A Liouville theorem for CR Yamabe type equation on Sasakian manifolds

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Abstract

In this paper, we study the CR Yamabe type equation

$$\Delta_b u + F(u) = 0$$

on complete noncompact $(2n+1)$-dimensional Sasakian manifolds with nonnegative curvature. Under some assumptions, we prove a rigidity result, that is, the manifold is CR isometric to Heisenberg group $\mathbb{H}^n$. The proofs are based on the Jerison-Lee's differential identity combining with integral estimates.

Keywords: Liouville theorem, CR Yamabe equation, semilinear elliptic equation

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1 Introduction

In this paper, we study the Liouville type theorems for solutions of semilinear equation

$$\Delta_b u + 2n^2 F(u) = 0 \tag{1.1}$$

on Sasakian manifolds.

The equation (1.1) is closely related to the CR Yamabe equation. Let Θ be the standard contact form on $\mathbb{H}^n$. We consider the conformal form $\theta = u^{\frac{2}{n}}\Theta$, where u is a smooth positive function on $\mathbb{H}^n$. Then the pseudo-Hermitian scalar curvature associated to θ is the positive constant $4n(n+1)$ if u satisfies the equation

$$\Delta_b u + 2n^2 u^{\frac{Q+2}{Q-2}} = 0, \tag{1.2}$$

where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$. By the work [14] of Jerison and Lee, there are nontrivial solutions as follows

$$u(z,t) := \frac{C}{|t + i|z|^2 + z\cdot\mu + \lambda|^n}, \tag{1.3}$$

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for some $C > 0$, $\lambda \in \mathbb{C}$, $\mu \in \mathbb{C}^n$ such that $\mathrm{Im}(\lambda) > \frac{|\mu|^2}{4}$. In particular, Jerison and Lee [14] obtained the uniqueness of the solutions in case of finite energy assumption, i.e., $u \in L^{\frac{2Q}{Q-2}}(\mathbb{H}^n)$. Garofalo and Vassilev [10] also got a uniqueness result under the assumption of cylindrical symmetry. In [3], Catino, Li, Monticelli, Roncoroni proved the classification of solutions, provided $n = 1$ or $n \geq 2$ and a suitable control at infinity holds (see also [9]). Recently, Prajapatl and Varghese [19] got the classification of solutions of (1.1) using the moving plane method. For general Sasakian manifolds, Catino, Monticelli, Roncoroni and Wang [6] generalized the result in [3] to complete noncompact $(2n+1)$-dimensional Sasakian manifolds with nonnegative curvature.

In the subcritical case, i.e.,

$$\Delta_b u + 2n^2 u^q = 0, \quad in \quad \mathbb{H}^n, \tag{1.4}$$

where $1 < q < \frac{Q+2}{Q-2}$, Ma and Ou [17] proved that the only nonnegative solution is the trivial one.

Before presenting our main results, we mention the corresponding equation in Euclidean space,

$$\Delta u + u^q = 0, \quad in \quad \mathbb{R}^n. \tag{1.5}$$

In the seminal paper [11], Gidas and Spruck proved that there are no positive solutions to (1.5), if $n \geq 3$ and $1 < q < \frac{n+2}{n-2}$. For the critical case $q = \frac{n+2}{n-2}$, Caffarelli, Gidas and Spruck [2] showed that any solution to (1.5) is given by

$$u(x) = \left(\frac{1}{a + b|x - x_0|^2} \right)^{\frac{n-2}{2}},$$

where $a, b > 0$, $ab = \frac{1}{n(n-2)}$ and $x_0 \in \mathbb{R}^n$. Under the finite energy assumption, Catino and Monticelli [4] proved the classification of solutions to (1.5) with $q = \frac{n+2}{n-2}$ in the Riemannian setting when the Ricci curvature is non-negative. Besides, Li and Zhang [16] studied the following equation

$$\Delta u + f(u) = 0, \quad in \quad \mathbb{R}^n, \tag{1.6}$$

and proved that: If f is locally bounded in $(0, \infty)$ and $f(s)s^{-\frac{n+2}{n-2}}$ is non-increasing in $(0, \infty)$. Let u be a positive classical solution of (1.6) with $n \geq 3$, then either for some $b > 0$

$$bu = \left(\frac{\mu}{1 + \mu^2 |x - \bar{x}|^2} \right)^{\frac{n-2}{2}}$$

for $\mu > 0, \bar{x} \in \mathbb{R}^n$ or $u \equiv a$ for some $a > 0$ such that $f(a) = 0$. Such Liouville type theorems have also been generalized for p-Laplacian. In [20], Serrin and Zou extended Gida and Spruck's result to quasilinear elliptic equations. We refer to [18, 12, 13, 5] and references therein for further results.

Returning to the equation (1.1), we first give some definitions. We say that $f \in C^0([0,\infty)) \cap C^1((0,\infty))$ is subcritical with exponent σ if

$$f(t) > 0, \quad \sigma f(t) - tf'(t) \geq 0, \quad \forall t > 0.$$

Different from [20, 24], we request that $f(t) > 0$. It is easy to verify that f is a subcritical function with exponent σ implying that $t^{-\sigma} f(t)$ is non-increasing in $(0,\infty)$. In the following, we denote by B_R the ball of radius R centered at a fixed point with respect to the Carathéodory distance (see Section 2) and denote $A_R = B_{2R} \setminus B_R$. The volume of B_R is denoted by $V(R)$.

In this paper, we study the equation (1.1) on complete noncompact Sasakian manifolds with nonnegative pseudo-Hermitian Ricci curvature Ric_b. Indeed, we have

Theorem 1.1. *Let (M^3, J, θ) be a 3-dimensional complete noncompact Sasakian manifold with nonnegative Tanaka-Webster scalar curvature and assume that F is a subcritical function with exponent 3. Let u be a positive solution to (1.1) such that u tends to zero at infinity. If there exists a constant $2 < \kappa \leq 3$ such that*

$$tF'(t) - \kappa F(t) \geq 0, \ \forall t > 0. \tag{1.7}$$

Then (M^3, J, θ) is isometric to $\mathbb{H}^1$ with its standard structures and u is the form of (1.3).

Theorem 1.2. *Let (M^5, J, θ) be a 5-dimensional complete noncompact Sasakian manifold with $Ric_b \geq 0$ and*

$$V(R) \leq CR^6$$

for some $C > 0$ and $R > 0$ large enough. Assume that F is a subcritical function with exponent 2. Let u be a positive solution to (1.1) such that u tends to zero at infinity. If there exist constants $\frac{3}{2} < \kappa \leq 2$ and $\sigma < 2$ such that

$$tF'(t) - \kappa F(t) \geq 0, \ \forall t > 0, \tag{1.8}$$

and

$$\int_{A_R} F^3(u) u^{-3} \leq CR^\sigma \tag{1.9}$$

for $\sigma < 2$ and $R > 0$ large enough. Then (M^5, J, θ) is isometric to $\mathbb{H}^2$ with its standard structures and u is the form of (1.3).

For higher dimensional case, we give a similar result under the integral condition.

Theorem 1.3. *Let $(M^{2n+1}, HM, J, \theta)$, $n \geq 3$ be a complete noncompact Sasakian manifold with $Ric_b \geq 0$ and*

$$V(R) \leq CR^{2n+2}$$

for some $C > 0$ and $R > 0$ large enough. Assume that F is a subcritical function with exponent $1 + \frac{2}{n}$. Let u be a positive solution to (1.1) such that u tends to zero at infinity. If there exists constants $\frac{2}{n} < \kappa \leq 1 + \frac{2}{n}$

$$tF^{'}(t) - \kappa F(t) \geq 0, \ \forall t > 0, \tag{1.10}$$

and

$$\int_{A_R} F^{n+1}(u)u^{-n-1} \leq CR^2 \tag{1.11}$$

for $R > 0$ large enough. Then (M^{2n+1}, J, θ) is isometric to $\mathbb{H}^n$ with its standard structures and u is the form of (1.3).

We recall the following volume comparison theorem on Sasakian manifolds (cf. [15]).

Theorem 1.4. *Let $(M^{2n+1,}HM, J, \theta)$ be a complete noncompact Sasakian manifold.*
(1) If $n = 1$ and the Tanaka-Webster scalar curvature is nonnegative, then

$$V(R) \leq CR^4, \quad \forall R > 1,$$

for some $C > 0$.
(2) If $n \geq 2$ and the Tanaka-Webster curvature satisfies

$$Ric_b(X, X) \geq R(X, JX, JX, X) \geq 0, \quad \forall X \in HM, \tag{1.12}$$

then

$$V(R) \leq CR^{2n+2}, \quad \forall R > 1$$

for some $C > 0$.

From Theorem 1.4, we have the following corollary.

Corollary 1.5. *Let $(M^{2n+1}, HM, J, \theta)$, $n \geq 2$ be a complete noncompact Sasakian manifold with curvature condition (1.12). Replacing the assumption on Ric_b and on $Vol(B_R)$ by (1.12) in Theorem 1.2 or Theorem 1.3, then (M^{2n+1}, J, θ) is isometric to $\mathbb{H}^n$ with its standard structures and u is the form of (1.3).*

Notation Throughout the paper, we adopt Einstein summation convention over repeated indices. Since the constant C is not important, it may vary at different occurrences.

