Extending Ghouila-Houri's Characterization of Comparability Graphs to Temporal Graphs[⋆]

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Abstract. An orientation of a given static graph is called *transitive* if for any three vertices a, b, c, the presence of arcs (a, b) and (b, c) forces the presence of the arc (a, c). If only the presence of an arc between a and c is required, but its orientation is unconstrained, the orientation is called *quasi-transitive*. A fundamental result presented by Ghouila-Houri [7] guarantees that any static graph admitting a quasi-transitive orientation also admits a transitive orientation. In a seminal work [16], Mertzios et al. introduced the notion of temporal transitivity in order to model information flows in simple temporal networks. We revisit the model introduced by Mertzios et al. and propose an analogous to Ghouila-Houri's characterization for the temporal scenario. We present a structure theorem that will allow us to express by a 2-SAT formula all the constraints imposed by temporal transitive orientations. The latter produces an efficient recognition algorithm for graphs admitting such orientations, that we will call T-comparability graphs. Inspired by the lexicographic strategy presented by Hell and Huang in [10] to transitively orient static graphs, we then propose an algorithm for constructing a temporal transitive orientation of a YES instance. This algorithm is straightforward and has a running-time complexity of $max\{O(nm), min\{O(kn), O(m^2)\}\}$, with k being the number of monolabel triangles in the temporal graph, i.e. triangles having the same unique time-label on their edges. This represents an improvement compared to the algorithm presented in [16]. Additionally, we extend the temporal transitivity model to temporal graphs having multiple time-labels associated to their edges and claim that the previous results hold in the multilabel setting. Finally, we propose a characterization of T-comparability graphs via forbidden temporal ordered patterns.

Keywords: Temporal graphs · transitive orientation · lexicographic strategy · forbidden patterns

1 Introduction

Context. In the static case, a graph is a comparability graph if and only if it admits a *transitive orientation*, that is, if it is possible to orient its edges such that, whenever an edge ab is oriented from a towards b and an edge bc from b towards c, a third edge ac exists and is oriented from a towards c. If only the existence of the third edge ac is required, but its orientation is unconstrained, the orientation is called *quasi-transitive*. Consider a static graph $G = (V, E)$ and a quasi-transitive orientation O of G. We can start by observing that whenever $ab \in E$ and $bc \in E$ but $ac \notin E$, orienting ab from a towards b in O will force edge bc to be oriented from c towards b. Analogously, orienting b towards c will force the orientation of b towards a. One can define the Boolean variable x_{ab} for each edge $ab \in E$ and interpret the truth assignment 1 to the variable as the orientation from a towards b, and the truth assignment 0 as the opposite orientation. Forcing relations between arcs can be therefore expressed by clauses of a Boolean formula. Then, deciding whether a graph admits a specific type of orientation can be formalized as a SAT problem. Now, we observe that the constraints imposed by quasi-transitive orientations can be thought of as local constraints, as they correspond to the forcing exerted by a single arc over another arc, necessarily adjacent. Then, it is easy to see that these the constraints can be expressed by 2-CNF formulas, i.e. clauses with at most two variables, and therefore formalized as a 2-SAT problem and solved by the famous Tarjan's algorithm [1], linear in the size of the set of clauses. It is important to emphasize

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that this is not the case for transitive orientations, where the arc constraints do not translate into local forcings, i.e. one arc forcing another, as the orientation of an edge ab belonging to a given triangle abc is determined by the orientations of both edges bc and ca. Note that two 3-SAT clauses are necessary to capture triangle constraints in transitive orientations. In this context, the following result proposed by Ghouila-Houri is particularly interesting as it allows the recognition problem of comparability graphs to be solved by 2-SAT in $O(nm)$.

Theorem 1. *[7] A graph G admits a transitive orientation if and only if it admits a quasi-transitive orientation.*

As we anticipated above, a quasi-transitive orientation is not necessarily a transitive orientation, since it may contain cyclically oriented triangles, or *directed triangles* as we will call them. Therefore, the theorem proposed by Ghouila-Houri does not solve the problem of obtaining such an orientation. Hence, how to efficiently obtain a transitive orientation of a given comparability graph was extensively studied. We will present a brief survey of the algorithms proposed to solve the problem.

In his proof, published in the early 1960s, Ghouila-Houri implicitly formulates an algorithm for constructing a transitive orientation of a comparability graph from a quasi-transitive orientation. Given a static graph $G = (V, E)$ and a quasi-transitive orientation of its edges O, the strategy is to break the cyclical orientation of triangles in O by identifying a module M containing some, or all, of the triangle vertices, transitively orienting the module, recursively orienting the subgraph induced by $V - M$ and orienting all edges between M and $V - M$ in the same direction. This approach laid the groundwork for subsequent techniques using modular decomposition, as we will see below.

Recall the local forcings between the arcs in a quasi-transitive orientation. When considering chains of adjacent arcs, the forcing relation can be propagated. The latter was studied by Golumbic [8] at the beginning of the 1980s and form the basis of the algorithm the author proposes for constructing a transitive orientation of a comparability graph. Golumbic considers a partition of all possible orientations of the graph's edges by this forcing relation. Since the relation between two arcs is symmetrical, each partition will correspond to an equivalence class where the forcing is pairwise satisfied. He will refer to these classes as *implication classes*. The algorithm will then construct the orientation by selecting an arbitrary unoriented edge, fixing its orientation, propagating it through its implication class, and then repeating the procedure on the subgraph induced by the not yet oriented edges. Note that the implication classes are re-computed at each iteration as only unoriented edges will be used to identify forcings, this is the key to avoid orienting triangles cyclically. The algorithm proposed by Golumbic has a running time of $O(nm)$. Later on, Spinrad [19] proposed an algorithm with an $O(n^2)$ running time, relying on a modular decomposition technique based on the same principles proposed by Ghouila-Houri in his proof. Along the same lines, in the late 1990's McConnell and Spinrad [15] proposed linear time algorithms based on efficiently computing the modular decomposition of a given graph. We can also reference the strategy proposed by McConnell and Montgolfier [14], consisting of a linear time algorithm for obtaining the modular decomposition of a directed graph. This way, we can consider the digraph obtained by quasi-transitively orienting a comparability graph. A straightforward observation shows that each strongly connected component of this digraph is a module. Then, it suffices to independently transitively re-orient the strongly connected components, as it can be proven that the global resulting orientation will be transitive.

In his seminal article [7], Ghouila-Houri proposed another important characterization of comparability graphs. Let $G = (V, E)$ be a static graph and G^+ the graph obtained by taking as vertices both possible orientations (a, b) and (b, a) for each edge $ab \in E$, and by linking all vertices (a, b) to (b, a) and to all (b, c) such that $ac \notin E$. One can think of G^+ as the "conflict" graph of G as it links incompatible edge orientations. Then, it is easy to see that G admits a transitive orientation if and only if G^+ admits a 2-coloring of its vertices. Once again, this characterization only provides a straightforward recognition algorithm, leaving unsolved the problem of obtaining a transitive orientation of G in the case of a positive instance. In the 1990s, Hell and Huang [10] used the previous characterization and provided a lexicographic argument to greedily 2-color the components of G^+ in order to obtain a transitive orientation of a comparability graph. The technique, as well as the correctness proof, are very straightforward and result in an algorithm having $O(nm)$ time complexity.

Although obtaining a transitive orientation can be done in linear time, if the graph was a comparability graph, up to our knowledge there is no linear time algorithm to verify if a given orientation is transitive, or even quasi-transitive, as this problem is closely related to the Boolean Matrix Multiplication one. Therefore, the recognition of comparability graphs still requires $O(nm)$ running-time in the worst case.

Temporal (or time-varying, or evolving) graphs were introduced with the aim of modeling dynamic networks, that is, networks whose connections evolve over time. They are of great use in modeling a large number of real-world systems, such as information, traffic and social networks, see [3, 13]. Our work uses the formalism presented in the foundational paper of Kempe et al. [12], where a *temporal network* is defined as a pair (G, λ), with λ a time-labelling function over the edges of a static graph G, specifying the discrete time at which its endpoints interact. With the objective of modeling the flow of information on paths whose time-labels respect the ordering of time, Kempe et al. introduced the notion of *(strict) time-respecting path*, where edges $e_1, e_2, ..., e_k$ of a given path in G satisfy $\lambda(e_1) \leq \lambda(e_2) \leq ... \leq \lambda(e_k)$ (resp. $\lambda(e_1) < \lambda(e_2) < ... < \lambda(e_k)$). As in these paths the edges model undirected interactions between the network nodes, Mertzios et al. [16] explored directed communications, extending the classical notion of transitive orientations to the temporal scenario. As different versions of temporal transitivity were previously considered in diverse areas such as medical data treatment [17] or text processing [20], Mertzios et al. propose a new definition which can be justified in the context of confirmation and verification of information in a temporal network. One can consider a scenario where an important information is sent from node a to node b at time t_1 and then this intermediary node sends the information to node c at t_2, with $t_1 \leq t_2$. Subsequently, node c might want to verify the information by querying directly from node a at a time $t_3 \geq t_2$. Before formally defining temporal transitive orientations, let us make a parallelism with the static case by introducing quasi-temporal transitive orientations. Whenever a directed time-respecting path abc is formed, temporal transitive orientations will ask for a third edge ac to exist, to have a greater time-label than the arc bc and to be oriented from a towards c. As for quasi-transitive orientations in the static scenario, the orientation of this third arc will not be restricted by quasi-temporal transitive orientations, unlike its existence and time-label value.

