TWO IRRATIONALLY ELLIPTIC CLOSED ORBITS OF REEB FLOWS ON THE BOUNDARY OF STAR-SHAPED DOMAIN IN $\mathbb{R}^{2n}$

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ABSTRACT. There are two long-standing conjectures in Hamiltonian dynamics concerning Reeb flows on the boundaries of star-shaped domains in $\mathbb{R}^{2n}$ $(n \geq 2)$. One conjecture states that such a Reeb flow possesses either n or infinitely many prime closed orbits; the other states that all the closed Reeb orbits are irrationally elliptic when the domain is convex and the flow possesses finitely many prime closed orbits. In this paper, we prove that for dynamically convex Reeb flow on the boundary of a star-shaped domain in $\mathbb{R}^{2n}$ $(n \geq 2)$ with exactly n prime closed orbits, at least two of them must be irrationally elliptic.

CONTENTS

1. INTRODUCTION

1.1. **Set up and background.** In this paper, we focus on the stability problem for closed characteristics, interpreted as closed orbits of the Reeb flow on the boundary of a star-shaped domain in $\mathbb{R}^{2n}$. Specifically, let Σ^{2n-1} bound a star-shaped domain in $\mathbb{R}^{2n}$. Throughout this paper, we assume Σ is smooth. The restriction of the Liouville form

$$\lambda_{\rm std} = \frac{1}{2}\sum_{i=1}^{n}(y_i \mathrm{d}x_i - x_i \mathrm{d}y_i)$$

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to Σ yields a contact form α, with $\xi = \ker\alpha$ defining the contact structure. The Reeb vector field R of α on Σ satisfies

$$\alpha(R) = 1, \quad \mathrm{d}\alpha(R, \cdot) = 0.$$

Denote its flow by ϕ_α^t. For a closed orbit x of period T, we call x *prime* if T is its minimal period. Two fundamental problems in Hamiltonian dynamics concern the multiplicity of prime closed Reeb orbits and their stability. We always identify the closed orbits $t \mapsto x(t)$ and $t \mapsto x(t+\theta)$ for all $\theta \in \mathbb{R}$.

The stability of a closed Reeb orbit x is characterized by its *Floquet multipliers*—the eigenvalues of the restricted linearized Poincaré return map

$$(\mathrm{d}\phi_\alpha^T)_{x(0)}|_\xi : \xi_{x(0)} \to \xi_{x(0)}.$$

Let $\sigma(x)$ denote this eigenvalue set and $\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. We classify x as:

- *hyperbolic* if $\sigma(x) \cap \mathbb{U} = \emptyset$;
- *elliptic* if $\sigma(x) \subset \mathbb{U}$;
- *irrationally elliptic* if $\mathrm{d}\phi_\alpha^T|_\xi$ decomposes into a direct sum of $n-1$ rotations
$$\begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$
on 2-dimensional symplectic subspaces, with rotation angles θ_i being irrational multiples of π for $1 \le i \le n-1$;
- *non-degenerate* if $1 \notin \sigma(x)$; in addition, the Reeb flow on Σ is called non-degenerate if all closed orbits, prime or not, are non-degenerate.

Denote by $\mathcal{T}(\Sigma)$ the set of prime closed Reeb orbits on Σ (with respect to the contact form $\alpha = \lambda_{\mathrm{std}}|_\Sigma$). There is a long-standing conjecture concerning the multiplicity of closed Reeb orbits on compact star-shaped hypersurfaces in $\mathbb{R}^{2n}$:

Conjecture 1.1. *For each Σ^{2n-1} bounding a star-shaped domain in $\mathbb{R}^{2n}$, ${}^\#\mathcal{T}(\Sigma) \ge n$.*

Since the pioneering works of Rabinowitz in [R78] of 1978 (for star-shaped hypersurfaces) and Weinstein in [W78] of 1978 (for convex hypersurfaces) showing ${}^\#\mathcal{T}(\Sigma) \ge 1$, the existence of multiple closed Reeb orbits has been deeply studied. For any convex hypersurface in $\mathbb{R}^{2n}$ with $n \ge 2$, Ekeland-Lassoued, Ekeland-Hofer and Szulkin in [EL87] of 1987, [EH87] of 1987 and [S88] of 1988 proved ${}^\#\mathcal{T}(\Sigma) \ge 2$; Long-Zhu in [LZ02] of 2002 further established the breakthrough result that ${}^\#\mathcal{T}(\Sigma) \ge \lfloor n/2 \rfloor + 1$ and ${}^\#\mathcal{T}(\Sigma) \ge n$ under non-degeneracy condition; Wang-Hu-Long in [WHL07] of 2007 proved ${}^\#\mathcal{T}(\Sigma) \ge 3$ for $n = 3$; Wang in [W16b] of 2016 showed ${}^\#\mathcal{T}(\Sigma) \ge \lfloor (n+1)/2 \rfloor + 1$ and in [W16a] of 2016 demonstrated ${}^\#\mathcal{T}(\Sigma) \ge 4$ for $n = 4$. Recent progress by Çineli-Ginzburg-Gürel in [ÇGG24] of 2024 confirmed the conjecture for Σ bounding star-shaped domains with dynamically convex Reeb flows.

For star-shaped domains, we summarize some other key results: In [G84] of 1984 and [B+85] of 1985, ${}^\#\mathcal{T}(\Sigma) \ge n$ was established under some pinching conditions. Viterbo in [V89] of 1989 proved generic existence of infinitely many closed Reeb orbits. Hu-Long in [HL02] of 2002 showed ${}^\#\mathcal{T}(\Sigma) \ge 2$ under non-degeneracy condition. Cristofaro-Gardiner-Hutchings in [CH16] of 2016 proved ${}^\#\mathcal{T}(\Sigma) \ge 2$ for $n = 2$ without pinching or non-degeneracy conditions, with alternative proofs in [G+13] of 2013, [GG15] of 2015 and [LL16] of 2016. Duan-Liu-Long-Wang in[D+18] and [D+24] established ${}^\#\mathcal{T}(\Sigma) \ge n$ under index and non-degeneracy conditions, yielding at least n $(n-1)$ non-hyperbolic orbits for even (odd) n. Gutt-Kang in [GK16] of 2016 proved ${}^\#\mathcal{T}(\Sigma) \ge n$ requiring non-degeneracy and Conley-Zehnder indices not less than $n-1$. Duan-Liu in [DL17] of 2017 and Ginzburg-Gürel in [GG20] of 2020 showed ${}^\#\mathcal{T}(\Sigma) \ge \lfloor (n+1)/2 \rfloor + 1$ for dynamically convex flows independently. Additionally, Ginzburg-Gürel-Macarini in [GGM18] of 2018 studied prequantization bundles, while Abreu-Macarini in [AM17] of 2017 investigated dynamically convex contact forms, yielding various multiplicity results.

Note that in [HWZ98] of 1998, Hofer-Wysocki-Zehnder proved that ${}^{\#}\mathcal{T}(\Sigma) = 2$ or ∞ holds for every Σ bounding a convex domain in $\mathbb{R}^4$. Subsequently in [HWZ03] of 2003, they proved the same result holds when Σ bounds a star-shaped domain in $\mathbb{R}^4$ with non-degeneracy and transversality assumptions, and conjectured that:

For each Σ bounding a star-shaped domain in $\mathbb{R}^4$, there holds ${}^{\#}\mathcal{T}(\Sigma) \in \{2\} \cup \{\infty\}$.

This conjecture was confirmed in [C+23b], by proving a stronger theorem that the "two or infinity" result holds for any closed contact 3-manifold whose contact structure possessing torsion first Chern class. For $n \geq 3$, the analogous conjecture is still widely open:

Conjecture 1.2 (cf. [WHL07; LL17; ÇGG24])**.** *For each Σ^{2n-1} bounding a star-shaped domain in $\mathbb{R}^{2n}$, there holds*

$$ {}^{\#}\mathcal{T}(\Sigma) \in \{n\} \cup \{\infty\}. $$

It seems interesting to study the property of closed Reeb orbits for the case ${}^{\#}\mathcal{T}(\Sigma) = n$. Note that we have the following example of Harmonic Oscillator (cf. [E90, Section 1.7]):

Example 1.3. Let $r = (r_1, \ldots, r_n)$ with $r_i > 0$ for $1 \leq i \leq n$. Define

$$ \mathcal{E}_n(r) = \left\{ z = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \,\middle|\, \frac{1}{2}\sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{r_i} = 1 \right\}. $$

In this case, the corresponding system is linear and all the solutions can be computed explicitly. Suppose $\frac{r_i}{r_j} \notin \mathbb{Q}$ whenever $i \neq j$, then ${}^{\#}\mathcal{T}(\mathcal{E}_n(r)) = n$ and all the closed Reeb orbits on $\mathcal{E}_n(r)$ are irrationally elliptic. On the other hand, if $\frac{r_i}{r_j} \in \mathbb{Q}$ for some $i \neq j$, then ${}^{\#}\mathcal{T}(\mathcal{E}_n(r)) = \infty$.

In [L00] of 2000, Long proved that for Σ bounding a convex domain in $\mathbb{R}^4$, ${}^{\#}\mathcal{T}(\Sigma) = 2$ implies that both of the closed Reeb orbits must be elliptic. In [WHL07], the authors proved further that both of the two closed Reeb orbits must be irrationally elliptic. (This result is generalized to any closed contact 3-manifold in [C+23a] of 2023.)

Based on these facts, Wang-Hu-Long in [WHL07] made the following conjecture:

Conjecture 1.4. *For each Σ^{2n-1} bounding a convex domain in $\mathbb{R}^{2n\geq 4}$, assume ${}^{\#}\mathcal{T}(\Sigma) < \infty$, then all the closed Reeb orbits on Σ are irrationally elliptic.*

Remark 1.5. Note that in Example 1.3, if $r_1 = r_2 = \cdots = r_n = 1$, then every trajectory of the Reeb vector field is closed on $\mathcal{E}_n(r)$, thus ${}^{\#}\mathcal{T}(\mathcal{E}_n(r)) = \infty$. Moreover, 1 is a Floquet multiplier with multiplicity $2n - 2$ for any closed Reeb orbit on $\mathcal{E}_n(r)$. Thus one can not hope to find irrationally elliptic closed Reeb orbit on arbitrary Σ.

In [W09] of 2009, Wang proved that if Σ bounds a convex domain in $\mathbb{R}^6$ with ${}^{\#}\mathcal{T}(\Sigma) = 3$, then there exist at least two elliptic closed Reeb orbits on Σ. This result was improved in [W14] of 2014, where it was shown that there exist at least two irrationally elliptic closed Reeb orbits on Σ under the same assumption. In [HO17] of 2017, Hu-Ou established the existence of at least two elliptic closed Reeb orbits on Σ under the assumption of Conjecture 1.4. In [W22] of 2022, Wang demonstrated that under the condition "$\rho_n(\Sigma) = n$" (slightly weaker than non-degeneracy), if Σ bounds a convex domain in $\mathbb{R}^{2n}$ with ${}^{\#}\mathcal{T}(\Sigma) = n$, then there are at least two irrationally elliptic closed Reeb orbits; this result also holds for star-shaped domains with dynamically convex Reeb flows. Our main result improves upon [W22] by eliminating the technical condition "$\rho_n(\Sigma) = n$", as stated in Theorem 1.7.

There are some related results considering the stability problem. A long-standing conjecture concerning the stability of closed Reeb orbits on compact convex hypersurfaces in $\mathbb{R}^{2n}$ (cf. [E90, p. 235]) states:

Conjecture 1.6. *For each Σ^{2n-1} bounding a strictly convex domain in $\mathbb{R}^{2n}$, there exists at least one elliptic closed Reeb orbit on Σ.*

This conjecture has been verified under the following assumptions:

- Ekeland (1986) [E86]: $\sqrt{2}$-pinched convex domains;
- Dell'Antonio-D'Onofrio-Ekeland (1992) [DDE92]: centrally symmetric domains;
- Long-Zhu (2002) [LZ02]: finite set of closed orbits ($^{\#}\mathcal{T}(\Sigma) < +\infty$);
- Arnaud (1999) [A99], Liu-Wang-Zhang (2020) [LWZ20]: domains invariant under specific rotation direct sums;
- Abreu-Macarini (2022) [AM22]: relaxing strict convexity to convexity in [LWZ20]).

To the best of our knowledge, no lower bound for multiple elliptic closed Reeb orbits is known under only the strictly convexity hypothesis.

1.2. **Main results.** To state our results, we first briefly review some key terminology, which will be formally defined in Sections 2 and 3.

Let (Σ^{2n-1}, α) be a contact manifold bounding a star-shaped domain in $\mathbb{R}^{2n}$, as introduced in the previous section. For a closed Reeb orbit x on Σ with period T, viewed as an immersion $S^1 = \mathbb{R}/T\mathbb{Z} \to \Sigma$, we define its action by

$$\mathcal{A}(x) := \int_{S^1} x^*\alpha.$$

By the definition of Reeb vector field, $\mathcal{A}(x)$ equals the period of x. For each $k \in \mathbb{N}$, the k-th iteration of x is the closed Reeb orbit y satisfying $\mathcal{A}(y) = k\mathcal{A}(x)$ and $y|_{[0,T]} = x$, denoted by $y = x^k$. The integer k or iteration x^k is called *admissible* if the algebraic multiplicity of eigenvalue 1 in $\sigma(x)$ and $\sigma(x^k)$ coincides. For instance, x^k is admissible when the set $\{\lambda \in \sigma(x) : |\lambda|^k = 1\}$ is either $\{1\}$ or empty.

For each closed Reeb orbit x on Σ, we associate two integer-valued Maslov-type indices $\mu_\pm(x)$, obtained via upper and lower semi-continuous extensions of the Conley-Zehnder index (see Section 3.1), and a $\mathbb{R}$-valued index $\hat{\mu}(x)$, known as the *mean index* (see Section 3.1). The Reeb flow on Σ^{2n-1} is *dynamically convex* if $\mu_-(x) \geq n+1$ for all closed Reeb orbits x. This notion was originally introduced in [HWZ98] and subsequently generalized by several authors, cf. Abreu-Macarini in [AM22], Duan-Liu in [DL17], Gutt-Kang in [GK16], etc. All these definitions are equivalent when restricted to the case where Σ is the boundary of a star-shaped domain in $\mathbb{R}^{2n}$.

The following theorem is the main result of this paper:

Theorem 1.7. *Let $\Sigma^{2n-1} \subset \mathbb{R}^{2n\geq 4}$ be the boundary of a star-shaped domain. Assume that the Reeb flow on Σ is dynamically convex and has exactly n prime closed orbits, then at least two of them are irrationally elliptic.*

The proof is given in Section 5 and the main ingredients include:

- Iteration theory of Maslov-type indices (see Section 3.2);
- Commutative property of closed orbits in the common index jump intervals originally established in [W16a] for $n = 4$ and generalized to arbitrary n in [W22] (see Proposition 5.2);
- Key descriptions of local Floer-theoretic invariants recently given in [ÇGG24] (see Section 5.1);
- The following crucial observation (Theorem 1.8) proved in Section 4.