2 Preliminaries

In this section, we first give a brief introduction to pseudo-Hermitian geometry. We refer to [8, 23, 22] for a complete overview of the material presented here. A smooth manifold M of real dimension $(2n+1)$ is said to be a CR manifold if there exists a rank n complex subbundle $H^{1,0}M$ of $TM \otimes \mathbb{C}$ satisfying

$$H^{1,0}M \cap H^{0,1}M = \{0\}, \quad [\Gamma(H^{1,0}M), \Gamma(H^{1,0}M)] \subseteq \Gamma(H^{1,0}M), \tag{2.1}$$

where $H^{0,1}M = \overline{H^{1,0}M}$. Equivalently, the CR structure may also be described by (HM, J), where $HM = Re\{H^{1,0}M \oplus H^{0,1}M\}$ and J is an almost complex structure on HM such that

$$J(X + \overline{X}) = \sqrt{-1}(X - \overline{X}), \quad \forall X \in H^{1,0}M.$$

Since HM is naturally oriented by J, M is orientable if and only if there exists a global nowhere vanishing 1-form θ such that $ker\theta = HM$. Any such θ is called a pseudo-Hermitian structure on M. The Levi form L_θ of θ is defined by

$$L_\theta(X, Y) = d\theta(X, JY)$$

for any $X, Y \in HM$. If the Levi form L_θ is positive definite for some choice of θ, we call such (M, HM, J, θ) a pseudo-Hermitian manifold. Noting that θ is a contact form, thus there exists a unique Reeb vector field ξ satisfying

$$\theta(\xi) = 1, \;\; d\theta(\xi, \cdot) = 0. \tag{2.2}$$

It leads a decomposition of the tangent bundle $TM = HM \oplus R\xi$, which induces a natural projection $\pi_b : TM \to HM$. This yields a natural Riemannian metric g_θ, called the Webster metric,

$$g_\theta = G_\theta + \theta \otimes \theta,$$

where $G_\theta = d\theta(\cdot, J\cdot)$. The volume form of g_θ is defined by $dV_\theta = \theta \wedge (d\theta)^n/n!$, and is often omitted when we integrate.

On a pseudo-Hermitian manifold $(M^{2n+1}, HM, J, \theta)$, a Lipschitz curve $\gamma : [0, l] \to M$ is called horizontal if $\gamma^{'} \in H_{\gamma(t)}M$ a.e. in $[0, l]$. By the well-known work of Chow [7], there always exists a horizontal curve joining two points $p, q \in M$. The Carnot-Carathéodory distance is defined as

$$d(p, q) = inf \left\{ \int_0^l \sqrt{g_\theta(\gamma', \gamma')} dt \mid \gamma \in \Gamma(p, q) \right\},$$

where $\Gamma(p, q)$ denotes the set of all horizontal curves joining p and q. Clearly, Carnot-Carathéodory is indeed a distance function and induces the same topology on M (cf. [1, 21]).

It is known that there exists a canonical connection ∇ on a pseudo-Hermitian manifold, called the Tanaka-Webster connection, preserving the horizontal bundle, the CR structure and the Webster metric. Moreover, its torsion T_∇ satisfies

$$T_\nabla(X, Y) = 2d\theta(X, Y)\xi \; and \; T_\nabla(\xi, JX) + JT_\nabla(\xi, X) = 0$$

for any $X, Y \in HM$. The pseudo-Hermitian torsion τ of ∇ is defined by $\tau(X) = T_\nabla(\xi, X)$ for any $X \in TM$. We say that M is Sasakian if $\tau = 0$. In this paper, we only consider Sasakian manifolds.

Assume that $\{\eta_\alpha\}_{\alpha=1}^n$ is a local unitary frame field of $H^{1,0}M$ and let $\{\theta^\alpha\}_{\alpha=1}^n$ be the dual coframe. Then

$$d\theta = 2\sqrt{-1}\theta^\alpha \wedge \theta^{\bar{\alpha}}.$$

Given a smooth function f on the Sasakian manifold $(M^{2n+1}, HM, J, \theta)$, its differential df and covariant derivatives ∇df can be expressed as

$$\begin{aligned} df =& f_0\theta + f_\alpha\theta^\alpha + f_{\bar{\alpha}}\theta^{\bar{\alpha}}, \\ \nabla df =& f_{\alpha\beta}\theta^\alpha \otimes \theta^\beta + f_{\alpha\bar{\beta}}\theta^\alpha \otimes \theta^{\bar{\beta}} + f_{\bar{\alpha}\beta}\theta^{\bar{\alpha}} \otimes \theta^\beta + f_{\bar{\alpha}\bar{\beta}}\theta^{\bar{\alpha}} \otimes \theta^{\bar{\beta}} \\ &+ f_{0\alpha}\theta \otimes \theta^\alpha + f_{0\bar{\alpha}}\theta \otimes \theta^{\bar{\alpha}} + f_{\alpha 0}\theta^\alpha \otimes \theta + f_{\bar{\alpha}0}\theta^{\bar{\alpha}} \otimes \theta, \end{aligned}$$

where

$$f_0 = \xi(f),\ f_\alpha = \eta_\alpha(f),\ f_{\bar{\alpha}} = \eta_{\bar{\alpha}}(f),\ f_{\alpha\beta} = \eta_\beta(\eta_\alpha f) - (\nabla_{\eta_\beta}\eta_\alpha)f,\ f_{0\alpha} = \eta_\alpha(\xi f),$$

and so on. Then the horizontal gradient of f is given by

$$\nabla_b f = f_\alpha \eta_{\bar{\alpha}} + f_{\bar{\alpha}} f_\alpha.$$

Since $\tau = 0$, we have (cf. [8])

$$\begin{aligned} &f_{\alpha\beta} - f_{\beta\alpha} = 0, \quad f_{\alpha\bar{\beta}} - f_{\bar{\beta}\alpha} = 2\sqrt{-1}f_0\delta_\alpha^\beta, \\ &f_{0\alpha} - f_{\alpha 0} = 0, \quad f_{\alpha\beta 0} - f_{\alpha 0\beta} = f_{\alpha\bar{\beta}0} - f_{\alpha 0\bar{\beta}} = 0, \\ &f_{\alpha\beta\bar{\gamma}} - f_{\alpha\bar{\gamma}\beta} = 2\sqrt{-1}\delta_\gamma^\beta f_{\alpha 0} + R_{\bar{\mu}\alpha\beta\bar{\gamma}} f_\mu, \quad f_{\alpha\bar{\beta}\bar{\gamma}} - f_{\alpha\bar{\gamma}\bar{\beta}} = 0, \end{aligned}$$

where $R_{\bar{\mu}\alpha\beta\bar{\gamma}}$ are the components of curvature of the Tanaka-Webster connection. Moreover, $R_{\alpha\bar{\beta}} = R_{\bar{\mu}\alpha\mu\bar{\beta}}$ are the components of the pseudo-Hermitian Ricci curvature Ric_b and $R = R_{\alpha\bar{\alpha}}$ is the Tanaka-Webster scalar curvature. Finally, we define the horizontal energy density by

$$|\nabla_b f|^2 = 2|\partial f|^2 = 2f_\alpha f_{\bar{\alpha}}$$

and the sub-Laplacian of f by

$$\Delta_b f = f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha}.$$

Next we are aim to give the Jerison-Lee type inequality for the positive solution u of (1.1). Let $e^f = u^{\frac{1}{n}}$, then we have

$$-\Delta_b f = 2n|\partial f|^2 + 2n\frac{F(u)}{u} = 2n|\partial f|^2 + 2n\frac{F(e^{nf})}{e^{nf}}.$$

Let $H(t) = \frac{F(e^{nt})}{e^{nt}}$, then $0 < H \in C^1$. Thus, we obtain

$$-\Delta_b f = 2n|\partial f|^2 + 2nH(f). \tag{2.3}$$

Assuming that F is a subcritical function with exponent $1+\frac{2}{n}$, we have

$$2H - H^{'} \geq 0. \tag{2.4}$$

As done in [14, 17], we define the following

$$\begin{aligned} &D_{\alpha\beta} = f_{\alpha\beta} - 2f_\alpha f_\beta, \quad D_\alpha = D_{\alpha\beta} f_{\bar\beta}, \\ &E_{\alpha\bar\beta} = f_{\alpha\bar\beta} - \frac{1}{n} f_{\gamma\bar\gamma}\delta_\alpha^\beta, \quad E_\alpha = E_{\alpha\bar\beta} f_\beta, \\ &G_\alpha = \sqrt{-1} f_{0\alpha} - \sqrt{-1} f_0 f_\alpha + H(f) f_\alpha + |\partial f|^2 f_\alpha, \\ &g = |\partial f|^2 + H(f) - \sqrt{-1} f_0. \end{aligned} \tag{2.5}$$

Then the equation (2.3) can be written as

$$f_{\alpha\bar\alpha} = -ng. \tag{2.6}$$

A direct calculation shows that

$$\begin{aligned} &E_{\alpha\bar\beta} = f_{\alpha\bar\beta} + g\delta_\alpha^\beta, \quad E_\alpha = f_{\alpha\bar\beta} f_\beta + g f_\alpha, \\ &D_\alpha = f_{\alpha\bar\beta} f_\beta - 2|\partial f|^2 f_\alpha, \quad G_\alpha = \sqrt{-1} f_{0\alpha} + g f_\alpha. \end{aligned} \tag{2.7}$$

Moreover, we obtain

$$\begin{aligned} |\partial f|^2_{\bar\alpha} =& D_{\bar\alpha} + E_{\bar\alpha} + \bar g f_{\bar\alpha} - 2 f_{\bar\alpha} H(f), \\ g_{\bar\alpha} =& D_{\bar\alpha} + E_{\bar\alpha} + G_{\bar\alpha} + (H^{'}(f) - 2H(f)) f_{\bar\alpha}, \\ \bar g_{\bar\alpha} =& D_{\bar\alpha} + E_{\bar\alpha} - G_{\bar\alpha} + 2\bar g f_{\bar\alpha} + (H^{'}(f) - 2H(f)) f_{\bar\alpha}, \\ g_0 =& \sqrt{-1} f_\alpha G_{\bar\alpha} - \sqrt{-1} f_{\bar\alpha} G_\alpha + 2 f_0 |\partial f|^2 + H^{'}(f) f_0 - \sqrt{-1} f_{00}. \end{aligned} \tag{2.8}$$

In view of the above observations, we give the following inequality ([24]).