Definition 1. *An orientation O of a temporal graph $\mathcal{G} = (G, \lambda)$ is a temporal transitive orientation, or* TTO*, (resp.* Quasi-TTO *or* QTTO*) if whenever $(a, b) \in O$ and $(b, c) \in O$, with $\lambda(ab) \leq \lambda(bc)$ then $(a, c) \in O$ (resp. (a, c) or $(c, a) \in O$) with $\lambda(ac) \geq \lambda(bc)$.*

By the previous definition, we observe that only monolabel triangles can be directed in a QTTO. This will be the equivalent to directed triangles in the static case, as it corresponds to the only restriction imposed by temporal transitive orientations that cannot be captured by local restrictions, i.e. a forcing from one arc to another. We will refer to temporal graphs admitting a temporal transitive orientation as *temporal comparability graphs*, or *T-comparability graphs* for short. One can note that this definition is a true generalization of comparability graphs in the static case, since if all edges share the same time-label, then $\mathcal{G} = (G, \lambda)$ is a T-comparability temporal graph if and only if G is a comparability graph. Given a temporal graph $\mathcal{G}$, the TTO Problem consists of deciding whether $\mathcal{G}$ admits a temporal transitive orientation of its edges. Mertzios et al. presented an algorithm to solve this problem, inspired in the strategy proposed by Golumbic [8] for the static case. This way, they extend the forcing relation to consider the temporal labels of the edges and express the temporal transitive constraints by a Boolean formula formed by the conjunction of a 3-NAE and a 2-SAT formula. Recall that 3-NAE stands for 3-Not-All-Equal, and is formed by a conjunction of clauses, with three literals each, satisfied when at least one of the literals receives 1 as truth assignment and another one 0. The goal of these clauses is to capture the constraint of orienting monolabel triangles in a non-cyclical way. As the problem of Non-All-Equal-3-SAT is NP-complete, the authors need to use structural arguments to prove that the overall running-time of their algorithm is polynomial, although they do not explicitly state its complexity and the correctness proof is rather technically involved.

Different variants to the TTO problem were defined by Mertzios et al. Given an orientation O of a temporal graph $\mathcal{G}$, if we only ask for $ac \in O$ with $\lambda(ac) \geq \lambda(bc)$ whenever abc is a strict time-respecting directed path, i.e. $\lambda(ab) < \lambda(bc)$, then the problem is defined as STRICT TTO. Similarly, if one asks the transitive arc from a to c to have a time-label such that $\lambda(ac) > \lambda(bc)$, the orientation is called strongly temporal transitive, and the associated problem STRONG TTO. By combining the two requirements, one gets the Strong Strict TTO problem. Mertzios et al. proved in [16] that the TTO, Strong TTO and Strong Strict TTO problems can be solved in polynomial time while deciding if a temporal graph admits a Strict TTO is NP-hard.

On another note, hereditary properties can be characterized by their set of minimal obstructions. However, for most hereditary properties these sets are often unknown, or known but infinite. The use of additional structures, such as an ordering of the graph's vertices, can allow to describe these properties by finite sets of forbidden structures. The study of characterizations via forbidden ordered patterns was proposed by Damaschke [5]. Since then, multiple studies have been carried out, allowing the description of many hereditary graph classes. In [11], Hell and al. show that all graph classes characterized by sets of three-vertex ordered patterns can be recognized in polynomial time. In [6], the authors refine the previous analysis presenting a detailed characterization of all 24 graph classes that can be described by a set of three-vertex ordered patterns, showing that all classes except two of them can be recognized in linear time. Moreover, [9] and [18] propose extensions of this technique by considering other types of structures for the order relation: exploring circular orderings in the first case and considering the ancestor relation defined by tree layouts in the latter. The growing development of this field of study made it possible to obtain multiple characterizations of hereditary classes via ordered patterns such as bipartite graphs, interval graphs, chordal graphs and, in particular, comparability graphs. In [4], they introduce the usage of forbidden ordered patterns for temporal graphs. In the temporal case, a pattern is a temporal subgraph that represents a specific arrangement with respect to the order not only of its vertices but also of its edges based on their time-labels.

Our contributions and structure of the paper. In section 2, we present the main notations and definitions that will be used throughout the paper and state some preliminar properties on temporal graphs admitting a temporal transitive orientation. Then, we begin section 3 by observing that Ghouila Houri's result, stated in Theorem 1, is not directly adaptable to the temporal case, since it is not true that every temporal graph admitting a QTTO is a T-comparability graph. Based on this observation, we define almost-temporal transitive orientations, or ATTO. For a given temporal graph $\mathcal{G}$, these orientations are slightly more restrictive than QTTO ones, but still containing the set of all TTO of $\mathcal{G}$. Then, we establish the exact analogue of Ghouila-Houri's theorem by proving that a temporal graph $\mathcal{G}$ admits an ATTO if and only if it admits a TTO. Inspired by the lexicographic strategy presented by Hell and Huang in [10], we present a lexicographic simple approach to solve the TTO problem which runs in $max\{O(nm), min\{O(kn), O(m^2)\}\}$ running-time, with k being the number of monolabel triangles in the temporal graph, i.e. triangles having the same unique time-label on their edges. As most temporal networks allow resources to traverse their arcs at multiple given times and therefore result in multilabel temporal graphs, we finish this section by extending the temporal transitivity notion to the multilabel setting and stating that the proposed results hold as well. In section 4, we introduce the notations we will use to work with temporal forbidden ordered patterns and we provide a characterization for the class of temporal graphs admitting a temporal transitive orientation by the means of these structures, both for TTO and Strict TTO. We conclude in section 5 by presenting a series of open problems, suggesting interesting directions for future research.

2 Preliminaries

2.1 Definitions and notations

Graphs. A graph is a pair $G = (V, E)$ where V is the vertex set and $E \subseteq V^2$ is the set of edges. Usually, for algorithmic complexity analysis, we take $n = |V|$ and $m = |E|$. If we consider $G = (V, A)$ with A a set of ordered pairs of vertices, called arcs, we say G is a *directed graph* or a *digraph*. Note that a directed graph might contain 2-cycles, i.e. bidirected edges. A *directed cycle abc* is a cycle where all edges are oriented in the same direction, with arcs $(a, b), (b, c)$ and (c, a). We will denote as *directed triangle* the particular case of the three-vertex directed cycle. If u and v are vertices of a graph (resp. digraph), we denote by uv the edge between u and v (resp. (u, v) the arc from u to v).

Temporal graphs. A *temporal graph* is a graph whose vertex set is fixed while its edge set changes over time. Given a classical graph $G = (V, E)$, we can obtain a temporal graph $\mathcal{G}$ by assigning a set of time-labels to its edges such that $\mathcal{G} = (G, \lambda)$ with $\lambda : E \to 2^{\mathbb{N}}$. These time-labels indicate the discrete time steps in which a given edge is active. We say G is the *underlying graph* of $\mathcal{G} = (G, \lambda)$. A *temporal subgraph* (G', λ') of a temporal graph (G, λ) is a temporal graph such that $G' = (V', E')$ is a subgraph of G and $\lambda'(e) \subseteq \lambda(e)$ for every $e \in E'$. A temporal graph is *simple* if $\lambda : E \to \mathbb{N}$, that is, every edge has a single presence time. As we will mostly reference simple temporal graphs, we will refer to them as temporal graphs and when considering temporal graphs admitting multiple time-labels per edge, we will explicitly call them *multilabel* temporal graphs. From [2], a multilabel temporal graph $\mathcal{G}$ is *proper* if $\lambda(e) \cap \lambda(e') = \emptyset$ whenever e and e' are incident to a common vertex. We say that a temporal graph is *monolabel* if all of its arcs share the same time-label set. In the temporal scenario, a classic graph might be referred to as *static graph*.

Orientations. Following [7, 10], computing an *orientation* O of an undirected graph $G = (V, E)$ consists of choosing exactly one of the pair (a, b), (b, a) for each edge $ab \in E$. We refer to a graph equipped with an orientation as an *oriented graph*. Let us denote by $AP(G)$ the set of all possible orientations of edges in G, that is $\{(u, v),\ uv \in E \text{ or } vu \in E\}$. For a subset O of $AP(G)$ we denote by its *reverse* the set defined by $O^R = \{(v, u),\ (u, v) \in O\}$. Note that an orientation O, with $O \subset AP(G)$, does not contain bidirected edges, or equivalently $O \cap O^R = \emptyset$. A *partial orientation* O of G, with $O \subset AP(G)$, is the orientation of a subset of E satisfying $O \cap O^R = \emptyset$.

2.2 Preliminar properties of T-comparability graphs

The aim of this subsection is to provide greater insight into T-comparability temporal graphs and to lay the groundwork for some open problems that will be raised in Section 5. Let $\mathcal{G} = (G, \lambda)$ be a temporal graph and O a TTO of $\mathcal{G}$. Observe that, unlike for the static case, O^R is not necessarily a TTO of $\mathcal{G}$. We can take the odd cycle in Figure 1 (*ii*) as an example. Of course, not all temporal graphs are T-comparability, as we can take any non-comparability static graph and assign an unique time-label to its edges. What is more, for a non T-comparability temporal graph $\mathcal{G} = (G, \lambda)$, the underlying graph G might be a comparability graph, as shown in Figure 1 (*i*). The opposite situation being also possible, i.e. a non-comparability static graph as underlying graph of a T-comparability temporal graph, as we see in Figure 1 (*ii*).

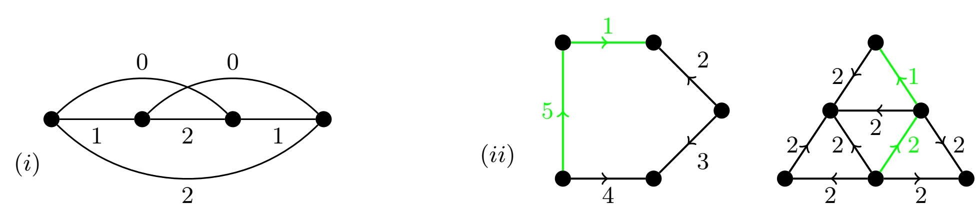


Fig. 1. (*i*) Comparability graph K_4 as underlying static graph of a non T-comparability temporal graph. (*ii*) Non-comparability graphs as underlying static graphs of T-comparability temporal graphs.