Fix a ground field $\mathbb{F}$. For each isolated closed Reeb orbit x, we define its *equivariant local symplectic homology* $\mathrm{CH}_*(x; \mathbb{F})$, which serves as the Morse-theoretic counterpart of critical modules introduced in [GM69a; GM69b]. The following theorem demonstrates that for certain closed Reeb orbit x, the support of the equivariant local symplectic homology concentrates at the Maslov-type index μ_+.

Theorem 1.8. *Let $\Sigma^{2n-1} \subset \mathbb{R}^{2n\geq 4}$ be the boundary of a star-shaped domain. Suppose x is an isolated closed Reeb orbit on Σ satisfying:*

- $\mathrm{CH}_{\mu_+(x)}(x; \mathbb{Q}) \neq 0$;

- $\mu_+(x) = d + n - 1$ *for some integer* d *with* $|d - \hat{\mu}(x)| < \frac{1}{2}$.

Then

$$\mathrm{CH}_*(x;\mathbb{Q}) = 0 \quad \textit{whenever} \quad * \neq \mu_+(x).$$

Additionally, if $x = z^k$ *is an admissible iteration of a closed Reeb orbit* z*, then for any admissible iteration* $z^{k'}$*,*

$$\mathrm{CH}_*(z^{k'};\mathbb{Q}) = 0 \quad \textit{whenever} \quad * \neq \mu_+(z^{k'}).$$

Remark 1.9. In fact, our proof in Section 4 shows that Theorem 1.8 holds whenever the absolute grading of $\mathrm{CH}_*(x;\mathbb{Q})$ is well-defined. This includes contact manifolds (Σ^{2n-1},ξ) satisfying $c_1(\xi) = 0 \in H^2(\Sigma;\mathbb{Z})$ and $H_1(\Sigma;\mathbb{Q}) = 0$; see Remark 3.5. The requirement of $\mathbb{Q}$-coefficients is essential and does not extend to fields of positive characteristic $p \geq 2$; see Remark 2.5.

Remark 1.10. We conjecture that the second condition $\mu_+(x) = d + n - 1$ can be eliminated. This conjecture is motivated by an analogous result for equivariant critical modules of closed characteristics established in [W09, Proposition 2.6]. Some alternative attempt is discussed in Remark 4.2.

1.3. Organization and notation.

1.3.1. *Organization.* The paper is organized as follows. Sections 2 and 3 present preliminary material, which is well-known to experts but included for the reader's convenience. In Sections 2.1 and 2.2, we review local Floer-theoretic invariants for closed orbits and their properties, including local Floer homology, equivariant and non-equivariant local symplectic homology. Section 2.3 recalls the definition of generating functions in $\mathbb{R}^{2n}$.

The main focus of Section 3 are Maslov-type indices for symplectic paths, extending from the Conley-Zehnder index, and their iteration theory. This includes basic definitions and properties in Section 3.1, abstract precise iteration formulas in Section 3.2, and the Common Index Jump Theorem in Section 3.3.

Section 4 contains the proof of Theorem 1.8 via an analogue for local Floer homology. In Section 5.1, we summarize key arguments from [ÇGG24], and Section 5.2 presents the proof of Theorem 1.7.

1.3.2. *Conventions and notation.* For the reader's convenience, we summarize key notations here. Some will be restated in the context when they appear.

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, rational numbers, real numbers, and complex numbers respectively. Both S^1 and $\mathbb{U}$ denote the unit circle. We use $\mathbb{U}$ when it is viewed as a subset of $\mathbb{C}$. Let $\mathbb{F}$ denote an arbitrary field, and $\mathbb{F}_p$ denote a field with positive characteristic $p \geq 2$. Let $\mathbb{Z}_p$ denote the field given by $\mathbb{Z}/p\mathbb{Z}$ when integer $p \geq 2$ is prime.

For a linear transformation M, let $\sigma(M)$ denote its eigenvalue set. Both id and Id denote the identity map. We consistently use Id when referring to the identity linear transformation. When the dimension of the underlying vector space on which the identity transformation acts needs to be specified, we use the notation Id_m, where the subscript m explicitly indicates this dimension.

The standard symplectic vector space $(\mathbb{R}^{2m}, \omega_{\mathrm{std}})$ has coordinates $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ with symplectic form

$$\omega_{\mathrm{std}} := \sum_{i=1}^{m} \mathrm{d}y_i \wedge \mathrm{d}x_i.$$

Denote by $\mathrm{Sp}(2m)$ the symplectic group equipped with the subspace topology from $\mathbb{R}^{4m^2}$, i.e.,

$$\mathrm{Sp}(2m) = \{M \in \mathrm{GL}(2m;\mathbb{R}) : M^T J M = J\},$$

where $\mathrm{GL}(2m;\mathbb{R})$ denotes the set of all the linear transformations on $\mathbb{R}^{2m}$ and $J := \begin{pmatrix} 0 & -\mathrm{Id}_m \\ \mathrm{Id}_m & 0 \end{pmatrix}$. For two symplectic matrices in block form:

$$M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \in \mathrm{Sp}(2m_k), \quad k = 1, 2,$$

their $\diamond$-product $M_1 \diamond M_2 \in \mathrm{Sp}(2m_1 + 2m_2)$ is defined as:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix},$$

representing the *symplectic direct sum* in the canonical coordinates of $\mathbb{R}^{2(m_1+m_2)}$.

We use the floor and ceiling functions defined by:

$$\lfloor a \rfloor = \max\{k \in \mathbb{Z} \mid k \leq a\}, \quad \lceil a \rceil = \min\{k \in \mathbb{Z} \mid k \geq a\}.$$

2. Preliminary

2.1. **Local Floer homology.**

2.1.1. *Basic definition and properties.* Fix a ground field $\mathbb{F}$. Let (W^{2n}, ω) be a symplectic manifold, and $H : S^1 \times W^{2n} \to \mathbb{R}$ be a smooth 1-periodic Hamiltonian on W^{2n}. Let x be a 1-periodic orbit of H, i.e., a 1-periodic solution of the system

$$\dot{x}(t) = X_{H_t}(x(t)),$$

where X_{H_t} denotes the Hamiltonian vector field defined by $\omega(X_{H_t}, \cdot) = -\mathrm{d}H_t$. Suppose x is isolated. Let φ_H^t be the Hamiltonian flow of H. Choose a sufficiently small tubular neighborhood U of x. As in [SZ92, Theorem 9.1] we can take a C^2-small perturbation $\tilde{H}$ of H supported in U such that all the 1-periodic orbits $\tilde{x}$ of $\tilde{H}$ entering U are *non-degenerate*, that is, $\mathrm{d}\varphi^1_{\tilde{H}}(\tilde{x}(0)) : T_{\tilde{x}(0)}W \to T_{\tilde{x}(0)}W$ has no eigenvalue equal to 1. When $\tilde{H}$ is sufficiently C^2-close to H and U is sufficiently small, every Floer trajectory u connecting two 1-periodic orbits of $\tilde{H}$ lying in U, whether broken or not, is also contained in U. (This follows from the energy separation argument induced by Gromov's compactness, see [S99, Section 1.5].)

Choose a generic almost complex structure J such that the regularity conditions are satisfied. Then we can construct a Floer complex over $\mathbb{F}$ on U in the standard way: the generators are given by the 1-periodic orbits of $\tilde{H}$ contained in U (hence close to x), graded by the Conley-Zehnder index, and the differential is given by counting Floer trajectories. The standard continuation argument in Floer theory shows that the homology of Floer complex is independent of the choice of $\tilde{H}$, the almost complex structure, and the sufficiently small neighborhood U; see [SZ92; S99; G10] for more details. We refer to the resulting homology group $\mathrm{HF}_*(H, x)$ as the *local Floer homology* of H at x. We omit the coefficient filed $\mathbb{F}$ unless there is a specific notation. Sometime we also denote the local Floer homology by $\mathrm{HF}_*(\varphi, x)$, where $\varphi = \varphi_H^1$ is the time-1 map.

The absolute grading of the local Floer homology is determined by fixing a symplectic trivialization of the tangent bundle TW along the orbit x. This trivialization induces corresponding trivializations along all non-degenerate closed orbits $\tilde{x}$ that x splits into, and yields the associated linearized symplectic paths used to define the Conley-Zehnder index (will be explained in Section 3.1).

Furthermore, when the first Chern class $c_1(TW) = 0$, there exists a canonical construction of symplectic trivializations for contractible orbits, see Subsection 3.1.4. This ensures the Conley-Zehnder index is well-defined for such orbits. Under this condition, we always assume the grading is given by these Conley-Zehnder indices.

Example 2.1 (cf. [G10, Example 3.4])**.** Assume $c_1(TW) = 0$, x is non-degenerate and contractible. Then the local Floer homology of H at x is given by

$$\mathrm{HF}_*(H, x) = \begin{cases} \mathbb{F}, & * = \mu(x), \\ 0 & \text{otherwise}, \end{cases}$$

where $\mu(x)$ denotes the Conley-Zehnder index defined in Subsection 3.1.4. In fact, due to the non-degeneracy of x, we can find a neighborhood U of x such that the Floer complex on U has exactly one generator x.

In general, the single isolated 1-periodic orbit x in this construction can be replaced by an isolated connected compact set of 1-periodic orbits of H; see [M12].

We list some basic properties of local Floer homology we need in this paper. For a more detailed account, we refer the reader to [G10, Section 3.2], [GG10, Section 3.2] and their reference.

(LF1) (Homotopy invariance) Let H^s, $s \in [0, 1]$, be a family of Hamiltonians such that x is a uniformly isolated one-periodic orbit for all H^s, i.e., x is the only periodic orbit of H^s for all s, in some open set independent of s. Then $\mathrm{HF}_*(H^s, x)$ is constant throughout the family. In particular, $\mathrm{HF}_*(H^0, x) = \mathrm{HF}_*(H^1, x)$.

(LF2) (Künneth formula) Let x_1 and x_2 be 1-periodic orbits of the Hamiltonians H_1 and H_2, respectively, on the sympletic manifolds M_1 and M_2. Then

$$\mathrm{HF}_*(H_1 + H_2, (x_1, x_2)) = \bigoplus_{*' + *'' = *} \mathrm{HF}_{*'}(H_1, x_1) \otimes \mathrm{HF}_{*''}(H_2, x_2),$$

where $H_1 + H_2$ is the naturally defined Hamiltonian on $M_1 \times M_2$.

When G and H are two 1-periodic Hamiltonians, the composition $G\#H$ is defined by

$$(G\#H)_t = G_t + H_t \circ (\varphi_G^t)^{-1}.$$

The flow of $G\#H$ is $\varphi_G^t \circ \varphi_H^t$. In general, $G\#H$ is not 1-periodic. However, when $\varphi_G^1 = \mathrm{id}$, $G\#H$ is 1-periodic automatically. Consider $\overline{H}$ given by $\overline{H}_t := -H_t \circ \varphi_H^t$, then we have $\varphi_{\overline{H}}^t = (\varphi_H^t)^{-1}$ and $\overline{H}\#H = 0 = H\#\overline{H}$.

(LF3) Let $x(t) \equiv p$ be a constant orbit of the Hamiltonian H. Let φ_G^t with $\varphi_G^0 = \varphi_G^1 = \mathrm{id}$ be a loop of Hamiltonian diffeomorphisms defined on a neighborhood of p and fixing p. Then

$$\mathrm{HF}_{*+2\mu_{\mathrm{Maslov}}}(G\#H, p) = \mathrm{HF}_*(H, p),$$

where μ_{Maslov} denotes the Maslov index of the loop $t \mapsto \mathrm{d}(\varphi_G^t)_p \in \mathrm{Sp}(2n)$.

(For a loop $\gamma(t)$ in $\mathrm{Sp}(2n)$, the *Maslov index* $\mu_{\mathrm{Maslov}}(\gamma)$ corresponds to its homotopy class in $\pi_1(\mathrm{Sp}(2n)) \simeq \mathbb{Z}$; see [MS17, Section 2.2] for details.)

(LF4) Define the support of $\mathrm{HF}_*(H, x)$ by $\mathrm{suppHF}_*(H, x) := \{m \in \mathbb{Z} : \mathrm{HF}_m(H, x) \neq 0\}$. Then we have

$$\mathrm{suppHF}_*(H, x) \subset [\mu_-(x), \mu_+(x)] \subset [\hat{\mu}(x) - n, \hat{\mu}(x) + n],$$

where $\mu_\pm$ denote the upper and lower Conley-Zehnder index and $\hat{\mu}$ denotes the mean index, which we will introduce in Section 3.

2.1.2. *Local Floer homology via local Morse homology.* Let $f : W^m \to \mathbb{R}$ be a smooth function on a manifold W (not necessarily symplectic) and $p \in W$ be an isolated critical point of f. The *local Morse homology* $\mathrm{HM}_*(f, p)$ is defined by choosing an isolating neighborhood U of p and a small Morse perturbation $\tilde{f}$ of f. Specifically, the chain complex is generated by critical points of $\tilde{f}$ in U, with the differential defined via counting trajectories of the negative gradient flow between critical points; see [S93; G10; AD14] for details.

Compared to Floer homology, local Morse homology is more accessible due to its foundation in classical Morse theory, which provides an intuitive geometric interpretation of the homology groups through gradient flow dynamics. A fundamental property of Floer homology is that, for a C^2-small autonomous Hamiltonian H, its Floer homology is equal to its Morse homology. In [G10], Ginzburg proved a slightly more general version of this fact, which allows us to understand the local Floer homology of a "relatively autonomous" Hamiltonian (this concept was introduced in [H09, Lemma 4]) by the Morse homology of a reference function. This argument will play a crucial role in our poof of Theorem 1.8.

Lemma 2.2 ([G10, Lemma 3.6])**.** *Let F be a smooth function and let K be a T-periodic Hamiltonian, both defined on a neighborhood of a point p in W^{2n}. Assume that p is an isolated critical point of F, and the following conditions are satisfied:*

- *The inequalities $\|X_{K_t}(z) - X_F(z)\| \leq \epsilon \|X_F(z)\|$ and $\|\dot{X}_{K_t}(z)\| \leq \epsilon \|X_F(z)\|$ hold pointwise for z near p for all $t \in S^1$. (The dot stands for the derivative with respect to time.)*
- *The Hessian $\mathrm{d}^2(K_t)_p$ and $\mathrm{d}^2 F_p$ and the constant $\epsilon \geq 0$ are sufficiently small. Namely, $\epsilon < 1$ and*
$$T \cdot (\epsilon(1-\epsilon)^{-1} + \max_t \|\mathrm{d}^2(K_t)_p\| + \|\mathrm{d}^2 F_p\|) < 2\pi.$$

Then p is an isolated T-periodic orbit of K. Futhermore,

(a) $\mathrm{HF}_*(K,p) = \mathrm{HM}_{*+n}(F,p)$*;*
(b) if $\mathrm{HF}_n(K,p) \neq 0$*, the functions K_t for all t and F have a strict local maximum at p.*

Remark 2.3. We can infer from (a) and (b) immediately that,
$$\mathrm{HF}_n(K,p) \neq 0 \Rightarrow \mathrm{HF}_*(K,p) = 0 \text{ whenever } * \neq n,$$
since in W^{2n} the Morse homology of a smooth function F at a strict local maximum point p is
$$\mathrm{HM}_*(F,p) = \begin{cases} \mathbb{F}, & * = 2n; \\ 0, & \text{otherwise}; \end{cases}$$
see [G10, Example 3.2].