Lemma 2.1. *Let $(M^{2n+1}, HM, J, \theta)$ be a complete noncompact Sasakian manifold with $Ric_b \geq 0$. Set*

$$\mathcal{M} = Re\ \nabla_{\eta_{\bar\alpha}} \{ e^{2(n-1)f} [(g + 3\sqrt{-1} f_0) E_\alpha + (g - \sqrt{-1} f_0) D_\alpha - 3\sqrt{-1} f_0 G_\alpha \}, \tag{2.9}$$

then

$$\begin{aligned} \mathcal{M} \geq & (|D_{\alpha\beta}|^2 + |E_{\alpha\bar\beta}|^2) e^{2(n-1)f} H(f) + (|G_\alpha|^2 + |D_{\alpha\beta} f_{\bar\gamma} + E_{\alpha\bar\gamma} f_\beta|^2) e^{2(n-1)f} \\ & + s_0 (|D_\alpha + G_\alpha|^2 + |E_\alpha - G_\alpha|^2) e^{2(n-1)f} \\ & + e^{2(n-1)f} |\sqrt{1-s_0}(D_\alpha + G_\alpha) + \frac{1}{2\sqrt{1-s_0}} f_\alpha (H^{'}(f) - 2H(f))|^2 \\ & + e^{2(n-1)f} |\sqrt{1-s_0}(E_\alpha - G_\alpha) + \frac{1}{2\sqrt{1-s_0}} f_\alpha (H^{'}(f) - 2H(f))|^2 \\ & - (H^{'}(f) - 2H(f)) e^{2(n-1)f} |\partial f|^2 \left((2n-1) H(f) + \frac{H^{'}(f) - 2H(f)}{2(1-s_0)} \right) \end{aligned}$$

$$-(2n-1)(H^{'}(f)-2H(f))e^{2(n-1)f}|\partial f|^4-3n(H^{'}(f)-2H(f))e^{2(n-1)f}f_0^2, \tag{2.10}$$

where $0<s_0<1$ *is a constant.*

This is exactly Lemma 2.1 in [24] for $p=0$. As shown in [24], if there exists a constant $\kappa\in(\frac{4}{n}-3,1+\frac{2}{n}]$ such that

$$tF^{'}(t)-\kappa F(t)\geq 0,\quad \forall t>0,$$

then we have

$$H^{'}(f)\geq(n\kappa-n)H(f)\geq 2(2-2n)H(f). \tag{2.11}$$

This implies that

$$\left((2n-1)H(f)+\frac{H^{'}(f)-2H(f)}{2(1-s_0)}\right)\geq\left(2n-1+\frac{n\kappa-n-2}{2(1-s_0)}\right)H(f)\geq 0$$

for some $0<s_0<min\{1,\frac{n\kappa-4+3n}{4n-2}\}$. In this case, all coefficients on the right side of (2.10) are positive. It is easy to see that we can find a suitable s_0 under the condition in Theorem 1.1, Theorem 1.2 and Theorem 1.3. In the sequel, we assume that $0<s_0<min\{1,\frac{n\kappa-4+3n}{4n-2}\}$, i.e., H satisfies (2.11). In particular, we have $\mathcal{M}\geq 0$.

Given $R>0$, there exists a cut-off function ϕ such that

$$\begin{cases}\phi=1 & in\ B_R,\\ 0\leq\phi\leq 1, & in\ B_{2R},\\ \phi=0, & in\ M\backslash B_{2R},\\ |\partial\phi|\leq\frac{C}{R}, & in\ M.\end{cases}$$

Let $A_R=B_{2R}\setminus B_R$. As in [3, 6], we obtain a lemma in terms of the test functions.

Lemma 2.2. *Let* $s>2$ *and* $\beta\geq 0$ *small enough. Then we have the following estimates*

$$\begin{aligned}&\int_M\Psi^{-\beta}\mathcal{M}\phi^s\\ \leq & C\left(\int_{A_R}\Psi^{-\beta}\mathcal{M}\phi^s\right)^{\frac{1}{2}}\cdot\left(\frac{1}{R^2}\int_{A_R}e^{2(n-1)f}|g|^2\Psi^{-\beta}\phi^{s-2}\right)^{\frac{1}{2}}\end{aligned} \tag{2.12}$$

and

$$\int_M\Psi^{-\beta}\mathcal{M}\phi^s\leq\frac{C}{R^2}\int_{A_R}e^{2(n-1)f}|g|^2\Psi^{-\beta}\phi^{s-2}, \tag{2.13}$$

where $\Psi=|g|^2e^{-2f}$.

Proof. By multiplying (2.9) by $\Psi^{-\beta}\phi^s$ and integrating by parts, we have

$$\begin{aligned}
&\int_M \Psi^{-\beta}\mathcal{M}\phi^s\\
=&\int_M Re\nabla_{\eta_{\bar\alpha}}\{e^{2(n-1)f}[(g+3\sqrt{-1}f_0)E_\alpha+(g-\sqrt{-1}f_0)D_\alpha-3\sqrt{-1}f_0G_\alpha]\}\Psi^{-\beta}\phi^s\\
=&\beta\int_M Re\{e^{2(n-1)f}[(g+3\sqrt{-1}f_0)E_\alpha+(g-\sqrt{-1}f_0)D_\alpha-3\sqrt{-1}f_0G_\alpha]\Psi_{\bar\alpha}\}\Psi^{-\beta-1}\phi^s\\
&-s\int_M Re\{e^{2(n-1)f}[(g+3\sqrt{-1}f_0)E_\alpha+(g-\sqrt{-1}f_0)D_\alpha-3\sqrt{-1}f_0G_\alpha]\phi_{\bar\alpha}\}\Psi^{-\beta}\phi^{s-1}.
\end{aligned}$$

From the definition of g, we obtain that

$$\begin{aligned}
&|(g+3\sqrt{-1}f_0)E_\alpha+(g-\sqrt{-1}f_0)D_\alpha-3\sqrt{-1}f_0G_\alpha|\\
\le&|D_\alpha+E_\alpha|(|\partial f|^2+H(f))+|f_0|\cdot|2D_\alpha-2E_\alpha+3G_\alpha|.
\end{aligned}$$

Since

$$\begin{aligned}
&|D_\alpha+E_\alpha|\le|D_\alpha+G_\alpha|+|E_\alpha-G_\alpha|\le C\sqrt{\mathcal{M}}e^{-(n-1)f},\\
&|2D_\alpha-2E_\alpha+3G_\alpha|\le 2|D_\alpha+G_\alpha|+2|E_\alpha-G_\alpha|+|G_\alpha|\le C\sqrt{\mathcal{M}}e^{-(n-1)f},\\
&|\partial f|^2+H(f)\le|g|,\quad |f_0|\le|g|,
\end{aligned}$$

we have

$$|(g+3\sqrt{-1}f_0)E_\alpha+(g-\sqrt{-1}f_0)D_\alpha-3\sqrt{-1}f_0G_\alpha|\le C|g|\sqrt{\mathcal{M}}e^{-(n-1)f}. \tag{2.14}$$

Next we compute $\Psi_{\bar\alpha}$. By (2.8), we have

$$\begin{aligned}
\Psi_{\bar\alpha}=&(|g|^2e^{-2f})_{\bar\alpha}=e^{-2f}(g_{\bar\alpha}\bar g+g\bar g_{\bar\alpha})-2|g|^2e^{-2f}f_{\bar\alpha}\\
=&e^{-2f}[\bar g(D_{\bar\alpha}+E_{\bar\alpha}+G_{\bar\alpha}+(H^{'}(f)-2H(f))f_{\bar\alpha})\\
&+g(D_{\bar\alpha}+E_{\bar\alpha}-G_{\bar\alpha}+2\bar g f_{\bar\alpha}+(H^{'}(f)-2H(f))f_{\bar\alpha})]-2|g|^2e^{-2f}f_{\bar\alpha}\\
=&2e^{-2f}[(D_{\bar\alpha}+G_{\bar\alpha})(|\partial f|^2+H(f))+(E_{\bar\alpha}-G_{\bar\alpha})(|\partial f|^2+H(f))+\sqrt{-1}f_0G_{\bar\alpha}]\\
&+2e^{-2f}(H^{'}(f)-2H(f))(|\partial f|^2+H(f))f_{\bar\alpha}.
\end{aligned}$$

Noting that $4(1-n)H(f)\le H^{'}(f)\le 2H(f)$, we have

$$|H^{'}(f)-2H(f)|\le CH(f).$$

In particular, we find that

$$|(H^{'}(f)-2H(f))(|\partial f|^2+H(f))f_{\bar\alpha}|\le C|g|\sqrt{\mathcal{M}}e^{-(n-1)f}.$$

Thus

$$|\Psi_{\bar\alpha}|\le C|g|\sqrt{\mathcal{M}}e^{-(n+1)f}. \tag{2.15}$$

From (2.14) and (2.15), we obtain

$$\begin{aligned}
&\int_M \Psi^{-\beta}\mathcal{M}\phi^s \\
\leq & C\beta\int_M \mathcal{M}|g|^2 e^{-2f}\Psi^{-\beta-1}\phi^s + Cs\int_M |g|\sqrt{\mathcal{M}}e^{(n-1)f}\Psi^{-\beta}|\partial\phi|\phi^{s-1} \\
= & C\beta\int_M \Psi^{-\beta}\mathcal{M}\phi^s + Cs\int_M |g|\sqrt{\mathcal{M}}e^{(n-1)f}\Psi^{-\beta}|\partial\phi|\phi^{s-1}.
\end{aligned}$$

Choosing $0 \leq \beta < \frac{1}{C}$, we conclude that

$$\begin{aligned}
\int_M \Psi^{-\beta}\mathcal{M}\phi^s \leq & C\int_M |g|\sqrt{\mathcal{M}}e^{(n-1)f}\Psi^{-\beta}|\partial\phi|\phi^{s-1} \\
\leq & C\left(\int_{A_R} \Psi^{-\beta}\mathcal{M}\phi^s\right)^{\frac{1}{2}} \cdot \left(\frac{1}{R^2}\int_{A_R} e^{2(n-1)f}|g|^2\Psi^{-\beta}\phi^{s-2}\right)^{\frac{1}{2}}.
\end{aligned}$$

The inequality (2.13) is a direct consequence of (2.12). □

It is easy to see that we need to prove the following inequality

$$\int_{A_R} e^{2(n-1)f}|g|^2\Psi^{-\beta}\phi^{s-2} \leq C$$

for $R > 0$ large enough. Finally, we give a result which will be used in the proof of Theorem 1.1.