However, one can prove that every static graph G admits a transitive temporalization, that is, the assignation of time-labels to its edges resulting in a T-comparability temporal graph. It is sufficient to consider an arbitrary ordering σ of the vertices of G and defining λ as:

$$\lambda(v_i v_j) = \begin{cases} n - i & \text{if } \ v_i \prec_\sigma v_j \\ n - j & \text{otherwise} \end{cases} \qquad \text{for every } \ v_i v_j \in E \ \text{ with } \ i, j \in \{1, ..., n\}, i \neq j$$

If we define O by orienting every edge from left to right according to σ, all directed paths in O will have strictly decreasing time-labels in $\mathcal{G}$. Therefore, all arcs will be unconstrained and the temporal transitivity conditions will be trivially satisfied. Moreover, we can verify that O is not only a TTO of $\mathcal{G}$ but a Strict TTO, Strong TTO and Strong Strict TTO as well. We also note the existence of static graphs for which any temporal labelling will produce a T-comparability temporal graph, such is the case for comparability triangle-free graphs, as trees and even cycles.

3 Recognition of T-comparability graphs and TTO construction

3.1 Structural characterization of temporal transitivity

In this section, we use Ghouila-Houri's celebrated result [7] for static graphs as a starting point to guide a structural study of temporal transitive orientations. We present our main result, which will lay the theoretical foundations for our T-comparability recognition and TTO construction algorithm.

We start by observing that any TTO is also a QTTO, the converse not being true since a QTTO may contain directed monolabel triangles. In fact, the following holds.

Lemma 1. *Let O be a QTTO of a temporal graph $\mathcal{G} = (G, \lambda)$. If O contains a directed cycle, then O contains a monolabel directed triangle.*

Proof. Take a minimal such cycle C. There must exist along the cycle two consecutive arcs (a, b) and (b, c) such that $\lambda(ab) \leq \lambda(bc)$. Since O is QTTO, either $(c, a) \in O$ with $\lambda(ab) = \lambda(bc) = \lambda(ca)$ and we have found our monolabel directed triangle or $(a, c) \in O$. But in the latter case, as $(a, c) \in O$ we find a shorter directed cycle, a contradiction if C was not a triangle. $\square$

As a direct result, we can state Lemma 2, and conclude that all TTO are acyclic.

Lemma 2. *An orientation O of a temporal graph $\mathcal{G}$ is TTO if and only if O is QTTO with no monolabel directed triangle.*

For the static case, recall Ghouila-Houri's theorem stating that any graph admitting a quasi-transitive orientation also admits a transitive orientation. As this result was at the origin of efficient recognition algorithms, it was natural to try to extend it to the temporal scenario. Unfortunately, in the temporal case Theorem 1 is not true, as witnessed by the example shown on Figure 2 where the temporal graph is oriented by a QTTO but one can prove it does not admit any TTO.

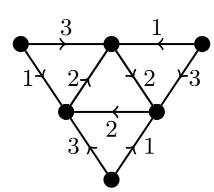


Fig. 2. A non T-comparability graph admitting a QTTO

However, we can state a simple generalization to proper temporal graphs using Lemma 2.

Corollary 1. *A proper temporal graph G admits a TTO if and only if it admits a QTTO.*

For the general case of temporal graphs, let us introduce a fundamental four-vertex configuration, in order to handle precisely monolabel triangles.

Definition 2. *A quadruple of vertices (a, b, c, d) of $\mathcal{G}$ forms a **correlated monolabel triangle** if:*

- *abc form a monolabel triangle of time-label t*
- *$bd \in E$ with $\lambda(bd) \leq t$*
- *$cd \in E$ with $\lambda(cd) > t$*
- *if $ad \in E$, then $\lambda(ad) < t$*

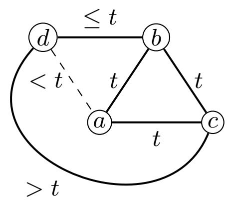


Fig. 3. Correlated monolabel triangle (a, b, c, d)

Note that when depicting a temporal graph, dashed labelled edges represent either a non edge or an edge satisfying the label condition.

Definition 3. *The implication digraph of $\mathcal{G}$, denoted **Imp($\mathcal{G}$)**, is the digraph with vertex set $AP(G)$ and with arcs from (a, b) to (c, b) and from (b, c) to (b, a) if $\lambda(ab) \leq \lambda(bc)$ with $ac \notin E$ or $\lambda(ac) < \lambda(bc)$.*

Observe that whenever an arc from (a, b) to (c, b) belongs to $Imp(\mathcal{G})$, it means that for any O TTO of $\mathcal{G}$, $(a, b) \in O$ implies $(c, b) \in O$, and similarly $(b, c) \in O$ implies $(b, a) \in O$. For any correlated monolabel triangle (a, b, c, d), it is not difficult to see that there is a directed path $(b, c)(d, c)(d, b)(a, b)$ in $Imp(\mathcal{G})$. Therefore, in any TTO, the presence of (b, c) will force the presence of (a, b). Since a TTO may not contain the directed triangle abc, it follows that if (b, c) is an arc of this orientation, then so will be (a, c). Based on this observation, we propose the following definition.

Definition 4. *The augmented implication digraph of $\mathcal{G}$, denoted **Aug($\mathcal{G}$)**, is the digraph obtained from $Imp(\mathcal{G})$ by adding all arcs $((b, c), (a, c))$ and $((c, a), (c, b))$, where (a, b, c, d) forms a correlated monolabel triangle.*

Note that the additional restrictions captured by $Aug(\mathcal{G})$ do not follow directly from the definition of temporal transitive orientations but are satisfied by any TTO of $\mathcal{G}$.

We say that an arc (x, y) *forces* an arc (u, v) if there is a directed path from (x, y) to (u, v) in $Aug(\mathcal{G})$. By the discussion above, if (x, y) forces (u, v), then any TTO of $\mathcal{G}$ that contains (x, y) must also contain (u, v). If O is a partial orientation, we denote by O^+ (resp. O^-) the set of vertices (u, v) of $Aug(\mathcal{G})$ that are forced by (resp. that force) some arc in O.

Definition 5. *We will say an arc (x, y) is **necessary** if (y, x) forces (x, y). We will denote by **Ness($\mathcal{G}$)** the set of necessary arcs of $\mathcal{G}$.*

Consider $(x, y) \in Ness(\mathcal{G})$ and (u, v) an arc forced by (x, y). Necessarily, (v, u) forces (y, x). As (y, x) forced (x, y), we conclude that $(u, v) \in Ness(\mathcal{G})$ and therefore $Ness(\mathcal{G})^+ = Ness(\mathcal{G})$. Observe that if $\mathcal{G}$ is a T-comparability temporal graph, then all arcs in $Ness(\mathcal{G})$ must be contained in any TTO of $\mathcal{G}$. The next lemma follows.

Lemma 3. *If $\mathcal{G}$ is a T-comparability temporal graph, the arcs in $Ness(\mathcal{G})$ induce an acyclic digraph. In particular, $Ness(\mathcal{G}) \cap (Ness(\mathcal{G}))^R = \emptyset$.*

Definition 6. *An orientation O is an Almost-TTO (or **ATTO**) of $\mathcal{G}$ if $O \cap O^R = \emptyset$ and $O^+ = O$.*

Lemma 4. *Let $\mathcal{G} = (G, \lambda)$ be a temporal graph and let O be an ATTO of $\mathcal{G}$. Then, $Ness(\mathcal{G}) \subseteq O$.*

Proof. Suppose $(a, b) \in Ness(\mathcal{G})$ such that $(a, b) \notin O$, then $(b, a) \in O$. By definition of $Ness(\mathcal{G})$, $(a, b) \in \{(b, a)\}^+$. Since O is ATTO, it follows that $(a, b) \in O$, a contradiction to O being an orientation. □

The definition of the augmented implication digraph $Aug(\mathcal{G})$ immediately implies the following useful lemma.

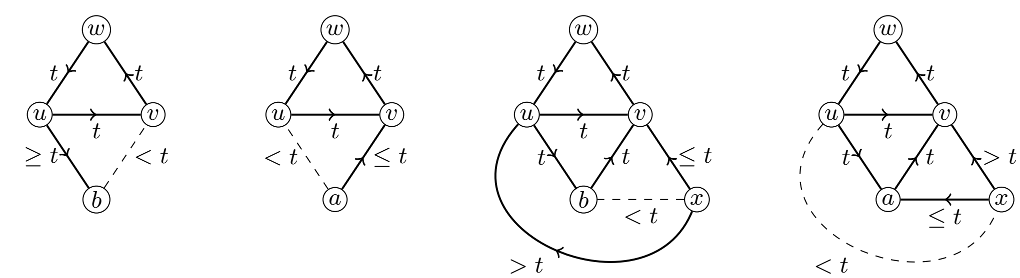
Lemma 5. *Let $\mathcal{G} = (G, \lambda)$ be a temporal graph and let O be an ATTO of $\mathcal{G}$. If (a, b, c, d) is a correlated monolabel triangle, then triangle abc cannot be directed in O.*

All ATTO of a temporal graph $\mathcal{G}$ are QTTO. Note that the converse is not true, as shown in Figure 2: on one hand it is easy to see that $(a, b), (b, c), (c, a) \subseteq Ness(\mathcal{G})$. On the other hand, as (a, b, c, d) form a correlated triangle, by Lemma 5 triangle abc cannot be directed in any ATTO of $\mathcal{G}$. However, not all ATTO are TTO, as a monolabel triangle not forming any correlated monolabel triangle with a fourth vertex can be directed in an ATTO. Then, we state that an orientation is TTO if and only if it is ATTO and contains no directed triangle.