2.2. **Local symplectic homology.**

2.2.1. *Basic definitions and facts.* Let (Σ, α) be a contact manifold with contact structure $\xi = \ker \alpha$, and x be an isolated closed Reeb orbit, not necessarily prime, on Σ. Fix an isolating neighborhood U of $x(\mathbb{R})$ in Σ, which means that U contains no other closed orbit of period close to $\mathcal{A}(x)$. Consider the symplectization $W = (1-\delta, 1+\delta) \times U$ with coordinates (r,z) and the symplectic form $\omega = \mathrm{d}(r\alpha)$. Take a Hamiltonian $h(r) : W \to \mathbb{R}$ only depending on the r-coordinate with $h'(1) = \mathcal{A}(x)$ and $h'' > 0$. The Hamiltonian vector field of h on W is given by
$$X_h(r,z) = (0, h'(r) R_\alpha).$$
Then x gives rise to an isolated set S of 1-periodic orbits of h with initial conditions on $x(\mathbb{R})$, which admits an S^1-action given by translation on time. Fix a ground field $\mathbb{F}$, the *local symplectic homology* $\mathrm{SH}_*(x;\mathbb{F})$ is defined by the local Floer homology $\mathrm{HF}_*(h;\mathbb{F})$ of h at S.

More explicitly, we can describe $\mathrm{SH}_*(x;\mathbb{F})$ as follows. Consider a small non-degenerate perturbation of α. Then x splits into a finite collection of non-degenerate closed Reeb orbits x_i. By a Morse-Bott construction as in [BO09], $\mathrm{SH}_*(x;\mathbb{F})$ is the homology of a certain complex with generators $\check{x}_i$ and $\hat{x}_i$ which are born from x_i and are of degrees $\mu(x_i)$ and $\mu(x_i)+1$ respectively, and the differential is given by counting the Morse-Bott broken trajectories; see also [GG20; Ç+23].

The absolute grading of the local symplectic homology is determined by fixing a symplecic trivialization of the contact structure $\xi = \ker \alpha$ along x. (Note that such a trivialization gives rise to trivializations along all closed orbits x_i that x splits into.) Furthermore, when $c_1(\xi) = 0$

and $H^1(\Sigma;\mathbb{Q}) = 0$, there exists a canonical construction of symplectic trivializations for each closed orbits, which ensures the Conley-Zehnder index is well-defined; see Subsection 3.1.4 and Remark 3.5. Under this condition, we always assume the grading is given by these Conley-Zehnder indices.

The local symplectic homology has an S^1-equivariant counterpart, which we denoted by $\mathrm{CH}_*(x;\mathbb{F})$. We refer the reader to [BO13; GG20] for the details of the definition. When $\mathbb{F} = \mathbb{Q}$, consider a small perturbation of α such that x splits into non-degenerate closed orbits x_i again. Then $\mathrm{CH}_*(x;\mathbb{Q})$ can be described as the homology of a complex generated by the orbits x_i and graded by the Conley-Zehnder index; see [GG20, Section 2.3].

Example 2.4. Assume x is non-degenerate, then we have

$$\mathrm{SH}_*(x;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & * = \mu(x) \text{ or } \mu(x)+1; \\ 0, & \text{otherwise}; \end{cases}$$

see [C+96, Proposition 2.2]. In addition, assume $x = y^k$, $k \in \mathbb{N}$, where y is the underlying prime closed orbit. Non-degenerate orbit x is called *bad* if $\mu(x)$ and $\mu(y)$ have different parity, or equivalently, y has an odd number of Floquet multipliers (counted with algebraic multiplicity) in $(-1, 0)$ and k is even; otherwise, x is called *good*. Then for a good orbit x, we have

$$\mathrm{CH}_*(x;\mathbb{Q}) = \begin{cases} \mathbb{Q}, & * = \mu(x); \\ 0, & \text{otherwise}; \end{cases}$$

while for a bad orbit x, $\mathrm{CH}_*(x;\mathbb{Q}) = 0$ in all degrees; see [GG20, Example 2.19].

Remark 2.5. As mentioned in [ÇGG24, Remark 2.6], the description for $\mathrm{CH}_*(x;\mathbb{Q})$ does not carry over to a field of positive characteristic $p \geq 2$. For instance, for a non-degenerate orbit $x = y^p$, y is prime, $\mathrm{CH}_*(x;\mathbb{F}_p)$ is isomorphic to the homology of a infinite-dimensional lens space $S^\infty/\mathbb{Z}_p$ over $\mathbb{F}_p$.

As in (LF4) of Section 2.1, the supports of $\mathrm{SH}_*(x;\mathbb{F})$ or $\mathrm{CH}_*(x;\mathbb{Q})$ are also bounded by the Maslov-type indices. Namely, we have

$$\operatorname{supp} \mathrm{SH}_*(x;\mathbb{F}) \subset [\mu_-(x), \mu_+(x)+1] \subset [\hat{\mu}(x) - n + 1, \hat{\mu}(x) + n] \tag{2.1}$$

and

$$\operatorname{supp} \mathrm{CH}_*(x;\mathbb{Q}) \subset [\mu_-(x), \mu_+(x)] \subset [\hat{\mu}(x) - n + 1, \hat{\mu}(x) + n - 1]. \tag{2.2}$$

Furthermore, it is shown in [GG20] that equivariant and non-equivariant local symplectic homology fit into the Gysin exact sequence

$$\cdots \to \mathrm{SH}_*(x;\mathbb{F}) \to \mathrm{CH}_*(x;\mathbb{F}) \xrightarrow{\mathcal{D}} \mathrm{CH}_{*-2}(x;\mathbb{F}) \to \mathrm{SH}_{*-1}(x;\mathbb{F}) \to \cdots.$$

And by [GG20, Proposition 2.21], the shift operator $\mathcal{D}$ vanishes when $\mathbb{F} = \mathbb{Q}$. Consequently, we have

$$\mathrm{SH}_*(x;\mathbb{Q}) = \mathrm{CH}_*(x;\mathbb{Q}) \oplus \mathrm{CH}_{*-1}(x;\mathbb{Q}) \tag{2.3}$$

since the Gysin sequence splits into short exact sequences.

A closed Reeb orbit x is called $\mathbb{F}$-*visible* if $\mathrm{SH}(x;\mathbb{F}) \neq 0$, is called *equivariantly* $\mathbb{F}$-*visible* if $\mathrm{CH}(x;\mathbb{F}) \neq 0$. By the Gysin exact sequence, a orbit is $\mathbb{F}$-visible implies that it is equivariantly $\mathbb{F}$-visible. And $\mathbb{Q}$-visiblity is equivalent to equivariantly $\mathbb{Q}$-visiblity due to (2.3).

The *equivariant Euler characteristic* of x is defined as

$$\chi^{\mathrm{eq}}(x) = \sum_m (-1)^m \dim \mathrm{CH}_m(x;\mathbb{Q}).$$

For instance, $\chi^{\mathrm{eq}}(x) = 0$ if x is not $\mathbb{Q}$-visible. According to Example 2.4, if x is non-degenerate, then $\chi^{\mathrm{eq}}(x)$ is either $(-1)^{\mu(x)}$ or 0 depending on whether x is good or bad.

The following result from [ÇGG24] will help us to determine $\mathbb{Q}$-visibility for certain orbit iterations in Section 5.1.

Lemma 2.6 ([ÇGG24, Lemma 2.8]). *Let x be an isolated (not necessarily prime) closed Reeb orbit. Assume that k is odd and iteration x^k is admissible. Then*

$$\chi^{\mathrm{eq}}(x^k) = \chi^{\mathrm{eq}}(x).$$

In particular, x^k is $\mathbb{Q}$-visible when $\chi^{\mathrm{eq}}(x) \neq 0$.

Moreover, the condition k is odd can be dropped provided x has an even number of Floquet multipliers (counted with algebraic multiplicity) in $(-1, 0)$.

2.2.2. *Specific local model.* We introduce a local model for what a contact manifold looks like along a closed Reeb orbit, which is given in [HM15]:

Lemma 2.7 ([HM15, Lemma 5.2]). *Let x be a prime T-period closed Reeb orbit of a contact manifold (Y, α) of dimension $2n-1$. Then there exists a tubular neighborhood $U = S^1 \times B^{2n-2}$ of $x(\mathbb{R})$, where $B^{2n-2} \subset \mathbb{R}^{2n-2}$ is a small open ball centered at the origin, with coordinates $(t, q_1, \ldots, q_{n-1}, p_1, \ldots, p_{n-1})$, such that $x(\mathbb{R}) = S^1 \times \{0\}$, $\alpha \simeq H\mathrm{d}t + \lambda_0$, where $H : U \to \mathbb{R}$ satisfies $H_t(0) \equiv T$, $\mathrm{d}H_t(0) \equiv 0$, and $\lambda_0 = \frac{1}{2}\sum_1^{n-1} q_i \mathrm{d}p_i - p_i \mathrm{d}q_i$.*

Such a specific local model allows us to give an explicit construction for *the associated Poincaré return map* of the Reeb flow near $x(0)$, with respect to a local cross section given by the slice $\{0\} \times B^{2n-2}$: Firstly we look at the Reeb vector field on our local model. Note that $\alpha = H\mathrm{d}t + \lambda_0$ and that $\mathrm{d}\alpha = \mathrm{d}H \wedge \mathrm{d}t + \omega_0$, where $\omega_0 = \mathrm{d}\lambda_0$ is the standard symplectic form on B^{2n-2}. Then one can verify that the Reeb vector field of α is given by

$$R = \frac{1}{F}\left(\frac{\partial}{\partial t} - X_H\right),$$

where $\iota_{X_H}\omega_0 = -\mathrm{d}H$ and $F = H - \lambda_0(X_H)$ (which is nonzero at least in a small neighborhood of x). Then we see that R has the same integral curve as the vector field $\frac{\partial}{\partial t} - X_H$ on $U = S^1 \times B^{2n-2}$. This implies that

$$(t, \varphi_{-H}^t(z)) = \phi_R^{\tau(t,z)}(0, z), \quad \forall t \in (0, 1],\ z \in B^{2n-2},$$

where φ_{-H}^t denotes the Hamiltonian flow of $-H$ on B^{2n-2}, and $\tau(t, z)$ is defined by

$$\tau(t, z) = \min\{s > 0 : \phi_R^s(0, z) \in \{t\} \times B^{2n-2} \subset U\}.$$

We conclude that the time-1 map $\varphi = \varphi_{-H}^1$ is exactly the associated Poincaré return map of $x(0)$, following the Reeb flow from the slice $\{0\} \times B^{2n-2}$ back to itself.

It is shown in [F20] that the local symplectic homology of an isolated closed Reeb orbit can be computed by the local Floer homology of the associated return map, provided the closed Reeb orbit is prime. Note that $0 \in B^{2n-2}$ is a constant periodic orbit of the Hamiltonian flow of $-H$ since $\mathrm{d}H_t(0) \equiv 0$. For each isolated and prime closed Reeb orbit x,

$$\mathrm{SH}_*(x, \mathbb{F}) \cong \mathrm{HF}_*(\varphi, 0; \mathbb{F}) \oplus \mathrm{HF}_{*-1}(\varphi, 0; \mathbb{F}), \tag{2.4}$$

see [F20, Theorem 1.1].

Remark 2.8. The tubular neighborhood $U = S^1 \times B^{2n-2}$ in Lemma 2.7 corresponds to a symplectic trivialization of the contact structure $\xi = \ker \alpha$. (Note that ξ can be identified with the normal bundle of x in Y.) Under this trivialization, the Reeb flow along x can be linearized and restricted to ξ, yielding a symplectic path used to define the Maslov-type indices of x; see Subsection 3.1.4.

On the other hand, consider the Hamiltonian flow φ_{-H}^t on the extended phase space $S^1 \times B^{2n-2}$, viewing φ_{-H}^t as a map $\{0\} \times B^{2n-2} \to \{t\} \times B^{2n-2}$. Since 0 is a constant orbit of φ_{-H} in B^{2n-2}, we can linearize φ_{-H}^t along $(t, 0)$ and restrict it to B^{2n-2}, obtaining a symplectic path that defines the Maslov-type indices of 0. As the Reeb vector field R shares integral curves with the vector field $\frac{\partial}{\partial t} - X_H$, these two symplectic paths—associated with x and 0 respectively—coincide up to reparameterization and consequently have identical Maslov-type indices.

For the k-th iteration x^k, consider the k-fold covering

$$\pi : U_k = S^1 \times B^{2n-2} \to U, \quad (e^{\sqrt{-1}\theta}, z) \mapsto (e^{\sqrt{-1}k\theta}, z),$$

and the pull-back of the contact form from U to U_k. This pullback and the Reeb vector field are invariant under the action of $\mathbb{Z}_k$ on U_k given by deck transformations with generator $(e^{\sqrt{-1}\theta}, z) \mapsto (e^{\sqrt{-1}(\theta+\frac{2\pi}{k})}, z)$. Provided x^k is isolated, then x^k can be lifted to an isolated closed Reeb orbit on U_k, with up to k distinct choices of initial values (which correspond to the same closed Reeb orbit under different initial conditions), denoted by kx. The associated return map of kx is φ^k. More precisely, assume $\mathcal{A}(x) = T$ and the minimal period of x is T/l for some $l \in \mathbb{N}$, we have

$$x^k : [0, kT] \to U = S^1 \times B^{2n-2}, \quad t \mapsto (e^{\frac{2\pi\sqrt{-1}lt}{T}}, 0).$$

And the lift kx is given by

$$kx : [0, kT] \to U_k = S^1 \times B^{2n-2}, \quad t \mapsto (e^{\frac{2\pi\sqrt{-1}(lt+mT)}{kT}}, 0), \tag{2.5}$$

for some $m \in \{0, \dots, k-1\}$.

Lemma 2.9 ([ÇGG24, Lemma 2.9])**.** *There is a natural $\mathbb{Z}_k$-action on* $\mathrm{SH}(kx; \mathbb{F})$*, and*

$$\mathrm{SH}_*(x^k; \mathbb{Q}) = \mathrm{SH}_*(kx; \mathbb{Q})^{\mathbb{Z}_k}, \tag{2.6}$$

where the superscript refers to the $\mathbb{Z}_k$-invariant part of the homology.

According to (2.5), kx is prime whenever x is prime. Then combining (2.4) and (2.6) we get

$$\dim \mathrm{SH}_*(x^k; \mathbb{Q}) \le \dim \mathrm{HF}_*(\varphi^k, 0; \mathbb{Q}) + \dim \mathrm{HF}_{*-1}(\varphi^k, 0; \mathbb{Q}). \tag{2.7}$$

2.3. **Generating functions.** In this subsection we recall the definition of the generating function on $\mathbb{R}^{2n}$, which will be used to construct the reference function for applying Lemma 2.2 in our proof of Theorem 1.8. We closely follow the description in [G10, Section 6.1] and the references therein.