Corollary 2.3. *([6]) Let (M^{2n+1}, θ, J) be a $(2n+1)$-dimensional complete Sasakian manifold satisfying the curvature assumptions in Theorem 1.4. Let u be a positive superharmonic function in M, i.e. $u \in C^2(M)$ and*

$$\Delta_b u \leq 0.$$

Then there exists a constant $C > 0$ such that

$$u(x) \geq \frac{C}{r(x)^{max\{3,n+1\}}}$$

for any $x \in M$ with $r(x) > 1$.

3 Proof of Theorem 1.1

Suppose that (M^{2n+1}, J, θ) is a complete noncompact Sasakian manifold with $Ric_b \geq 0$. Let f, g, H, ϕ and $\mathcal{M}$ be defined as in Section 2. In this section, we assume that there exist constants $1 + \frac{1}{n} < \kappa \leq 1 + \frac{2}{n}$ such that

$$tF^{'}(t) - \kappa F(t) \geq 0, \ \forall t > 0.$$

In particular, we have

$$H^{'}(f) \geq (n\kappa - n)H(f) > H(f).$$

First, we give some integral estimates which are useful in the proof.

Lemma 3.1. *Let $n \geq 1$ and $s \geq 4$. If $\beta \geq 0$ is small enough, there exist a constant C and a small $\epsilon > 0$ such that*

$$\int_M \Psi^{-\beta}\mathcal{M}\phi^s \leq \frac{C}{R^4}\int_{A_R} e^{2(n-1)f}|\partial f|^2\Psi^{-\beta}\phi^{s-4} + \frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^4\Psi^{-\beta}\phi^{s-2} \tag{3.1}$$

and

$$\begin{aligned}\int_{A_R} e^{2(n-1)f}|g|^2\Psi^{-\beta}\phi^{s-2} \leq &\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^2\Psi^{-\beta}\phi^{s-4} + \epsilon R^2\int_M \Psi^{-\beta}\mathcal{M}\phi^s \\ &+ C\int_{A_R} e^{2(n-1)f}|\partial f|^4\Psi^{-\beta}\phi^{s-2}\end{aligned} \tag{3.2}$$

for any $R > 0$.

Proof. From (2.13) and integrating by parts, we have

$$\begin{aligned}&\int_M \Psi^{-\beta}\mathcal{M}\phi^s \\ \leq&\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|g|^2\Psi^{-\beta}\phi^{s-2} \\ =&-\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}Re(f_{\alpha\bar{\alpha}}\bar{g})\Psi^{-\beta}\phi^{s-2} \\ =&\frac{C}{R^2}\int_{A_R}\{e^{2(n-1)f}Re(f_\alpha\bar{g}_{\bar{\alpha}})\Psi^{-\beta}\phi^{s-2} + (s-2)e^{2(n-1)f}Re(f_\alpha\phi_{\bar{\alpha}}\bar{g})\Psi^{-\beta}\phi^{s-3} \\ &+ 2(n-1)e^{2(n-1)f}|\partial f|^2Re(\bar{g})\Psi^{-\beta}\phi^{s-2} - \beta e^{2(n-1)f}Re(f_\alpha\Psi_{\bar{\alpha}}\bar{g})\Psi^{-\beta-1}\phi^{s-2}\}.\end{aligned} \tag{3.3}$$

Similar to get (2.14), we obtain

$$|D_{\bar{\alpha}} + E_{\bar{\alpha}} - G_{\bar{\alpha}}| \leq C\sqrt{\mathcal{M}}e^{-(n-1)f}.$$

Thus

$$\begin{aligned}Re(f_\alpha\bar{g}_{\bar{\alpha}}) =&Re((D_{\bar{\alpha}} + E_{\bar{\alpha}} - G_{\bar{\alpha}})f_\alpha + 2\bar{g}|\partial f|^2 + (H^{'}(f) - 2H(f))|\partial f|^2) \\ \leq&C|\partial f|\sqrt{\mathcal{M}}e^{-(n-1)f} + 4|\partial f|^4 + \frac{1}{2}H^2(f).\end{aligned}$$

Using Young's inequality, we have

$$\begin{aligned}&\int_{A_R} e^{2(n-1)f}Re(f_\alpha\bar{g}_{\bar{\alpha}})\Psi^{-\beta}\phi^{s-2} \\ \leq&\int_{A_R} e^{2(n-1)f}(C|\partial f|\sqrt{\mathcal{M}}e^{-(n-1)f} + 4|\partial f|^4 + \frac{1}{2}H^2(f))\Psi^{-\beta}\phi^{s-2} \\ \leq&\frac{\epsilon R^2}{2}\int_M \Psi^{-\beta}\mathcal{M}\phi^s + \frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^2\Psi^{-\beta}\phi^{s-4}\end{aligned}$$

$$+\frac{1}{2}\int_{A_R} e^{2(n-1)f}H^2(f)\Psi^{-\beta}\phi^{s-2}+4\int_{A_R} e^{2(n-1)f}|\partial f|^4\Psi^{-\beta}\phi^{s-2}.$$

Similarly, we find that

$$\begin{aligned}
&(s-2)\int_{A_R} e^{2(n-1)f}Re(f_\alpha\phi_{\bar\alpha}\bar g)\Psi^{-\beta}\phi^{s-3}\\
\leq&\epsilon\int_{A_R} e^{2(n-1)f}|g|^2\Psi^{-\beta}\phi^{s-2}+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^2\Psi^{-\beta}\phi^{s-4},\\
&2(n-1)\int_{A_R} e^{2(n-1)f}|\partial f|^2Re(\bar g)\Psi^{-\beta}\phi^{s-2}\\
=&2(n-1)\int_{A_R} e^{2(n-1)f}|\partial f|^4\Psi^{-\beta}\phi^{s-2}+2(n-1)\int_{A_R} e^{2(n-1)f}|\partial f|^2H(f)\Psi^{-\beta}\phi^{s-2}\\
\leq&\epsilon\int_{A_R} e^{2(n-1)f}H^2(f)\Psi^{-\beta}\phi^{s-2}+C\int_{A_R} e^{2(n-1)f}|\partial f|^4\Psi^{-\beta}\phi^{s-2}.
\end{aligned}$$

Moreover, from (2.15) and $\Psi=|g|^2e^{-2f}$, we have

$$\begin{aligned}
&-\beta\int_{A_R} e^{2(n-1)f}Re(f_\alpha\Psi_{\bar\alpha}\bar g)\Psi^{-\beta-1}\phi^{s-2}\\
\leq&C\int_{A_R} e^{(n-3)f}|\partial f||g|^2\sqrt{\mathcal{M}}\Psi^{-\beta-1}\phi^{s-2}\\
\leq&\frac{\epsilon R^2}{2}\int_M \Psi^{-\beta}\mathcal{M}\phi^s+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^2\Psi^{-\beta}\phi^{s-4}.
\end{aligned}$$

Hence, we derive that

$$\begin{aligned}
&\int_{A_R} e^{2(n-1)f}|g|^2\Psi^{-\beta}\phi^{s-2}\\
\leq&\epsilon R^2\int_M \Psi^{-\beta}\mathcal{M}\phi^s+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^2\Psi^{-\beta}\phi^{s-4}\\
&+(\frac{1}{2}+\epsilon)\int_{A_R} e^{2(n-1)f}H^2(f)\Psi^{-\beta}\phi^{s-2}+C\int_{A_R} e^{2(n-1)f}|\partial f|^4\Psi^{-\beta}\phi^{s-2}.
\end{aligned}$$

Notice that

$$\int_{A_R} e^{2(n-1)f}|g|^2\Psi^{-\beta}\phi^{s-2}\geq\int_{A_R} e^{2(n-1)f}H^2(f)\Psi^{-\beta}\phi^{s-2},$$

since $|g|^2\geq H^2(f)$. Therefore, we obtain (3.2) by choosing a small ϵ. Combining (3.2) and (3.3), we have

$$\begin{aligned}
&\int_M \Psi^{-\beta}\mathcal{M}\phi^s\leq\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|g|^2\Psi^{-\beta}\phi^{s-2}\\
\leq&\epsilon C\int_M \Psi^{-\beta}\mathcal{M}\phi^s+\frac{C}{R^4}\int_{A_R} e^{2(n-1)f}|\partial f|^2\Psi^{-\beta}\phi^{s-4}
\end{aligned}$$

$$+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^4\Psi^{-\beta}\phi^{s-2}.$$

Similarly, we obtain (3.1) by choosing a small ϵ.

$\square$

Lemma 3.2. *Let $n = 1$ or $n = 2$. If $s > 0$ is large enough and $\beta > 0$ is small enough, then there exists a constant $C > 0$ such that*

$$\int_{A_R} e^{2(n-1)f}|\partial f|^2\Psi^{-\beta}\phi^{s-4} \le \frac{C}{R^{2(n-\beta)}}Vol(B_{2R})$$

for $R > 0$ large enough.

