \frac{1}{2}\Delta_{g_\psi}\operatorname{Rm}(g_\psi) + \mathcal{L}_{\frac{X}{2}}\operatorname{Rm}(g_\psi) - \operatorname{Rm}(g_\psi) + \operatorname{Rm}(g_\psi)*\operatorname{Rm}(g_\psi).$$

□

Corollary 4.24. *There exists a dimensional constant $C(n) > 0$ such that*

$$\begin{aligned}\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)|\operatorname{Rm}(g_\psi)|^2 \leq{}& -|\nabla\operatorname{Rm}(g_\psi)|^2 - |\overline{\nabla}\operatorname{Rm}(g_\psi)|^2\\ &+ C(n)|\operatorname{Rm}(g_\psi)|^3 + 2|\operatorname{Rm}(g_\psi)|^2,\end{aligned}\tag{4.38}$$

and for points $(x,\tau)\in M\times[0,T_{\max})$ such that $|\operatorname{Rm}(g_\psi)|(x,\tau) > 0$,

$$\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)|\operatorname{Rm}(g_\psi)|(x,\tau) \leq \frac{C(n)}{2}|\operatorname{Rm}(g_\psi)|^2(x,\tau) + |\operatorname{Rm}(g_\psi)|(x,\tau).\tag{4.39}$$

Proof. First, we prove that (4.38) implies (4.39). Note that

$$\begin{aligned}\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)|\operatorname{Rm}(g_\psi)| ={}& \frac{1}{2|\operatorname{Rm}(g_\psi)|^2}\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)|\operatorname{Rm}(g_\psi)|^2\\ &+ \frac{1}{4|\operatorname{Rm}(g_\psi)|^3}\left|\nabla|\operatorname{Rm}(g_\psi)|^2\right|^2,\end{aligned}\tag{4.40}$$

and

$$\left|\nabla|\operatorname{Rm}(g_\psi)|^2\right|^2 \leq 2|\operatorname{Rm}(g_\psi)|^2\left(|\nabla\operatorname{Rm}(g_\psi)|^2 + |\overline{\nabla}\operatorname{Rm}(g_\psi)|^2\right).\tag{4.41}$$

Then (4.39) comes from (4.40) and (4.41).

Now we prove (4.38). In local holomorphic coordinates, we compute

$$\begin{aligned}\frac{\partial}{\partial\tau}|\operatorname{Rm}(g_\psi)|^2 ={}& \frac{\partial}{\partial\tau}\left(g_\psi^{i\bar{p}}g_\psi^{q\bar{j}}g_\psi^{k\bar{r}}g_\psi^{s\bar{l}}\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}}\operatorname{Rm}(g_\psi)_{\bar{p}q\bar{r}s}\right)\\ ={}& 2\operatorname{Re}\left(<\frac{1}{2}\Delta_{g_\psi}\operatorname{Rm}(g_\psi) + \mathcal{L}_{\frac{X}{2}}\operatorname{Rm}(g_\psi), \operatorname{Rm}(g_\psi)>\right)\\ &+ 2\operatorname{Re}\left(<-\operatorname{Rm}(g_\psi) + \operatorname{Rm}(g_\psi)*\operatorname{Rm}(g_\psi), \operatorname{Rm}(g_\psi)>\right)\\ &+ \frac{\partial}{\partial\tau}g_\psi^{i\bar{p}}g_\psi^{q\bar{j}}g_\psi^{k\bar{r}}g_\psi^{s\bar{l}}\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}}\operatorname{Rm}(g_\psi)_{\bar{p}q\bar{r}s}\\ &+ g_\psi^{i\bar{p}}\frac{\partial}{\partial\tau}g_\psi^{q\bar{j}}g_\psi^{k\bar{r}}g_\psi^{s\bar{l}}\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}}\operatorname{Rm}(g_\psi)_{\bar{p}q\bar{r}s}\\ &+ g_\psi^{i\bar{p}}g_\psi^{q\bar{j}}\frac{\partial}{\partial\tau}g_\psi^{k\bar{r}}g_\psi^{s\bar{l}}\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}}\operatorname{Rm}(g_\psi)_{\bar{p}q\bar{r}s}\\ &+ g_\psi^{i\bar{p}}g_\psi^{q\bar{j}}g_\psi^{k\bar{r}}\frac{\partial}{\partial\tau}g_\psi^{s\bar{l}}\operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}}\operatorname{Rm}(g_\psi)_{\bar{p}q\bar{r}s}.\end{aligned}$$

Moreover,

$$\begin{aligned}\Delta_{\omega_\psi,X}|\operatorname{Rm}(g_\psi)|^2 = {} & 2\operatorname{Re}\left(< \frac{1}{2}\Delta_{g_\psi}\operatorname{Rm}(g_\psi) + \mathcal{L}_{\frac{X}{2}}\operatorname{Rm}(g_\psi), \operatorname{Rm}(g_\psi) >\right) \\ & + |\nabla\operatorname{Rm}(g_\psi)|^2 + |\overline{\nabla}\operatorname{Rm}(g_\psi)|^2 \\ & - \mathcal{L}_{\frac{X}{2}} g_\psi^{i\bar{p}} g_\psi^{q\bar{j}} g_\psi^{k\bar{r}} g_\psi^{s\bar{l}} \operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}} \operatorname{Rm}(g_\psi)_{\bar{p}q\bar{r}s} \\ & - g_\psi^{i\bar{p}} \mathcal{L}_{\frac{X}{2}} g_\psi^{q\bar{j}} g_\psi^{k\bar{r}} g_\psi^{s\bar{l}} \operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}} \operatorname{Rm}(g_\psi)_{\bar{p}q\bar{r}s} \\ & - g_\psi^{i\bar{p}} g_\psi^{q\bar{j}} \mathcal{L}_{\frac{X}{2}} g_\psi^{k\bar{r}} g_\psi^{s\bar{l}} \operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}} \operatorname{Rm}(g_\psi)_{\bar{p}q\bar{r}s} \\ & - g_\psi^{i\bar{p}} g_\psi^{q\bar{j}} g_\psi^{k\bar{r}} \mathcal{L}_{\frac{X}{2}} g_\psi^{s\bar{l}} \operatorname{Rm}(g_\psi)_{i\bar{j}k\bar{l}} \operatorname{Rm}(g_\psi)_{\bar{p}q\bar{r}s}.\end{aligned}$$

Since

$$\frac{\partial}{\partial\tau} g_\psi^{i\bar{p}} = -\mathcal{L}_{\frac{X}{2}} g_\psi^{i\bar{p}} + \operatorname{Ric}(g_\psi)^{i\bar{q}} + g_\psi^{i\bar{p}}.$$

Then we get

$$\begin{aligned}\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)|\operatorname{Rm}(g_\psi)|^2 = {} & 2|\operatorname{Rm}(g_\psi)|^2 + \operatorname{Ric}(g_\psi) * \operatorname{Rm}(g_\psi) * \operatorname{Rm}(g_\psi) \\ & + 2\operatorname{Re}\left(< \operatorname{Rm}(g_\psi) * \operatorname{Rm}(g_\psi), \operatorname{Rm}(g_\psi) >\right).\end{aligned}$$

Thus (4.38) holds as desired. □

The following corollary tells us the Riemann curvature is uniformly bounded along the normalized Kähler-Ricci flow.

Corollary 4.25. *There exists a constant $C > 0$ such that on $M \times [0, T_{\max})$*

$$|\operatorname{Rm}(g_\psi)|_{g_\psi} \leq C.$$

Proof. Notice that

$$\nabla_{\bar{b}}\Psi_{ip}^k = \partial_{\bar{b}}(\Gamma(g_\psi)_{ip}^k - \Gamma(g)_{ip}^k) = \operatorname{Rm}(g)_{i\bar{b}p}^k - \operatorname{Rm}(g_\psi)_{i\bar{b}p}^k.$$

Since g_ψ and g are uniformly comparable on $M \times [0, T_{\max})$ by Proposition 4.16, there exists a constant $C_1 > 0$ such that

$$|\overline{\nabla}\Psi|^2 \geq \frac{1}{2}|\operatorname{Rm}(g_\psi)|^2 - C_1|\operatorname{Rm}(g)|^2.$$

By Proposition 4.15, there exists $C_2 > 0$ such that $|\operatorname{Rm}(g)| \leq \frac{C_2}{f_\psi}$. As a consequence, we conclude that there exists a constant $C_3 > 0$ such that

$$|\overline{\nabla}\Psi|^2 \geq \frac{1}{2}|\operatorname{Rm}(g_\psi)|^2 - \frac{C_3}{f_\psi^2}.$$

Recall [(4.23), Proposition 4.20] and Proposition 4.21 and the fact that f_ψ is bounded from below, there exists a constant $C_4 > 0$ such that

$$\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right) S \leq -\frac{1}{2}|\operatorname{Rm}(g_\psi)|^2 + C_4.$$

Recall [(4.39), Corollary 4.24],

$$\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)|\operatorname{Rm}(g_\psi)| \leq \frac{C(n)}{2}|\operatorname{Rm}(g_\psi)|^2 + |\operatorname{Rm}(g_\psi)|.$$

Combining these inequalities, we claim that there exists a constant $C_5 > 0$ such that

$$\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi,X}\right)(|\operatorname{Rm}(g_\psi)| + (C(n)+1)S) \leq -\frac{1}{2}|\operatorname{Rm}(g_\psi)|^2 + |\operatorname{Rm}(g_\psi)| + C_5.$$

Recall the evolution equation of $e^\tau f_\psi$: $\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)e^\tau f_\psi=0$. Thus, for all $\eta>0$, we have

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(|\operatorname{Rm}(g_\psi)|+(C(n)+1)S-\eta e^\tau f_\psi)\le -\frac{1}{2}|\operatorname{Rm}(g_\psi)|^2+|\operatorname{Rm}(g_\psi)|+C_5.$$