Consider the symplectic vector space $\mathbb{R}^{2n} \times \bar{\mathbb{R}}^{2n} := (\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \omega_{\mathrm{std}} \oplus -\omega_{\mathrm{std}})$, and identify $\mathbb{R}^{2n}$ with the Lagrangian diagonal $\triangle \subset \mathbb{R}^{2n} \times \bar{\mathbb{R}}^{2n}$. Fix a Lagrangian complement N to $\triangle$. Then $\mathbb{R}^{2n} \times \bar{\mathbb{R}}^{2n}$ can be treated as $T^*\triangle$ via a symplectomorphisms sending $\triangle$ to the zero section and identifying N with each cotangent fiber by a canonical isomorphism $\eta \mapsto (\omega_{\mathrm{std}} \oplus -\omega_{\mathrm{std}})(\eta, \cdot)$. (This identification is a special case of Weinstein's Lagrangian Neighborhood Theorem; see [W71; MS17] for details.)

Let φ be a Hamiltonian diffeomorphism defined on a neighborhood of the origin p in $\mathbb{R}^{2n}$ such that $\|\varphi - \mathrm{id}\|_{C^1}$ is sufficiently small. Then the graph Γ of φ is C^1-close to $\triangle$. Consequently, Γ is transverse to every fiber and can be viewed as the graph in $T^*\triangle$ of an exact 1-form $\mathrm{d}F$ near $p \in \triangle = \mathbb{R}^{2n}$. The function F, normalized by $F(p) = 0$, is called the *generating function* of φ and possesses the following properties:

- p is an isolated critical point of F if and only if p is an isolated fixed point of φ,
- $\|F\|_{C^2} = O(\|\varphi - \mathrm{id}\|_{C^1})$ and $\|\mathrm{d}^2F_p\| = \|\mathrm{d}\varphi_p - \mathrm{Id}\|$.

In general the function F depends on the choice of the Lagrangian complement N to $\triangle$. In this paper we take as N the specific linear subspace

$$N_0 := \{(x, 0, 0, y)\} \subset \mathbb{R}^{2n} \times \bar{\mathbb{R}}^{2n},$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are the standard canonical coordinates on $\mathbb{R}^{2n}$, i.e., $\omega_{\mathrm{std}} = \sum \mathrm{d}y_i \wedge \mathrm{d}x_i$. Then the symplectomorphisms between $\mathbb{R}^{2n} \times \bar{\mathbb{R}}^{2n}$ and $T^*\triangle$ can be explicitly expressed as $\Psi_{N_0} : \mathbb{R}^{2n} \times \bar{\mathbb{R}}^{2n} \to T^*\triangle$,

$$\Psi_{N_0}(x, y, x', y') := (x', y, y - y', x' - x) = (q, q', p, p'),$$

where (q, q', p, p') are the canonical coordinates in $T^*\triangle = T^*\mathbb{R}^{2n}$ such that the symplectic form $\omega_{\mathrm{can}} = \sum(\mathrm{d}q_i \wedge \mathrm{d}p_i + \mathrm{d}q'_i \wedge \mathrm{d}p'_i)$. Then F is determined by the equation

$$\Psi_{N_0}((x, y), \varphi(x, y)) = ((q, q'), \mathrm{d}F(q, q')), \tag{2.8}$$

where the left-hand side represents the graph of φ mapped by the symplectomorphism Ψ_{N_0}, and the right-hand side is the graph of a 1-form in $T^*\mathbb{R}^{2n}$.

Set $z = (x, y)$, then (2.8) can be rewritten as

$$\varphi(z) - z = X_F(\psi(z)),$$

where X_F is the Hamiltonian vector field of F and $\psi(z) := (x\text{-component of } \varphi(z), y)$.

3. Maslov-type indices: definition and iteration properties

3.1. **Maslov-type indices and dynamically convexity.** Let $\mathrm{Sp}(2m)$ denote the group of symplectic linear transformations on $\mathbb{R}^{2m}$ (equipped with the standard symplectic form). In this section we recall the definition and some basic properties of the various Maslov-type indices for sympletcic paths, i.e. paths in $\mathrm{Sp}(2m)$. The index theory was introduced by Conley-Zehnder in [CZ84] for the non-degenerate case with $m \geq 2$, Long-Zehnder in [LZ90] for the non-degenerate case with $m = 1$, and Long in [L90] and Viterbo in [V90] independently for the degenerate case. For a more thorough treatment, we refer the reader to [L02], [SZ92, Section 3] or [GG20, Section 4].

3.1.1. *The mean index.* Let $\widetilde{\mathrm{Sp}}(2m)$ denote the universal covering of $\mathrm{Sp}(2m)$. An element $\Phi \in \widetilde{\mathrm{Sp}}(2m)$ can be viewed as a path $\Phi(t) : [0, 1] \to \mathrm{Sp}(2m)$ with $\Phi(0) = \mathrm{Id}$, up to homotopy with fixed end-points.

Remark 3.1. For Φ and $\Psi \in \widetilde{\mathrm{Sp}}(2m)$, consider the matrix product path $\Psi(t)\Phi(t) : [0, 1] \to \mathrm{Sp}(2m)$. One can construct a homotopy with fixed end-points, such that

$$\Psi(t)\Phi(t) \sim \Phi(t) * \Psi(t)\Phi(1) := \begin{cases} \Phi(2t), & 0 \leq t \leq \dfrac{1}{2}, \\ \Psi(2t-1)\Phi(1), & \dfrac{1}{2} \leq t \leq 1. \end{cases} \tag{3.1}$$

More precisely, the homotopy is given by $\Gamma : [0, 1] \times [0, 1] \to \mathrm{Sp}(2m)$,

$$\Gamma(s, t) := \begin{cases} \Phi\left(\dfrac{2t}{2-s}\right), & 0 \leq t \leq \dfrac{s}{2}, \\ \Psi\left(\dfrac{2t-s}{2-s}\right)\Phi\left(\dfrac{2t}{2-s}\right), & \dfrac{s}{2} \leq t \leq \dfrac{2-s}{2}, \\ \Psi\left(\dfrac{2t-s}{2-s}\right)\Phi(1), & \dfrac{2-s}{2} \leq t \leq 1. \end{cases}$$

Therefore, the multiplication in $\widetilde{\mathrm{Sp}}(2m)$ can be viewed as the concatenation of paths. From this opinion, for each $k \in \mathbb{N}$, $\Phi^k \in \widetilde{\mathrm{Sp}}(2m)$ is identified with the k-th iteration of symplectic path $\Phi(t)$ defined in, for example, [LZ02, Page 322].

For each $\Phi \in \widetilde{\mathrm{Sp}}(2m)$, one can define its *mean index* $\hat{\mu}(\Phi)$, which measures the total rotation angle of certain unit eigenvalues of $\Phi(t)$, as follows. Following [SZ92], one can construct a continuous function $\rho : \mathrm{Sp}(2m) \to S^1$, uniquely characterized by the following properties:

- (Naturality) If A and $B \in \mathrm{Sp}(2m)$, then $\rho(BAB^{-1}) = \rho(A)$.
- (Product) If $A \in \mathrm{Sp}(2m_1)$ and $B \in \mathrm{Sp}(2m_2)$ with $m_1 + m_2 = m$, then

$$\rho(A \diamond B) = \rho(A)\rho(B).$$

- (Determinant) If $A \in \mathrm{U}(m) = \mathrm{Sp}(2m) \cap \mathrm{O}(2m)$, then

$$\rho(A) = \det_{\mathbb{C}}(X + \sqrt{-1}Y), \text{ where } A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}, X, Y \in \mathbb{R}^{m\times m}.$$

In particular, ρ induces an isomorphism $\rho_* : \pi_1(\mathrm{Sp}(2m)) \to \pi_1(S^1) \simeq \mathbb{Z}$.

- (Normalization) If $\sigma(A)\cap S^1\subset\mathbb{R}$, then
$$\rho(A)=(-1)^{m_0/2},$$
where m_0 is the total multiplicity of the negative real eigenvalues.

Consider the path $\rho(\Phi(t)):[0,1]\to S^1$ and lift it to a path $\theta:[0,1]\to\mathbb{R}$ such that $\rho(\Phi(t))=\exp(\sqrt{-1}\theta(t))$. We set
$$\hat{\mu}(\Phi):=\frac{\theta(1)-\theta(0)}{\pi}.$$
The properties of ρ imply the following properties of mean index:

(MI1) (Additivity) $\hat{\mu}(\Phi\diamond\Psi)=\hat{\mu}(\Phi)+\hat{\mu}(\Psi)$, where $\Phi\in\widetilde{\mathrm{Sp}}(2m_1)$ and $\Psi\in\widetilde{\mathrm{Sp}}(2m_2)$.

(MI2) (Maslov index) $\hat{\mu}$ restricts to an isomorphism $\pi_1(\mathrm{Sp}(2m))\to 2\mathbb{Z}$. Indeed,
$$\hat{\mu}(\Phi)=2\mu_{\mathrm{Maslov}}(\Phi)$$
provided Φ is a loop. Moreover, for each $\Psi\in\widetilde{\mathrm{Sp}}(2m)$, if ψ is a loop in $\mathrm{Sp}(2m)$ with $\psi(0)=\Psi(1)$, then we have
$$\hat{\mu}(\Psi*\psi)=\hat{\mu}(\Psi)+2\mu_{\mathrm{Maslov}}(\psi),$$
where $*$ denotes the concatenation of paths.

(MI3) (Integer case) $\hat{\mu}(\Phi)\in\mathbb{Z}$ provided $\sigma(\Phi(1))\cap S^1\subset\mathbb{R}$. In particular, $\hat{\mu}(\Phi)\in 2\mathbb{Z}$ provided $\sigma(\Phi(1))=\{1\}$.

(MI4) (Homogeneity) $\hat{\mu}(\Phi^k)=k\hat{\mu}(\Phi)$, for each $k\in\mathbb{N}$. (cf. [SZ92, Lemma 3.4].)

Furthermore, by the explicit construction of ρ in [SZ92, Page 1316] (see also [L02, Page 60]), $\rho(A)$ is totally determined by $\sigma(A)$ and the Krein type of eigenvalues in $\sigma(A)\cap S^1$, which, as mentioned in [GG20, Section 4.1.1], implies that:

(MI5) $\hat{\mu}(\Phi_s)\equiv$ *const* for a family of elements Φ_s in $\widetilde{\mathrm{Sp}}(2m)$ as long as $\sigma(\Phi_s(1))\equiv$ *const* for all s.

(MI6) $\hat{\mu}(\Phi)=0$, provided $\sigma(\Phi(t))\equiv\{1\}$ for $t\in[0,1]$.

Remark 3.2. In fact, $\hat{\mu}:\widetilde{\mathrm{Sp}}(2m)\to\mathbb{R}$ is a quasimorphism between Lie groups, that is,
$$|\hat{\mu}(\Phi\Psi)-\hat{\mu}(\Phi)-\hat{\mu}(\Psi)|<\mathit{const},$$
where the constant is independent of Φ and Ψ. And one can prove that $\hat{\mu}$ is the unique quasimorphism $\widetilde{\mathrm{Sp}}(2m)\to\mathbb{R}$ which is continuous, homogeneous (see (MI4)), and normalized by
$$\hat{\mu}(\Phi_0)=2\ \text{ for }\Phi_0(t)=\exp(2\pi\sqrt{-1}t)\diamond\mathrm{Id}_{2m-2};$$
see [BG92]. Although $\hat{\mu}$ fails to be a homomorphism in general, according to (3.1), we still have
$$\hat{\mu}(\Psi\Phi)=\hat{\mu}(\Psi)+\hat{\mu}(\Phi),\quad\text{provided }\Psi\in\widetilde{\mathrm{Sp}}(2m)\text{ is a loop.}\tag{3.2}$$

3.1.2. *Maslov-type indices.* Recall that $\Phi\in\widetilde{\mathrm{Sp}}(2m)$ is called *non-degenerate* if $1\notin\sigma(\Phi(1))$; *totally degenerate* if $\sigma(\Phi(1))=\{1\}$; *strongly non-degenerate* if Φ^k is non-degenerate for each $k\in\mathbb{N}$, i.e., none of the eigenvalues of $\Phi(1)$ is a root of unity. Let $\mathrm{Sp}(2m)^*$ denote the set of $A\in\mathrm{Sp}(2m)$ such that $1\notin\sigma(A)$, and $\widetilde{\mathrm{Sp}}(2m)^*$ denote the set of $\Phi\in\widetilde{\mathrm{Sp}}(2m)$ such that Φ is non-degenerate. For each $\Phi\in\widetilde{\mathrm{Sp}}(2m)$, one can connect $\Phi(1)$ to some symplectic transformations M with $\rho(M)\in\{\pm1\}$ by a path lying entirely in $\mathrm{Sp(2m)}^*$. Concatenating this path with Φ to get a new path $\Phi'\in\widetilde{\mathrm{Sp}}(2m)$ such that $\hat{\mu}(\Phi')\in\mathbb{Z}$. Then the *Conley-Zehnder index* of Φ is defined by $\mu(\Phi):=\hat{\mu}(\Phi')$. One can show that $\mu(\Phi)$ is well-defined and $\mu:\widetilde{\mathrm{Sp}}(2m)^*\to\mathbb{Z}$ is constant on connected components of $\widetilde{\mathrm{Sp}}(2m)^*$, namely, it is locally constant. (cf. [SZ92, Theorem 3.3]) Also we have
$$(-1)^{\mu(\Phi)-m}=\operatorname{sign}\det(\mathrm{Id}-\Phi(1)),\tag{3.3}$$

which implies that the parity of $\mu(\Phi)$ is determined by the number of eigenvalues of $\Phi(1)$ in $(-1,0)$, counted with algebraic multiplicity. And we list here a specific example that for the path $\Phi(t)=\exp(2\pi\sqrt{-1}\lambda t)$, $t\in[0,1]$, in $\mathrm{Sp}(2)$ we have

$$\mu(\Phi)=\mathrm{sign}(\lambda)(2\lfloor|\lambda|\rfloor+1)\quad\text{when } \lambda\notin\mathbb{Z}.$$

In general, for a not necessarily non-degenerate $\Phi\in\widetilde{\mathrm{Sp}}(2m)$, one can define its *upper and lower Conley-Zehnder indices* as

$$\mu_+(\Phi):=\limsup_{\Psi\to\Phi}\mu(\Psi)\quad\text{and}\quad\mu_-:=\liminf_{\Psi\to\Phi}\mu(\Psi),$$

where in both cases the limit is taken over $\Psi\in\widetilde{\mathrm{Sp}}(2m)^*$ converging to Φ, see [L97, Pages 63-64]. The indices $\mu_\pm$ are the upper semi-continuous and lower semi-continuous extensions of μ from $\widetilde{\mathrm{Sp}}(2m)^*$ to $\widetilde{\mathrm{Sp}}(2m)$ respectively. Clearly, $\mu_\pm(\Phi)=\mu(\Phi)$ provided Φ is non-degenerate. Moreover, we have

$$\mu_+(\Phi)-\mu_-(\Phi)=\nu(\Phi):=\dim\ker(\Phi(1)-\mathrm{Id}),\ \forall\Phi\in\widetilde{\mathrm{Sp}}(2m),\tag{3.4}$$

see [L02, Page 142].