Proof. Note that

$$\Psi^{-\beta} = |g|^{-2\beta}e^{2\beta f} = e^{2\beta f}(|\partial f|^4 + H^2(f) + |f_0|^2)^{-\beta} \le e^{2\beta f}H^{-2\beta}(f). \tag{3.4}$$

Then we have

$$\begin{aligned}
&\int_{A_R} e^{2(n-1)f}|\partial f|^2\Psi^{-\beta}\phi^{s-4}\\
\le&\int_{A_R} e^{2(n-1)f}|\partial f|^2 e^{2\beta f}H^{-2\beta}(f)\phi^{s-4}\\
=&\int_{A_R} e^{2(n+\beta-1)f}(Re(g) - H(f))H^{-2\beta}(f)\phi^{s-4}\\
=&-\frac{1}{n}\int_{A_R} e^{2(n+\beta-1)f}Re(f_{\alpha\bar{\alpha}})H^{-2\beta}(f)\phi^{s-4} - \int_{A_R} e^{2(n+\beta-1)f}H^{1-2\beta}(f)\phi^{s-4}\\
=&\frac{2(n+\beta-1)}{n}\int_{A_R} e^{2(n+\beta-1)f}|\partial f|^2H^{-2\beta}(f)\phi^{s-4}\\
&-\frac{2\beta}{n}\int_{A_R} e^{2(n+\beta-1)f}|\partial f|^2H^{-1-2\beta}(f)H^{'}\phi^{s-4}\\
&+\frac{s-4}{n}\int_{A_R} e^{2(n+\beta-1)f}Re(f_\alpha\phi_{\bar{\alpha}})H^{-2\beta}(f)\phi^{s-5} - \int_{A_R} e^{2(n+\beta-1)f}H^{1-2\beta}(f)\phi^{s-4}\\
\le&\frac{2(n+\beta-1)-2(n\kappa-n)\beta}{n}\int_{A_R} e^{2(n+\beta-1)f}|\partial f|^2H^{-2\beta}(f)\phi^{s-4}\\
&+\frac{s-4}{n}\int_{A_R} e^{2(n+\beta-1)f}Re(f_\alpha\phi_{\bar{\alpha}})H^{-2\beta}(f)\phi^{s-5} - \int_{A_R} e^{2(n+\beta-1)f}H^{1-2\beta}(f)\phi^{s-4},
\end{aligned}$$

since $H^{'}(f) \ge (n\kappa - n)H(f) > H(f)$. Moreover, for small β, we have

$$\frac{2(n+\beta-1)-2(n\kappa-n)\beta}{n} < 1$$

whenever $n = 1$ or $n = 2$. Hence,

$$\int_{A_R} e^{2(n+\beta-1)f}|\partial f|^2H^{-2\beta}(f)\phi^{s-4}$$

$$
\begin{aligned}
\leq & C\int_{A_R} e^{2(n+\beta-1)f}|\partial f|\cdot|\partial\phi|H^{-2\beta}(f)\phi^{s-5} - C\int_{A_R} e^{2(n+\beta-1)f}H^{1-2\beta}(f)\phi^{s-4} \\
\leq & \epsilon\int_{A_R} e^{2(n+\beta-1)f}|\partial f|^2 H^{-2\beta}(f)\phi^{s-4} + \frac{C}{R^2}\int_{A_R} e^{2(n+\beta-1)f}H^{-2\beta}(f)\phi^{s-6} \\
& - C\int_{A_R} e^{2(n+\beta-1)f}H^{1-2\beta}(f)\phi^{s-4}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& \int_{A_R} e^{2(n+\beta-1)f}|\partial f|^2 H^{-2\beta}(f)\phi^{s-4} \\
\leq & \frac{C}{R^2}\int_{A_R} e^{2(n+\beta-1)f}H^{-2\beta}(f)\phi^{s-6} - C\int_{A_R} e^{2(n+\beta-1)f}H^{1-2\beta}(f)\phi^{s-4}.
\end{aligned}
$$

According to the fact that $H^{'}(f) \leq 2H(f)$, we can assume that there exists a f_0 such that for any $f \leq f_0$

$$
H(f) \geq \frac{H(f_0)}{e^{2f_0}}e^{2f} = Ce^{2f}.
$$

On the other hand, u tends to zero at infinity. In this case, there exists a R_0 such that if $R \geq R_0$, we have $f \leq f_0$. Hence, we get $H(f) \geq Ce^{2f}$ for R large enough. Let $a = \frac{n+\beta-1}{n-\beta}$, then

$$
e^{2(n+\beta-1)f}H^{-2\beta}(f) \leq Ce^{2(n+\beta-1-a)f}H^{a-2\beta}(f).
$$

Using Young's inequality, we obtain

$$
\begin{aligned}
& \int_{A_R} e^{2(n+\beta-1)f}|\partial f|^2 H^{-2\beta}(f)\phi^{s-4} \\
\leq & \frac{C}{R^2}\int_{A_R} e^{2(n+\beta-1-a)f}H^{a-2\beta}(f)\phi^{s-6} - C\int_{A_R} e^{2(n+\beta-1)f}H^{1-2\beta}(f)\phi^{s-4} \\
\leq & \frac{C}{R^{\frac{2(1-2\beta)}{1-a}}}\int_{A_R} \phi^{s-4-\frac{2(1-2\beta)}{1-a}} + (\delta - C)\int_{A_R} e^{2(n+\beta-1)f}H^{1-2\beta}(f)\phi^{s-4}.
\end{aligned}
$$

Letting $s > 4 + \frac{2(1-2\beta)}{1-a}$ and δ sufficiently small, we have

$$
\int_{A_R} e^{2(n+\beta-1)f}|\partial f|^2 H^{-2\beta}(f)\phi^{s-4} \leq \frac{C}{R^{\frac{2(1-2\beta)}{1-a}}}Vol(B_{2R}). \tag{3.5}
$$

This complete the proof since

$$
\int_{A_R} e^{2(n-1)f}|\partial f|^2\Psi^{-\beta}\phi^{s-4} \leq \int_{A_R} e^{2(n+\beta-1)f}|\partial f|^2 H^{-2\beta}(f)\phi^{s-4}.
$$

$\square$

Lemma 3.3. *Let $n=1$ and s, β be as in Lemma 3.2. Then there exist constants $C>0$ and $\epsilon>0$ such that*

$$\int_{A_R}|\partial f|^4\Psi^{-\beta}\phi^{s-2}\leq \epsilon R^2\int_{A_R}\mathcal{M}\Psi^{-\beta}\phi^s+\frac{C}{R^2}Vol(B_{2R})$$

for $R>0$ large enough.

Proof. From (3.4) and $n=1$, we have

$$\begin{aligned}
&\int_{A_R}|\partial f|^4\Psi^{-\beta}\phi^{s-2}\\
\leq&\int_{A_R}|\partial f|^4e^{2\beta f}H^{-2\beta}(f)\phi^{s-2} \qquad (3.6)\\
=&-\int_{A_R}|\partial f|^2Re(f_{\alpha\bar{\alpha}})e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}-\int_{A_R}|\partial f|^2e^{2\beta f}H^{1-2\beta}(f)\phi^{s-2}\\
=&2\beta\int_{A_R}|\partial f|^4e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}-2\beta\int_{A_R}|\partial f|^4e^{2\beta f}H^{-1-2\beta}(f)H^{'}\phi^{s-2}\\
&+(s-2)\int_{A_R}|\partial f|^2Re(f_\alpha\phi_{\bar{\alpha}})e^{2\beta f}H^{-2\beta}(f)\phi^{s-3}-\int_{A_R}|\partial f|^2e^{2\beta f}H^{1-2\beta}(f)\phi^{s-2}\\
&+\int_{A_R}Re(f_\alpha|\partial f|^2_{\bar{\alpha}})e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}.
\end{aligned}$$

By (2.8) and $|D_{\bar{\alpha}}+E_{\bar{\alpha}}|\leq C\sqrt{\mathcal{M}}$, we compute that

$$\begin{aligned}
&\int_{A_R}Re(f_\alpha|\partial f|^2_{\bar{\alpha}})e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}\\
=&\int_{A_R}Re(f_\alpha(D_{\bar{\alpha}}+E_{\bar{\alpha}}))e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}+\int_{A_R}|\partial f|^4e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}\\
&-\int_{A_R}|\partial f|^2e^{2\beta f}H^{1-2\beta}(f)\phi^{s-2}\\
\leq&C\int_{A_R}|\partial f|\sqrt{\mathcal{M}}e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}+\int_{A_R}|\partial f|^4e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}\\
&-\int_{A_R}|\partial f|^2e^{2\beta f}H^{1-2\beta}(f)\phi^{s-2}.
\end{aligned}$$

Since $H^{'}(f)\geq(\kappa-1)H(f)>H(f)$ and $2<\kappa\leq 3$, we obtain

$$\begin{aligned}
&2\beta(\kappa-2)\int_{A_R}|\partial f|^4e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}\\
\leq&(s-2)\int_{A_R}|\partial f|^2Re(f_\alpha\phi_{\bar{\alpha}})e^{2\beta f}H^{-2\beta}(f)\phi^{s-3}+C\int_{A_R}|\partial f|\sqrt{\mathcal{M}}e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}.
\end{aligned}$$

By Young's inequality, we have

$$2\beta(\kappa-2-\epsilon)\int_{A_R}|\partial f|^4e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}$$

$$
\begin{aligned}
\leq & \frac{C}{R^2}\int_{A_R}|\partial f|^2 e^{2\beta f}H^{-2\beta}(f)\phi^{s-4}+C\int_{A_R}|\partial f|\sqrt{\mathcal{M}}e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}\\
= & \frac{C}{R^2}\int_{A_R}|\partial f|^2 e^{2\beta f}H^{-2\beta}(f)\phi^{s-4}+C\int_{A_R}|\partial f|\sqrt{\mathcal{M}}\Psi^{-\beta}|g|^{2\beta}H^{-2\beta}(f)\phi^{s-2}\\
\leq & \frac{C}{R^2}\int_{A_R}|\partial f|^2 e^{2\beta f}H^{-2\beta}(f)\phi^{s-4}+\epsilon R^2\int_{A_R}\mathcal{M}\Psi^{-\beta}\phi^{s}\\
& +\frac{C}{R^2}\int_{A_R}H^{-4\beta}(f)|g|^{4\beta}|\partial f|^2\Psi^{-\beta}\phi^{s-4}, \qquad (3.7)
\end{aligned}
$$

where ϵ is a small constant. Using Young's inequality again, we derive that

$$
\begin{aligned}
& H^{-4\beta}(f)|g|^{4\beta}|\partial f|^2\Psi^{-\beta}\phi^{s-4}\\
\leq & 2\beta|g|^2\Psi^{-\beta}\phi^{s-2}+(1-2\beta)H^{-\frac{4\beta}{1-2\beta}}(f)|\partial f|^{\frac{2}{1-2\beta}}\Psi^{-\beta}\phi^{s-\frac{4(1-\beta)}{1-2\beta}}\\
\leq & 2\beta|g|^2\Psi^{-\beta}\phi^{s-2}+\frac{1}{2}|\partial f|^4 H^{-8\beta}(f)\Psi^{-\beta-\beta(1-4\beta)}\phi^{s-2}+\frac{1-4\beta}{2}\phi^{s-\frac{6-8\beta}{1-4\beta}}. \qquad (3.8)
\end{aligned}
$$