Let us now propose a lemma, reminiscent of a statement called the *Triangle Lemma* in [8], that will be the key argument for our main theorem. Let us notice that the lemma proposed as such a temporal analogue in [16] is not completely true, see section 6.1 of the appendix for a counterexample.

Lemma 6. *Let O be an ATTO of $\mathcal{G} = (G, \lambda)$. If uvw is a directed triangle in O and there is an arc (a, b) such that $((a, b), (u, v)) \in Aug(\mathcal{G})$, then abw is also a directed triangle in O.*

Proof.
Recall that if uvw is a directed triangle in an ATTO, then it is necessarily monolabel and let t be the common time-label. We will distinguish cases on whether the arc from (a, b) to (u, v) is also an arc in $Imp(\mathcal{G})$ or not, and whether $a = u$ or $b = v$.



(i) (ii) (iii) (iv)

i) Let $a = u$ and $((u, b), (u, v))$ be an arc of $Imp(\mathcal{G})$. Then, $\lambda(ub) \geq t$ and either $bv \notin E$ or $\lambda(bv) < t$. But now, wub is a directed path with $\lambda(wu) \leq \lambda(ub)$. Since O is ATTO, we must have $wb \in E$ with $\lambda(wb) \geq \lambda(ub) \geq t$. It is not possible to orient wb from w towards b in O, as it would create the time-respecting directed path vwb and by hypothesis $bv \notin E$ or $\lambda(bv) < t$, which would contradict O being ATTO. We conclude that $(b, w) \in O$, which implies triangle $ubw = abw$ is directed.

ii) Let $b = v$ and $((a, v), (u, v))$ be an arc of $Imp(\mathcal{G})$. For this second case, $\lambda(av) \leq t$ and either $au \notin E$ or $\lambda(au) < t$. Since O is ATTO and avw is a time-respecting directed path, we must have $aw \in E$ and $\lambda(aw) \geq t$. Suppose aw is oriented from a towards w in O and consider the directed path awu. Since $au \notin E$ or $\lambda(au) < t$ and, once again, O is ATTO, we must have $\lambda(aw) > t$. Then, the quadruple (u, v, w, a) forms a correlated monolabel triangle. But this is a contradiction by Lemma 5 since triangle uvw is directed in O, an ATTO. Hence, aw must be oriented from w towards a in O, forming the directed triangle $avw = abw$.

iii) Let $a = u$ and $((u, b), (u, v))$ be an arc of $Aug(\mathcal{G})$ but not of $Imp(\mathcal{G})$. For this case to be produced, we claim the existence of a vertex x such that (b, v, u, x) is a correlated monolabel triangle: as (v, u) forced (b, v) in $Imp(G)$, arcs $((v, u), (b, u))$ and $((u, b), (u, v))$ were added to $Aug(G)$. Suppose $(v, x) \in O$. Since O is ATTO and vxu form a directed path with $\lambda(xu) > \lambda(vu)$, this forces $(u, x) \in O$ which in turn forces $(w, x) \in O$ with $\lambda(wx) > t$. But then, the existence of the non-monolabel directed triangle vwx contradicts the hypothesis of O being ATTO. Hence, $(x, v) \in O$. Then, necessarily $(b, v) \in O$ and $wx \in E$ with $\lambda(wx) \geq t$. Suppose $(w, x) \in O$, then vwx must form a monolabel triangle. That is, $\lambda(xv) = \lambda(wx) = t$. But then, both possible orientations (u, x) and (x, u) of xu lead to a contradiction as the triplet uwx would fail to satisfy the ATTO conditions. Therefore, we claim $(x, w) \in O$. Also, we deduce $(x, u) \in O$ as otherwise xuw would form a non-monolabel directed triangle. As wub form a monolabel directed path, either (w, b) or (b, w) belong to O, with $\lambda(bw) \geq t$. If $(w, b) \in O$, we obtain a correlated monolabel triangle (b, v, w, x) directed in an ATTO, a contradiction by Lemma 5. Finally, if $(b, w) \in O$ we have found our directed triangle $ubw = abw$.

iv) Let $b = v$ and $((a, v), (u, v))$ be an arc of $Aug(\mathcal{G})$ but not of $Imp(\mathcal{G})$. Similarly to the previous case, there must exist a vertex x such that (u, a, v, x) is a correlated monolabel triangle. As observed when such quadruples were defined, since (a, v) belongs to O, then so do (x, v), (x, a) and (u, a). Consider the monolabel directed path avw. As O is ATTO, $aw \in E$ with $\lambda(aw) \geq t$. Assume by contradiction that aw is oriented from a towards w. Then, the directed path xaw, with $\lambda(x, a) \leq \lambda(a, w)$, implies the existence of an edge xw such that $\lambda(xw) \geq t$. It cannot be oriented from w towards x because it would create a non-monolabel directed triangle wxv. Therefore, $(x, w) \in O$. Since by hypothesis $ux \notin E$ or $\lambda(ux) < t$, if $\lambda(xw) \leq t$, O would fail being ATTO. Hence, $\lambda(xw) > t$. But now, (u, a, w, x) form a correlated monolabel triangle directed in O, a contradiction by Lemma 5.

□

Lemma 7. *Let O be an ATTO of $\mathcal{G} = (G, \lambda)$. If uvw is a directed triangle in O and there is an arc (a, b) forcing (u, v) in $Aug(\mathcal{G})$, then abw is also a directed triangle in O.*

Proof. As (a, b) forces (u, v), there exists a directed path from (a, b) to (u, v) in $Aug(\mathcal{G})$. Then, we can repeatedly apply Lemma 6 starting from the second to last arc in the directed path, that is arc directly forcing (u, v), until reaching arc (a, b). At each step, we get a new directed triangle containing an arc forced by the immediately previous arc in the path, this allows us to continue to apply the lemma. When reaching arc (a, b), Lemma 6 will finally guarantee that abw is also a directed triangle in O. □

Lemma 8. *Let $\mathcal{G}$ be a temporal graph admitting an ATTO, then $Ness(\mathcal{G})$ is necessarily acyclic.*

Proof. Let O be an ATTO of $\mathcal{G}$. Suppose $Ness(\mathcal{G})$ induces a directed cycle in O and take C as the minimal one. Assume for contradiction C is not monolabel. Since O is ATTO, necessarily $|C| > 3$. Then, there must exist two consecutive arcs (a, b) and (b, c) in C forming a directed path of strictly increasing time-labels and as O is ATTO, we get $(a, c) \in O$. Moreover, as $\lambda(ab) < \lambda(bc)$, arc (b, c) will force arc (a, c) in $Aug(\mathcal{G})$. And since $(b, c) \in Ness(\mathcal{G})$, we get $(a, c) \in Ness(\mathcal{G})$. Hence, we reached a smaller directed cycle induced by $Ness(\mathcal{G})$, which contradicts C being the minimal one. Then, we state that whenever $Ness(\mathcal{G})$ induces a directed cycle, it induces a monolabel directed cycle as well. We will denote this monolabel directed cycle C', with $C' \subseteq Ness(\mathcal{G})$. Now, we claim the existence of a directed triangle, such that one of its arcs is contained in C', and therefore in $Ness(\mathcal{G})$. Let $v_1, ..., v_k$ be the vertices of C'. We take $k > 3$, as otherwise C' is a directed triangle and trivially satisfies the property. Assume there is no such directed triangle in O formed by, at least, one arc in C'. As O is ATTO and $v_1v_2v_3$ for a monolabel directed path in O, then necessarily $v_1v_3 \in E$. If $(v_3, v_1) \in O$, we have found a directed triangle $v_1v_2v_3$ such that $(v_2, v_3) \in C'$, a contradiction. Then, $(v_1, v_3) \in O$, and since $(v_1, v_3) \in O$ and $(v_3, v_4) \in O$ we can repeat the same argument with vertices v_1, v_3 and v_4. Hence, in order to avoid creating a directed triangle $v_1v_iv_{i+1}$ with $(v_i, v_{i+1}) \in C'$, necessarily $(v_1, v_i) \in O$ for all i such that $1 < i < k$. But then,

$v_{k-1}v_kv_1$ form a directed triangle with $(v_{k-1}, v_k) \in C'$, contradicting the hypothesis. On the other hand, one can prove that no directed triangle can have an arc in $Ness(\mathcal{G})$. Suppose uvw is a directed triangle in O of arcs $(u, v), (v, w), (w, u)$ and let $(u, v) \in Ness(\mathcal{G})$. Then, as (v, u) forces (u, v) in $Aug(\mathcal{G})$, by Lemma 7 $(v, u), (u, w)$ and (w, v) form a directed triangle in O, a contradiction. Supposing $Ness(\mathcal{G})$ induced a directed cycle in O lead us to a contradiction and as $Ness(\mathcal{G}) \subseteq O$ we conclude that $Ness(\mathcal{G})$ is necessarily acyclic. ☐

We are now ready to prove our main result.

Theorem 2. *Let $\mathcal{G} = (G, \lambda)$ be a temporal graph. The following assertions are equivalent:*

i) *$\mathcal{G}$ is a T-comparability temporal graph*
ii) *$\mathcal{G}$ admits an ATTO*
iii) *$Ness(\mathcal{G})$ induces an acyclic digraph*

Proof. $i) \Rightarrow ii)$ is straightforward as all TTO are ATTO and $ii) \Rightarrow iii)$ is proved by Lemma 8. To prove $iii) \Rightarrow i)$, we will mimic the proof provided by Hell et al. in [10] for the static case, by using an analogous lexicographic argument. The proof will provide a simple algorithm to decide whether a given temporal graph $\mathcal{G}$ is a T-comparability temporal graph and construct a TTO of $\mathcal{G}$ if the answer is YES, as detailed in Subsection 3.2.