For all $0<T<T_{\max}$, $|\operatorname{Rm}(g_\psi)|+(C(n)+1)S$ is bounded on $M\times[0,T]$. Hence, the function $|\operatorname{Rm}(g_\psi)|+(C(n)+1)S-\eta e^\tau f_\psi$ tends to $-\infty$ at spatial infinity uniformly. This implies the existence of maximum point of $|\operatorname{Rm}(g_\psi)|+(C(n)+1)S-\eta e^\tau f_\psi$. We denote it by (x_0,τ_0). If $\tau_0>0$, then by the weak maximum principle, we have

$$\begin{aligned}0&\le\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(|\operatorname{Rm}(g_\psi)|+(C(n)+1)S-\eta e^\tau f_\psi)(x_0,\tau_0)\\&\le -\frac{1}{2}|\operatorname{Rm}(g_\psi)|^2(x_0,\tau_0)+|\operatorname{Rm}(g_\psi)|(x_0,\tau_0)+C_5.\end{aligned}$$

Therefore, we get $|\operatorname{Rm}(g_\psi)|(x_0,\tau_0)\le 1+\sqrt{2C_5+1}$. For all $(x,\tau)\in M\times[0,T]$, since S is non negative and bounded uniformly by Proposition 4.21, it follows that

$$\begin{aligned}|\operatorname{Rm}(g_\psi)|(x,\tau)-\eta e^\tau f_\psi(x,\tau)&\le|\operatorname{Rm}(g_\psi)|(x_0,\tau_0)+(C(n)+1)S(x_0,\tau_0)\\&\le 1+\sqrt{2C_5+1}+\sup_{M\times[0,T_{\max})}S(C(n)+1).\end{aligned}$$

If $\tau_0=0$, then for all $(x,\tau)\in M\times[0,T]$, we have

$$|\operatorname{Rm}(g_\psi)|(x,\tau)-\eta e^\tau f_\psi(x,\tau)\le\sup_M(|\operatorname{Rm}(g_\psi)|(0)+(C(n)+1)S(0))<\infty.$$

Let η tend to 0, and we conclude that there exists a constant $C>0$ such that on $M\times[0,T_{\max})$, $\sup_M|\operatorname{Rm}(g_\psi))|(\tau)\le C$ holds. □

Corollary 4.26. *Let $T_{\max}$ be the maximal time as in Theorem 3.6, then $T_{\max}=\infty$.*

Proof. Recall in Theorem 3.6, if $T_{\max}<\infty$, then

$$\limsup_{\tau\to T_{\max}}\sup_M|\operatorname{Rm}(g_\psi(\tau))|_{g_\psi(\tau)}=+\infty,$$

this gives a contradiction to Corollary 4.25. Therefore $T_{\max}=\infty$. □

Now to prove Theorem C, it suffices to prove the uniform bound for all orders' covariant derivatives of Riemann curvature operators.

Proof of Theorem C. This proof relies on Shi's global estimates on curvature along Ricci flow (see [19]). Recall $g_\varphi(t)_{t\ge1}$ is the solution to Kähler-Ricci flow corresponding to solution $g_\psi(\tau)$ as in Theorem 3.1. The curvature bound of $g_\psi(\tau)$ in Corollary 5.2 implies that there exists a constant $C>0$ such that for all $t\ge1$

$$\sup_M|\operatorname{Rm}(g_\varphi(t))|_{g_\varphi(t)}\le\frac{C}{t}. \tag{4.42}$$

Now we take $\alpha>1$, for all $t\ge\alpha$, we apply Shi's global estimates on the time interval $[\frac{t}{\alpha},t]$. Since on $M\times[\frac{t}{\alpha},t]$, we have

$$\sup_{M\times[\frac{t}{\alpha},t]}|\operatorname{Rm}(g_\varphi(\cdot))|_{g_\varphi(\cdot)}\le\frac{\alpha C}{t}.$$

Thanks to Shi's global estimates, for any $m\in\mathbb{N}_0$, there exists a constant $C_m=C(\dim M,m)$ such that for all $s\in[\frac{t}{\alpha},t]$

$$\sup_M|(\nabla^{g_\varphi(s)})^m\operatorname{Rm}(g_\varphi(s))|_{g_\varphi(s)}\le C_m\frac{\alpha C}{t}(s-\frac{t}{\alpha})^{-\frac{m}{2}}.$$

In particular, there exists a constant $C_m(\alpha)>0$ such that,

$$\sup_M|(\nabla^{g_\varphi(t)})^m\operatorname{Rm}(g_\varphi(t))|_{g_\varphi(t)}\le C_m\frac{\alpha C}{t}(t-\frac{t}{\alpha})^{-\frac{m}{2}}=C_m(\alpha)t^{-1-\frac{m}{2}}.$$

By the correspondence of Kähler-Ricci flow and normalized Kähler-Ricci flow, therefore, for all $m \geq 1$,

$$\sup_M |(\nabla^{g_\psi(\tau)})^m \operatorname{Rm}(g_\psi(\tau))|_{g_\psi(\tau)} \leq C_m(\alpha),$$

holds for all $\tau \geq \log \alpha$. □

5. Proof of convergence theorem A

Let (M, g, X) be an asymptotically conical gradient Kähler-Ricci expander and let ψ_0 be a smooth function defined on M satisfying Condition II. Since Condition II is stronger than Condition I, all the previous uniform estimates along the normalized Kähler-Ricci flow still hold. In summary, we have

Proposition 5.1. *There exists a real-valued smooth function $\psi \in C^\infty(M \times [0,\infty))$ such that $g_\psi := g + \partial\bar{\partial}\psi$ is an immortal complete solution to normalized Kähler-Ricci flow starting from g_{ψ_0} and $\psi(\tau)_{\tau\in[0,\infty)}$ satisfies*

$$\begin{aligned}
&\frac{\partial}{\partial \tau}\psi(\tau) = \log \frac{\omega_\psi(\tau)^n}{\omega^n} + \frac{X}{2}\cdot\psi(\tau) - \psi(\tau),\\
&JX \cdot \psi = 0,\\
&\psi(0) = \psi_0.
\end{aligned}$$

Moreover, there exists a constant $C > 1$ such that for all $\tau \geq 0$

(i) $\frac{1}{C}g \leq g_\psi \leq Cg$;
(ii) $|\frac{X}{2}\cdot\psi - \psi| \leq C$;
(iii) $\frac{1}{C}f \leq f_\psi \leq Cf$;
(iv) $|\dot{\psi}(\tau)| \leq Ce^{-\tau}, f_\psi|\partial\dot{\psi}|^2_{g_\psi} \leq Ce^{-\tau}$;
(v) $f_\psi S \leq C$;
(vi) $|\operatorname{Rm}(g_\psi)| \leq C$.

5.1. A priori estimates.

To study the convergence of flow, we need better uniform estimates on the Riemann curvature $\operatorname{Rm}(g_\psi)$. In this subsection we omit all the time dependence of $g_\psi(\tau)$, we denote it simply by g_ψ. Moreover, we use ∇(resp. $\overline{\nabla}$) to denote the holomorphic (resp. anti-holomorphic) part of ∇^{g_ψ}, we also use $|\cdot|$ to denote $|\cdot|_{g_\psi}$ and $<>$ to denote the scalar product of g_ψ.

Proposition 5.2. *There exists a constant $C > 0$ such that on $M \times [0,\infty)$,*

$$|\operatorname{Rm}(g_\psi)| \leq C\frac{1}{f_\psi},$$

holds uniformly.

Proof. We consider $f_\psi|\operatorname{Rm}(g_\psi)|$, at a point $(x,\tau)\in M\times[0,\infty)$ such that $|\operatorname{Rm}(g_\psi)|(x,\tau)>0$. Due to [(4.39), Corollary 4.24], there exist constants $C_1, A>0$ such that

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(f_\psi|\operatorname{Rm}(g_\psi)|) &= -f_\psi|\operatorname{Rm}(g_\psi)|+f_\psi\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)|\operatorname{Rm}(g_\psi)|\\ &\quad -2\operatorname{Re}<\partial f_\psi,\bar\partial|\operatorname{Rm}(g_\psi)|>_{g_\psi}\\ &\le \frac{C_1}{2}f_\psi|\operatorname{Rm}(g_\psi)|^2-X\cdot|\operatorname{Rm}(g_\psi)|\\ &= \frac{C_1}{2}f_\psi|\operatorname{Rm}(g_\psi)|^2-\frac{X}{f_\psi}\cdot(f_\psi|\operatorname{Rm}(g_\psi)|)+\frac{|X|^2}{f_\psi}|\operatorname{Rm}(g_\psi)|\\ &\le \frac{C_1}{2}f_\psi|\operatorname{Rm}(g_\psi)|^2-\frac{X}{f_\psi}\cdot(f_\psi|\operatorname{Rm}(g_\psi)|)+A|\operatorname{Rm}(g_\psi)|\\ &\le \frac{C_1}{2}f_\psi|\operatorname{Rm}(g_\psi)|^2-\frac{X}{f_\psi}\cdot(f_\psi|\operatorname{Rm}(g_\psi)|)\\ &\quad +Af_\psi|\operatorname{Rm}(g_\psi)|^2+\frac{A}{f_\psi}\end{aligned}$$

Here the fourth line inequality is ensured by Corollary 4.11, and the last line inequality comes from the Cauchy-Schwarz inequality. If we denote $\frac{X}{2}-\frac{X}{f_\psi}$ by $\tilde X$, and we still use $\frac{C_1}{2}$ to denote $\frac{C_1}{2}+A$, then we get

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi}-\tilde X\right)(f_\psi|\operatorname{Rm}(g_\psi)|)\le\frac{C_1}{2}f_\psi|\operatorname{Rm}(g_\psi)|^2+\frac{A}{f_\psi}.$$

Moreover, recall [(4.36), Lemma 4.22, there is a constant C_2 so that

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(f_\psi S)\le -f_\psi|\overline\nabla\Psi|^2-f_\psi|\nabla\Psi|^2+C_2(S+\frac{\sqrt S}{f_\psi^{\frac12}})-X\cdot S.$$

Therefore,

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi}-\tilde X\right)(f_\psi S)\le -f_\psi|\overline\nabla\Psi|^2-f_\psi|\nabla\Psi|^2+C_2(S+\frac{\sqrt S}{f_\psi^{\frac12}})+\frac{|X|^2}{f_\psi}S.$$

Recall that there exists a constant $C_3>0$ so that $S\le\frac{C_3}{f_\psi}$ and $|\overline\nabla\Psi|^2\ge\frac12|\operatorname{Rm}(g_\psi)|^2-\frac{C_3}{f_\psi}$, we conclude that there exists a constant $C_4>0$ so that