The indices $\mu_\pm$ and μ possess the same additivity as $\hat\mu$ in (MI1), see [L97, Theorem 1.4] or [GG20, Lemma 4.3]. For each $\Phi\in\widetilde{\mathrm{Sp}}(2m)$, the mean index $\hat\mu$ and the indices $\mu_\pm$ are related by the inequalities

$$\hat\mu(\Phi)-m\le\mu_-(\Phi)\le\mu_+(\Phi)\le\hat\mu(\Phi)+m.\tag{3.5}$$

(cf. [L02, Page 213].) Consequently,

$$\lim_{k\to\infty}\frac{\mu_\pm(\Phi^k)}{k}=\hat\mu(\Phi).\tag{3.6}$$

Furthermore, suppose ψ is a loop in $\mathrm{Sp}(2m)$ with $\psi(0)=\Phi(1)$, then one can prove that

$$\mu_\pm(\Phi*\psi)-\mu_\pm(\Phi)=\hat\mu(\Phi*\psi)-\hat\mu(\Phi)=2\mu_{\mathrm{Maslov}}(\psi),\tag{3.7}$$

where the second equality is from (MI2). (Indeed, the first equality can be derived from Corollary 3.10 of Section 3.2.)

3.1.3. *Dynamical convexity.* As mentioned before, the notion of dynamical convexity, originally introduced in [HWZ98] for the Reeb flow on the standard contact sphere S^3, is a condition arguing some lower bound of the indices μ_- for closed Reeb orbits. In this section, essentially focus on the symplectic path aspect, we adopt the following definition.

Definition 3.3. A path $\Phi\in\widetilde{\mathrm{Sp}}(2m)$ is said to be *dynamically convex* if $\mu_-(\Phi)\ge m+2$.

The following lemma, as a consequence of the abstract precise iteration formula (3.9) we will introduce in Section 3.2, asserts that the indices $\mu_-(\Phi^k)$ are increasing provided Φ is dynamically convex.

Lemma 3.4 (cf. [LZ02, Corollary 3.1] or [GG20, Lemma 4.8])**.** *For each $\Phi\in\widetilde{\mathrm{Sp}}(2m)$, we have*

$$\mu_-(\Phi^{k+1})\ge\mu_-(\Phi^k)+(\mu_-(\Phi)-m)$$

for all $k\in\mathbb{N}$. In particular,

$$\mu_-(\Phi^k)\ge(\mu_-(\Phi)-m)k+m.$$

Assume furthermore, that Φ is dynamically convex. Then the function $\mu_-(\Phi^k)$ of $k\in\mathbb{N}$ is strictly increasing,

$$\mu_-(\Phi^k)\ge 2k+m,$$

and $\hat\mu(\Phi)\ge\mu_-(\Phi)-m\ge 2$. Thus $\mu_-(\Phi^k)\ge m+2$ and all the iterations Φ^k are also dynamically convex.

3.1.4. *Maslov-type indices for closed orbits.* Now we briefly recall how to translate the definition of the Maslov-type indices introduced above for elements of $\widetilde{\mathrm{Sp}}(2m)$ to closed orbits of Hamiltonian flow or Reeb flow.

We begin with the case of Hamiltonian flow. Consider a symplectic manifold (W^{2n},ω). Following [SZ92, Section 5], let x be a contractible 1-periodic orbit of a Hamiltonian $H : S^1\times W^{2n}\to\mathbb{R}$. Choose a *capping disk* for x, i.e., a map $v : D^2\to W^{2n}$ satisfying $v(e^{2\pi it}) = x(t)$ for $t\in[0,1]$, where D^2 denotes the unit disk in $\mathbb{R}^2=\mathbb{C}$. Fix a symplectic trivialization of the pullback bundle v^*TW:

$$\psi_v : (v^*TW,\omega)\to D^2\times(\mathbb{C}^n,\omega_{\mathrm{std}}).$$

Linearizing the Hamiltonian flow along x with respect to ψ_v gives rise to a symplectic path

$$\Phi_v(t) = \psi_v(x(t))\circ \mathrm{d}\varphi_H^t(x(0))\circ\psi_v^{-1}(x(0)),\quad t\in[0,1],$$

with $\Phi_v(0)=\mathrm{Id}$, so $\Phi_v\in\widetilde{\mathrm{Sp}}(2n)$. For a fixed capping disk v, any two such trivializations yield the same symplectic path up to homotopy with fixed endpoints. We then define

$$\mu_\pm(x;v) := \mu_\pm(\Phi_v)\quad\text{and}\quad \hat\mu(x;v) := \hat\mu(\Phi_v).$$

If x is non-degenerate, set $\mu(x;v):=\mu(\Phi_v)$. Moreover, assuming $c_1(TW)=0\in H^2(W;\mathbb{Z})$, any two capping disks for x produce the same homotopy class $\Phi\in\widetilde{\mathrm{Sp}}(2n)$ (see [SZ92, Lemma 5.2]). Thus, in this case, we define

$$\mu_\pm(x) := \mu_\pm(\Phi),\quad \hat\mu(x) := \hat\mu(\Phi),$$

and $\mu(x):=\mu(\Phi)$ for non-degenerate x. We call Φ the *associated symplectic path* of x. We sometimes use notations $\mu_\pm(x,H)$, $\hat\mu(x,H)$, $\mu(x,H)$ or $\mu_\pm(x,\varphi_H)$, $\hat\mu(x,\varphi_H)$, $\mu(x,\varphi_H)$ to emphasize the Hamiltonian H, depending on the context.

Now consider the case of Reeb flow. Let x be a closed Reeb orbit on a contact manifold (Σ^{2n-1},α) with contact structure $\xi=\ker\alpha$. Assume x is contractible with action $\mathcal{A}(x)=T$. Rescale α to $\frac{1}{T}\alpha$ (still denoted by α) so that $\mathcal{A}(x)=1$. The definitions for Hamiltonian flow then extend analogously to the Reeb flow. Specifically, take a capping disk $w : D^2\to\Sigma^{2n-1}$ satisfying $w(e^{2\pi it})=x(t)$. Fix a symplectic trivialization ψ_w of $w^*\xi$, linearize the Reeb flow ϕ_α^t along x and restrict to ξ, obtaining a path $\Phi_w\in\widetilde{\mathrm{Sp}}(2n-2)$:

$$\Phi_w(t) = \psi_w(x(t))\circ \mathrm{d}\phi_\alpha^t(x(0))|_\xi\circ\psi_w^{-1}(x(0)),\quad t\in[0,1].$$

Define $\mu_\pm(x;w):=\mu_\pm(\Phi_w)$ and $\hat\mu(x;w):=\hat\mu(\Phi_w)$. When x is non-degenerate, set $\mu(x;w):=\mu(\Phi_w)$. If $c_1(\xi)=0$, any two capping disks yield identical $\Phi\in\widetilde{\mathrm{Sp}}(2n-2)$, so we define

$$\mu_\pm(x) := \mu_\pm(\Phi),\quad \hat\mu(x) := \hat\mu(\Phi),$$

and $\mu(x):=\mu(\Phi)$ for non-degenerate x. As in the case of Hamiltonian flow, Φ is called the *associated symplectic path* of x.

For iterated orbits, define $w^k : D^2\to\Sigma^{2n-1}$ by $w^k(z)=w(z^k/|z|^{k-1})$ for $z\neq0$ and $w^k(0)=w(0)$. This provides a capping disk for x^k. Any symplectic trivialization ψ_w of $w^*\xi$ extends to $(w^k)^*\xi$ analogously, satisfying $\Phi_{w^k}=\Phi_w^k$. Consequently, if $c_1(\xi)=0$,

$$\mu_\pm(x^k) = \mu_\pm(\Phi^k),\quad \hat\mu(x^k) = \hat\mu(\Phi^k),$$

and $\mu(x^k)=\mu(\Phi^k)$ for non-degenerate x.

These definitions apply directly to our setting where Σ^{2n-1} bounds a star-shaped domain in $\mathbb{R}^{2n\geq4}$, as this implies that Σ is simply connected and $H^2(\Sigma;\mathbb{Z})=0$.

Remark 3.5. Although the above definition suffices for our purposes, we note that [M16] introduced an alternative definition of Maslov-type indices for closed Reeb orbits x that eliminates the contractibility assumption while coinciding with the standard definition in the contractible case. This approach has been adopted in recent works such as [AM22; ÇGG24].

The construction proceeds as follows. Assuming $c_1(\xi)=0$, the top complex wedge power $L_\xi := \bigwedge_{\mathbb{C}}^{n-1}\xi$ forms a trivial complex line bundle. Choose either a trivialization of this bundle or equivalently, a nonvanishing section $\mathfrak{s}$ of L_ξ. Let $\psi : x^*\xi \to S^1 \times \mathbb{C}^{n-1}$ be a Hermitian trivialization whose induced trivialization on $\bigwedge_{\mathbb{C}}^{n-1} x^*\xi$ matches $\mathfrak{s}$. This condition uniquely determines the homotopy class of ψ. Furthermore, the induced trivialization on x^k coincides with the k-th iteration of ψ up to homotopy. Using this trivialization, we convert the linearized Reeb flow along x into a path $\Phi \in \widetilde{\mathrm{Sp}}(2n-2)$. When $H_1(\Sigma;\mathbb{Q}) = 0$, the homotopy class $\Phi \in \widetilde{\mathrm{Sp}}(2n-2)$ is independent of the section $\mathfrak{s}$, resulting in well-defined Maslov-type indices for x. Complete details can be found in [M16, Section 4].

3.2. **Splitting number and abstract precise iteration formula.** In [L99], to develop the Bott-type formula of the Maslov-type indices, Long generalized the definition of $\mu_\pm$ to each $\omega \in \mathbb{U}$ (where $\mu_\pm$ corresponds to the case $\omega = 1$) and introduced the splitting number theory for the symplectic matrix, which induces a precise description of $\mu_-(\Phi^k)$ for arbitrary $k \in \mathbb{N}$.

3.2.1. *Splitting number.* Consider the following real function introduced in [L99]:

$$D_\omega(M) = (-1)^{n-1}\bar{\omega}^n \det(M - \omega \mathrm{Id}), \ \forall \omega \in \mathbb{U}, \ M \in \mathrm{Sp}(2m).$$

Then for each $\omega \in \mathbb{U}$ the following codimension 1 hypersurface in $\mathrm{Sp}(2m)$ is defined:

$$\mathrm{Sp}(2m)^0_\omega = \{M \in \mathrm{Sp}(2m) \mid D_\omega(M) = 0\}.$$

Let

$$\mathrm{Sp}(2m)^*_\omega := \mathrm{Sp}(2m) \setminus \mathrm{Sp}(2m)^0_\omega,$$
$$\widetilde{\mathrm{Sp}}(2m)^*_\omega := \{\Phi \in \widetilde{\mathrm{Sp}}(2m) : \Phi(1) \in \mathrm{Sp}(2m)^*_\omega\}.$$

Following [L99], the ω-*index* i_ω is defined as follows. Firstly, for $\Phi \in \widetilde{\mathrm{Sp}}(2m)^*_\omega$, $i_\omega(\Phi)$ is defined analogously as the Conley-Zehnder index introduced in the previous subsection via replacing $\mathrm{Sp}(2m)^*$ and $\widetilde{\mathrm{Sp}}(2m)^*$ by $\mathrm{Sp}(2m)^*_\omega$ and $\widetilde{\mathrm{Sp}}(2m)^*_\omega$ respectively. Then, for an arbitrary $\Phi \in \widetilde{\mathrm{Sp}}(2m)$, define

$$i_\omega(\Phi) := \liminf_{\Psi \to \Phi} i_\omega(\Psi),$$

where the limit is taken over $\Psi \in \widetilde{\mathrm{Sp}}(2m)^*_\omega$ converging to Φ. Like (3.4), there also holds

$$i_\omega(\Phi) + \nu_\omega(\Phi) = \limsup_{\Psi \to \Phi} i_\omega(\Psi),$$

where $\nu_\omega(\Phi) := \dim_{\mathbb{C}} \ker_{\mathbb{C}}(\Phi(1) - \omega \mathrm{Id})$, and is called the ω-*nullity* of Φ.

The splitting number is defined by the "jump" of ω-indices.

Definition 3.6. For each $M \in \mathrm{Sp}(2m)$ and $\omega \in \mathbb{U}$, *the splitting numbers* $S^\pm_M(\omega)$ of M at ω are defined by

$$S^\pm_M(\omega) = \lim_{\epsilon \to 0^+} i_{\omega \exp(\pm\sqrt{-1}\epsilon)}(\Phi) - i_\omega(\Phi), \tag{3.8}$$

for any $\Phi \in \widetilde{\mathrm{Sp}}(2m)$ satisfying $\Phi(1) = M$.

Remark 3.7. The ω-nullity and the splitting number can be defined for a periodic orbit of Hamiltonian flow or Reeb flow by its (restricted) linearized Poincaré return map. For instance, let x be a closed orbit of a Hamiltonian (or Reeb) flow and $\Phi \in \widetilde{\mathrm{Sp}}(2m)$ be its associated symplectic path defined in Subsection 3.1.4, then we define

$$\nu_\omega(x) := \nu_\omega(\Phi(1)), \qquad S^\pm_x(\omega) := S^\pm_{\Phi(1)}(\omega).$$

And we always omit the subscript ω of $\nu_\omega(x)$ when $\omega = 1$.

Lemma 3.8 ([L99])**.** *For each $M \in \mathrm{Sp}(2m)$ and $\omega \in \mathbb{U}$, the splitting numbers $S^\pm_M(\omega)$ are well defined, i.e., they are independent of the choice of the path $\Phi \in \widetilde{\mathrm{Sp}}(2m)$ with $\Phi(1) = M$ appeared in* (3.8)*. And the following properties hold:*

(S1) $S^+_M(\omega) = S^-_M(\bar{\omega})$;

(S2) $S^{\pm}_{M_1\diamond M_2}(\omega) = S^{\pm}_{M_1}(\omega) + S^{\pm}_{M_2}(\omega)$;
(S3) $S^{\pm}_M(\omega) = 0$ *whenever* $\omega \notin \sigma(M)$;
(S4) $0 \le S^{\pm}_M(\omega) \le \nu_\omega(M)$, *where* $\nu_\omega(M) := \dim_{\mathbb{C}} \ker_{\mathbb{C}}(M - \omega \mathrm{Id})$.

3.2.2. *Abstract precise iteration formula.* For each $\Phi \in \widetilde{\mathrm{Sp}}(2m)$, the iteration behavior of indices $\mu_-(\Phi^k)$, $k \in \mathbb{N}$, is completely determined by $\mu_-(\Phi)$ and the splitting numbers of the end point $\Phi(1)$. Indeed, we have the following abstract precise iteration formula proved in [LZ02]:

Proposition 3.9 ([LZ02, Theorem 2.1])**.** *For each $\Phi \in \widetilde{\mathrm{Sp}}(2m)$ with $M = \Phi(1)$, and $k \in \mathbb{N}$, we have*

$$\begin{aligned}\mu_-(\Phi^k) &= k(\mu_-(\Phi) + S^+_M(1) - C(M))\\ &\quad + 2\sum_{\theta\in(0,2\pi)} \lceil \frac{k\theta}{2\pi} \rceil S^-_M(e^{\sqrt{-1}\theta}) - (S^+_M(1) + C(M)),\end{aligned} \tag{3.9}$$

where $C(M) := \sum_{\theta\in(0,2\pi)} S^-_M(e^{\sqrt{-1}\theta})$.