Let O be a partial ATTO of $\mathcal{G}$ containing $Ness(\mathcal{G})$. We start by showing that taking any edge ab not yet oriented in O, ensures that $O' = O \cup \{(a, b)\}^+$ will also be an ATTO of $\mathcal{G}$. As O is ATTO, it is clear that $O'^+ = O'$ holds. Assume for contradiction that $O' \cap O'^R \neq \emptyset$, i.e. there exists a pair of vertices u, v such that $\{(u, v), (v, u)\} \subset O'$. Since O is ATTO, both arcs cannot belong to O, so assume $(u, v) \in \{(a, b)\}^+$. It is not possible that $(v, u) \in \{(a, b)\}^+$ as well, as this would imply $(b, a) \in \{(u, v)\}^+$ and thus $(b, a) \in \{(a, b)\}^+$, meaning $(b, a) \in Ness(G)$ which contradicts the assumption of edge ab being not yet oriented in O. Hence $(v, u) \in O$. Observe that $(u, v) \in \{(a, b)\}^+$ implies $(b, a) \in \{(v, u)\}^+$. But now, since $O^+ = O$, we must have $(b, a) \in O$, again contradicting the assumption of ab unoriented in O.

As $Ness(\mathcal{G})$ induces an acyclic digraph, there exists an order σ on the vertices of $\mathcal{G}$ such that all arcs in $Ness(\mathcal{G})$ are oriented forward, i.e. if $(u, v) \in Ness(\mathcal{G})$ then $u \prec_\sigma v$. We propose to construct an orientation O in a greedy lexicographic fashion, following the order defined by σ. We initially set $O = Ness(\mathcal{G})$. Then, while O is not yet a complete orientation of $\mathcal{G}$, we repeat what follows. We consider the unoriented edge ab, with $a \prec_\sigma b$, such that it is lexicographically minimal with respect to σ, and we add $\{(a, b)\}^+$ to O. By the observation above, as O was initially an ATTO of $\mathcal{G}$ containing $Ness(\mathcal{G})$, the final orientation obtained by this procedure will also be ATTO. It remains to show that O does not contain directed triangles. Assume by contradiction it contains at least one, and consider the monolabel directed triangle uvw that is lexicographically minimal with respect to the position in σ of its three vertices. Then, uvw contains at least one arc oriented backwards by σ, without loss of generality take (u, v) as this backward arc with $v \prec_\sigma u$. Since it was oriented backwards, $(u, v) \notin Ness(\mathcal{G})$ and it is not possible that (u, v) was chosen as next unoriented edge ab in the algorithm above. Hence, the orientation of (u, v) was assigned because $(u, v) \in \{(a, b)\}^+$ for some edge ab, chosen as minimal unoriented edge at some step in our greedy algorithm. As at this step uv was also an unoriented edge, this implies that ab is lexicographically smaller that uv. But now, since there is a directed path from (a, b) to (u, v) in $Aug(\mathcal{G})$, by applying Lemma 7, we get that abw is also a directed triangle in O, contradicting the lexicographic minimality of uvw. ☐

Remarks: Note that not all ATTO are TTO, as an ATTO might contain directed monolabel triangles, that do not form any correlated monolabel triangle with a forth vertex. Take a single monolabel triangle as an example where a directed orientation of its arcs is an ATTO. Exactly like in the static case, where it is important to note that the existence of a quasi-transitive orientation O of G, proves the existence of a transitive orientation O' of G, although O might not be transitive as it can contain directed triangles. This is why the lexicographic strategy is the key argument in our method, as it was in [11] for the static case. Additionally, we ensure that Theorem 2 also holds for the static case. Note that static graphs can be considered as monolabel temporal graphs. As no correlated monolabel triangle can be induced by a monolabel temporal graph $\mathcal{G}$, it follows that in that case $Aug(\mathcal{G})$ is equal to $Imp(\mathcal{G})$. Then, it is easy to see that $i) \Leftrightarrow ii)$ is the Ghouila-Houri result [7]. Moreover, in the static scenario the forcing relations are symmetric, i.e. if (a, b) forces (b, a) in $Aug(\mathcal{G})$, then (b, a) forces (a, b) in return. Hence, if $(a, b) \in Ness(\mathcal{G})$ then (b, a) also belongs to $Ness(\mathcal{G})$. Therefore, $Ness(\mathcal{G}) \neq \emptyset$ implies that $\mathcal{G}$ is not a comparability graph as $Ness(\mathcal{G})$ will necessarily induce a cycle, in particular every necessary arc will be contained in a 2-cycle.

3.2 Algorithmic aspects

The proof presented for Theorem 2 establishes the algorithm we will propose for deciding whether a given temporal graph $\mathcal{G}$ is T-comparability, and if so, obtaining a temporal transitive orientation of its edges.

The strategy will start by obtaining the set $Ness(\mathcal{G})$ of necessary arcs of $\mathcal{G}$. For this, it is first required to construct the implication digraph $Imp(\mathcal{G})$ and then to add the appropriate constraints to obtain the augmented digraph $Aug(\mathcal{G})$. We start by observing that the size of both digraphs is in $O(nm)$, which is the order of the maximum number of triplets of vertices, forming a triangle or a three vertex path and therefore potentially producing an implication constraint. Consider $f(n, m)$ and $g(n, m)$ the complexity functions of computing digraphs $Imp(\mathcal{G})$ and $Aug(\mathcal{G})$ respectively. Then, by the previous observations, we claim that deciding whether $\mathcal{G}$ admits a QTTO (resp. ATTO) can be solved in $max\{f(n, m), O(nm)\}$ (resp. $max\{g(n, m), O(nm)\}$) by a 2-SAT linear time algorithm. To obtain $Imp(\mathcal{G})$, only triangles and three-vertex paths need to be evaluated as no other structure can produce an implication, i.e. an arc in the digraph. Then, it is easy to see that $f(n, m) \in O(nm)$. However, for $Aug(\mathcal{G})$, we need to additionally consider correlated monolabel triangles, formed by a specific quadruple of vertices. To do this, two different strategies can be implemented, and the structure of the graph will determine which of the two is most efficient. First, one can consider for each monolabel triangle abc, all other vertex d and verify whether the quadruple forms a correlated monolabel triangle. If we consider k as the number of monolabel triangles, then clearly this procedure is in $O(kn)$. Alternatively, one can consider every pair of edges ac, bd in E and evaluate whether (a, b, c, d) satisfies the desired configuration. This can be done in $O(m^2)$. When k is small, the first strategy will be more efficient, however this is not always the case as $k \in O(nm)$. Therefore, we state that $g(n, m) \in min\{O(kn), O(m^2)\}$, and as Theorem 2 proves that any temporal graph $\mathcal{G}$ admitting an ATTO is a T-comparability graph, the following theorem holds.

Theorem 3. *The recognition problem for T-comparability temporal graphs can be solved in* $max\{O(nm), min\{O(kn), O(m^2)\}\}$.

Once the digraph $Aug(\mathcal{G})$ is constructed, it is straightforward to obtain $Ness(\mathcal{G})$ by Tarjan's [21] linear strategy in the size of the digraph, that is $O(nm)$. Again by Theorem 2, we know that $Ness(\mathcal{G})$ is a directed acyclic digraph, or DAG, and we can obtain in $O(nm)$ an order σ such that all necessary arcs are oriented from left to right according to σ, i.e. all $(a, b) \in Ness(\mathcal{G})$ satisfy $a \prec_\sigma b$. We can extend σ with the rest of the vertices in V in an arbitrary way. Then, we consider $Ness(\mathcal{G})$ as the initial orientation O and we greedily orient the remaining edges by selecting at every step the unoriented edge ab, with $a \prec_\sigma b$, such that a is lexicographically minimum. We then fix orientations $\{(a, b)\}^+$ in O. Using the same arguments as in proof of Theorem 2, we state that this results in a TTO of $\mathcal{G}$. Note that adding (a, b) to O, will fix the orientation of all the arcs forced in $\{(a, b)\}^+$. Then, after orienting $\{(a, b)\}^+$ we can remove all these vertices from $Aug(\mathcal{G})$, as all vertices in $\{(b, a)\}^-$, and claim the complexity of the greedy orientation construction can be done in $O(nm)$. To conclude, we state that the overall time complexity of the algorithm is of $max\{O(nm), min\{O(kn), O(m^2)\}\}$.

Theorem 4. *Deciding if $\mathcal{G}$ is a T-comparability temporal graph and constructing a TTO if the answer is YES can be done in* $max\{O(nm), min\{O(kn), O(m^2)\}\}$.

Remarks: The bottleneck of our algorithm can be the computation of the correlated monolabel triangles when $k \in O(mn)$ as then the construction of the digraph $Aug(\mathcal{G})$ will require $O(m^2)$. On the other hand, since up to our knowledge there is not yet a linear time algorithm to verify whether an orientation is transitive in the static case, and temporal transitive orientations generalize transitive ones, this supposes a lower bound for the T-comparability recognition problem.

3.3 Multilabel temporal transitive orientations

Since most temporal networks allow resources to traverse their arcs at multiple given times, it is natural to try to formulate an extension to the notion of temporal transitivity to multilabel temporal graphs, i.e. temporal graphs $\mathcal{G} = (G, \lambda)$ with $G = (V, E)$ and $\lambda : E \to 2^{\mathbb{N}}$. We shortly introduce the motivation and main definitions, further clarifications can be found in the appendix section 6.4. Let us recall the initial motivation behind the model proposed by Mertzios et al. [16]. The authors introduced temporal transitive orientations as a way to avoid "shortcuts" in the transmission of information by nodes in a communication network. This way, if node a sends information to node b and node b then sends it to node c, the latter should be able to verify this information from source a at a later, or equal, time than when receiving the message via b. In the multilabel setting, it is considered that new information can reach node a and be transmitted to b multiple times, at the moments determined by the time-label set associated with the arc connecting both nodes. Therefore, the arc connecting node a with node c should not allow the information to be transmitted at an earlier time to the one of b sending the information to c. We formalized this idea as follows. First, we introduce the operator that will allow us to compare two time-label sets in the multilabel scenario.