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi}-\tilde X\right)(f_\psi S)\le -\frac14 f_\psi|\operatorname{Rm}(g_\psi)|^2+C_4\frac{1}{f_\psi}.$$

Now we use $e^{2\tau}f_\psi^2$ as a barrier function, we compute:

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi}-\tilde X\right)(e^{2\tau}f_\psi^2) &= 2e^{2\tau}f_\psi^2+e^{2\tau}(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi}-\tilde X)(f_\psi^2)\\ &= 2e^{2\tau}f_\psi^2+e^{2\tau}(-2f_\psi^2+2|\partial f_\psi|^2)\ge 0.\end{aligned}$$

Hence we get there exists a constant $C_5>0$ such that for all $\eta>0$,

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi}-\tilde X\right)\left(f_\psi|\operatorname{Rm}(g_\psi)+(2C_1+4)f_\psi S-\eta e^{2\tau}f_\psi^2\right)\le -|\operatorname{Rm}(g_\psi)|^2f_\psi+\frac{C_4}{f_\psi}.$$

For all $T<\infty$, there exists a constant $B>0$ such that $f_\psi|\operatorname{Rm}(g_\psi)+(2C_1+4)f_\psi S\le Bf$ by the rough bound of Riemann curvature in Theorem 3.6, therefore, $f_\psi|\operatorname{Rm}(g_\psi)+(2C_1+4)f_\psi-\eta e^{2\tau}f_\psi^2<0$ tends to $-\infty$ at spatial infinity uniformly so that the maximum point (x_0,τ_0) exists. If $\tau_0>0$, by the weak maximum principle, therefore,

$$0\le -|\operatorname{Rm}(g_\psi)|^2f_\psi(x_0,\tau_0)+\frac{C_4}{f_\psi(x_0,\tau_0)}.$$

This implies that $|\operatorname{Rm}(g_\psi)|(x_0,\tau_0) \leq \sqrt{C_4} f_\psi(x_0,\tau_0)^{-1}$, and therefore for all $\tau \in [0,T]$

$$f_\psi|\operatorname{Rm}(g_\psi)| + (2C_1+4)f_\psi S - \eta e^{2\tau} f_\psi^2 \leq f_\psi|\operatorname{Rm}(g_\psi)|(x_0,\tau_0) + (2C_1+4)f_\psi S(x_0,\tau_0).$$

Since $|\operatorname{Rm}(g_\psi)|(x_0,\tau_0) \leq \sqrt{C_4} f_\psi(x_0,\tau_0)^{-1}$ and $S \leq \frac{C}{f_\psi}$ for some C on $M \times [0,\infty)$, we conclude that there exists a constant $C_5 > 0$ independent of T and η such that for all $\tau \in [0,T]$

$$f_\psi|\operatorname{Rm}(g_\psi)| + (2C_1+4)f_\psi S - \eta e^{2\tau} f_\psi^2 \leq C_5.$$

If $\tau_0 = 0$, we have for all $\tau \in [0,T]$,

$$f_\psi|\operatorname{Rm}(g_\psi)| + (2C_1+4)f_\psi S - \eta e^{2\tau} f_\psi^2 \leq \sup_M f_\psi(0)|\operatorname{Rm}(g_\psi)|(0) + (2C_1+4)f_\psi(0)S(0) < \infty.$$

Take $C_6 = C_5 + \sup_M f_\psi(0)|\operatorname{Rm}(g_\psi)(0) + (2C_1+4)f_\psi(0)S(0)$, this constant is independent of T and η, hence we have for all $\tau \in [0,\infty)$, for all $\eta > 0$,

$$f_\psi|\operatorname{Rm}(g_\psi)| \leq f_\psi|\operatorname{Rm}(g_\psi) + (2C_1+4)f_\psi S \leq C_6 + \eta e^{2\tau} f_\psi^2. \tag{5.1}$$

Let η in (5.1) tend to 0 to get $f_\psi|\operatorname{Rm}(g_\psi)| \leq C_6$ as desired. □

Now we use ∇ to denote ∇^{g_ψ}, the real covariant derivative with respect to g_ψ, and we investigate to estimate higher order covariant derivatives of $\operatorname{Rm}(g_\psi)$.

Before developing the evolution equation of high order covariant derivatives of the Riemann curvature tensor, we need a technical lemma as follows:

Lemma 5.3. *Let $k \in \mathbb{N}^*$ and let $R \in C^\infty(\otimes^k T^*M)$ be a $k-$tensor that may depend on time. Then along the normalized Kähler-Ricci flow*

$$\frac{\partial}{\partial\tau}\nabla R - \mathcal{L}_{\frac{X}{2}}(\nabla R) = \nabla\left(\frac{\partial}{\partial\tau}R - \mathcal{L}_{\frac{X}{2}}R\right) + \nabla\operatorname{Ric} * R.$$

Proof. We compute it in local real coordinates, let $\{a, i_0, i_1, ..., i_k\}$ be any real index.

$$\begin{aligned}\frac{\partial}{\partial\tau}\nabla_{i_0}R_{i_1\dots i_k} &= \nabla_{i_0}\left(\frac{\partial}{\partial\tau}R\right)_{i_1\dots i_k} - \sum_j \frac{\partial}{\partial\tau}\Gamma(g_\psi)^a_{i_0 i_j}R_{i_1\dots a\dots i_k}\\ &= \nabla_{i_0}\left(\frac{\partial}{\partial\tau}R\right)_{i_1\dots i_k} - \sum_j A^a_{i_0 i_j}R_{i_1\dots a\dots i_k}.\end{aligned}\tag{5.2}$$

Here

$$\begin{aligned}A^a_{i_0 i_j} &= \frac{1}{2}g_\psi^{ab}\left(\nabla_{i_0}(\mathcal{L}_{\frac{X}{2}}g_\psi)_{i_j b} + \nabla_{i_j}(\mathcal{L}_{\frac{X}{2}}g_\psi)_{i_0 b} - \nabla_b(\mathcal{L}_{\frac{X}{2}}g_\psi)_{i_0 i_j}\right)\\ &\quad - \frac{1}{2}g_\psi^{ab}\left(\nabla_{i_0}\operatorname{Ric}(g_\psi)_{i_j b} + \nabla_{i_j}\operatorname{Ric}(g_\psi)_{i_0 b} - \nabla_b\operatorname{Ric}(g_\psi)_{i_0 i_j}\right).\end{aligned}$$

Moreover the Lie derivative of ∇R in local holomorphic coordinates is given by:

$$\begin{aligned}\mathcal{L}_{\frac{X}{2}}(\nabla R)_{i_0 i_1\dots i_k} &= \nabla_{i_0}\left(\mathcal{L}_{\frac{X}{2}}R\right)_{i_1\dots i_k} - \sum_j\left(\mathcal{L}_{\frac{X}{2}}\Gamma(g_\psi)\right)^a_{i_0 i_j}R_{i_1\dots a\dots i_k}\\ &= \nabla_{i_0}\left(\mathcal{L}_{\frac{X}{2}}R\right)_{i_1\dots i_k} - \sum_j B^a_{i_0 i_j}R_{i_1\dots a\dots i_k},\end{aligned}\tag{5.3}$$

where $B^a_{i_0 i_j} = \frac{1}{2}g_\psi^{ab}\left(\nabla_{i_0}(\mathcal{L}_{\frac{X}{2}}g_\psi)_{i_j b} + \nabla_{i_j}(\mathcal{L}_{\frac{X}{2}}g_\psi)_{i_0 b} - \nabla_b(\mathcal{L}_{\frac{X}{2}}g_\psi)_{i_0 i_j}\right)$.

Combing (5.2) and (5.3), we have that $\frac{\partial}{\partial\tau}\nabla R - \mathcal{L}_{\frac{X}{2}}(\nabla R) = \nabla\left(\frac{\partial}{\partial\tau}R - \mathcal{L}_{\frac{X}{2}}R\right) + \nabla\operatorname{Ric} * R$ holds as desired. □

Lemma 5.4. *Along the normalized Kähler-Ricci flow, the $m-th$ covariant derivative curvature tensor evolves by*

$$\begin{aligned}\frac{\partial}{\partial\tau}\nabla^m \operatorname{Rm}(g_\psi) = &\frac{1}{2}\Delta_{g_\psi}\nabla^m \operatorname{Rm}(g_\psi) + \mathcal{L}_{\frac{X}{2}}\nabla^m \operatorname{Rm}(g_\psi) - \nabla^m \operatorname{Rm}(g_\psi) \\ &+ \sum_{p+q=m} \nabla^p \operatorname{Rm}(g_\psi) * \nabla^q \operatorname{Rm}(g_\psi).\end{aligned} \tag{5.4}$$

Proof. This result is proved by induction. For $m = 0$, (5.4) holds thanks to Proposition 4.23. We suppose that (5.4) holds for all integers $k \leq m$. By applying Lemma 5.3 for tensor $\nabla^m \operatorname{Rm}(g_\psi)$, we have

$$\begin{aligned}\frac{\partial}{\partial\tau}\nabla^{m+1} \operatorname{Rm}(g_\psi) - \mathcal{L}_{\frac{X}{2}}\nabla^{m+1} \operatorname{Rm}(g_\psi) = &\nabla\left(\frac{\partial}{\partial\tau}\nabla^m \operatorname{Rm}(g_\psi) - \mathcal{L}_{\frac{X}{2}}\nabla^m \operatorname{Rm}(g_\psi)\right) \\ &+ \nabla \operatorname{Ric}(g_\psi) * \nabla^m \operatorname{Rm}(g_\psi).\end{aligned} \tag{5.5}$$