As a consequence of Proposition 3.9 and (3.6), we have

Corollary 3.10 ([LZ02, Corollary 2.1])**.** *For each $\Phi \in \widetilde{\mathrm{Sp}}(2m)$ with $M = \Phi(1)$, we have*

$$\hat{\mu}(\Phi) = \mu_-(\Phi) + S^+_M(1) - C(M) + \sum_{\theta\in(0,2\pi)} \frac{\theta}{\pi} S^-_M(e^{\sqrt{-1}\theta}). \tag{3.10}$$

3.2.3. *Basic normal forms.* For an arbitrary $M \in \mathrm{Sp}(2m)$, the splitting numbers is subtle to compute. To overcome this difficulty, Long introduced in [L99] the *homotopy component* $\Omega^0(M)$, which is defined as the path connected component containing M of the set

$$\Omega(M) = \{N \in \mathrm{Sp}(2m) : \sigma(N) \cap \mathbb{U} = \sigma(M) \cap \mathbb{U} \text{ and } \nu_\omega(N) = \nu_\omega(M),\ \forall \omega \in \sigma(M) \cap \mathbb{U}\},$$

and the following *basic normal forms* :

$$\begin{aligned}
D(\lambda) &:= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, && \lambda = \pm 2,\\
N_1(\lambda, b) &:= \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, && \lambda = \pm 1, a = \pm 1, 0,\\
R(\theta) &:= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, && \theta \in (0,\pi) \cup (\pi, 2\pi),\\
N_2(\omega, b) &:= \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, && \omega = e^{\sqrt{-1}\theta},\ \theta \in (0,\pi) \cup (\pi, 2\pi),
\end{aligned} \tag{3.11}$$

where $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbb{R}$ and $b_2 \neq b_3$. In addition, $N_2(\omega, b)$ is *non-trivial* if $(b_2 - b_3)\sin\theta < 0$, and *trivial* if $(b_2 - b_3)\sin\theta > 0$.

With these preparations, the computation for the splitting numbers can be reduced to the cases of basic normal forms. Indeed, we have the following lemma, the proof of the assertions can be found in [L99].

Lemma 3.11. *Let $\omega \in \mathbb{U}$ and $M \in \mathrm{Sp}(2m)$. The following statements hold:*

(i) there exists $N^* \in \Omega^0(M)$ such that N^* is a symplectic direct sum (see Subsection 1.3.2) of some basic normal forms.
(ii) the splitting numbers $S^{\pm}_N(\omega)$ are constant for all $N \in \Omega^0(M)$;

(iii) the splitting numbers of the basic normal forms is given by :

$$(S^+_{N_1(1,a)}(1), S^-_{N_1(1,a)}(1)) = \begin{cases}(1,1), & \text{if } a \geq 0,\\ (0,0), & \text{if } a < 0.\end{cases}$$
$$(S^+_{N_1(-1,a)}(-1), S^-_{N_1(-1,a)}(-1)) = \begin{cases}(1,1), & \text{if } a \leq 0,\\ (0,0), & \text{if } a > 0.\end{cases}$$
$$(S^+_{R(\theta)}(e^{\sqrt{-1}\theta}), S^-_{R(\theta)}(e^{\sqrt{-1}\theta})) = (0,1), \ \theta \in (0,\pi) \cup (\pi, 2\pi).$$
$$(S^+_{N_2(\omega,b)}(\omega), S^-_{N_2(\omega,b)}(\omega)) = \begin{cases}(1,1), & \text{if } N_2(\omega,b) \text{ is non-trivial},\\ (0,0), & \text{if } N_2(\omega,b) \text{ is trivial}.\end{cases}$$

3.3. **Common index jump theorem.** We state the following Common Index Jump Theorem, originally from [LZ02, Theorem 4.3], with an enhanced version in [DLW16, Theorem 3.5]. To maintain notational consistency throughout this article, our formulation here follows [GG20, Theorem 5.2].

Theorem 3.12. *Let $\Phi_1, \dots, \Phi_r$ be a finite collection of elements in $\widetilde{\mathrm{Sp}}(2m)$ satisfying $\hat{\mu}(\Phi_i) > 0$ for $1 \leq i \leq r$. Then for any $\eta > 0$, there exists an integer sequence $d_j \to \infty$ and r integer sequences $k_{ij} \to \infty$ such that for $1 \leq i \leq r$, $j \in \mathbb{N}$, we have*

$$|\hat{\mu}(\Phi_i^{k_{ij}}) - d_j| < \eta, \tag{3.12}$$
$$\mu_\pm(\Phi_i^{k_{ij}+1}) = d_j + \mu_\pm(\Phi_i), \tag{3.13}$$
$$\mu_+(\Phi_i^{k_{ij}-1}) = d_j - (\mu_-(\Phi_i) + 2S^+_{\Phi_i(1)}(1) - \nu(\Phi_i)), \tag{3.14}$$

Moreover, for any $N \in \mathbb{N}$, we can make all d_j and k_{ij} divisible by N.

In what follows, we will refer to the collection $\{k_{ij}, d_j\}$ for a fixed j as a *common index jump event.*

Under the assumption of dynamically convexity, we have the following proposition as a consequence of Lemma 3.4 and Theorem 3.12.

Proposition 3.13. *Suppose the paths $\Phi_1, \dots, \Phi_r$ in Theorem 3.12 are dynamically convex. Then for all $l \in \mathbb{N}$, we have*

$$\mu_-(\Phi_i^{k_{ij}+l}) \geq d_j + 2 + m, \tag{3.15}$$
$$\mu_+(\Phi_i^{k_{ij}-l}) \leq d_j - 2. \tag{3.16}$$

Proof. By the index increasing property stated in Lemma 3.4, it is sufficient to consider the case $l = 1$. Then we conclude

$$\mu_-(\Phi_i^{k_{ij}+1}) \geq d_j + 2 + m,$$

from (3.13) and the dynamically convexity, and (3.15) follows.

Due to (3.14), to prove (3.16), it remains to show that

$$\mu_-(\Phi_i) + 2S^+_{\Phi_i(1)}(1) - \nu(\Phi_i) \geq 2. \tag{3.17}$$

Fix $i \in \{1 \dots, r\}$. According to (i) of Lemma 3.11, we can find $M' \in \Omega^0(\Phi_i(1))$ such that

$$M' = N(1,1)^{\diamond p_+} \diamond \mathrm{Id}_2^{\diamond p_0} \diamond N(1,-1)^{\diamond p_-} \diamond G,$$

where $p_\pm$, p_0 are nonnegative integers with $p_- + p_0 + p_+ \leq m$ and G is a symplectic matrix with $1 \notin \sigma(G)$. Then we have

$$\nu(\Phi_i) = p_+ + 2p_0 + p_- \quad \text{and} \quad S^+_{\Phi_i(1)}(1) = p_+ + p_0.$$

Consequently,

$$2S^+_{\Phi_i(1)}(1) - \nu(\Phi_i) = p_+ - p_- \geq -p_- \geq -m. \tag{3.18}$$

Then (3.17) follows from (3.18) and the dynamically convexity. □

4. Proof of Theorem 1.8

4.1. **Analogue for local Floer homology.** Our strategy to prove Theorem 1.8 involves translating the equivariant local symplectic homology of a closed Reeb orbit to the local Floer homology via its Poincaré return map and then establishing an analogous argument. Based on the discussion of the local model in Subsection 2.2.2, it suffices to consider the local Floer homology in the case where $W^{2n} = \mathbb{R}^{2n}$ and $x(t) \equiv p = 0$ is a constant one-periodic orbit of a Hamiltonian H defined in a neighborhood of p.

Theorem 4.1. *Let p be an isolated fixed point of the Hamiltonian diffeomorphism $\varphi = \varphi^1_H$. Assume that*

- $\mathrm{HF}_{\mu_+}(H,p) \neq 0$, *where μ_+ is abberiviation of $\mu_+(p,H)$;*
- $\mu_+(p,H) = d + n$, *where d is some integer satisfying $|d - \hat{\mu}(p,H)| < \frac{1}{2}$;*

then we have

$$\mathrm{HF}_*(H,p) = 0, \quad \textit{whenever } * \neq \mu_+(p,H).$$

4.1.1. *Particular case 1: p is non-degenerate.* In this case, the assertion holds obviously. Indeed, since p is non-degenerate, we have $\mu_+(p,H) = \mu(p,H)$ and

$$\mathrm{HF}_*(H,p) = \begin{cases} \mathbb{F}, & * = \mu(p,H), \\ 0, & \text{otherwise}, \end{cases}$$

by Example 2.1.

4.1.2. *Particular case 2: p is totally degenerate.* In this case, all the eigenvalues of $\mathrm{d}\varphi_p$ are equal to one, then $\hat{\mu}(p,H)$ is an integer by (MI3) of Section 3.1. Hence we have

$$\hat{\mu}(p,H) = d \tag{4.1}$$

by the second hypothesis. Let $\Phi = \mathrm{d}\varphi_p$ and $V = T_pM \simeq \mathbb{R}^{2n}$. By [G10, Lemma 5.5], we can find a decomposition of $V = L \oplus L'$, where both L and L' are Lagrangian subspaces of V and $\Phi(L) = L$. Moreover, we can choose a linear canonical coordinate system (x,y) on V, which is compatible with the decomposition, i.e., such that the x-coordinates span L and the y-coordinates span L'. [G10, Lemma 5.5] asserts that we can make $\|\Phi - \mathrm{Id}\|$ arbitrarily small by a suitable choice of such coordinate system. As a consequence, the generating function F of φ can be defined in a neighborhood U of p as in Section 2.3. Set $z = (x,y) \in \mathbb{R}^{2n}$, then F is determined by the equation

$$\varphi(z) - z = X_F(\psi(z)), \tag{4.2}$$

where X_F is the Hamiltonian vector field of F, and $\psi : U \to U$ is a diffeomorphism defined by

$$\psi(z) = (x\text{-component of } \varphi(z), y).$$

As in [G10, Section 6], starting with F, one can construct near p a 1-periodic Hamiltonian K_t with time-1 map φ, such that the pair (K,F) satisfies the hypotheses of Lemma 2.2. Then by Remark 2.3 we can conclude that

$$\begin{cases} \mathrm{HF}_*(K,p) = \mathrm{HM}_{*+n}(F,p), \\ \mathrm{HF}_n(K,p) \neq 0 \Rightarrow \mathrm{HF}_*(K,p) = 0, \forall * \neq n. \end{cases} \tag{4.3}$$

On the other hand, consider the composition $H\#\overline{K}$ introduced in Section 2.1, whose Hamiltonian flow $\varphi^t_H \circ (\varphi^t_K)^{-1}$ fix p. Since both the Hamiltonians K and H generate the same time-1 map φ, such flow is a loop of Hamiltonian diffeomorphisms starting at id in a neighborhood of p, fixing p. Note that $H = (H\#\overline{K})\#K$. According to (LF3) of Section 2.1, we have

$$\mathrm{HF}_{*+m}(H,p) = \mathrm{HF}_*(K,p), \quad m = 2\mu_{\mathrm{maslov}}(\psi), \tag{4.4}$$

where ψ is a loop in $\mathrm{Sp}(2n)$ given by $\psi(t) := (\mathrm{d}\varphi_H^t)_p \circ (\mathrm{d}\varphi_K^t)_p^{-1}$. By Remark 3.1, $(\mathrm{d}\varphi_H^t)_p = \psi(t)\circ(\mathrm{d}\varphi_K^t)_p$ can be viewed as concatenation of $(\mathrm{d}\varphi_K^t)_p$ and $\psi\circ(\mathrm{d}\varphi_K^1)_p$ up to a homotopy with fixed end. Then by (3.7), we have

$$\begin{aligned} m = 2\mu_{\mathrm{maslov}}(\psi) &= 2\mu_{\mathrm{maslov}}(\psi\circ(\mathrm{d}\varphi_K^1)_p) \\ &= \hat{\mu}(\psi(t)\circ(\mathrm{d}\varphi_K^t)_p) - \hat{\mu}((\mathrm{d}\varphi_K^t)_p) \\ &= \hat{\mu}(p,H) - \hat{\mu}(p,K), \end{aligned} \tag{4.5}$$

where the second equality holds by the homotopy invariance of the Maslov index and the fact $\mathrm{Sp}(2n)$ is connected, see [MS17, Theorem 2.2.12].

Assume we have verified that $\mu_+(p,H) = n + \hat{\mu}(p,H) - \hat{\mu}(p,K)$, then by (4.4) and (4.5), $\mathrm{HF}_{\mu+}(H,p) \neq 0$ implies that $\mathrm{HF}_n(K,p) \neq 0$. Thus, Theorem 4.1 is established for the totally degenerate case via (4.3).

To complete the proof, we verify $\mu_+(p,H) = n + \hat{\mu}(p,H) - \hat{\mu}(p,K)$. In view of (4.1), this reduces to showing $\hat{\mu}(p,K) = 0$.

Now we recall the construction of the Hamiltonian K. (For more details, see [G10, Section 6].) Let $\widetilde{K}$ be a Hamiltonian defined in some neighborhood of p, satisfying

$$\varphi_{\widetilde{K}}^t(z) - z = tX_F(\psi^t(z)), \tag{4.6}$$

where

$$\psi^t(z) = (x\text{-component of } \varphi_{\widetilde{K}}^t(z), y).$$

Note that $\varphi_{\widetilde{K}}^1 = \varphi$ and $\psi^1 = \psi$.

Consider a monotone increasing smooth function $\lambda : [0,1] \to [0,1]$ with $\lambda(t) \equiv 0$ when t is near 0 and $\lambda(t) \equiv 1$ when t is near 1. The Hamiltonian K is given by

$$K_t(z) = (1-\lambda'(t))F(z) + \lambda'(t)\widetilde{K}_{\lambda(t)}(\varphi_F^{\lambda(t)-t}(z)), \tag{4.7}$$

where φ_F denotes the Hamiltonian flow of F. Consequently,

$$\varphi_K^t = \varphi_F^{t-\lambda(t)}\varphi_{\widetilde{K}}^{\lambda(t)}. \tag{4.8}$$

It is clearly that $\varphi_K^1 = \varphi$.

Let $\tilde{\Phi}_t, \Phi_t \in \widetilde{\mathrm{Sp}}(2n)$ be the linearizations of $\varphi_{\widetilde{K}}^t$ and φ_K^t at p respectively. Denote by Q the Hessian of F at p and let X_Q be the linear Hamiltonian vector field of Q on V. Linearizing (4.2), (4.6) and (4.8), we see that

$$\Phi - \mathrm{Id} = X_Q P(\Phi), \tag{4.9}$$

$$\tilde{\Phi}_t = \mathrm{Id} + tX_Q P(\tilde{\Phi}_t), \tag{4.10}$$

$$\Phi_t = \exp((t-\lambda(t))X_Q)\tilde{\Phi}_{\lambda(t)}, \tag{4.11}$$

where $P(\Phi) : V \to V$ is defined by

$$\Phi : (x,y) \mapsto (\bar{x},\bar{y}), \quad P(\Phi)(x,y) := (\bar{x},y),$$

with respect to the decomposition $V = L \oplus L'$. ($P(\tilde{\Phi}_t)$ is defined analogously.)