As discussed in Lemma 3.2, we have $H(f)\geq Ce^{2f}$ for R large enough. Together with Corollary 2.3 and (3.4), we get

$$
\begin{aligned}
H^{-8\beta}(f)\Psi^{-\beta(1-4\beta)} \leq & CH^{-8\beta}(f)e^{2\beta(1-4\beta)f}H^{-2\beta(1-4\beta)}(f)\\
\leq & Ce^{-18\beta f+8\beta^2 f}\leq CR^{54\beta} \qquad (3.9)
\end{aligned}
$$

by choosing $\beta<\frac{1}{4}$. Substituting (3.8) and (3.9) into (3.7), we have

$$
\begin{aligned}
& \int_{A_R}|\partial f|^4 e^{2\beta f}H^{-2\beta}(f)\phi^{s-2}\\
\leq & \frac{C}{R^2}\int_{A_R}|\partial f|^2 e^{2\beta f}H^{-2\beta}(f)\phi^{s-4}+\epsilon R^2\int_{A_R}\mathcal{M}\Psi^{-\beta}\phi^{s}\\
& +\frac{C}{R^{2-54\beta}}\int_{A_R}|\partial f|^4\Psi^{-\beta}\phi^{s-2}+\frac{C}{R^2}\int_{A_R}\phi^{s-\frac{6-8\beta}{1-4\beta}}+\frac{C}{R^2}\int_{A_R}|g|^2\Psi^{-\beta}\phi^{s-2}\\
\leq & \epsilon R^2\int_{A_R}\mathcal{M}\Psi^{-\beta}+\frac{C}{R^{2-54\beta}}\int_{A_R}|\partial f|^4\Psi^{-\beta}\phi^{s-2}+\frac{C}{R^2}Vol(B_{2R}),
\end{aligned}
$$

where we use (3.5), Lemma 3.1 and Lemma 3.2. Then the result follows from (3.6) for large R and $\beta<\frac{1}{27}$. □

Now we are in the place to proof the Theorem 1.1.

Proof of Theorem 1.1. Let f, g, H, ϕ and $\mathcal{M}$ be defined as in above. Assume that $R>0$ large enough such that $H(f)\geq Ce^{2f}$. From Lemma 3.1, Lemma 3.2 and Lemma 3.3, we have

$$
\int_M \Psi^{-\beta}\mathcal{M}\phi^s\leq\frac{C}{R^4}\int_{A_R}|\partial f|^2\Psi^{-\beta}\phi^{s-4}+\frac{C}{R^2}\int_{A_R}|\partial f|^4\Psi^{-\beta}\phi^{s-2}
$$

$$\leq \frac{C}{R^{6-2\beta}} Vol(B_{2R}) + C\epsilon \int_M \Psi^{-\beta} \mathcal{M} \phi^s + \frac{C}{R^4} Vol(B_{2R}).$$

Choosing ϵ small enough and using Theorem 1.4, we obtain

$$\int_M \Psi^{-\beta} \mathcal{M} \phi^s \leq C.$$

Combining with (3.2), we have

$$\int_{A_R} |g|^2 \Psi^{-\beta} \phi^{s-2} \leq CR^2.$$

Then Lemma 2.2 yields that

$$\int_M \Psi^{-\beta} \mathcal{M} \phi^s \leq C \left(\int_{A_R} \Psi^{-\beta} \mathcal{M} \phi^s \right)^{\frac{1}{2}}.$$

Hence

$$\int_M \Psi^{-\beta} \mathcal{M} = 0,$$

i.e. $\mathcal{M} = 0$. By Lemma 2.1, we have $|\partial f| = 0$ or $H^{'}(f) = 2H(f)$. Since f satisfies (2.3) and $0 < H \in C^1$, we have $H^{'}(f) = 2H(f)$. In particular,

$$F(u) = Cu^{1+\frac{2}{n}}.$$

The conclusion then follows from [6]. $\square$

4 Proof of Theorem 1.2

In this section, we assume that $n = 2$. Let f, g, H, ϕ and $\mathcal{M}$ be defined as in Section 2. In this section, we assume that there exist constants $\frac{3}{2} < \kappa \leq 2$ such that

$$tF^{'}(t) - \kappa F(t) \geq 0, \ \forall t > 0.$$

In particular, we have

$$H^{'}(f) \geq (2\kappa - 2)H(f) > H(f).$$

First, we give some integral estimates.

Lemma 4.1. *Let $s > 0$ be large enough and $\beta > 0$ be small enough. Then there exist a constant $C > 0$ and a small constant $\epsilon > 0$ such that the following inequalities hold*

$$\int_{A_R} e^{2f} H(f) |\partial f|^2 \Psi^{-\beta} \phi^{s-2}$$

$$\leq \frac{C}{R^2}\int_{A_R} e^{2(1-\beta)f}H(f)\phi^{s-4} + C\int_{A_R} e^{2(1-\beta)f}H^2(f)\phi^{s-2} \tag{4.1}$$

and

$$\begin{aligned}
&\int_{A_R} e^{2f}|\partial f|^4\Psi^{-\beta}\phi^{s-2}\\
\leq&\frac{C}{R^2}\int_{A_R} e^{2f}|\partial f|^2\Psi^{-\beta}\phi^{s-4} + C\int_{A_R} e^{2f}H(f)|\partial f|^2\Psi^{-\beta}\phi^{s-2}\\
&+\epsilon R^2\int_{A_R}\Psi^{-\beta}\mathcal{M}\phi^s
\end{aligned} \tag{4.2}$$

for $R > 0$ large enough.

Proof. Note that $R > 0$ large enough, we have $H(f) \geq Ce^2 f$. From (3.4), we have

$$\int_{A_R} e^{2f}H(f)|\partial f|^2\Psi^{-\beta}\phi^{s-2} \leq C\int_{A_R} e^{2(1-\beta)f}H(f)|\partial f|^2\phi^{s-2}. \tag{4.3}$$

A direct computation shows that

$$\begin{aligned}
&\int_{A_R} e^{2(1-\beta)f}H(f)|\partial f|^2\phi^{s-2} = \int_{A_R} e^{2(1-\beta)f}H(f)(Re(g)-H(f))\phi^{s-2}\\
=&-\frac{1}{2}\int_{A_R} e^{2(1-\beta)f}H(f)Re(f_{\alpha\bar{\alpha}})\phi^{s-2} - \int_{A_R} e^{2(1-\beta)f}H^2(f)\phi^{s-2}\\
=&(1-\beta)\int_{A_R} e^{2(1-\beta)f}H(f)|\partial f|^2\phi^{s-2} + \frac{s-2}{2}\int_{A_R} e^{2(1-\beta)f}H(f)Re(f_\alpha\phi_{\bar{\alpha}})\phi^{s-3}\\
&+\frac{1}{2}\int_{A_R} e^{2(1-\beta)f}H^{'}(f)Re(f_{\alpha\bar{\alpha}})\phi^{s-2} - \int_{A_R} e^{2(1-\beta)f}H^2(f)\phi^{s-2}.
\end{aligned}$$

Since $H^{'}(f) > H(f)$, we have

$$\begin{aligned}
&(\frac{1}{2}-\beta)\int_{A_R} e^{2(1-\beta)f}H(f)|\partial f|^2\phi^{s-2}\\
\leq&\int_{A_R} e^{2(1-\beta)f}H^2(f)\phi^{s-2} - \frac{s-2}{2}\int_{A_R} e^{2(1-\beta)f}H(f)|\partial f|\cdot|\partial\phi|\phi^{s-3}\\
\leq&\epsilon\int_{A_R} e^{2(1-\beta)f}H(f)|\partial f|^2\phi^{s-2} + \frac{C}{R^2}\int_{A_R} e^{2(1-\beta)f}H(f)\phi^{s-4}\\
&+C\int_{A_R} e^{2(1-\beta)f}H^2(f)\phi^{s-2}.
\end{aligned}$$

Letting $0 < \epsilon < \frac{1}{2}-\beta$, we derive that

$$\int_{A_R} e^{2(1-\beta)f}H(f)|\partial f|^2\phi^{s-2}$$

$$\leq \frac{C}{R^2}\int_{A_R} e^{2(1-\beta)f}H(f)\phi^{s-4} + C\int_{A_R} e^{2(1-\beta)f}H^2(f)\phi^{s-2}.$$

Then (4.1) follows from (4.3). In order to obtain (4.2), we compute that

$$\begin{aligned}
&\int_{A_R} e^{2f}|\partial f|^4\Psi^{-\beta}\phi^{s-2} \\
=&\frac{1}{2}\int_{A_R} e^{2f}|\partial f|^2 Re(f_{\alpha\bar{\alpha}})\Psi^{-\beta}\phi^{s-2} - \int_{A_R} e^{2f}|\partial f|^2 H(f)\Psi^{-\beta}\phi^{s-2} \\
=&\int_{A_R} e^{2f}|\partial f|^4\Psi^{-\beta}\phi^{s-2} + \frac{s-2}{2}\int_{A_R} e^{2f}|\partial f|^2 Re(f_\alpha\phi_{\bar{\alpha}})\Psi^{-\beta}\phi^{s-3} \\
&+\frac{1}{2}\int_{A_R} e^{2f} Re(f_\alpha|\partial f|^2_{\bar{\alpha}})\Psi^{-\beta}\phi^{s-2} - \frac{\beta}{2}\int_{A_R} e^{2f}|\partial f|^2 Re(f_\alpha\Psi_{\bar{\alpha}})\Psi^{-\beta-1}\phi^{s-2} \\
&-\int_{A_R} e^{2f}|\partial f|^2 H(f)\Psi^{-\beta}\phi^{s-2},
\end{aligned}$$

that is,

$$\begin{aligned}
0 =&\frac{s-2}{2}\int_{A_R} e^{2f}|\partial f|^2 Re(f_\alpha\phi_{\bar{\alpha}})\Psi^{-\beta}\phi^{s-3} + \frac{1}{2}\int_{A_R} e^{2f} Re(f_\alpha|\partial f|^2_{\bar{\alpha}})\Psi^{-\beta}\phi^{s-2} \\
&-\frac{\beta}{2}\int_{A_R} e^{2f}|\partial f|^2 Re(f_\alpha\Psi_{\bar{\alpha}})\Psi^{-\beta-1}\phi^{s-2} - \int_{A_R} e^{2f}|\partial f|^2 H(f)\Psi^{-\beta}\phi^{s-2} \\
=&\frac{s-2}{2}\int_{A_R} e^{2f}|\partial f|^2 Re(f_\alpha\phi_{\bar{\alpha}})\Psi^{-\beta}\phi^{s-3} + \frac{1}{2}\int_{A_R} e^{2f} Re(f_\alpha(D_{\bar{\alpha}} + E_{\bar{\alpha}}))\Psi^{-\beta}\phi^{s-2} \\
&-\frac{\beta}{2}\int_{A_R} e^{2f}|\partial f|^2 Re(f_\alpha\Psi_{\bar{\alpha}})\Psi^{-\beta-1}\phi^{s-2} - 2\int_{A_R} e^{2f}|\partial f|^2 H(f)\Psi^{-\beta}\phi^{s-2} \\
&+\frac{1}{2}\int_{A_R} e^{2f}|\partial f|^2 Re(\bar{g})\Psi^{-\beta}\phi^{s-2}.
\end{aligned}$$