And by high order Bochner's formula

$$\begin{aligned}\frac{1}{2}\Delta_{g_\psi}\nabla^{m+1} \operatorname{Rm}(g_\psi) = &\nabla\left(\frac{1}{2}\Delta_{g_\psi}\nabla^m \operatorname{Rm}(g_\psi)\right) \\ &+ \nabla \operatorname{Rm}(g_\psi) * \nabla^m \operatorname{Rm}(g_\psi) + \operatorname{Rm}(g_\psi) * \nabla^{m+1} \operatorname{Rm}(g_\psi).\end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6) and the induction hypothesis, we have that

$$\begin{aligned}&\frac{\partial}{\partial\tau}\nabla^{m+1} \operatorname{Rm}(g_\psi) - \frac{1}{2}\Delta_{g_\psi}\nabla^{m+1} \operatorname{Rm}(g_\psi) - \mathcal{L}_{\frac{X}{2}}\nabla^{m+1} \operatorname{Rm}(g_\psi) \\ = &\nabla\left(\frac{\partial}{\partial\tau}\nabla^m \operatorname{Rm}(g_\psi) - \frac{1}{2}\Delta_{g_\psi}\nabla^m \operatorname{Rm}(g_\psi) - \mathcal{L}_{\frac{X}{2}}\nabla^m \operatorname{Rm}(g_\psi)\right) \\ &- \nabla \operatorname{Rm}(g_\psi) * \nabla^m \operatorname{Rm}(g_\psi) + \operatorname{Rm}(g_\psi) * \nabla^{m+1} \operatorname{Rm}(g_\psi) \\ = &-\nabla^{m+1} \operatorname{Rm}(g_\psi) + \nabla\left(\sum_{p+q=m} \nabla^p \operatorname{Rm}(g_\psi) * \nabla^q \operatorname{Rm}(g_\psi)\right) \\ &- \nabla \operatorname{Rm}(g_\psi) * \nabla^m \operatorname{Rm}(g_\psi) + \operatorname{Rm}(g_\psi) * \nabla^{m+1} \operatorname{Rm}(g_\psi).\end{aligned}$$

Thus, (5.4) holds for all $m \in \mathbb{N}_0$. ☐

Corollary 5.5. *For $m \in \mathbb{N}_0$, there exists a constant $C(m, n) > 0$ such that*

$$\begin{aligned}&\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi, X}\right)\left(|\nabla^m \operatorname{Rm}(g_\psi)|^2\right) \\ \leq &-|\nabla^{m+1} \operatorname{Rm}(g_\psi)|^2 + (m+2)|\nabla^m \operatorname{Rm}(g_\psi)|^2 \\ &+ C(m, n)\sum_{p+q=m} |\nabla^p \operatorname{Rm}(g_\psi)||\nabla^q \operatorname{Rm}(g_\psi)||\nabla^m \operatorname{Rm}(g_\psi)|.\end{aligned} \tag{5.7}$$

Proof. This proof is analogous to the proof of [(4.38), Corollary 4.24] and will therefore be omitted. ☐

Corollary 5.6. *For all $m \in \mathbb{N}_0$ there exists a constant $C_m > 0$ such that on $M \times [0, \infty)$,*

$$|\nabla^m \operatorname{Rm}(g_\psi)| \leq C_m \frac{1}{f_\psi^{\frac{m}{2}+1}}. \tag{5.8}$$

To obtain this estimate, we need a rough estimate on higher order covariant derivatives of $\operatorname{Rm}(g_\psi)$. Now we consider $g_\varphi(t)_{t\in[1,\infty)}$ as the solution to Kähler-Ricci flow corresponding to $g_\psi(\tau)_{\tau\in[1,\infty)}$. The following lemma asserts that $\nabla^m \operatorname{Rm}(g_\psi)$ remains uniformly bounded on every compact time interval $[0, T]$. This follows from Shi's local derivative estimates for the Ricci flow on non-compact manifolds.

Lemma 5.7. *For any $0 < T < \infty$, for all $m \in \mathbb{N}_0$ there exists a constant $B_m = B(m, T) > 0$ such that on $M \times [0, T]$,*

$$|\nabla^m \operatorname{Rm}(g_\psi)| \leq C_m. \tag{5.9}$$

Proof. Recall that $g_\varphi(t) = t\Phi_t^* g_\psi(\log t)$ for all $t \geq 1$ is the solution to Kähler-Ricci flow as in Proposition 2.3. It suffices to prove that there exists a constant $B_m > 0$ such that on $M \times [1, e^T]$,

$$|(\nabla^{g_\varphi})^m \operatorname{Rm}(g_\varphi)|_{g_\varphi} \leq B_m.$$

By Proposition 5.2, on $M \times [0, \infty)$, there exists a constant $C > 0$ such that

$$|\operatorname{Rm}(g_\psi)| \leq \frac{C}{f}.$$

Hence on $M \times [1, e^T]$, there exists a constant $B = B(T)$ such that

$$|\operatorname{Rm}(g_\varphi)|_{g_\varphi} \leq \frac{B(T)}{f}.$$

Moreover, by our initial condition $|(\nabla^g)^k(g_{\psi_0} - g)| = O(f^{-\frac{k}{2}})$ as in Definition 1.6, for all $k \leq m$, there exists a constant $A_k > 0$ such that

$$|(\nabla^{g_\varphi})^k \operatorname{Rm}(g_\varphi)|(0) \leq \frac{A_k}{f^{1+\frac{m}{2}}}.$$

Thus by Perelman's pseudolocality theorem (see [21, Lemma A.4] or [12]) , we conclude that there exists a constant B_m such that

$$|(\nabla^{g_\varphi})^m \operatorname{Rm}(g_\varphi)|_{g_\varphi} \leq B_m.$$

□

With the above Lemma, we can prove Corollary 5.6.

Proof of Corollary 5.6. For $m \in \mathbb{N}_0$, we compute,

$$\begin{aligned}\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi, X}\right)(f_\psi^{m+2}|\nabla^m \operatorname{Rm}(g_\psi)|^2) = {}& |\nabla^m \operatorname{Rm}(g_\psi)|^2 \left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi, X}\right) f_\psi^{m+2} \\ & + f_\psi^{m+2}\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi, X}\right)|\nabla^m \operatorname{Rm}(g_\psi)|^2 \\ & - 2(m+2) f_\psi^{m+1} \operatorname{Re}(< \partial f_\psi, \bar{\partial}|\nabla^m \operatorname{Rm}(g_\psi)|^2 >).\end{aligned} \tag{5.10}$$

On the one hand,

$$\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi, X}\right) f_\psi^{m+2} = -(m+2) f_\psi^{m+2} - (m+2)(m+1)|\partial f_\psi|^2 f_\psi^m. \tag{5.11}$$

On the other hand, by Corollary 5.5,

$$\begin{aligned}&\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi, X}\right)\left(|\nabla^m \operatorname{Rm}(g_\psi)|^2\right) \\ &\leq -|\nabla^{m+1} \operatorname{Rm}(g_\psi)|^2 + (m+2)|\nabla^m \operatorname{Rm}(g_\psi)|^2 \\ &\quad + C(m,n) \sum_{p+q=m} |\nabla^p \operatorname{Rm}(g_\psi)||\nabla^q \operatorname{Rm}(g_\psi)||\nabla^m \operatorname{Rm}(g_\psi)|.\end{aligned} \tag{5.12}$$

By combining (5.10), (5.11) and (5.12), we get,

$$\begin{aligned}&\left(\frac{\partial}{\partial\tau} - \Delta_{\omega_\psi, X}\right)(f_\psi^{m+2}|\nabla^m \operatorname{Rm}(g_\psi)|^2) \\ &\leq -f_\psi^{m+2}|\nabla^{m+1} \operatorname{Rm}(g_\psi)|^2 \\ &\quad + f_\psi^{m+2} C(n,m) \sum_{p+q=m} |\nabla^p \operatorname{Rm}(g_\psi)||\nabla^q \operatorname{Rm}(g_\psi)||\nabla^m \operatorname{Rm}(g_\psi)| \\ &\quad - (m+2) f_\psi^{m+1} X \cdot |\nabla^m \operatorname{Rm}(g_\psi)|^2.\end{aligned}$$

By Cauchy-Schwarz inequality, for any $\sigma > 0$, we have

$$\begin{aligned} X \cdot |\nabla^m \operatorname{Rm}(g_\psi)|^2 &\leq 2|X||\nabla^{m+1} \operatorname{Rm}(g_\psi)||\nabla^m \operatorname{Rm}(g_\psi)| \\ &\leq \sigma |X|^2 |\nabla^{m+1} \operatorname{Rm}(g_\psi)|^2 + \frac{1}{\sigma}|\nabla^m \operatorname{Rm}(g_\psi)|^2. \end{aligned}$$

Choose $\sigma > 0$ such that $(m+2)\sigma|X|^2 \leq \frac{1}{2} f_\psi$ thanks to Corollary 4.11, to get

$$\begin{aligned} &\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(f_\psi^{m+2}|\nabla^m \operatorname{Rm}(g_\psi)|^2) \\ &\leq -\frac{1}{2} f_\psi^{m+2}|\nabla^{m+1}\operatorname{Rm}(g_\psi)|^2 + (m+2)\frac{1}{\sigma} f_\psi^{m+1}|\nabla^m \operatorname{Rm}(g_\psi)|^2 \\ &\quad + f_\psi^{m+2} C(n,m) \sum_{p+q=m} |\nabla^p \operatorname{Rm}(g_\psi)||\nabla^q \operatorname{Rm}(g_\psi)||\nabla^m \operatorname{Rm}(g_\psi)|. \end{aligned} \tag{5.13}$$

Now we use induction to prove [(5.8), Corollary 5.6], since it holds when $m = 0$ by Proposition 5.2, we suppose that (5.8) holds for all $k \leq m \in \mathbb{N}_0$, now for $k = m+1$, (5.13) implies that there exists a constant $C' > 0$ such that

$$\begin{aligned} \left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(f_\psi^{m+3}|\nabla^{m+1} \operatorname{Rm}(g_\psi)|^2) &\leq C' f_\psi^{m+2}|\nabla^{m+1}\operatorname{Rm}(g_\psi)|^2 \\ &\quad + C' f_\psi^{\frac{m+1}{2}} |\nabla^{m+1}\operatorname{Rm}(g_\psi)|. \end{aligned}$$

By Cauchy-Schwarz inequality, there exists a constant $C'' > 0$ such that

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(f_\psi^{m+3}|\nabla^{m+1} \operatorname{Rm}(g_\psi)|^2) \leq C'' f_\psi^{m+2}|\nabla^{m+1}\operatorname{Rm}(g_\psi)|^2 + C''\frac{1}{f_\psi}.$$

Now for m, by induction hypothesis and (5.13), we can find a constant $C_0 > 0$ such that