Definition 7. *Let S_1, S_2 be two sets in $2^{\mathbb{N}}$, $S_1 \preceq S_2$ (resp. $S_1 \prec S_2$) if there exist $x \in S_1$ and $y \in S_2$ such that $x \leq y$ (resp. $x < y$).*

Let us denote by $\npreceq$ (resp. $\nprec$) the negations. Notice that $S_1 \npreceq S_2$ (resp. $S_1 \nprec S_2$) implies that all elements in S_1 are strictly greater (resp. greater or equal) than all elements in S_2.

Definition 8. *Let $\mathcal{G} = (G, \lambda)$ be a multilabel temporal graph and O an orientation of $\mathcal{G}$. We will say O is a multilabel temporal transitive orientation (MTTO) if whenever $(a, b), (b, c) \in O$ with $\lambda(ab) \preceq \lambda(bc)$, then necessarily $(a, c) \in O$ with $\lambda(ac) \nprec \lambda(bc)$.*

In the multilabel scenario, the equivalent to the monolabel triangles considered in the previous subsections will be monolabel triangles with singleton time-label sets. Then, we can define *almost-multilabel temporal transitive orientations* (AMTTO) as the analogous to the simple temporal case. This leads to an equivalent characterization of T-comparability multilabel temporal graphs and lexicographic strategy for constructing a MTTO of a YES instance.

Theorem 5. *Let $\mathcal{G} = (G, \lambda)$ be a multilabel temporal graph. Then, the following assertions are equivalent:*

i) $\mathcal{G}$ is a T-comparability multilabel temporal graph
ii) $\mathcal{G}$ admits an AMTTO
iii) $Ness(\mathcal{G})$ induces an acyclic digraph

Furthermore, there is a $max\{O(nm), min\{O(kn), O(m^2)\}\}$-time algorithm to decide if $\mathcal{G}$ is a T-comparability multilabel temporal graph and construct a MTTO if the answer is YES, with k being the number of monolabel triangles with a singleton time-label set associated to their edges.

4 Characterizations via forbidden ordered patterns

All proofs in this section are included in the appendix, sections 6.2 and 6.3.

Temporal ordered patterns. An *hereditary property* $\mathcal{P}$ satisfies that whenever $G \in \mathcal{P}$ then $H \in \mathcal{P}$ for all H induced subgraph of G. Hereditary properties can be characterized by a, possibly infinite, set of forbidden structures, in this case we will consider forbidden temporal ordered patterns. A *temporal pattern* is a temporal graph $\mathcal{H} = (H, \lambda')$. A temporal graph $\mathcal{G} = (G, \lambda')$ *avoids* $\mathcal{H}$ if no temporal subgraph $\mathcal{H}'$ of $\mathcal{G}$ is isomorphic to $\mathcal{H}$. An *ordered temporal graph* $(\mathcal{G}, \sigma)$ is a graph G equipped with an order σ on its vertices. The definition of *ordered temporal pattern* $(\mathcal{H}, \sigma)$ is analogous. An ordered temporal graph $(\mathcal{G}, \sigma)$ *avoids* an ordered temporal pattern $(\mathcal{H}, \sigma')$ when there is no temporal subgraph $\mathcal{H}'$ of $\mathcal{G}$ isomorphic to $\mathcal{H}$ through an isomorphism ϕ such that $\phi(u) \prec_{\sigma'} \phi(v)$ whenever $u \prec_{\sigma} v$. Given a family of temporal patterns $\mathcal{F}$, we define the class $\mathcal{C}_{\mathcal{F}}$ as the set of connected temporal graphs admitting an order of their vertices avoiding all temporal patterns in $\mathcal{F}$. We use the classic pattern representation, see [6], and we depict by dashed conditional edges either the absence of an edge or an edge whose time-label meets the specified condition.

4.1 Temporal transitive orientations

Consider the ordered pattern TTO illustrated in Figure 4.

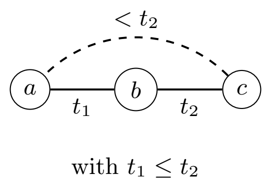


Fig. 4. TTO temporal ordered pattern

Theorem 6. *The temporal graph $\mathcal{G} = (G, \lambda)$ admits a temporal transitive orientation if and only if it belongs to $\mathcal{C}_{TTO}$.*

Observe that when a temporal graph is monolabel, the pattern corresponds to the usual pattern for characterizing the class of comparability graphs, see [6] for a survey on three-vertex patterns. These proves the characterization via forbidden temporal patterns of the T-comparability temporal graphs works as an extension of the static case.

4.2 Strict temporal transitive orientations

Consider the set of ordered patterns *Strict-*TTO illustrated in Figure 5.

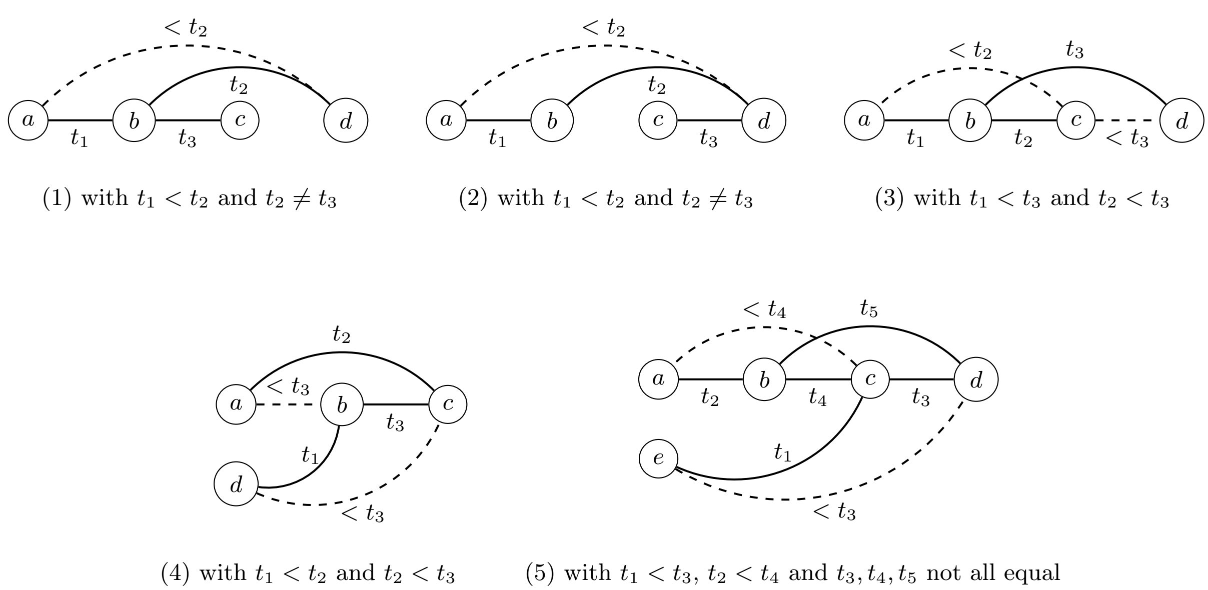


Fig. 5. *Strict-*TTO forbidden temporal ordered patterns

Theorem 7. *The temporal graph $\mathcal{G}$ admits a strict temporal transitive orientation if and only if it belongs to $\mathcal{C}_{Strict-TTO}$.*

Recall that deciding whether a temporal graph is T-comparability can be solved in polynomial time whereas the recognition problem for graphs admitting a Strict TTO was proved NP-hard. We can see how this difference in the complexity of both problems also translates into a complexity gap when proposing characterizations via temporal ordered patterns.

Analogous characterizations can be formulated for the Strong TTO and Strong Strict TTO versions.

5 Perspectives

In section 3, to prove our main result, Theorem 2, we used the lexicographic argument proposed by Hell and Huang for determining if a static graph is a comparability graph. The question of whether it is possible to directly prove that any graph admitting an ATTO is a T-comparability graph by using a similar approach to Ghouila-Houri's original proof, i.e. by module contraction and re-orientation of an ATTO, remains unresolved. For this, the concept of module should be extended to the temporal setting. Then, a set of vertices M having the same neighbors outside of M and connected to them by exactly the same time-labels could be considered as a temporal module. What is more, in the static case all possible transitive orientations of a graph can be described in a compact way by the modular decomposition tree, see [8]. In the temporal setting, recall that our algorithm constructs a temporal transitive orientation of a temporal graph $\mathcal{G}$ by computing $Ness(\mathcal{G})$, choosing any linear extension that puts all necessary arcs forward and greedily completing the orientation in a lexicographic way. We observe that any TTO O of $\mathcal{G}$ can in fact be obtained by this algorithm by choosing the appropriate linear extension of $Ness(\mathcal{G})$: it suffices to start with the linear extension that puts all arcs of O forward. However, it remains unanswered whether a compact representation of all TTO can be proposed, as for the static case. It is in this context that we raise the following open problem.

Open Problem 1 *For a given* T*-comparability temporal graph $\mathcal{G} = (G, \lambda)$, is it possible to obtain a TTO by a strategy based on temporal modular decomposition? Can this decomposition provide a compact structure to describe all possible TTO of $\mathcal{G}$?*

In Section 2 we showed that every static graph $G = (V, E)$ admits a transitive temporalization, i.e. the assignation of time-labels to its edges, resulting in a T-comparability temporal graph. This observation leads to the following interesting relabelling problems.

Open Problem 2 *For a given temporal graph $\mathcal{G} = (G, \lambda)$, how to compute the minimum number of edges for which we need to modify the time-label in order to obtain a T-comparability graph?*

Open Problem 3 *Consider a graph $\mathcal{G} = (G, \lambda)$, partially time-labelled, can the labelling be completed in order to obtain a T-comparability graph?*

And finally, is there a link between problems 2 and 3? As an application of these relabelling problems, one can consider a transportation network modeled by a temporal graph $\mathcal{G}$ and evaluate how to transform the transportation time tables, i.e. the time-labels of $\mathcal{G}$, for the transportation network to satisfy temporal transitivity.