Noting that $Re(\bar{g}) = |\partial f|^2 + H(f)$ and $\Psi^{-2}|\partial f|^4|g|^2 \leq e^{4f}$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{A_R} e^{2f}|\partial f|^4\Psi^{-\beta}\phi^{s-2} \\
\leq&\epsilon^{'}\int_{A_R} e^{2f}|\partial f|^4\Psi^{-\beta}\phi^{s-2} + \frac{C}{R^2}\int_{A_R} e^{2f}|\partial f|^2\Psi^{-\beta}\phi^{s-4} + \epsilon R^2\int_{A_R}\Psi^{-\beta}\mathcal{M}\phi^s \\
&+\frac{C}{R^2}\int_{A_R} e^{-2f}|\partial f|^6|g|^2\Psi^{-\beta-2}\phi^{s-4} + \frac{3}{2}\int_{A_R} e^{2f}|\partial f|^2 H(f)\Psi^{-\beta}\phi^{s-2} \\
\leq&\epsilon^{'}\int_{A_R} e^{2f}|\partial f|^4\Psi^{-\beta}\phi^{s-2} + \frac{C}{R^2}\int_{A_R} e^{2f}|\partial f|^2\Psi^{-\beta}\phi^{s-4} + \epsilon R^2\int_{A_R}\Psi^{-\beta}\mathcal{M}\phi^s \\
&+\frac{3}{2}\int_{A_R} e^{2f}|\partial f|^2 H(f)\Psi^{-\beta}\phi^{s-2}.
\end{aligned}$$

By choosing $\epsilon^{'} < \frac{1}{2}$, we obtain (4.2). $\square$

Now we are ready to prove the Theorem 1.2.

Proof of Theorem 1.2. Let f, g, H, ϕ and $\mathcal{M}$ be defined as in above. Assume that $R>0$ large enough such that $H(f)\geq Ce^{2f}$. From Lemma 3.1 and Lemma 4.1, we have

$$\begin{aligned}&\int_M \Psi^{-\beta}\mathcal{M}\phi^s \leq \frac{C}{R^4}\int_{A_R} e^{2f}|\partial f|^2\Psi^{-\beta}\phi^{s-4}+\frac{C}{R^2}\int_{A_R} e^{2f}|\partial f|^4\Psi^{-\beta}\phi^{s-2}\\ \leq&\frac{C}{R^4}\int_{A_R} e^{2f}|\partial f|^2\Psi^{-\beta}\phi^{s-4}+\epsilon\int_{A_R}\Psi^{-\beta}\mathcal{M}\phi^s+\frac{C}{R^4}\int_{A_R} e^{2(1-\beta)f}H(f)\phi^{s-4}\\ &+\frac{C}{R^2}\int_{A_R} e^{2(1-\beta)f}H^2(f)\phi^{s-2}.\end{aligned}$$

Applying Lemma 3.2 and letting $0<\epsilon<1$, we have

$$\begin{aligned}&\int_M \Psi^{-\beta}\mathcal{M}\phi^s\\ \leq&\frac{C}{R^{8-2\beta}}Vol(B_{2R})+\frac{C}{R^4}\int_{A_R} e^{2(1-\beta)f}H(f)\phi^{s-4}+\frac{C}{R^2}\int_{A_R} e^{2(1-\beta)f}H^2(f)\phi^{s-2}.\end{aligned}$$

Noting that $H(f)\geq Ce^{2f}$ and using Hölder's inequality, we derive that

$$\begin{aligned}&\int_M \Psi^{-\beta}\mathcal{M}\phi^s\\ \leq&\frac{C}{R^{8-2\beta}}Vol(B_{2R})+\frac{C}{R^4}\int_{A_R} H^{2-\beta}(f)\phi^{s-4}+\frac{C}{R^2}\int_{A_R} H^{3-\beta}(f)\phi^{s-2}\\ \leq&\frac{C}{R^{8-2\beta}}Vol(B_{2R})+\frac{C}{R^4}\left(\int_{A_R} H^3(f)\right)^{\frac{3-\beta}{3}}\cdot(Vol(B_{2R}))^{\frac{\beta}{3}}\\ &+\frac{C}{R^2}\left(\int_{A_R} H^3(f)\right)^{\frac{2-\beta}{3}}\cdot(Vol(B_{2R}))^{\frac{1+\beta}{3}}\\ \leq&\frac{C}{R^{2-2\beta}}+\frac{C}{R^{2-\sigma-(2-\frac{\sigma}{3})\beta}}+\frac{C}{R^{2-\frac{2\sigma}{3}-(2-\frac{\sigma}{3})\beta}},\end{aligned}$$

where the last inequality follows from (1.9). Letting $R\to\infty$, we have

$$\int_M \Psi^{-\beta}\mathcal{M}=0,$$

provided β is small enough. In particular, $\mathcal{M}=0$. Similar to the proof of Theorem 1.1, we have $H^{'}(f)=2H(f)$. The conclusion then follows from [6]. $\square$

5 Proof of Theorem 1.3

Let (M^{2n+1},HM,J,θ), $n\geq 3$ be a complete noncompact Sasakian manifold with $Ric_b\geq 0$ and

$$V(R)\leq CR^{2n+2}$$

for some $C>0$ and every $R>0$ large enough. Let f, g, H, ϕ and $\mathcal{M}$ be defined as in Section 2. In this section, we assume that there exist constants $\frac{2}{n}<\kappa\leq 1+\frac{2}{n}$ such that

$$tF^{'}(t)-\kappa F(t)\geq 0,\ \forall t>0.$$

In particular, we have

$$H^{'}(f)\geq (n\kappa-n)H(f)>(2-n)H(f).$$

We first state the following inequalities.

Lemma 5.1. *Let $\epsilon>0$ small enough and $s>0$ large enough. There exists a constant $C>0$ such that*

$$\int_{A_R} e^{2(n-1)f}|\partial f|^2\phi^{s-4}\leq C\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4}+\frac{C}{R^{2n}}Vol(B_{2R}), \tag{5.1}$$

$$\int_{A_R} e^{2(n-1)f}H(f)|\partial f|^2\phi^{s-2}\leq C\int_{A_R} e^{2(n-1)f}H^2(f)\phi^{s-2}+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4} \tag{5.2}$$

and

$$\begin{aligned}&\int_{A_R} e^{2(n-1)f}|\partial f|^4\phi^{s-2}\\ \leq& C\int_{A_R} e^{2(n-1)f}H(f)|\partial f|^2\phi^{s-2}+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^2\phi^{s-4}+\epsilon R^2\int_{A_R}\mathcal{M}\phi^s\end{aligned} \tag{5.3}$$

for $R>0$ large enough.

Proof. Observe that

$$\begin{aligned}&\int_{A_R} e^{2(n-1)f}|\partial f|^2\phi^{s-4}=\int_{A_R} e^{2(n-1)f}(Re(g)-H(f))\phi^{s-4}\\ =&-\frac{1}{n}\int_{A_R} e^{2(n-1)f}Re(f_{\alpha\bar{\alpha}})\phi^{s-4}-\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4}\\ =&\frac{2(n-1)}{n}\int_{A_R} e^{2(n-1)f}|\partial f|^2\phi^{s-4}+\frac{s-4}{n}\int_{A_R} e^{2(n-1)f}Re(f_\alpha\phi_{\bar{\alpha}})\phi^{s-5}\\ &-\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4}.\end{aligned}$$

This yields that

$$\begin{aligned}&\frac{n-2}{n}\int_{A_R} e^{2(n-1)f}|\partial f|^2\phi^{s-4}\\ \leq& C\int_{A_R} e^{2(n-1)f}|\partial f|\cdot|\partial\phi|\phi^{s-5}+\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4}\end{aligned}$$

$$\leq\epsilon\int_{A_R} e^{2(n-1)f}|\partial f|^2\phi^{s-4}+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}\phi^{s-6}+\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4}.$$

Letting $0<\epsilon<\frac{n-2}{n}$, (5.1) follows from

$$\begin{aligned}
&\int_{A_R} e^{2(n-1)f}|\partial f|^2\phi^{s-4}\\
\leq&\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}\phi^{s-6}+C\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4}\\
\leq&\frac{C}{R^2}\int_{A_R} e^{2(n-1-\frac{n-1}{n})f}H^{\frac{n-1}{n}}(f)\phi^{s-6}+C\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4}\\
\leq&C\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4}+\frac{C}{R^{2n}}\int\phi^{s-4-2n}.
\end{aligned}$$

Here we assume that $R>0$ large enough such that $H(f)\geq Ce^{2f}$. Similarly,

$$\begin{aligned}
&\int_{A_R} e^{2(n-1)f}H(f)|\partial f|^2\phi^{s-2}\\
=&\frac{2(n-1)}{n}\int_{A_R} e^{2(n-1)f}H(f)|\partial f|^2\phi^{s-2}+\frac{s-2}{n}\int_{A_R} e^{2(n-1)f}H(f)Re(f_\alpha\phi_{\bar\alpha})\phi^{s-3}\\
&+\frac{1}{n}\int_{A_R} e^{2(n-1)f}H^{'}(f)|\partial f|^2\phi^{s-2}-\int_{A_R} e^{2(n-1)f}H^2(f)\phi^{s-2}.
\end{aligned}$$