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(f_\psi^{m+2}|\nabla^{m} \operatorname{Rm}(g_\psi)|^2) \leq -\frac{1}{2} f_\psi^{m+2}|\nabla^{m+1}\operatorname{Rm}(g_\psi)|^2 + C_0\frac{1}{f_\psi}.$$

Hence

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(f_\psi^{m+3}|\nabla^{m+1} \operatorname{Rm}(g_\psi)|^2 + 2C'' f_\psi^{m+2}|\nabla^m \operatorname{Rm}(g_\psi)|^2) \leq (2C''C_0 + C'')\frac{1}{f_\psi}.$$

Thanks to Lemma 5.7, we can apply Lemma 4.13 on $f_\psi^{m+3}|\nabla^m \operatorname{Rm}(g_\psi)|^2 + 2C'' f_\psi^{m+2}|\nabla^{m+1}\operatorname{Rm}(g_\psi)|^2$, there exists a constant $C_1 > 0$ such that on $M \times [0, \infty)$ such that

$$\sup_{M\times[0,\infty)} f_\psi^{m+3}|\nabla^m \operatorname{Rm}(g_\psi)|^2 + 2C'' f_\psi^{m+2}|\nabla^{m+1}\operatorname{Rm}(g_\psi)|^2 \leq C_1 + C_2,$$

with $C_2 = \sup_M f_\psi^{m+3}(0)|\nabla^{m+1}\operatorname{Rm}(g_\psi)|^2(0) + 2C'' f_\psi^{m+2}(0)|\nabla^{m+1}\operatorname{Rm}(g_\psi)|^2(0)$ which is finite by Condition II in Definition 1.6, thus [(5.8), Corollary 5.6] holds for $m+1$, therefore (5.8) holds for all integers as expected. □

With these crucial curvature estimates in hand, one can investigate other geometric tensors along the normalized Kähler–Ricci flow. Of particular importance is the tensor

$$\mathcal{L}_{\frac{X}{2}} g_\psi - \operatorname{Ric}(g_\psi) - g_\psi,$$

which measures whether a metric is an expanding Kähler–Ricci soliton. By the normalized Kähler–Ricci flow equation, this tensor is in fact given by $\partial\bar{\partial}\dot{\psi}$.

Definition 5.8 (Obstruction tensor)**.** Define $T := \partial\bar{\partial}\dot{\psi}$, or equivalently, by normalized Kähler-Ricci flow equation

$$T = \mathcal{L}_{\frac{X}{2}} g_\psi - \operatorname{Ric}(g_\psi) - g_\psi.$$

This obstruction tensor plays a crucial role in analyzing the long time behavior of solutions to the normalized Kähler-Ricci flow. Establishing precise estimates for this tensor will enable us to prove

Theorem A. Before giving uniform control of $\nabla^k T$ for all $k \in \mathbb{N}_0$, we present some rough estimates of $\nabla^k T$.

Lemma 5.9 (Rough estimates on $\nabla^k T$)**.** *For all $0 < T < \infty$ $k \in \mathbb{N}_0$, there exists a constant $B_k = B(k, T) > 0$ such that on $M \times [0, T]$, we have*

$$|\nabla^k T| \leq B_k.$$

Proof. Recall that $T = \mathcal{L}_{\frac{X}{2}} g_\psi - \operatorname{Ric}(g_\psi) - g_\psi$. Now $\nabla^k(\operatorname{Ric}(g_\psi) + g_\psi)$ is uniformly bounded on space-time by Corollary 5.6. It suffices to show that $(\nabla^{g_\psi})^k \mathcal{L}_{\frac{X}{2}} g_\psi$ is bounded on $M \times [0, T]$. Let g_φ be the solution to Kähler-Ricci flow corresponding to g_ψ as in Proposition 2.3, it suffices for us to control $(\nabla^{g_\varphi})^k \mathcal{L}_{\frac{X}{2}} g_\varphi$ on $M \times [0, T]$.

Thanks to Kähler-Ricci flow equation

$$\begin{aligned}
\frac{\partial}{\partial t} (\nabla^{g_\varphi})^k \mathcal{L}_{\frac{X}{2}} g_\varphi &= -(\nabla^{g_\varphi})^k \mathcal{L}_{\frac{X}{2}} \operatorname{Ric}(g_\varphi) + \sum_{\substack{p+q=k \\ q<k}} (\nabla^{g_\varphi})^p \operatorname{Ric}(g_\varphi) * (\nabla^{g_\varphi})^q \mathcal{L}_{\frac{X}{2}} g_\varphi \\
&= (\nabla^{g_\varphi})^k \left(\nabla^{g_\varphi} \operatorname{Ric}(g_\varphi) * X + \nabla^{g_\varphi} X * \operatorname{Ric}(g_\varphi) \right) \\
&\quad + \sum_{\substack{p+q=k \\ q<k}} (\nabla^{g_\varphi})^p \operatorname{Ric}(g_\varphi) * (\nabla^{g_\varphi})^q \mathcal{L}_{\frac{X}{2}} g_\varphi.
\end{aligned} \tag{5.14}$$

Thanks to Corollary 5.6, $(\nabla^{g_\varphi})^k \left(\nabla^{g_\varphi} \operatorname{Ric}(g_\psi) * X + \nabla^{g_\varphi} X * \operatorname{Ric}(g_\psi) \right)$ is bounded on $M \times [0, T]$ for all $k \in \mathbb{N}_0$. In particular, $\mathcal{L}_{\frac{X}{2}} g_\varphi$ is bounded on $M \times [0, T]$. Now we assume that $(\nabla^{g_\varphi})^l \mathcal{L}_{\frac{X}{2}} g_\varphi$ is bounded for all $l \leq k$ on $M \times [0, T]$, then for $l = k + 1$, the induction hypothesis, together with curvature bound and ODE method imply that $(\nabla^{g_\varphi})^{l+1} \mathcal{L}_{\frac{X}{2}} g_\varphi$ is bounded on $M \times [0, T]$.

Thus, on $M \times [0, T]$, $\nabla^k T$ is bounded by some positive constant $B_k = B(k, T)$ as expected. ☐

Lemma 5.10. *The $k-$th covariant derivative $\nabla^k T$ evolves along the normalized Kähler-Ricci flow by*

$$\left(\frac{\partial}{\partial \tau} - \frac{1}{2} \Delta_{g_\psi} - \mathcal{L}_{\frac{X}{2}} \right) \nabla^k T = -\nabla^k T + \sum_{p+q=k} \nabla^p \operatorname{Rm}(g_\psi) * \nabla^q T. \tag{5.15}$$

Proof. Let us prove this Lemma by induction, (5.15) holds for $k = 0$ because

$$\begin{aligned}
\frac{\partial}{\partial \tau} T = \partial \bar{\partial} \left(\frac{\partial}{\partial \tau} \dot{\psi} \right) &= \partial \bar{\partial} \left(\Delta_{\omega_\psi, X} \dot{\psi} - \dot{\psi} \right) \\
&= -T + \mathcal{L}_{\frac{X}{2}} T + \partial \bar{\partial} (\Delta_{\omega_\psi} \dot{\psi}) \\
&= \left(\frac{1}{2} \Delta_{g_\psi} + \mathcal{L}_{\frac{X}{2}} \right) T + \operatorname{Rm}(g_\psi) * T - T.
\end{aligned}$$

Here the last line is ensured by

$$\partial \bar{\partial} (\Delta_{\omega_\psi} \dot{\psi}) = \Delta_{\omega_\psi} \partial \bar{\partial} \dot{\psi} + \operatorname{Rm}(g_\psi) * \partial \bar{\partial} \dot{\psi},$$

and,

$$\Delta_{\omega_\psi} \partial \bar{\partial} \dot{\psi} = \overline{\Delta}_{\omega_\psi} \partial \bar{\partial} \dot{\psi} + \operatorname{Rm}(g_\psi) * \partial \bar{\partial} \dot{\psi}. \tag{5.16}$$

Suppose that (5.15) holds for all integers $m \leq k$. For $\nabla^{k+1} T$, by Lemma 5.3, we compute

$$\frac{\partial}{\partial \tau} \nabla^{k+1} T - \mathcal{L}_{\frac{X}{2}} \nabla^{k+1} T = \nabla \left(\frac{\partial}{\partial \tau} \nabla^k T - \mathcal{L}_{\frac{X}{2}} \nabla^k T \right) + \nabla \operatorname{Ric}(g_\psi) * \nabla^k T. \tag{5.17}$$

And by Bochner's formula, we have

$$\frac{1}{2} \Delta_{g_\psi} \nabla^{k+1} T = \nabla \left(\frac{1}{2} \Delta_{g_\psi} \nabla^k T \right) + \nabla \operatorname{Rm}(g_\psi) * \nabla^k T + \operatorname{Rm}(g_\psi) * \nabla^{k+1} T. \tag{5.18}$$

Combining (5.17) and (5.18), we have

$$\begin{aligned}&\left(\frac{\partial}{\partial\tau}-\frac{1}{2}\Delta_{g_\psi}-\mathcal{L}_{\frac{X}{2}}\right)\nabla^{k+1}T\\&=\nabla\left(\left(\frac{\partial}{\partial\tau}-\frac{1}{2}\Delta_{g_\psi}-\mathcal{L}_{\frac{X}{2}}\right)\nabla^{k}T\right)+\mathrm{Rm}(g_\psi)*\nabla^{k+1}T+\nabla\,\mathrm{Rm}(g_\psi)*\nabla^kT.\end{aligned}$$

By the induction hypothesis, we have that (5.15) holds for $k+1$.

□

Proposition 5.11. *There exists a constant $C(k,n)>0$ such that on $M\times[0,\infty)$,*

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)|\nabla^kT|^2\le-|\nabla^{k+1}T|^2+k|\nabla^kT|^2+C(k,n)\sum_{p+q=k}|\nabla^p\,\mathrm{Rm}(g_\psi)||\nabla^qT||\nabla^kT|.$$

Proof. This proof is analogous to the proof of Corollary 5.5 and will therefore be omitted.. □

We observe that ∇^kT satisfies an evolution equation analogous to that of $\nabla^k\mathrm{Rm}(g_\psi)$. Our goal is therefore to establish that ∇^kT decays at a polynomial rate in f, comparable to the decay of its initial data $\nabla^kT(0)$.