Claim 1: $\sigma(X_Q) = \{0\}$.

Proof of Claim 1. Representing (4.9) as a block matrix relative to the decomposition $V = L\oplus L'$, and noting that $\Phi(L) = L$, we have $\Phi = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. Then (4.9) is equivalent to

$$\begin{pmatrix} A - \mathrm{Id} & B \\ 0 & D - \mathrm{Id} \end{pmatrix} = X_Q \begin{pmatrix} A & B \\ 0 & \mathrm{Id} \end{pmatrix}.$$

This yields

$$X_Q = \begin{pmatrix} \mathrm{Id} - A^{-1} & A^{-1}B \\ 0 & D - \mathrm{Id} \end{pmatrix}, \tag{4.12}$$

where the invertibility of A is guaranteed by the fact $\sigma(\Phi) = \{1\}$. Therefore,

$$\sigma(X_Q) = \sigma(\mathrm{Id} - A^{-1}) \cup \sigma(D - \mathrm{Id}) = \{0\},$$

since $\sigma(\Phi) = \{1\}$. □

Claim 2: $\hat{\mu}(\tilde{\Phi}_t) = 0$.

Proof of Claim 2. Expressing (4.10) in block matrix form relative to $V = L \oplus L'$, let $\tilde{\Phi}_t = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}$. Then by (4.12) and a direct computation, (4.10) becomes

$$\begin{aligned}\begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} &= \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} + t \begin{pmatrix} \mathrm{Id} - A^{-1} & A^{-1}B \\ 0 & D - \mathrm{Id} \end{pmatrix} \begin{pmatrix} A(t) & B(t) \\ 0 & \mathrm{Id} \end{pmatrix}. \\ &= \begin{pmatrix} \mathrm{Id} + t(\mathrm{Id} - A^{-1})A(t) & t(B(t) - A^{-1}B(t) + A^{-1}B) \\ 0 & tD + (1-t)\mathrm{Id} \end{pmatrix}. \end{aligned} \tag{4.13}$$

This implies that

$$\begin{cases} C(t) \equiv 0, \\ D(t) = tD + (1-t)\mathrm{Id}, \\ A(t) = [(1-t)\mathrm{Id} + tA^{-1}]^{-1}. \end{cases} \tag{4.14}$$

Consequently,

$$\sigma(\tilde{\Phi}_t) = \sigma(A(t)) \cup \sigma(D(t)) \equiv \{1\}, \quad \forall t \in [0, 1], \tag{4.15}$$

where the second equality uses (4.14) and the fact $\sigma(D) = \{1\} = \sigma(A^{-1})$. It follows immediately that $\hat{\mu}(\tilde{\Phi}_t) = 0$ from (4.15) and (MI6) of Section 3.1. □

Combining Claim 1, Claim 2, (3.2) and (4.11), we conclude $\hat{\mu}(p, K) = 0$ and therefore Theorem 4.1 holds for the totally degenerate case.

Remark 4.2. Conjecturally, Theorem 4.1 holds without the condition $\mu_+(p, H) = d + n$. Following the idea in this section, it sufficiently to prove this conjecture for totally degenerate case. By our discussion above, this reduces to find the pair (K, F) in Lemma 2.2 such that

$$\hat{\mu}(p, K) = n + \hat{\mu}(p, H) - \mu_+(p, H),$$

which is equivalent to

$$\hat{\mu}(p, K) = n + S^+_\Phi(1) - \nu_1(\Phi), \text{ where } \Phi = \mathrm{d}\varphi_p,$$

due to Corollary 3.10.

4.1.3. *Particular case 3: split diffeomorphisms.* Assume that $V = \mathbb{R}^{2n}$ is decomposed as a product of two symplectic vector spaces V_0 and V_1, $p = (p_0, p_1)$ in this decomposition, and the Hamiltonian H is split, i,e., $H = H_0 + H_1$ where H_0 and H_1 are Hamiltonians on V_0 and V_1 respectively, whose flows fix p_0 and p_1, correspondingly. Futhermore, assume that the time-1 map φ_{H_0} of H_0 is non-degenerate at p_0 and the time-1 map φ_{H_1} of H_1 is totally degenerate at p_1. By (LF2) of Section 2.1 we have

$$\mathrm{HF}_*(H,p) = \bigoplus_{*'+*''=*} \mathrm{HF}_{*'}(H_0,p_0) \otimes \mathrm{HF}_{*''}(H_1,p_1). \tag{4.16}$$

Since φ_{H_0} is nondegenerate at p_0, From our particular case 1 we obtain

$$\mathrm{HF}_*(H_0,p_0) = \begin{cases} \mathbb{F}, & * = \mu(p_0,H_0), \\ 0, & \text{otherwise.} \end{cases} \tag{4.17}$$

Thus $\mathrm{HF}_{\mu_+(p,H)}(H,p) \neq 0$ implies that $\mathrm{HF}_{\mu_+(p_1,H_1)}(H_1,p_1) \neq 0$, where

$$\mu_+(p,H) = \mu(p_0,H_0) + \mu_+(p_1,H_1).$$

To apply our particular case 2 to $\mathrm{HF}_*(H_1,p_1)$, we need to verify that

$$\mu_+(p_1,H_1) - \hat{\mu}(p_1,H_1) = \frac{1}{2}\dim V_1. \tag{4.18}$$

By (3.5),

$$\begin{aligned} \mu(p_0,H_0) - \hat{\mu}(p_0,H_0) &\leq \frac{1}{2}\dim V_0, \\ \mu_+(p_1,H_1) - \hat{\mu}(p_1,H_1) &\leq \frac{1}{2}\dim V_1. \end{aligned} \tag{4.19}$$

Observe that both $\mu_+(p_1,H_1)$ and $\hat{\mu}(p_1,H_1)$ are integers. Assuming for contradiction that

$$\mu_+(p_1,H_1) - \hat{\mu}(p_1,H_1) \leq \frac{1}{2}\dim V_1 - 1. \tag{4.20}$$

and combining (4.20) with (4.19), we see that

$$\begin{aligned} \mu_+(p,H) - \hat{\mu}(p,H) &= \mu(p_0,H_0) + \mu_+(p_1,H_1) - (\hat{\mu}(p_0,H_0) + \hat{\mu}(p_1,H_1)) \\ &\leq \frac{1}{2}(\dim V_0 + \dim V_1) - 1 \\ &= n-1. \end{aligned}$$

which contradicts the hypothesis $\mu_+(p,H) - d = n$ since $|d - \hat{\mu}(p,H)| < \frac{1}{2}$. This establishes (4.18).

According to our particular case 2,

$$\mathrm{HF}_*(H_1,p_1) = \begin{cases} \mathbb{F}, & * = \mu_+(p_1,H_1), \\ 0, & \text{otherwise.} \end{cases} \tag{4.21}$$

Therefore Theorem 4.1 holds for split Hamiltonian H by (4.16), (4.17) and (4.21).

More generally, assume that the time-1 map $\varphi = \varphi_H$, but not necessarily H, is split, i.e., $\varphi = (\varphi_0, \varphi_1)$, where φ_0 and φ_1 are the germs of symplectomorphisms fixing p_0 in V_0 and, respectively, fixing p_1 in V_1. In fact both φ_0 and φ_1 are Hamiltonian, see [GG10, Remark 4.1]. Denote by H_0 and H_1 some Hamiltonians generating φ_0 and φ_1 respectively. As above, in addition we assume that the time-1 map φ_0 of H_0 is non-degenerate at p_0 and the time-1 map φ_1 of H_1 is totally degenerate at p_1.

Note that we do not necessarily have $H = H_0 + H_1$, but both H and $H_0 + H_1$ generate the same time-1 map φ. Therefore, similarly with (4.4) and (4.5), by (LF3) of Section 2.1 and (3.7), we have

$$\begin{cases} \mathrm{HF}_{*+m}(H,p) = \mathrm{HF}_*(H_0+H_1,p), \\ m = \hat{\mu}(p,H) - \hat{\mu}(p,H_0+H_1), \\ \mu_+(p,H) = \mu_+(p,H_0+H_1) + m, \end{cases} \tag{4.22}$$

where m is an even integer. Consequently, $\mathrm{HF}_{\mu_+(p,H)}(H,p) \neq 0$ implies

$$\mathrm{HF}_{\mu_+}(H_0+H_1,p) \neq 0, \text{ where } \mu_+ = \mu_+(p,H_0+H_1). \tag{4.23}$$

Moreover, $\mu_+(p,H) - d = n$ and $|d - \hat{\mu}(p,H)| < \frac{1}{2}$ combined with (4.22) yields

$$\mu_+(p,H_0+H_1) - (d-m) = n, \ |d - m - \hat{\mu}(p,H_0+H_1)| < \frac{1}{2}. \tag{4.24}$$

Note that (4.23) and (4.24) precisely match the hypotheses in Theorem 4.1 for the split Hamiltonian $H_0 + H_1$. From our previous discussion, we conclude that

$$\mathrm{HF}_*(H_0+H_1,p) = 0 \ \text{ whenever } * \neq \mu_+(p,H_0+H_1).$$

Finally, by (4.22), we deduce that Theorem 4.1 remains valid for H even when only its time-1 map is assumed to be split.

4.1.4. *The general case.* Let $\varphi = \varphi_H$ be the Hamiltonian diffeomorphsim fixing p in $\mathbb{R}^{2n}$ and $\mathrm{d}\varphi_p$ denote the linearization of φ at p. Choose a symplectic decomposition of $T_p\mathbb{R}^{2n} \simeq \mathbb{R}^{2n} = V_0 \times V_1$ such that all eigenvalues of $\mathrm{d}\varphi_p|_{V_0}$ are different from 1 and all eigenvalues of $\mathrm{d}\varphi_p|_{V_1}$ are equal to 1. As in [GG10, Page 343], one can construct a homotopy of Hamiltonian diffeomorphisms φ_s, $s \in [0,1]$ such that

- $\varphi_0 = \varphi$, and φ_1 is split with respect to the decomposition $\mathbb{R}^{2n} = V_0 \times V_1$ as in Subsection 4.1.3;
- p is a uniformly isolated fixed point of φ_s, and $(\mathrm{d}\varphi_s)_p \equiv \mathrm{d}\varphi_p$, $\forall s \in [0,1]$.

Let K_s be the Hamiltonian generating φ_s as its time-1 map, obtained by concatenating the flow φ_H^t, $t \in [0,1]$ with the homotopy φ_ζ, $\zeta \in [0,s]$, up to an obvious reparametrization. Then $K_0 = H$ and p is a uniformly isolated fixed point of φ_{K_s} for all $s \in [0,1]$. Thus, by (LF1) of Section 2.1, we have

$$\mathrm{HF}_*(H,p) = \mathrm{HF}_*(K_1,p). \tag{4.25}$$

Moreover, since $(\mathrm{d}\varphi_s)_p$ is constant, the homotopy invariance of the Maslov-type index yields

$$\hat{\mu}(p,H) = \hat{\mu}(p,K_1), \quad \mu_+(p,H) = \mu_+(p,K_1). \tag{4.26}$$

As $\varphi_1 = \varphi_{K_1}$ is split, Theorem 4.1 applies to K_1 by Subsection 4.1.3. Through (4.25) and (4.26), we conclude that Theorem 4.1 holds for H as well.

This concludes the proof of Theorem 4.1. □

4.2. **From local Floer homology to equivariant local symplectic homology.**

Proof of Theorem 1.8. Assume $x = y^l$ is the l-th iteration of a prime closed Reeb orbit y. By (2.3) and (2.7), the condition $\mathrm{CH}_{\mu_+(x)}(x;\mathbb{Q}) \neq 0$ implies:

$$\begin{aligned} 1 &\leq \dim \mathrm{SH}_{\mu_+(x)+1}(y^l;\mathbb{Q}) \\ &\leq \dim \mathrm{HF}_{\mu_+(x)+1}(\varphi_H^l, 0;\mathbb{Q}) + \dim \mathrm{HF}_{\mu_+(x)}(\varphi_H^l, 0;\mathbb{Q}), \end{aligned} \tag{4.27}$$

where φ_H is the Poincaré return map of y defined on B^{2n-2}. From Remark 2.8, we have:

$$\mu_+(0,\varphi_H^l) = \mu_+(x) \quad \text{and} \quad \hat{\mu}(0,\varphi_H^l) = \hat{\mu}(x).$$

By (LF4) of Section 2.1, we obtain

$$\dim \mathrm{HF}_{\mu_+(x)+1}(\varphi_H^l, 0;\mathbb{Q}) = 0,$$

which combined with (4.27) leads to

$$\dim \mathrm{HF}_{\mu_+(x)}(\varphi_H^l, 0;\mathbb{Q}) \geq 1. \tag{4.28}$$

Under the assumptions of Theorem 1.8, we have:

$$\mu_+(0,\varphi_H^l) = d+n-1, \quad \text{where} \quad |d-\hat{\mu}(0,\varphi_H^l)|<\tfrac{1}{2}. \tag{4.29}$$

Since (4.28) and (4.29) satisfy the conditions of Theorem 4.1 on B^{2n-2}, we conclude

$$\mathrm{HF}_*(\varphi_H^l,0;\mathbb{Q})=0 \quad \text{for all } *\neq \mu_+(x). \tag{4.30}$$

Furthermore, from (2.3) and (2.7), we obtain:

$$\begin{aligned} 0 &\le \dim \mathrm{CH}_*(y^l;\mathbb{Q})+\dim \mathrm{CH}_{*-1}(y^l;\mathbb{Q}) \\ &\le \dim \mathrm{HF}_*(\varphi_H^l,0;\mathbb{Q})+\dim \mathrm{HF}_{*-1}(\varphi_H^l,0;\mathbb{Q}), \end{aligned}$$

which implies

$$\mathrm{CH}_*(x;\mathbb{Q})=0 \quad \text{for all } *\neq \mu_+(x).$$

For the additional part, assume both $x=z^k$ and $z^{k'}$ are admissible iterations of $z=y^m$, $l=mk$. By [GG10, Theorem 1.1], we have isomorphism

$$\mathrm{HF}_{*+s}(\varphi_H^l,0;\mathbb{Q})=\mathrm{HF}_*(\varphi_H^{l'},0;\mathbb{Q}), \tag{4.31}$$

where $l'=mk'$ and $s=\mu_+(x)-\mu_+(z^{k'})$.