One could also think of alternative definitions for temporal transitivity. For example, we could say that an orientation O of a temporal graph is $T\tilde{T}O$ if whenever $(a, b) \in O$ and $(b, c) \in O$ with $\lambda(ab) \leq \lambda(bc)$, then $(a, c) \in O$ but with an unrestricted time-label. We start by observing that, as for TTO, $T\tilde{T}O$ constraints are not directly expressable by 2-SAT. It is therefore natural to try to compare the complexity of the recognition problems for graphs admitting a TTO and a $T\tilde{T}O$ and the problems of obtaining such orientations. Let's denote by Quasi-$T\tilde{T}O$ (or $Q\tilde{T}TO$) all orientations O such that whenever $(a, b) \in O$ and $(b, c) \in O$ with $\lambda(ab) \leq \lambda(bc)$, then (a, c) or (c, a) belong O. Does the Ghouila-Houri result for static graphs holds for $T\tilde{T}O$? That is, is it true that a temporal graph admitting a Quasi-$T\tilde{T}O$ will necessarily be a $T\tilde{T}O$ graph? We observe that the lexicographic argument proposed by Hell and Huang cannot be directly adapted, as it is not true that each valid solution of the 2-coloring problem on the resulting conflict graph corresponds to a valid $T\tilde{T}O$ orientation. Furthermore, our method cannot be directly adapted either, as it relies heavily on the identification of correlated monolabel triangles, structures that do not imply a particular sequence of forcings in this new scenario.

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6 Appendix

6.1 About a lemma in [16]

In [16], Mertzios et al. use the following so called Temporal Triangle Lemma in order to establish their result. As stated it is incorrect, as shown by Figure 6, since $b'c \in A$ and $ab' \notin C$.

Lemma 9 (Temporal Triangle Lemma [16]). *Let (G, λ) be a temporal graph with a synchronous triangle on the vertices a, b, c, where $\lambda(a,b) = \lambda(b,c) = \lambda(c,a) = t$. Let A, B, C be three Λ-implication classes of (G, λ), where $ab \in C$, $bc \in A$, and $ca \in B$, where $A \neq B^{-1}$ and $A \neq C^{-1}$.*

1. *If some $b'c' \in A$, then $ab' \in C$ and $c'a \in B$.*
2. *If some $b'c' \in A$ and $a'b' \in C$, then $c'a' \in B$.*
3. *No edge of A touches vertex a.*

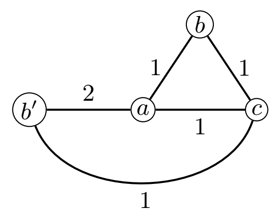


Fig. 6. Counterexample to Lemma 9

6.2 TTO characterization via ordered patterns

Proof of Theorem 6: *The temporal graph $\mathcal{G} = (G, \lambda)$ admits a temporal transitive orientation if and only if it belongs to $\mathcal{C}_{TTO}$.*

Proof. ($\Rightarrow$) Let O be a TTO of $\mathcal{G} = (G, \lambda)$. By a direct result of Lemma 2, O is acyclic and therefore defines a partial ordering σ of V. Let σ' be a linear extension of σ. Then, for every arc $(u, v) \in O$, $u \prec_{\sigma'} v$. Consider the ordered triplet a, b, c and let t_1, t_2 and t_3 be the time-labels associated to the potential edges ab, bc and ac respectively. Note that the only possible arcs between the triplet's nodes are $(a, b), (b, c)$ and (a, c). As the orientation was TTO, if arcs $(a, b), (b, c) \in O$ such that $t_1 \leq t_2$ then necessarily $(a, c) \in O$ such that $t_3 \geq t_2$. We conclude that the pattern is avoided by every ordered triplet a, b, c.

($\Leftarrow$) Let σ' be an ordering avoiding the pattern. We know that for every triplet a, b, c ordered by σ', if $ab \in E$ and $bc \in E$ then either $\lambda(ab) > \lambda(bc)$ or $ac \in E$ such that $\lambda(ac) \geq \lambda(bc)$. Then, if we orient all edges from left to right according to σ', we obtain a TTO of $\mathcal{G}$.

□

6.3 Strict TTO characterization via ordered patterns

Let O be a Strict TTO of a temporal graph $\mathcal{G} = (G, \lambda)$, with $G = (V, E)$. Note that unlike temporal transitive orientations, O can contain a directed cycle. However, the directed cycles must be monolabel, and even more we state the following.

Observation 1 *All strongly connected components in O are monolabel.*

Proof. All directed cycles in O are necessarily monolabel. Suppose this is not true and let C be the smallest non-monolabel directed cycle. Note that $|C| > 3$ as otherwise O would not be Strict TTO. By hypothesis, there must exist two consecutive arcs in C forming a strictly time-respecting directed path. But this implies the existence of a chord between these two arcs creating a strictly smaller directed cycle, a contradiction. Let S be a strongly connected component of O. Then, each arc induced by S will belong to a monolabel directed cycle of vertices all contained in S. Consider two directed cycles C_1 and C_2 both belonging to S. Note that the existence of C_1 and C_2 in S of different time-labels implies the existence of two directed cycles C_1' and C_2' in S of different time-labels t_1 and t_2 respectively and sharing a common vertex v. Let $t_1 < t_2$. Then, there necessarily exists an arc (v_1, v) belonging to C_1' and an arc (v, v_2) belonging to C_2', of time-labels t_1 and t_2 respectively. As these two arcs form a strictly time-respecting directed path, necessarily there exists a third arc (v_1, v_2) in O. But then, S induces a directed cycle $C_1 \cup C_2 \cup \{(v_1, v_2)\} \setminus \{(v_1, v), (v, v_2)\}$ with two distinct time-labels which implies a contradiction to directed cycles being monolabel in all Strict TTO. ◻

The following straightforward observations, related to the incoming and outcoming arcs of strongly connected components in O, will be useful when proposing a characterization via temporal ordered patterns, in Theorem 7.

Observation 2 *Let S be a strongly connected component of O with time-label t_1 and let (v, w) be an outcoming arc from S such that $v \in S, w \notin S$, $\lambda(vw) = t_2$ and $t_1 < t_2$. Then, for every other vertex $u \in S$, $(u, w) \in O$ with $\lambda(uw) = t_2$.*

Observation 3 *Let S be a strongly connected component of O with time-label t_2 and let (v, w) be an incoming arc to S such that $v \notin S, w \in S$, $\lambda(vw) = t_1$ and $t_1 < t_2$. Then, for every other vertex $w' \in S$ such that $(w, w') \in O$, $(v, w') \in O$ with $\lambda(vw') \geq t_2$.*

Proof of Theorem 7: *The temporal graph $\mathcal{G}$ admits a strict temporal transitive orientation if and only if it belongs to $\mathcal{C}_{Strict-TTO}$.*

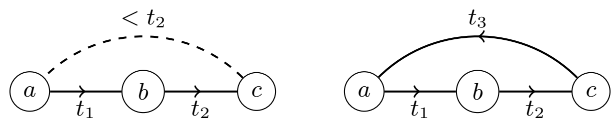
Proof. $\rightarrow$) Let O be a Strict TTO of $\mathcal{G}$ and let D' be the directed acyclic graph obtained by contracting its strongly connected components. Let σ' be the pre-order on the vertices of $\mathcal{G}$ defined by D' and take σ as a linear extension of σ' such that all strongly connected components in O are consecutive in σ. Observe that all arcs (u, v) between different strongly connected components will be oriented from left to right by σ, i.e. $u \prec_\sigma v$. Let a,b and c be a triplet of vertices in $\mathcal{G}$ ordered by σ. Note that if $\lambda(ab) < \lambda(bc)$ but $(a, c) \notin E$ or $\lambda(ac) < \lambda(bc)$, then necessarily there exists a strongly connected component S in O such that either $a, b \in S$ with $c \notin S$ or $b, c \in S$ with $a \notin S$, by Observation 2 we claim that only the second case is possible. Given five ordered vertices a, b, c, d and e, we will analyse the conditions for each pattern to arise and conclude that all patterns are avoided by σ.

- Patterns 1 and 2: Suppose pattern 1 is produced by the set of ordered vertices a, b, c, d. We deduce vertices b and d must belong to the same strongly connected component S in O. As vertices belonging to a given strongly connected component should be consecutive in σ, we state that c also belongs to S. However, we reach a contradiction by Observation 1 since $t_2 \neq t_3$. For pattern 2, the argument is analogous.
- Pattern 3: Assume vertices a, b, c, d produce pattern 3. Again, vertices b and c must belong to the same strongly connected component S in O. Additionally, since $t_2 \neq t_3$, we state that vertex $d \notin S$. Then, arc (b, d) is an outcoming arc from S such that $t_2 < t_3$. But since $(c, d) \notin E$ or $\lambda(cd) < t_3$ this contradicts Observation 2.
- Pattern 4: Suppose vertices a, b, c, d form pattern 4. Then, there must exist a strongly connected component S in O such that $b, c \in S$ and since $t_2 \neq t_3$, by Observation 1, $a \notin S$. Then, $(a, c) \in O$ but this contradicts Observation 3, as (a, b) should belong to O with $\lambda(ab) \geq t_3$.
- Pattern 5: Let the ordered vertices a, b, c, d, e form pattern 5. We argue the existence of two strongly connected components S_1 and S_2 in O such that $b, c \in S_1$ and $c, d \in S_2$. Since c belongs both to S_1 and S_2, necessarily $S_1 = S_2$. But then, by Observation 1, t_3, t_4 and t_5 should be equal, which contradicts the hypothesis.