Corollary 5.12. *There exists a constant $C_k>0$ such that on $M\times[0,\infty)$,*

$$|\nabla^kT|\le C_ke^{-\tau}f_\psi^{-\frac{k}{2}-1}.$$

Proof. First, we compute

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(e^{2\tau}f_\psi^{k+2}|\nabla^kT|^2)&=|\nabla^kT|^2\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(e^{2\tau}f_\psi^{k+2})\\&\quad+e^{2\tau}f_\psi^{k+2}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)|\nabla^kT|^2\\&\quad-2(k+2)e^{2\tau}f_\psi^{k+1}\,\mathrm{Re}(<\partial f_\psi,\bar\partial|\nabla^kT|^2>).\end{aligned}$$

On the one hand

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(e^{2\tau}f_\psi^{k+2})=-ke^{2\tau}f_\psi^{k+2}-(k+2)(k+1)e^{2\tau}f_\psi^k|\partial f_\psi|^2.$$

On the other hand

$$\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)|\nabla^kT|^2\le-|\nabla^{k+1}T|^2+k|\nabla^kT|^2+C(k,n)\sum_{p+q=k}|\nabla^p\,\mathrm{Rm}(g_\psi)||\nabla^qT||\nabla^kT|.$$

Hence we get

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(e^{2\tau}f_\psi^{k+2}|\nabla^kT|^2)&\le-e^{2\tau}f_\psi^{k+2}|\nabla^{k+1}T|^2\\&\quad+C(k,n)e^{2\tau}f_\psi^{k+2}\sum_{p+q=k}|\nabla^p\,\mathrm{Rm}(g_\psi)||\nabla^qT||\nabla^kT|\\&\quad-(k+2)e^{2\tau}f_\psi^{k+1}X\cdot|\nabla^kT|^2\end{aligned}$$

Now we notice that for all $\sigma>0$, by Cauchy-Schwarz inequality

$$X\cdot|\nabla^kT|^2\le2|X||\nabla^{k+1}T||\nabla^kT|\le\sigma|X|^2|\nabla^{k+1}T|^2+\frac{1}{\sigma}|\nabla^kT|^2.$$

Take $\sigma>0$ such that $(k+2)\sigma|X|^2\le\frac{1}{2}f_\psi$, we have

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}-\Delta_{\omega_\psi,X}\right)(e^{2\tau}f_\psi^{k+2}|\nabla^kT|^2)&\le-\frac{1}{2}e^{2\tau}f_\psi^{k+2}|\nabla^{k+1}T|^2+(k+2)\frac{1}{\sigma}e^{2\tau}f_\psi^{k+1}|\nabla^kT|^2\\&\quad+C(k,n)e^{2\tau}f_\psi^{k+2}\sum_{p+q=k}|\nabla^p\,\mathrm{Rm}(g_\psi)||\nabla^qT||\nabla^kT|.\end{aligned}\tag{5.19}$$

From now on, we use induction to prove Corollary 5.12. First, we prove there exists a constant $C_0 > 0$ such that $|T| \leq C_0 e^{-\tau} f_\psi^{-1}$. On the one hand we have by (5.19)

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(e^{2\tau} f_\psi^2 |T|^2) \leq C(0,n) e^{2\tau} f_\psi |T|^2.$$

Now recall [(4.11), Corollary 4.14], there exists a constant $A > 0$ such that

$$\begin{aligned}\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(e^{2\tau} f_\psi |\nabla \dot{\psi}|^2) &= -\frac{1}{2} e^{2\tau} f_\psi |\nabla^2 \dot{\psi}|^2 + A f_\psi^{-1} \\ &\leq -\frac{1}{2} e^{2\tau} f_\psi |T|^2 + A f_\psi^{-1}.\end{aligned}$$

Hence we get

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(e^{2\tau} f_\psi^2 |T|^2 + 2C(0,n) e^{2\tau} f_\psi |\nabla \dot{\psi}|^2) \leq 2C(0,n) A f_\psi^{-1}.$$

Thanks to Lemma 5.9 and initial condition $|\mathcal{L}_{\frac{X}{2}} g_\psi - g_{\psi_0}|_g = O(f^{-1})$ as in Definition 1.6, by Lemma 4.13, there exists a constant $C > 0$ such that

$$\sup_{M \times [0,\infty)} e^{2\tau} f_\psi^2 |T|^2 \leq C + \sup_M f_\psi^2(0) |T|^2(0) + 2C(0,n) f_\psi(0) |\nabla \dot{\psi}|^2(0) < \infty.$$

Hence there exists a constant $C_0 > 0$ such that for all $\tau \in [0, \infty)$,

$$|T| \leq C_0 e^{-\tau} f_\psi^{-1}.$$

Suppose that for all $0 \leq m \leq k$, there exists a constant $C_m > 0$ such that on $M \times [0, \infty)$,

$$|\nabla^m T| \leq C_m e^{-\tau} f_\psi^{-1-\frac{m}{2}}.$$

For $k+1$, by (5.19), there exists a constant $C(k+1) > 0$ such that

$$\begin{aligned}&\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(e^{2\tau} f_\psi^{k+3} |\nabla^{k+1} T|^2) \\ &\leq -\frac{1}{2} e^{2\tau} f_\psi^{k+3} |\nabla^{k+2} T|^2 + C(k+1) e^{2\tau} f_\psi^{k+2} |\nabla^{k+1} T|^2 \\ &\quad + C(k+1) e^{2\tau} f_\psi^{k+3} \sum_{p+q=k+1} |\nabla^p \operatorname{Rm}(g_\psi)| |\nabla^q T| |\nabla^{k+1} T|.\end{aligned}$$

By Corollary 5.4 and the induction hypothesis, we deduce that there exists a constant $C(k+1) > 0$ such that

$$\begin{aligned}&\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(e^{2\tau} f_\psi^{k+3} |\nabla^{k+1} T|^2) \\ &\leq C(k+1) e^{2\tau} f_\psi^{k+2} |\nabla^{k+1} T|^2 + C(k+1) e^{\tau} f_\psi^{\frac{k+1}{2}} |\nabla^{k+1} T|.\end{aligned}$$

By Cauchy-Schwarz inequality, there exists a constant $C(k+1) > 0$ such that

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(e^{2\tau} f_\psi^{k+3} |\nabla^{k+1} T|^2) \leq C(k+1) e^{2\tau} f_\psi^{k+2} |\nabla^{k+1} T|^2 + \frac{1}{f_\psi}.$$

By the induction hypothesis and (5.19) there exists a constant $C(k) > 0$ such that

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(e^{2\tau} f_\psi^{k+2} |\nabla^k T|^2) \leq -\frac{1}{2} e^{2\tau} f_\psi^{k+2} |\nabla^{k+1} T|^2 + C(k) \frac{1}{f_\psi}$$

Hence

$$\left(\frac{\partial}{\partial \tau} - \Delta_{\omega_\psi, X}\right)(e^{2\tau} f_\psi^{k+3} |\nabla^{k+1} T|^2 + 2C(k+1) e^{2\tau} f_\psi^{k+2} |\nabla^k T|^2) \leq (1 + 2C(k+1)C(k)) \frac{1}{f_\psi}.$$

Thanks to Lemma 5.9 and the initial condition $|(\nabla^g)^k(\mathcal{L}_{\frac{X}{2}} g_\psi - g_{\psi_0})|_g = O(f^{-\frac{k}{2}-1})$ in Definition 1.6, by Lemma 4.13, there exists a constant $A > 0$ such that

$$\begin{aligned}&\sup_{M\times[0,\infty)} e^{2\tau} f_\psi^{k+3}|\nabla^{k+1}T|^2\\ &\leq A + \sup_M f_\psi^{k+3}(0)|\nabla^{k+1}T|^2(0) + 2C(k+1)f_\psi^{k+2}(0)|\nabla^k T|^2(0) < \infty.\end{aligned}$$

Hence there exists a constant $C_{k+1} > 0$ such that for all $\tau \in [0,\infty)$,

$$|\nabla^{k+1}T| \leq C_{k+1} f_\psi^{-1-\frac{k+1}{2}}.$$

Therefore Corollary 5.12 holds for all $k \in \mathbb{N}_0$ □

Before proceeding to the proof of Theorem A, we require a technical lemma concerning covariant derivatives of tensors.

Lemma 5.13. *Let M be a Riemannian manifold with Riemannian metrics g_1, g_2. Let $k \in \mathbb{N}$ be an integer, and let $R \in C^\infty(\otimes^k T^*M)$ be a $k-$tensor. Then for any $m \in \mathbb{N}^*$, the difference of $(\nabla^{g_1})^m R$ and $(\nabla^{g_2})^m R$ is given by:*

$$(\nabla^{g_1})^m R - (\nabla^{g_2})^m R = \sum_{\substack{p+q=m\\ q<m}} \sum_{\substack{i_1+\cdots+i_l=p-l\\ l\leq p}} (\nabla^{g_1})^{i_1}\Gamma * \cdots (\nabla^{g_1})^{i_l}\Gamma * (\nabla^{g_2})^q R.$$

Here $\Gamma = \Gamma(g_1) - \Gamma(g_2)$.

Proof. This Lemma is proved by induction. When $m = 1$, by the definition of Christoffel symbols, we have

$$\nabla^{g_1} R - \nabla^{g_1} R = \Gamma * R.$$

Hence Lemma 5.13 holds for $m = 1$. We suppose that Lemma 5.13 holds for all integers $j \leq m$. Now for $j = m+1$, we have

$$\begin{aligned}(\nabla^{g_1})^{m+1}R - (\nabla^{g_2})^{m+1}R &= (\nabla^{g_1} - \nabla^{g_2})(\nabla^{g_2})^m R + \nabla^{g_1}\left((\nabla^{g_1})^m R - (\nabla^{g_2})^m R\right)\\ &= \Gamma * (\nabla^{g_2})^m R + \nabla^{g_1}\left((\nabla^{g_1})^m R - (\nabla^{g_2})^m R\right).\end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned}\nabla^{g_1}\left((\nabla^{g_1})^m R - (\nabla^{g_2})^m R\right) &= \nabla^{g_1} \sum_{\substack{p+q=m\\ q<m}} \sum_{\substack{i_1+\cdots+i_l=p-l\\ l\leq p}} (\nabla^{g_1})^{i_1}\Gamma * \cdots (\nabla^{g_1})^{i_l}\Gamma * (\nabla^{g_2})^q R\\ &= \sum_{\substack{p+q=m+1\\ q<m+1}} \sum_{\substack{i_1+\cdots+i_l=p-l\\ l\leq p}} (\nabla^{g_1})^{i_1}\Gamma * \cdots (\nabla^{g_1})^{i_l}\Gamma * (\nabla^{g_2})^q R.\end{aligned}$$

Therefore, we have

$$(\nabla^{g_1})^{m+1}R - (\nabla^{g_2})^{m+1}R = \sum_{\substack{p+q=m+1\\ q<m+1}} \sum_{\substack{i_1+\cdots+i_l=p-l\\ l\leq p}} (\nabla^{g_1})^{i_1}\Gamma * \cdots (\nabla^{g_1})^{i_l}\Gamma * (\nabla^{g_2})^q R, \tag{5.20}$$

as expected. Lemma 5.13 is true for all $m \in \mathbb{N}^*$. □

With uniform estimates on T, we can prove Theorem A.

