Comparison of typicality in quantum and classical many-body systems

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Quantum typicality refers to the phenomenon that the expectation values of any given observable are nearly identical for the overwhelming majority of all normalized vectors in a sufficiently high-dimensional Hilbert (sub-)space. As a consequence, we show that the thermal equilibrium fluctuations in many-body quantum systems can be very closely imitated by the purely quantum mechanical uncertainties (quantum fluctuations) of suitably chosen pure states. On the other hand, we find that analogous typicality effects of similar generality are not encountered in classical systems. The reason is that the basic mathematical structure, in particular the description of pure states, is fundamentally different in quantum and classical mechanics.

I. TYPICALITY IN QUANTUM SYSTEMS

To the best of our knowledge, the so-called typicality or concentration of measure phenomenon in quantum systems has been originally discovered in Ref. [1], and has later been independently rediscovered several times [2–7]. A more detailed review of the pertinent literature and of various modifications and generalizations can be found for instance in Ref. [8].

Essentially, quantum typicality refers to the following quite remarkable property of any sufficiently high-dimensional Hilbert space $\mathcal{H}$: Consider an arbitrary subspace $\mathcal{H}_D \subset \mathcal{H}$ of large but finite dimension D, and let us randomly sample vectors $|\psi\rangle$ from $\mathcal{H}_D$ which are normalized but otherwise unbiased (uniformly distributed [9]). Then the expectation values $\langle\psi|A|\psi\rangle$ of any given observable (Hermitian operator) A will assume with very high probability very similar values. More precisely speaking, if we denote by $\bar{A}$ the mean value (average) of all those expectation values, then the probability that $|\langle\psi|A|\psi\rangle - \bar{A}|$ exceeds an arbitrary but fixed threshold δ can be upper bounded by $\|A\|_{op}/\sqrt{D}\delta$, where $\|A\|_{op}$ denotes the operator norm of A (largest eigenvalue in modulus) [10]. Given that A models some experimental measurement device, the eigenvalues of A correspond to the possible measurement outcomes, hence $\|A\|_{op}$ will be bounded and usually can be set equal to unity without much loss of generality [11]. Choosing for δ the resolution limit of the given measurement device, and assuming that D is much larger than $1/\delta$, it follows that the expectation values $\langle\psi|A|\psi\rangle$ will be experimentally indistinguishable from $\bar{A}$ (and thus from each other) for the overwhelming majority of all normalized vectors (pure states) $|\psi\rangle \in \mathcal{H}_D$. In brief, $\langle\psi|A|\psi\rangle$ is typically very close to $\bar{A}$.

Moreover, the average expectation value $\bar{A}$ can also be readily rewritten as $\mathrm{Tr}\{\bar{\rho}A\}$, where the statistical operator $\bar{\rho}$ is obtained by averaging $|\psi\rangle\langle\psi|$ over all the randomly sampled pure states $|\psi\rangle$ from above. It follows that $\bar{\rho}$ is independent of A, is generically a mixed state ($\mathrm{Tr}\{\bar{\rho}^2\} < 1$), and can be imitated very well (as far as expectation values are concerned) by most pure states $|\psi\rangle \in \mathcal{H}_D$.

The physically most important examples are quantum many-body system with $N \gg 1$ degrees of freedom, Hamiltonian H, and concomitant eigenvectors $|n\rangle$ and eigenvalues E_n. As usual in the context of the microcanonical formalism, we furthermore consider some microcanonical energy window $I := [E, E + \Delta]$ of macroscopically small but microscopically large width Δ, meaning that Δ is experimentally unresolvably small while the number of states $|n\rangle$ with the property $E_n \in I$ is very large (exponentially growing with N). As our $\mathcal{H}_D$ we choose the corresponding subspace spanned by all those $|n\rangle$ with $E_n \in I$, which is often denoted as the microcanonical energy shell in this context. Hence, the above introduced statistical operator $\bar{\rho}$ can be identified with the microcanonical ensemble $\rho_{mc} := P/\mathrm{Tr}\{P\}$, where P is the projector onto $\mathcal{H}_D$. Furthermore, the dimensionality $D = \mathrm{Tr}\{P\}$ of the energy shell $\mathcal{H}_D$ is exponentially large in the system's degrees of freedom N, and the above introduced average $\bar{A}$ can be identified with the microcanonical expectation value

$$A_{mc} := \mathrm{Tr}\{\rho_{mc}A\} \ . \tag{1}$$

Typicality thus implies that for a randomly sampled $|\psi\rangle \in \mathcal{H}_D$ the expectation value $\langle\psi|A|\psi\rangle$ is practically indistinguishable from the thermal equilibrium value A_{mc} with overwhelming probability (exceptions are exponentially rare in N).