Noting that $H^{'}(f)\geq(n\kappa-n)H(f)>(2-n)H(f)$, we have

$$\begin{aligned}
&\frac{n\kappa-2}{n}\int_{A_R} e^{2(n-1)f}H(f)|\partial f|^2\phi^{s-2}\\
\leq&\int_{A_R} e^{2(n-1)f}H^2(f)\phi^{s-2}+\epsilon\int_{A_R} e^{2(n-1)f}H(f)|\partial f|^2\phi^{s-2}\\
&+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4}.
\end{aligned}$$

By choosing ϵ small enough, we obtain (5.2). Finally,

$$\begin{aligned}
&\int_{A_R} e^{2(n-1)f}|\partial f|^4\phi^{s-2}\\
=&\frac{2(n-1)}{n}\int_{A_R} e^{2(n-1)f}|\partial f|^4\phi^{s-2}+\frac{s-2}{n}\int_{A_R} e^{2(n-1)f}|\partial f|^2Re(f_\alpha\phi_{\bar\alpha})\phi^{s-3}\\
&+\frac{1}{n}\int_{A_R} e^{2(n-1)f}Re(f_\alpha|\partial f|^2_{\bar\alpha})\phi^{s-2}-\int_{A_R} e^{2(n-1)f}H(f)|\partial f|^2\phi^{s-2}.
\end{aligned}$$

A direct calculation implies that

$$\frac{n-2}{n}\int_{A_R} e^{2(n-1)f}|\partial f|^4\phi^{s-2}$$

$$
\begin{aligned}
=&\int_{A_R} e^{2(n-1)f}H(f)|\partial f|^2\phi^{s-2}-\frac{s-2}{n}\int_{A_R} e^{2(n-1)f}|\partial f|^2Re(f_\alpha\phi_{\bar\alpha})\phi^{s-3}\\
&-\frac{1}{n}\int_{A_R} e^{2(n-1)f}Re(f_\alpha(D_{\bar\alpha}+E_{\bar\alpha}))\phi^{s-2}-\frac{1}{n}\int_{A_R} e^{2(n-1)f}Re(\bar g)|\partial f|^2\phi^{s-2}\\
&+\frac{2}{n}\int_{A_R} e^{2(n-1)f}H(f)|\partial f|^2\phi^{s-2}\\
\le&(1+\frac{1}{n})\int_{A_R} e^{2(n-1)f}H(f)|\partial f|^2\phi^{s-2}+C\int_{A_R} e^{(n-1)f}|\partial f|\sqrt{\mathcal{M}}\phi^{s-2}\\
&-(\frac{1}{n}-\epsilon)\int_{A_R} e^{2(n-1)f}|\partial f|^4\phi^{s-2}+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^2\phi^{s-4}\\
\le&(1+\frac{1}{n})\int_{A_R} e^{2(n-1)f}H(f)|\partial f|^2\phi^{s-2}+\epsilon R^2\int_{A_R}\mathcal{M}\phi^s\\
&-(\frac{1}{n}-\epsilon)\int_{A_R} e^{2(n-1)f}|\partial f|^4\phi^{s-2}+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^2\phi^{s-4}.
\end{aligned}
$$

Hence for ϵ sufficiently small, we deduce (5.3). □

Noting that in the proof of Lemma 3.1, we do not use the condition $H^{'}(f)\geq H(f)$. Hence Lemma 3.1 is also valid under the conditions in Theorem 1.3. In this case, we assume that $\beta=0$.

Proof of Theorem 1.3. Let f, g, H, ϕ and $\mathcal{M}$ be defined as in above. Assume that $R>0$ large enough such that $H(f)\geq Ce^{2f}$. From Lemma 3.1 and Lemma 5.1, we have

$$
\begin{aligned}
&\int_M \mathcal{M}\phi^s\\
\le&\frac{C}{R^4}\int_{A_R} e^{2(n-1)f}|\partial f|^2\phi^{s-4}+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^4\phi^{s-2}\\
\le&\frac{C}{R^{2n+4}}Vol(B_{2R})+\frac{C}{R^4}\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4}\\
&+\epsilon\int_M\mathcal{M}\phi^s+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}H^2(f)\phi^{s-2}.
\end{aligned}
$$

By choosing $\epsilon>0$ small enough and using $H(f)\geq Ce^{2f}$, we find that

$$
\begin{aligned}
&\int_M \mathcal{M}\phi^s\\
\le&\frac{C}{R^{2n+4}}Vol(B_{2R})+\frac{C}{R^4}\int_{A_R} e^{2(n-1)f}H(f)\phi^{s-4}+\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}H^2(f)\phi^{s-2}\\
\le&\frac{C}{R^2}+\frac{C}{R^4}\int_{A_R} H^n(f)\phi^{s-4}+\frac{C}{R^2}\int_{A_R} H^{n+1}(f)\phi^{s-2}\\
\le&\frac{C}{R^2}+\frac{C}{R^4}\left(\int_{A_R} H^{n+1}(f)\right)^{\frac{n}{n+1}}\cdot(Vol(B_{2R}))^{\frac{1}{n+1}}+\frac{C}{R^2}\int_{A_R} H^{n+1}(f)
\end{aligned}
$$

$$\leq \frac{C}{R^2} + C + \frac{C}{R^{\frac{2}{n+1}}} \leq C.$$

It follows that

$$\int_M \mathcal{M} \leq C,$$

which implies that

$$\begin{aligned}
&\int_{A_R} e^{2(n-1)f}|g|^2\Psi^{-\beta}\phi^{s-2} \\
\leq &\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}|\partial f|^2\phi^{s-4} + \epsilon R^2\int_M \mathcal{M}\phi^s + C\int_{A_R} e^{2(n-1)f}|\partial f|^4\phi^{s-2} \\
\leq &CR^2.
\end{aligned}$$

Finally, by Lemma 2.2, we obtain

$$\int_M \mathcal{M} = 0,$$

that is, $\mathcal{M} = 0$. Hence we deduce the conclusion as in the proof of Theorem 1.1 and Theorem 1.2. □

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References

[1] A. Agrachev, D. Barilari, and U. Boscain, *A comprehensive introduction to sub-Riemannian geometry*, vol. 181 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2020. From the Hamiltonian viewpoint, With an appendix by Igor Zelenko.

[2] L. A. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math., 42 (1989), pp. 271–297.

[3] G. Catino, Y. Li, D. D. Monticelli, and A. Roncoroni, *A liouville theorem in the Heisenberg group*, J. Eur. Math. Soc., (2025).

[4] G. Catino and D. D. Monticelli, *Semilinear elliptic equations on manifolds with nonnegative ricci curvature*, Journal of the European Mathematical Society, (2024).

[5] G. Catino, D. D. Monticelli, and A. Roncoroni, *On the critical p-Laplace equation*, Adv. Math., 433 (2023), pp. Paper No. 109331, 38.

[6] G. Catino, D. D. Monticelli, A. Roncoroni, and X. Wang, *Liouville theorems on pseudohermitian manifolds with nonnegative tanaka-webster curvature*, arXiv preprint arXiv:2412.08500, (2024).

[7] W. L. Chow, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann., 117 (1939), pp. 98–105.

[8] S. Dragomir and G. Tomassini, *Differential geometry and analysis on CR manifolds*, vol. 246 of Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 2006.

[9] J. Flynn and J. Vétois, *Liouville-type results for the CR yamabe equation in the heisenberg group*, 2023.

[10] N. Garofalo and D. Vassilev, *Symmetry properties of positive entire solutions of Yamabe-type equations on groups of Heisenberg type*, Duke Math. J., 106 (2001), pp. 411–448.

[11] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math., 34 (1981), pp. 525–598.

[12] Q. Gu, Y. Sun, J. Xiao, and F. Xu, *Global positive solution to a semilinear parabolic equation with potential on Riemannian manifold*, Calc. Var. Partial Differential Equations, 59 (2020), pp. Paper No. 170, 24.

[13] J. He, L. Sun, and Y. Wang, *Optimal liouville theorems for the Lane-Emden equation on Riemannian manifolds*, arXiv preprint, arXiv:2411.06956, (2024).

[14] D. Jerison and J. M. Lee, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc., 1 (1988), pp. 1–13.

[15] P. W. Y. Lee and C. Li, *Bishop and Laplacian comparison theorems on Sasakian manifolds*, Comm. Anal. Geom., 26 (2018), pp. 915–954.

[16] Y. Li and L. Zhang, *Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations*, J. Anal. Math., 90 (2003), pp. 27–87.

[17] X.-N. Ma and Q. Ou, *A Liouville theorem for a class semilinear elliptic equations on the Heisenberg group*, Adv. Math., 413 (2023), pp. Paper No. 108851, 20.

[18] P. Mastrolia, D. D. Monticelli, and F. Punzo, *Nonexistence of solutions to parabolic differential inequalities with a potential on Riemannian manifolds*, Math. Ann., 367 (2017), pp. 929–963.

[19] J. V. Prajapat and A. S. Varghese, *Symmetry and classification of solutions to an integral equation in the Heisenberg group* $\mathbb{H}^n$, Math. Ann., 392 (2025), pp. 659–700.

[20] J. Serrin and H. Zou, *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math., 189 (2002), pp. 79–142.

[21] R. S. Strichartz, *Sub-Riemannian geometry*, J. Differential Geom., 24 (1986), pp. 221–263.

[22] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds*, Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 9, Kinokuniya Book Store Co., Ltd., Tokyo, 1975.

[23] S. M. Webster, *Pseudo-Hermitian structures on a real hypersurface*, J. Differential Geometry, 13 (1978), pp. 25–41.

[24] B. Zhao, *Liouville type results for semilinear elliptic equation and inequality on pseudo-hermitian manifolds*, 2025.

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