Proof of Theorem A. For any $k \in \mathbb{N}_0$, by Corollary 5.12, there exists a constant $C_k > 0$ such that

$$|(\nabla^{g_\psi})^k \partial\bar{\partial}\dot{\psi}| \leq C_k e^{-\tau} f_\psi^{-1-\frac{k}{2}}.$$

Therefore, thanks to Proposition 5.1 and Proposition 4.21, for any $k \in \mathbb{N}_0$, there exists a constant $C_k > 0$ such that on $M \times [0,\infty)$,

$$|(\nabla^{g_\psi})^k \partial\bar{\partial}\dot{\psi}|_g \leq C_k e^{-\tau} f^{-1-\frac{k}{2}}. \tag{5.21}$$

Claim 5.14. For any $k \in \mathbb{N}_0$, there exists a constant $C_k > 0$ such that on $M \times [0, \infty)$,

$$|(\nabla^g)^k \partial\bar{\partial}\psi|_g \leq C_k f^{-\frac{k}{2}},$$

and,

$$|(\nabla^g)^k \partial\bar{\partial}\dot{\psi}|_g \leq C_k e^{-\tau} f^{-1-\frac{k}{2}}.$$

Proof of Claim 5.14. We prove this claim with induction. When $k = 0$, according to (5.21), after integration, it is clear to see that Claim 5.14 holds for $k = 0$. Now we suppose that Claim 5.14 holds for all integers $l \leq k$. Now for $l = k + 1$, since g_ψ is uniformly bi-Lipschitz to g, by Lemma 5.13 we have

$$(\nabla^g)^{k+1}\partial\bar{\partial}\dot{\psi} - (\nabla^{g_\psi})^{k+1}\partial\bar{\partial}\dot{\psi} = \sum_{\substack{p+q=k+1 \\ q \leq k}} \sum_{\substack{i_1+\cdots+i_l=p \\ i_j \geq 1}} (\nabla^g)^{i_1} g_\psi * \cdots * (\nabla^g)^{i_l} g_\psi * (\nabla^{g_\psi})^q \partial\bar{\partial}\dot{\psi}.$$

By the induction hypothesis and (5.21), there exists a constant $D_{k+1} > 0$ such that

$$|(\nabla^g)^{k+1}\partial\bar{\partial}\dot{\psi}|_g \leq D_{k+1} e^{-\tau} f^{-1} \left(|(\nabla^g)^{k+1}\partial\bar{\partial}\psi|_g + f^{-\frac{k+1}{2}} \right). \tag{5.22}$$

We define $y(\rho) = |(\nabla^g)^{k+1}\partial\bar{\partial}\psi|_g(x, \rho)$ for all $\rho \leq \tau$ and fixed $x \in M$. Since f is bounded from below by $\varepsilon > 0$, then (5.14) implies

$$\begin{aligned} \frac{d}{d\rho}\left(y(\rho)\exp\{D_{k+1}e^{-\rho}f^{-1}(x)\}\right) &\leq \exp\{D_{k+1}e^{-\rho}f^{-1}(x)\}D_{k+1}e^{-\rho}f(x)^{-\frac{k+3}{2}} \\ &\leq \exp\{D_{k+1}\varepsilon^{-1}\}D_{k+1}e^{-\rho}f(x)^{-\frac{k+3}{2}} \end{aligned}$$

By integration and the initial condition $(\nabla^g)^{k+1} g_{\psi_0} = O(f^{-\frac{1+k}{2}})$, we conclude that there exists a constant $C_{k+1} > 0$ such that

$$|(\nabla^g)^{k+1}\partial\bar{\partial}\psi|_g \leq C_k f^{-\frac{k+1}{2}}.$$

This result together with (5.22), implies that

$$|(\nabla^g)^{k+1}\partial\bar{\partial}\dot{\psi}|_g \leq D_{k+1}(C_{k+1} + 1) e^{-\tau} f^{-\frac{k+3}{2}}.$$

Hence Claim 5.14 holds for all integers. □

Claim 5.15. The solution g_ψ converges uniformly and smoothly to a asymptotically conical gradient expander (M, g_∞, X). Moreover, as g_{ψ_0} is asymptotically conical with a unique asymptotic Kähler cone (M, g_0') as in Proposition 2.13, (M, g_∞, X) is the unique (up to biholomorphisms) asymptotically conical gradient expander with asymptotic cone C, g_0'.

Proof of Claim 5.15. By integration in time for $0 \leq \tau < \rho < \infty$,

$$|(\nabla^g)^k (g_\psi(\tau) - g_\psi(\rho))|_g \leq C_k (e^{-\tau} - e^{-\rho}) f^{-1-\frac{k}{2}}.$$

This implies that $g_\psi(\tau)$, as τ goes to ∞, converges uniformly smoothly to a Kähler metric g_∞ such that for all $k \in \mathbb{N}_0$,

$$|(\nabla^g)^k (g_\psi(\tau) - g_\infty)|_g \leq C_k e^{-\tau} f^{-1-\frac{k}{2}}.$$

At the same we have $|(\nabla^g)^k g_\psi|_g \leq C_k f^{-\frac{k}{2}}$ for all $k \in \mathbb{N}_0$, then we conclude that for all $k \in \mathbb{N}_0$,

$$|(\nabla^{g_\psi(\tau)})^k (g_\psi(\tau) - g_\infty)|_{g_\psi(\tau)} \leq C_k e^{-\tau} f^{-1-\frac{k}{2}}.$$

Moreover, since $\partial\bar{\partial}\dot{\psi} = \mathcal{L}_{\frac{X}{2}} g_\psi - \mathrm{Ric}(g_\psi) - g_\psi$, by letting τ tend to ∞, we have

$$\mathcal{L}_{\frac{X}{2}} g_\infty - \mathrm{Ric}(g_\infty) - g_\infty = 0.$$

As a consequence, the Hamiltonian potential f_ψ converges uniformly to a smooth function f_∞ as τ tends to ∞, and it turns out that f_∞ is a Hamiltonian potential of X with respect to X. Furthermore, by Proposition 4.16 and Corollary 5.6, we find that (M, g_∞, X) is an asymptotically conical gradient Kähler-Ricci expander which is bi-Lipschitz to (M, g, X).

In particular, $|g_{\psi_0} - g_\infty|_g \leq C_0 f^{-1}$. Let Φ_t denote the flow of $-\frac{1}{2t}X$ for $t > 0$, $g(t)$ be the self-similar solution of g as in Proposition 2.1, we have

$$|t\Phi_t^* g_{\psi_0} - t\Phi_t^* g_\infty|_{g(t)} \leq C_0 \frac{1}{\Phi_t^* f}.$$

Let E denote the exceptional set as in Theorem 2.4, then if $x \notin E$, $X(x) \neq 0$. Since Φ_t is the flow of $-\frac{1}{2t}X$ and f is proper, thus, as t tends to 0, $\Phi_t^* f$ tends to ∞. Moreover, we identify $C \setminus \{o\}$ with $M \setminus E$ via the Kähler resolution as in Theorem 2.4. Let r denote the radial function of the Kähler cone (C, g_0). Then for any $\varepsilon > 0$, $\Phi_t^* f$ tends to ∞ uniformly on $\{r(x) \geq \varepsilon\}$. This implies that when t tends to 0, $t\Phi_t^* g_{\psi_0}$ and $t\Phi_t^* g_\infty$ converge to the same limit metric. By Proposition 2.13, as t tends to 0, $t\Phi_t^* g_\infty$ converges locally to the conical metric g_0' in Proposition 2.13. Hence g_∞ is asymptotically conical with the unique asymptotic cone (C, g_0').

□

Now if the initial perturbation satisfies

$$\left|(\nabla^g)^k (g_{\psi_0} - g)\right|_g = o\left(f^{-\frac{k}{2}}\right) \quad \text{for all } k \in \mathbb{N}_0,$$

then we have that

$$\left|(\nabla^g)^k (g_\infty - g)\right|_g = o\left(f^{-\frac{k}{2}}\right) \quad \text{for all } k \in \mathbb{N}_0.$$

Let $g_\infty(t)_{t>0}$ be the self-similar solution to Kähler-Ricci flow associated to g_∞. Since $\left|(\nabla^g)^k (g_\infty - g)\right|_g = o\left(f^{-\frac{k}{2}}\right)$ for all $k \in \mathbb{N}_0$, thus for all $t > 0$,

$$\begin{aligned}\left|(\nabla^{g(t)})^k (g_\infty(t) - g(t))\right|_{g(t)}(x) &= t^{-\frac{k}{2}} \left|(\nabla^g)^k (g_\infty - g)\right|_g (\Phi_t(x)) \\ &= t^{-\frac{k}{2}} o(f \circ \Phi_t^{-\frac{k}{2}}) = o(1)(t\Phi_t^* f)^{-\frac{k}{2}},\end{aligned}$$

where $o(1)$ denotes a function which tends to 0 as $\Phi_t^* f$ tends to ∞. Since when t tends to 0, on $\{r \geq \varepsilon\}$, $\Phi_t^* f$ tends to ∞ uniformly, hence $o(1)$ converges to 0 uniformly when t converges to 0 on $\{r \geq \varepsilon\}$. Now we prove that on $\{r \geq \varepsilon\}$, function $t\Phi_t^* f$ is bounded from below. Recall the normalization of f in Lemma 2.5:

$$f = n + R_\omega + |\partial f|_g^2.$$

Now we compute,

$$\frac{\partial}{\partial t} t\Phi_t^* f = \Phi_t^* (f - |\partial f|^2) = \Phi_t^* (R_\omega + n) > 0.$$

Here the last inequality comes from the fact $R_\omega + n > 0$ on M. Moreover, for any $x \in C - \{o\}$,

$$\lim_{t \to 0^+} t\Phi_t^* f(x) = \lim_{t \to 0^+} \left(nt + \Phi_t^* R_\omega(x) + \frac{1}{2} g(x,t)(X, X)\right) = \frac{1}{2} r(x)^2.$$

We conclude that on $\{r \geq \varepsilon\}$, for all $t > 0$, $t\Phi_t^* f \geq \frac{1}{2} r^2 \geq \frac{1}{2}\varepsilon^2$. Function $t\Phi_t^* f$ is bounded from below on $\{r \geq \varepsilon\}$, and therefore, $g_\infty(t)$ converges locally smoothly to g_0 as $g(t)$ converges locally smoothly to g_0. The solution $g_\infty(t)$ to Kähler-Ricci flow satisfies conical condition in [9, Theorem 1.4].