While these results are by now widely known, we finally mention an immediate consequence which seems to be less known, but will play a key role later on, and also seems of notable physical interest in itself. Our first observation is that if A models some experimental measurement device, the same applies to A^2 (one simply has to square the outcome of every measurement of A), and analogously for $A - A_{mc}$ and thus for $(A - A_{mc})^2$. Typicality thus implies that the approximation

$$\langle\psi|(A - A_{mc})^2|\psi\rangle = \mathrm{Tr}\{\rho_{mc}(A - A_{mc})^2\} \tag{2}$$

is very well satisfied for most $|\psi\rangle \in \mathcal{H}_D$. Moreover, A_{mc} on the left-hand side can be very well approximated by $\langle\psi|A|\psi\rangle$ for most $|\psi\rangle \in \mathcal{H}_D$, as seen above. One readily can conclude that both the latter approximation as well as the approximation (2) are simultaneously satisfied very well for most $|\psi\rangle \in \mathcal{H}_D$, implying that the same applies to

$$\langle\psi|(A - \langle\psi|A|\psi\rangle)^2|\psi\rangle = \mathrm{Tr}\{\rho_{mc}(A - A_{mc})^2\} \; . \qquad (3)$$

The square root of the quantity on the left-hand side is usually considered to represent the quantum fluctuations or quantum uncertainty of A in the pure state $|\psi\rangle$. Indeed, it is exactly this type of expression that appears in Heisenberg's uncertainty principle. On the other hand, the square root of the right-hand side of (3) is commonly identified with the thermal fluctuations of A (statistical fluctuations of A in the thermal equilibrium ensemble ρ_{mc}). In other words, typicality implies that *thermal equilibrium fluctuations are very well imitated by the purely quantum mechanical uncertainties (quantum fluctuations) for most pure state $|\psi\rangle$ in the energy shell.*

Particularly interesting situations arise when the thermal fluctuations on the right-hand of (3) are *not* unobservably small. A first example are the well-known critical fluctuations arising in the context of phase transitions for systems at a critical point. A second example is the position of a Brownian particle suspended in a liquid. In such cases, also the quantum fluctuations on the left-hand side of (3) will be of macroscopic size for most pure states $|\psi\rangle$. In other words, they amount to so-called Schrödinger cat-states, which are generally considered not to be experimentally feasible. This raises the interesting, and to our knowledge previously never addressed issue of how many of the typical states $|\psi\rangle$ are in fact unfeasible in some given model system.

II. TYPICALITY IN CLASSICAL SYSTEMS

Next we turn to the question of whether analogous typicality phenomena as in quantum systems may also arise in classical systems. While statements of similar generality and rigour as in the quantum case are not (yet) available [12, 13], we will provide non-rigorous (physical) arguments implying that typicality properties are generally *not* to be expected in classical systems.

Similarly as in the quantum case, we thus focus on classical many-body systems, whose pure states are described by phase space points ϕ, and whose dynamics is governed by some Hamilton function $H(\phi)$, while observables are now modelled by phase-space functions $A(\phi)$. As before, $I := [E, E + \Delta]$ denotes some microcanonical energy window, while the concomitant energy shell is now defined as the subset of all phase space points ϕ with the property that $H(\phi) \in I$. Furthermore, the microcanonical ensemble $\rho_{mc}(\phi)$ is defined as some suitably normalized function which is constant within the energy shell and zero otherwise, giving rise to thermal (microcanonical) expectation values of the form

$$A_{mc} := \int d\Gamma \rho_{mc}(\phi) A(\phi) \; , \qquad (4)$$

where $\int d\Gamma$ indicates the integral over the entire phase space, and moreover incorporates some suitable (textbook) normalization constant, accounting for the correct dimensionality of the integral and the proper counting of indistinguishable particles. Accordingly, the thermal equilibrium fluctuations of $A(\phi)$ can be quantified by

$$\sigma_A^2 := \int d\Gamma \rho_{mc}(\phi)(A(\phi) - A_{mc})^2 \; . \qquad (5)$$

Overall, the similarities and differences compared to the quantum case (previous section) are obvious and well-known.

Let us first focus on observables $A(\phi)$ which depend non-trivially on only a few of the very many components of ϕ, and which therefore may be denoted as microscopic observables. A particularly simple example is one component of the momentum of one particle (in cartesians coordinates). In the latter special case, the fluctuations of the observable $A(\phi)$ are well-known to be governed (in very good approximation for sufficiently large systems) by the equipartition theorem, implying $\sigma_A^2 = mk_BT$, where m is the mass of the considered particle, k_B Boltzmann's constant, and T the temperature associated with the microcanonical ensemble ρ_{mc}. The salient point is that those fluctuations are generally *not* negligible on the characteristic (microscopic) scale of the considered observable. It seems reasonable to expect that a qualitatively similar behavior will also be recovered for more general microscopic observables $A(\phi)$: Whenever looking at some system properties on the microscopic scale, the statistical fluctuations (at thermal equilibrium) are expected to be non-negligible.