$\leftarrow$)

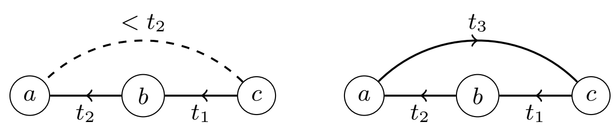
Let σ be an ordering of the vertices in $\mathcal{G}$ avoiding the patterns, we construct an orientation O as follows. Given a triplet a, b, c ordered by σ with $ab, bc \in E$, if $\lambda(ab) = t_1$, $\lambda(bc) = t_2$ with $t_1 < t_2$, and either $ac \notin E$ or $\lambda(ac) < t_2$, then we add arc (c, b) to O. We orient every other edge from left to right according to σ. Let's verify O is a Strict TTO of $\mathcal{G}$. That is, taking t_1, t_2 and t_3 such that $t_1 < t_2$ and $t_3 \geq t_2$, none of the following cases occur, for any triplet a, b, c ordered by σ and oriented by O.

- Case 1: $(a, b), (b, c) \in O$ with either $ac \notin E$, $\lambda(ac) < t_2$ or $(c, a) \in O$



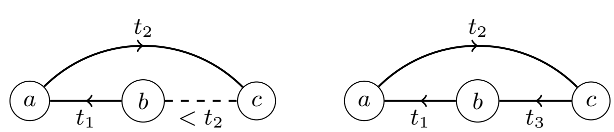
  If both $(a, b), (b, c) \in O$, then $ac \in E$ such that $\lambda(ac) \geq t_2$. Otherwise, the orientation construction rules would have fixed $(c, b) \in O$. Suppose now $(c, a) \in O$ with time label set t_3, this implies the existence of vertex a' such that $a'a \in E$ with $\lambda(a'a) < t_3$, and either $a'c \notin E$ or $\lambda(a'c) < t_3$. But then vertices a', a, b, c would form pattern 1, which is a contradiction.

- Case 2: $(b, a), (c, b) \in O$ with either $ac \notin E$, $\lambda(ac) < t_2$ or $(a, c) \in O$



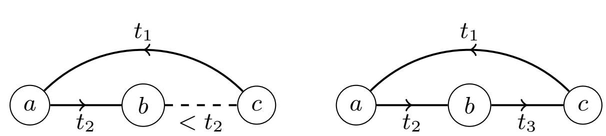
  If $(b, a), (c, b) \in O$, we can state the existence of both vertex a' such that $a'a \in E$ with $\lambda(a'a) < t_2$, and either $a'b \notin E$ or $\lambda(a'b) < t_2$, and vertex b' such that $b'b \in E$ with $\lambda(b'b) < t_1$, and either $b'c \notin E$ or with $\lambda(b'c) < t_1$. If $ac \notin E$, $\lambda(ac) < t_2$ or $(a, c) \in O$, then vertices a', a, b, c would form pattern 5.

- Case 3: $(b, a), (a, c) \in O$ with either $bc \notin E$, $\lambda(bc) < t_2$ or $(c, b) \in O$



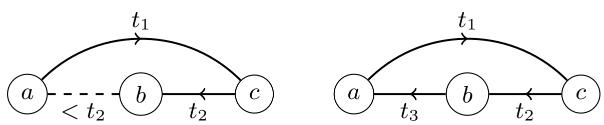
  If $(b, a) \in O$, there must exist a vertex a' such that $a'a \in E$, with $\lambda(a'a) < t_1$, and either $a'b \notin E$ or $\lambda(a'b) < t_1$, this corresponds to pattern 3. If $(c, b) \in O$, then vertices a, b, b', c would form pattern 5.

- Case 4: $(c, a), (a, b) \in O$ with either $bc \notin E$, $\lambda(bc) < t_2$ or $(b, c) \in O$



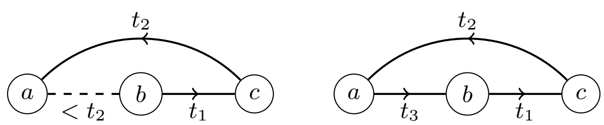
As for the previous case, orientation (c, a) implies the existence of a vertex a' such that $a'a \in E$ such that either $a'c \notin E$ or $\lambda(a'c) < t_1$. As $t_1 \neq t_2$, then vertices a', a, b, c reproduce pattern 1. If $(b, c) \in O$, the vertices will still produce the first pattern.

– <u>Case 5:</u> $(a, c), (c, b) \in O$ with either $ab \notin E$, $\lambda(ab) < t_2$ or $(b, a) \in O$



As $(c, b) \in O$, there must exist a vertex b' such that $b'b \in E$, with $\lambda(b', b) < t_2$ and either $b'c \notin E$ or $\lambda(b', c) < t_2$, this corresponds to pattern 4. Now, suppose $(b, a) \in O$, this corresponds to pattern 5.

– <u>Case 6:</u> $(b, c), (c, a) \in O$ with either $ab \notin E$, $\lambda(ab) < t_2$ or $(a, b) \in O$



We deduce the existence of a vertex a' such that $a'a \in E$, with $\lambda(a', a) < t_2$ and either $a'c \notin E$ or $\lambda(a', c) < t_2$, this corresponds to pattern 2. If $(a, b) \in O$, the vertices will produce pattern 1.

For each triplet a, b, $c \in V$ ordered by σ, we have analysed all six scenarios were O could fail to satisfy the Strict TTO constraints and proved the impossibility of them to occur. Therefore, we can state that if $(a, b) \in O$ and $(b, c) \in O$ with $\lambda(ab) < \lambda(bc)$ then by construction of O we ensure that (a, c) will also belong to the orientation with a time-label such that $\lambda(ac) \geq \lambda(bc)$. We conclude that O is a Strict TTO of $\mathcal{G}$. □

6.4 Multilabel temporal transitive orientations

We start by observing that orienting an edge of a multilabel temporal graph will fix the orientation for all its temporal time-labels. Let's now recall definition of multilabel temporal transitive orientations.

Definition 9. *Let $\mathcal{G} = (G, \lambda)$ be a multilabel temporal graph and O an orientation of $\mathcal{G}$. We will say O is a multilabel temporal transitive orientation (MTTO) if whenever $(a, b), (b, c) \in O$ with $\lambda(ab) \preceq \lambda(bc)$, then necessarily $(a, c) \in O$ with $\lambda(ac) \not\prec \lambda(bc)$.*

An alternative way to put it is that whenever $(a, b), (b, c) \in O$ with $min\{\lambda(ab)\} \leq max\{\lambda(bc)\}$, then $(a, c) \in O$ with $min\{\lambda(ac)\} \geq max\{\lambda(bc)\}$, taking as min and max the minimum and maximum elements of a given set. As for simple temporal graphs, one can define the Strict MTTO version by asking for the third arc (a, c) to belong to O only when $\lambda(ab) \prec \lambda(bc)$. One can verify that these definitions correspond to an extension of the original definitions for temporal transitivity by considering simple temporal graphs. Also, as for the static case, MTTO are acyclic and for the strict version, only monolabel strongly connected components having a singleton time-label set can be non-trivial. Next, we present some arguments to state that the characterization and algorithmic result presented for simple temporal graphs hold for the general case of multilabel temporal graphs as well. Complete proofs are omitted as they are analogous to the simple case.

An analogous definition for *quasi-multilabel temporal transitive orientations* can be proposed as follows.

Definition 10. *Let $\mathcal{G} = (G, \lambda)$ be a multilabel temporal graph and O an orientation of $\mathcal{G}$. We will say O is a quasi-multilabel temporal transitive orientation (QMTTO) if whenever $(a, b), (b, c) \in O$ with $\lambda(ab) \preceq \lambda(bc)$, then necessarily $(a, c) \in O$ or $(c, a) \in O$ with $\lambda(ac) \not\prec \lambda(bc)$.*

In the multilabel scenario, the equivalent to the monolabel triangles considered in Section 3 will be monolabel triangles with singleton time-label sets, as these will be the only directed triangles admitted by a QMTTO. Then, we can define the extension to correlated monolabel triangles as a quadruple of vertices (a, b, c, d) of $\mathcal{G}$ such that:

- triangle abc is monolabel with singleton time-label set T
- $bd \in E$ with $\lambda(bd) \preceq T$
- $cd \in E$ with $\lambda(cd) \not\preceq T$
- if $(a, d) \in E$, then $\lambda(ad) \prec T$

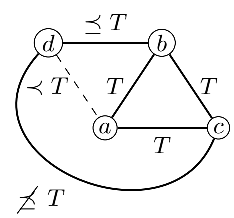


Fig. 7. Correlated monolabel triangle (a, b, c, d)

One can prove the same forcings as in the static case will be produced for a given correlated monolabel triangle in the multilabel setting. Therefore, we can consider an analogous definition of *almost-multilabel temporal transitive orientations* (AMTTO) leading to an equivalent characterization and lexicographic strategy. Note that only the minimum and maximum values of the time-label sets are considered when formulating the constraints. Then, if we considered multilabel temporal graphs with ordered time-label sets, the size of the label sets will not impact the algorithm's complexity. These arguments lead to the formulation of Theorem 5.

From [2], a multilabel temporal graph $\mathcal{G}$ is *proper* if $\lambda(e) \cap \lambda(e') = \emptyset$ whenever e and e' are incident to a common vertex. Then, we can state the following, which is particularly interesting for the Strict MTTO recognition problem, proved NP-hard for simple temporal graphs.

Theorem 8. *The MTTO and Strict MTTO recognition problem for proper multilabel temporal graphs can be solved in $O(mn)$.*

Proof. No proper graph can contain a triangle such that two of its arcs share the same time-label set. Then, the constraints for both MTTO and Strict MTTO will be entirely captured by $Imp(\mathcal{G})$. Deciding whether $Imp(\mathcal{G})$ is satisfiable can be solved in $O(mn)$ which is the cost of computing the implication graph, as well as its size. $\square$