Moreover, g_∞ and g are two asymptotically conical gradient Kähler-Ricci expander with bounded curvature, solution $g_\infty(t)$ satisfies cohomology condition and curvature condition in [9, Theorem 1.4]. For the Killing condition, JX is naturally a Killing vector field for g_∞ because (M, g_∞, X) is a gradient expander (see Lemma 2.8).

As a result of Theorem 1.4 of [9], we claim that $g_\infty(t) = g(t)$ for all $t > 0$, thus $g_\infty = g$. □

5.2. **Applications.** In this section, we characterize the limiting metric g_∞ obtained when perturbing the initial metric g by a term $\alpha\partial\bar{\partial}f$ (for some suitably chosen constant α) using Theorem A. This characterization applies in particular to the canonical examples of Cao [3].

Theorem 5.16. *Let (M, g, X) be an asymptotically conical gradient Kähler-Ricci expander. Let f be the normalized Hamiltonian potential of X with respect to g. For any $\alpha \in \mathbb{R}$ such that $(1+\alpha)g + \alpha \operatorname{Ric}(g) > 0$, the following holds:*

(i) *the symmetric two-tensor $g_\alpha := g + \alpha\partial\bar{\partial} f$ defines a Kähler metric;*
(ii) *the solution to the normalized Kähler-Ricci flow starting from g_α exists for all time;*
(iii) *as time goes to ∞, this solution converges to ${\Phi^*}_{\frac{1}{1+\alpha}} g$ smoothly uniformly, where Φ_t is the flow generated by $-\frac{1}{2t}X$.*

Proof. First, we prove that $1 + \alpha > 0$. Since $\operatorname{Ric}(g)$ converges to 0 at infinity, the condition $(1 + \alpha)g + \alpha \operatorname{Ric}(g) > 0$ implies $1 + \alpha > 0$. Let the initial Kähler potential be given by $\psi_0 = \alpha\partial\bar{\partial} f$. Then, as shown in Section 1.2, ψ_0 satisfies Condition II in the sense of Definition 1.6.

Thanks to Theorem C and Theorem A, the solution to the normalized Kähler-Ricci flow starting from g_α exists for all time and when time goes to ∞, this solution converges smoothly uniformly to some asymptotical conical gradient Kähler-Ricci expander (M, g^α_∞, X). Let $g_\alpha(\tau)_\tau$ denote this solution.

Now we consider $\tilde{g}_\alpha(\tau) := \Phi^*_{1+\alpha} g_\alpha(\tau)$, $\tilde{g}_\alpha(\tau)$ is a solution to Kähler-Ricci flow starting from $\Phi^*_{1+\alpha} g_\alpha$. Let $g(t)_{t>0}$ be the self-similar solution to Kähler-Ricci flow associated to g. Then we observe that

$$\begin{aligned}
\Phi^*_{1+\alpha} g_\alpha = g(1+\alpha) + \alpha\Phi^*_{1+\alpha} \operatorname{Ric}(g) &= g(1+\alpha) + \alpha \operatorname{Ric}(g(1+\alpha)) \\
&= g - \int_1^{1+\alpha} \operatorname{Ric}(g(s))ds + \alpha \operatorname{Ric}(g(1+\alpha)) \\
&= g + \alpha\partial\bar{\partial} \int_1^{1+\alpha} \log \frac{\omega(s)^n}{\omega(1+\alpha)^n} ds.
\end{aligned}$$

Let $\tilde{\psi}_0 = \alpha \int_1^{1+\alpha} \log \frac{\omega(s)^n}{\omega(1+\alpha)^n} ds$, now we check that $(\nabla^g)^k \tilde{\psi}_0 = O(f^{-\frac{k}{2}})$ for all $k \in \mathbb{N}_0$. It suffices to show that for any $k \in \mathbb{N}_0$, there exists a constant $C_k > 0$ such that for all $s \in [1, 1+\alpha]$

$$|(\nabla^g)^k g(s)|_g \leq C_k f^{-\frac{k}{2}},$$

and

$$|(\nabla^g)^k \operatorname{Ric}(g(s))|_g \leq C_k f^{-\frac{k+2}{2}}.$$

We proceed by induction. First, we prove that there exists a constant $A > 1$ such that for all $s \in [1, 1+\alpha]$, we have

$$\frac{1}{A} f \leq s\Phi^*_s f \leq Af.$$

Thanks to the soliton identity $f = R_\omega + n + |\partial f|^2_g$, we compute,

$$\frac{d}{ds} s\Phi^*_s f = \Phi^*_s (f - |\partial f|^2_g) = \Phi^*_s (n + R_\omega).$$

Let $A_1 := \sup_M |R_\omega| + n$, then we have $|\frac{d}{ds} s\Phi^*_s f| \leq A_1$. Thus, for all $s \in [1, 1+\alpha]$, we have

$$|s\Phi^*_s f - f| \leq A_1|s - 1| \leq A_1|\alpha|.$$

The potential f is bounded from below by $\varepsilon > 0$, so on the one hand

$$s\Phi^*_s f \leq f + A_1|\alpha| \leq \left(1 + \frac{A_1|\alpha|}{\varepsilon}\right) f.$$

On the other hand

$$s\Phi^*_s f \geq f - A_1|\alpha| \geq f - \frac{A_1|\alpha|}{s\varepsilon} s\Phi^*_s f \geq f - \frac{A_1|\alpha|}{\min\{1+\alpha, 1\}\varepsilon} s\Phi^*_s f.$$

If we define $A = 1 + \frac{A_1|\alpha|}{\min\{1+\alpha,1\}} + \frac{A_1|\alpha|}{\varepsilon}$, then we have

$$\frac{1}{A} f \leq s\Phi^*_s f \leq Af, \tag{5.23}$$

as expected.

Since g has bounded Riemannian curvature, there exists a constant $C_0 > 1$ such that for all $s \in [1, 1+\alpha]$,

$$\frac{1}{C_0} g \leq g(s) \leq C_0 g.$$

Moreover, there exists a constant $C' > 0$ such that $|\operatorname{Ric}(g)|_g \leq \frac{C'}{f}$. Thus, we have

$$|\operatorname{Ric}(g(s))|_g \leq C_0 |\operatorname{Ric}(g(s))|_{g(s)} = \frac{C_0 C'}{s\Phi_s^* f} \leq \frac{C_0 C' A}{f}.$$

Hence, the claim holds for the base case $k = 0$. We suppose this argument holds for all integers $l \leq k$. Now for $l = k+1$. By Lemma 5.13, we have

$$\begin{aligned}&(\nabla^g)^{k+1} \operatorname{Ric}(g(s)) - (\nabla^{g(s)})^{k+1} \operatorname{Ric}(g(s)) \\ &= \sum_{\substack{p+q=k+1 \\ q \leq k}} \sum_{i_1+\cdots i_l = p} (\nabla^g)^{i_1} g(s) * \cdots (\nabla^g)^{i_l} g(s) * (\nabla^{g(s)})^q \operatorname{Ric}(g(s)).\end{aligned}$$

By induction hypothesis and by the curvature bound of g, there exists a constant $D_{k+1} > 0$ such that

$$|(\nabla^g)^{k+1} \operatorname{Ric}(g(s))|_g \leq D_{k+1} f^{-1} \left(|(\nabla^g)^{k+1} g(s)|_g + f^{-\frac{k+1}{2}} \right). \tag{5.24}$$

Let us define $y(s) := |(\nabla^g)^{k+1} g(s)|_g(x)$ for fixed $x \in M$ and $s \in [1, 1+\alpha]$. Then we have the following ODE inequality:

$$\frac{d}{ds} y(s) \leq D_{k+1} f^{-1} \left(y(s) + f^{-\frac{k+1}{2}} \right)$$

By integration and the initial condition $y(1) = 0$, we can find a constant $C_{k+1} > 0$ such that $|(\nabla^g)^{k+1} g(s)|_g(x) \leq C_{k+1} f^{-\frac{k+1}{2}}$. This result together with (5.24) imply that this argument holds for all integers.

Then by Theorem A, $\tilde{g}_\alpha(\tau)$ converges smoothly uniformly to g. Therefore, $g_\alpha(\tau)$, as τ tends to ∞, converges smoothly uniformly to $\Phi^*_{\frac{1}{1+\alpha}} g$. □

Corollary 5.17. *Let (M, g, X) be the canonical examples of Cao* [3]. *Then for any $\alpha \geq 0$, the following holds:*

(i) *the form $g_\alpha := g + \alpha \partial\bar{\partial} f$ defines a Kähler metric;*
(ii) *the solution to the normalized Kähler-Ricci flow starting from g_α exists for all time;*
(iii) *as time goes to ∞, this solution converges to $\Phi^*_{\frac{1}{1+\alpha}} g$ smoothly uniformly, where Φ_t is the flow generated by $-\frac{1}{2t} X$.*

Proof. Since in Cao's models, the Ricci curvature is positive, hence for all $\alpha \geq 0$, $(1+\alpha)g + \alpha \operatorname{Ric}(g) > 0$. Hence we can use Theorem 5.16. □

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