On the other hand, for any given pure state ϕ the pertinent fluctuations are trivially zero: There is no "uncertainty" in the value of the observable (measurement outcome) $A(\phi)$ for any given ϕ. Hence an analogous line of reasoning as in the derivation of Eqs. (2) and (3) is generally impossible in the classical case. Since the main ingredient of this line of reasoning were typicality properties, it follows that such properties cannot be generally fulfilled in classical systems.

Let us finally turn to macroscopic (extensive or intensive) observables $A(\phi)$, for instance the total kinetic energy of all particles. Then it is generally expected and even provable in some cases [13] that the fluctuations in (5) become negligible for sufficiently large systems on the characteristic scale of the considered observable [14]. In turn, this means that $(A(\phi) - A_{mc})^2$ on the right-hand side of (5) must be negligibly small for most ϕ in the energy shell, i.e., we recover the hallmark of typicality [15].

In conclusion, in classical systems typicality is generally expected to be absent for microscopic and present for macroscopic observables. However, in both cases the essential underlying physics – thermal fluctuations are negligible on macroscopic but not on microscopic scales – may appear in some sense as rather obvious. Moreover, in both cases those tacitly assumed basic properties of the thermal fluctuations appear to be physically reasonable, but to our knowledge not yet rigorously provable in satisfying generality. In some cases, they actually may even be wrong, as exemplified by the critical (macroscopic) fluctuations of systems at a critical point. Indeed, the problem seems closely related to the derivation of general rigorous results regarding the occurrence (or not) of phase transitions.

III. DISCUSSION

As seen in the first part of Sec. I, the basic phenomenon of quantum typicality arises in any sufficiently high-dimensional subspace $\mathcal{H}_D$, independently of any further details of the considered quantum system. A particularly simple example is a harmonic oscillator in one dimension (albeit the physical relevance of typicality is not obvious in examples like this). In fact, it is not even necessary that $\mathcal{H}_D$ arises in the context of some quantum mechanical model system in the first place. It therefore appears questionable whether quantum typicality may generally be understood as a consequence of quantum mechanical entanglement effects, as it is sometimes suggested. Rather, elementary geometrical properties of high dimensional Hilbert spaces may be at the root. If so, there remains no obvious *a priori* reasons why similar typicality phenomena may not arise in classical systems as well. This question was at the focus of Sec. II, with the main result that classical typicality is generally expected to be encountered for macroscopic but not for microscopic observables.

A first important implication of the different typicality properties in quantum and classical systems is that pure states seem to be very different things in quantum and classical mechanics. In particular, most quantum pure states cannot be expected to approach some classical pure states in the classical limit. A second main conclusion is that typicality (in the original sense which applies to arbitrary observables) seems to be a purely quantum mechanical phenomenon, even though entanglement effects generally do not seem to be the basic physical origin.

From a more technical viewpoint, the main tool and common denominator of our present explorations was to take advantage not only of the usually considered mean values but also of the concomitant quantum and statistical fluctuations.

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[9] Given a orthonormalized basis $\{|n\rangle\}_{n=1}^D$ of $\mathcal{H}_D$, the considered vectors $|\psi\rangle$ can be written as $\sum_{n=1}^D c_n|n\rangle$, where $c_n = a_n + ib_n$ are complex coefficients with real and imaginary parts a_n and b_n, respectively, satisfying the normalization condition $\sum_{n=1}^D a_n^2+b_n^2 = 1$. Sampling 2D-dimensional vectors $(a_1, b_1, ..., a_D, b_D)$ according to a uniform distribution on the unit sphere in $\mathbb{R}^{2D}$ thus amounts to an explicit recipe of how uniformly distributed, normalized $|\psi\rangle \in \mathcal{H}_D$ may be generated.
[10] The mathematical derivation of these general findings can be readily inferred for instance from the results in Ref. [1] or [5] and Chebyshev's inequality.
[11] Moreover, the upper bound can be optimized by working with the restriction (projection) of A to $\mathcal{H}_D$ and adding a suitable constant so that the largest and smallest eigenvalues are of equal modulus and opposite sign.
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[14] Intuitively, such a behavior may often be understood as a consequence of the central limit theorem or some generalization thereof.
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