Combining (4.30) with (4.31) yields

$$\mathrm{HF}_*(\varphi_H^{l'},0;\mathbb{Q})=0 \quad \text{for all } *\neq \mu_+(z^{k'}).$$

Finally, applying (2.3) and (2.7) again we get

$$\mathrm{CH}_*(z^{k'};\mathbb{Q})=0 \quad \text{for all } *\neq \mu_+(z^{k'}).$$

□

Remark 4.3. The explicit expression of the shift s in (4.31), while not stated in the original formulation of [GG10, Theorem 1.1], can be extracted from the proof in [GG10, Section 4]. For clarity, we present the detailed explanation below. Our goal is to show that:

Assume z is an isolated closed Reeb orbit and z^k is an admissible iteration. Let φ_H be the Poincaré return map of z, then there holds isomorphism

$$\mathrm{HF}_{*+s}(\varphi_H^k,0;\mathbb{Q})=\mathrm{HF}_*(\varphi_H,0;\mathbb{Q}), \quad \textit{where } s=\mu_+(z^k)-\mu_+(z).$$

Proof. Recall the Bott-type formula for nullity (cf.[L02, Theorem 9.2.1]):

$$\nu(z^k)=\sum_{\omega^k=1}\nu_\omega(z),$$

where the ω-nullity $\nu_\omega(\cdot)$ of a orbit is defined in Remark 3.7. Since z^k is an admissbile iteration, we have $\nu(z^k)=\nu(z)$, and consequently by (3.4),

$$\mu_+(z^k)-\mu_+(z)=\mu_-(z^k)-\mu_-(z). \tag{4.32}$$

Now we divide the proof into three cases:

Case 1: *z is non-degenerate.* In this case, both z^k and z are non-degenerate, hence we apply Example 2.1 to get

$$s=\mu(0,\varphi_H^k)-\mu(0,\varphi_H)=\mu(z^k)-\mu(z),$$

where the second equality holds by Remark 2.8.

Case 2: *z is totally degenerate.* According to [GG10, Page 339], in this case

$$s=\hat{\mu}(0,\varphi_H^k)-\hat{\mu}(0,\varphi_H)=\mu_-(0,\varphi_H^k)-\mu_-(0,\varphi_H)=\mu_-(z^k)-\mu_-(z),$$

where the second equality follows from Proposition 3.9 and Corollary 3.10.

Case 3: *general case* As in [GG10, Section 4.5], choose a symplectic decomposition of $T_0B^{2n-2}\simeq\mathbb{R}^{2n-2}=V_0\times V_1$ such that all eigenvalues of $(\mathrm{d}\varphi_H)_0|_{V_0}$ are different from 1 and all eigenvalues of $(\mathrm{d}\varphi_H)_0|_{V_1}$ are equal to 1, then one can find a Hamiltonian K with time-1 map φ_K defined in a neighborhood of origin in $\mathbb{R}^{2n-2}$ such that:

- There exists a homotopy of Hamiltonian diffeomorphisms φ_s, $s\in[0,1]$, such that $\varphi_0=\varphi_H$, $\varphi_1=\varphi_K$ and $(\mathrm{d}\varphi_s)_0\equiv(\mathrm{d}\varphi_H)_0$;
- φ_K is split as $\varphi_K=(\varphi_{K_0},\varphi_{K_1})$, where φ_{K_0} (φ_{K_1} resp.) denotes the time-1 map of the Hamiltonian K_0 on V_0 (K_1 on V_1 resp.) defined in a neighborhood of the origin;
- $\hat\mu(0,\varphi_H)=\hat\mu(0,\varphi_K)$, and for each admissible iteration z^k, there holds isomorphism

$$\mathrm{HF}_*(\varphi_H^k,0;\mathbb{Q})=\mathrm{HF}_*(\varphi_K^k,0;\mathbb{Q}).$$

Following Subsection 4.1.3, by (4.22) and (MI4) of Section 3.1, we have

$$\mathrm{HF}_{*+m}(\varphi_K,0;\mathbb{Q})=\mathrm{HF}_*(\varphi_{K_0+K_1},0;\mathbb{Q}),\qquad m=\hat\mu(0,\varphi_K)-\hat\mu(0,\varphi_{K_0+K_1}),\tag{4.33}$$

$$\mathrm{HF}_{*+m_k}(\varphi_K^k,0;\mathbb{Q})=\mathrm{HF}_*(\varphi_{K_0+K_1}^k,0;\mathbb{Q}),\qquad m_k=km,\tag{4.34}$$

where m is even. Note that the flow $\varphi^t_{(K_0+K_1)^{\#k}}=(\varphi^t_{K_0^{\#k}},\varphi^t_{K_1^{\#k}})$ is split, according to (LF2) of Section 2.1, we have

$$\begin{aligned}\mathrm{HF}_*(\varphi_{K_0+K_1},0;\mathbb{Q})&=\bigoplus_{*'+*''=*}\mathrm{HF}_{*'}(\varphi_{K_0},0;\mathbb{Q})\otimes\mathrm{HF}_{*''}(\varphi_{K_1},0;\mathbb{Q})\\&=\bigoplus_{*'+*''=*}\mathrm{HF}_{*'+s_0}(\varphi_{K_0}^k,0;\mathbb{Q})\otimes\mathrm{HF}_{*''+s_1}(\varphi_{K_1}^k,0;\mathbb{Q})\\&=\mathrm{HF}_{*+s_0+s_1}(\varphi_{K_0+K_1}^k,0;\mathbb{Q}),\end{aligned}\tag{4.35}$$

where

$$s_0=\mu(0,\varphi_{K_0}^k)-\mu(0,\varphi_{K_0})\ \text{ and }s_1=\hat\mu(0,\varphi_{K_1}^k)-\hat\mu(0,\varphi_{K_1})\tag{4.36}$$

follow from Case 1 and Case 2. Summarizing (4.33)-(4.36) yields

$$\begin{aligned}s&=m_k+s_0+s_1-m\\&=(k-1)[(\hat\mu(0,\varphi_K)-\hat\mu(0,\varphi_{K_0+K_1})+\hat\mu(0,\varphi_{K_1})]+\mu(0,\varphi_{K_0}^k)-\mu(0,\varphi_{K_0}).\end{aligned}\tag{4.37}$$

Let $\Phi\in\widetilde{\mathrm{Sp}}(2n-2)$ denote the associated symplectic path of z as in Subsection 3.1.4. By Remark 2.8 and the fact that φ_K is homotopic to φ_H by φ_s with fixed $(\mathrm{d}\varphi_s)_0$, we conclude that

$$\hat\mu(0,\varphi_K)=\hat\mu(\Phi).$$

Let $n_0:=\dim V_0$, $n_1:=\dim V_1$, $\Psi_0\in\widetilde{\mathrm{Sp}}(n_0)$ and $\Psi_1\in\widetilde{\mathrm{Sp}}(n_1)$ denote the symplectic paths given by $(\mathrm{d}\varphi^t_{K_0})_0$ and $(\mathrm{d}\varphi^t_{K_1})_0$, $t\in[0,1]$ respectively. Note that Ψ_0 is non-degenerate while Ψ_1 is totally degenerate, and

$$\hat\mu(0,\varphi_{K_0+K_1})=\hat\mu(\Psi_0)+\hat\mu(\Psi_1),\quad \mu(0,\varphi_{K_0})=\mu(\Psi_0),\quad \hat\mu(0,\varphi_{K_1})=\hat\mu(\Psi_1).$$

Then (4.37) can be rewritten as

$$s=(k-1)[(\hat\mu(\Phi)-\hat\mu(\Psi_0)]+\mu(\Psi_0^k)-\mu(\Psi_0).\tag{4.38}$$

According to (3.7), we can modify Ψ_0 by concatenating at the end point a symplectic loop ψ with Maslov index $(\hat\mu(\Phi)-\hat\mu(\Psi_0))/2$ (such a loop can be constructed directly by rotation) so that the resulting path Ψ satisfying

$$\begin{cases}\hat\mu(\Psi)=\hat\mu(\Phi),\\ \Phi(1)=\Psi(1)\diamond\Psi_1(1),\end{cases}\tag{4.39}$$

Moreover,

$$\mu(\Psi)-\mu(\Psi_0)=\hat\mu(\Psi)-\hat\mu(\Psi_0),\tag{4.40}$$

$$\mu(\Psi^k)-\mu(\Psi_0^k)=k(\hat\mu(\Psi)-\hat\mu(\Psi_0)),\tag{4.41}$$

note that path Ψ^k can be viewed as Ψ_0^k concatenated with the symplectic loop $\psi_i := \Psi_0^{i-1}(1)\psi$ at the point $\Psi_0^i(1)$ for $i = 1, \ldots, k$, and all these k loops have the same Maslov index with ψ. Substituting (4.39)-(4.41) into (4.38) yields

$$s = \mu(\Psi^k) - \mu(\Psi). \tag{4.42}$$

According to Lemma 3.8 and Corollary 3.10, (4.39) implies

$$\mu(\Psi) = \mu_-(\Phi) + S^+_{\Phi(1)}(1),$$

which yields

$$s = \mu_-(\Phi^k) - \mu_-(\Phi) = \mu_-(z^k) - \mu_-(z)$$

by Proposition 3.9.

Summarizing these cases and (4.32), we conclude that

$$s = \mu_-(z^k) - \mu_-(z) = \mu_+(z^k) - \mu_+(z). \tag{4.43}$$

□

Remark 4.4. Let $\Phi \in \widetilde{\mathrm{Sp}}(2n)$. In [ÇGG24, Section 3.1.1], a non-degenerate $\Psi \in \widetilde{\mathrm{Sp}}(2m)$ with $0 \leq m \leq n$ satisfying (4.39) is called the *non-degenerate part* of Φ, i.e.,

$$\begin{cases} \hat{\mu}(\Psi) = \hat{\mu}(\Phi), \\ \Psi(1) \diamond A = \Phi(1), \text{ for some } A \in \mathrm{Sp}(2n-2m) \text{ with } \sigma(A) = \{1\}. \end{cases}$$

Such Ψ is uniquely determined as elements in $\Psi \in \widetilde{\mathrm{Sp}}(2m)$ since its Conley-Zehnder index and end point are determined. For instance, the non-degenerate part of Φ is given by $\Psi = \Phi$ if Φ is non-degenerate, and is trivial if Φ is totally degenerate. As already mentioned in the above remark, Proposition 3.9 together with Corollary 3.10 yields

$$\mu(\Psi^k) - \mu(\Psi) = \mu_-(\Phi^k) - \mu_-(\Phi), \quad \forall k \in \mathbb{N}.$$

whenever Ψ is the non-degenerate part of Φ.

5. Two irrationally elliptic closed Reeb orbits

5.1. **Common index jump intervals.** Let Σ^{2n-1} be the boundary of a star-shaped domain $W \subset \mathbb{R}^{2n}$. Assume that

(DF): *the Reeb flow on Σ^{2n-1} is dynamically convex and has finitely many prime closed orbits.*

Under this assumption, Çineli-Ginzburg-Gürel proved in [ÇGG24, Theorem D] that for each isolated closed Reeb orbit x (not necessarily prime) on Σ, its equivariant local symplectic homology $\mathrm{CH}_*(x;\mathbb{Q})$ satisfies

$$\dim \mathrm{CH}_*(x;\mathbb{Q}) \leq 1.$$

Denote by $\mathcal{P}$ the set of all $\mathbb{Q}$-visible isolated closed Reeb orbits. Then for each $x \in \mathcal{P}$, there exists some $h \in \mathbb{Z}$ such that

$$\mathrm{CH}_*(x;\mathbb{Q}) = \begin{cases} \mathbb{Q}, & * = h, \\ 0, & \text{otherwise}. \end{cases} \tag{5.1}$$

We call h from (5.1) *the degree of* x, denoted by $\deg(x)$. By (2.2), the degree is bounded by the Maslov-type indices, namely,

$$\deg(x) \in [\mu_-(x), \mu_+(x)] \subset [\hat{\mu}(x) - n + 1, \hat{\mu}(x) + n - 1]. \tag{5.2}$$

Furthermore, the degree map has the following bijective property:

Theorem 5.1 ([ÇGG24, Theorem E]). *Assume* ***(DF)*** *holds, then $\mathcal{P}$ has the following properties:*

- *(O1) The map $\deg : \mathcal{P} \to \mathbb{Z}$ is a bijection onto $n - 1 + 2\mathbb{N} = \{n+1, n+3, \ldots\}$, and orbits with larger degree have larger period, i.e. $\deg(x) > \deg(y)$ implies $\mathcal{A}(x) > \mathcal{A}(y)$ for each x, $y \in \mathcal{P}$.*

(O2) All closed orbits $x \in \mathcal{P}$ have the same ration $\mathcal{A}(x)/\hat{\mu}(x)$.

Let $x_0 \in \mathcal{P}$ be the orbit with the least action. (O1) implies $\deg(x_0) = n+1$. If $x_0 = z^k$ with $k \geq 2$, by Lemma 3.4 and dynamical convexity we have

$$\mu_-(x_0) = \mu_-(z^k) > n+1 = \deg(x_0),$$

which contradicts (5.2). Thus, x_0 must be prime and $\mu_-(x_0) = n+1 = \deg(x_0)$.

Let $\{x_0, \ldots, x_r\}$ be all the prime closed orbits which are *eventually* $\mathbb{Q}$*-visible*, i.e., $x_i^k \in \mathcal{P}$ for some $k \in \mathbb{N}$. Note that the orbits $x_{i\geq 1}$ might be $\mathbb{Q}$-invisible and/or have action below $\mathcal{A}(x_0)$. For each $i \geq 1$ we define s_i by

$$s_i := \begin{cases} \max\{s \in \mathbb{N} : \mathcal{A}(x_i^s) \leq \mathcal{A}(x_0)\}, & \text{if } \mathcal{A}(x_i) \leq \mathcal{A}(x_0); \\ 0, & \text{if } \mathcal{A}(x_i) > \mathcal{A}(x_0). \end{cases} \tag{5.3}$$

Consider common index jump events $\{k_{ij}, d_j\}_{j=1}^{\infty}$ as in Theorem 3.12, where we require $\eta > 0$ to be sufficiently small, all d_j and k_{ij} are divisible by $N \in \mathbb{N}$ specified as follows. Let

$$p := \operatorname{lcm}\{k \in \mathbb{N} : \exists\, 0 \leq i \leq r,\, \lambda \in \mathbb{C}, \text{ s.t. } \lambda \in \sigma(x_i),\, \lambda^k = 1\ \},$$

where $\sigma(x_i)$ denote the set of all eigenvalues of the linearized Poincaré map of x_i, and $\operatorname{lcm}\{\cdot\}$ denotes the least common multiple of the numbers in the set. Then we let

$$N = 2p \prod_{i=1}^{r} s_i! \tag{5.4}$$

where, as usual, we set $0! = 1$.

Concentrating on one event, we drop j from the notation to get $d = d_j$ and $k_i = k_{ij}$. The $\mathbb{Q}$-visible closed orbits x_j^k are divided into three collections:

- Γ_+ formed by the $\mathbb{Q}$-visible orbits $x_i^{k>k_i}$. Then $\mu_-(x_i^k) \geq d+n+1$ by (3.15) with $m = n-1$, and hence $\deg(x_i^k) \geq d+n+1$ by (5.2).
- Γ_0 comprising the $r+1$ $\mathbb{Q}$-visible orbits $x_i^{k_i}$. Then by (3.5) with $m = n-1$ and (3.12),

  $$d-n+1 \leq \mu_-(x_i^{k_i}) \leq \mu_+(x_i^{k_i}) \leq d+n-1,$$

  and hence

  $$\deg(x_i^{k_i}) \in \mathcal{I} := [d-n+1, d+n-1] \subset \mathbb{N} \tag{5.5}$$

  by (5.2).
- Γ_- formed by the $\mathbb{Q}$-visible orbits $x_i^{k<k_i}$. Then $\mu_+(x_i^k) \leq d-2$ by (3.16), and hence $\deg(x_i^k) \leq d-2$.

For each common index jump event, the interval $\mathcal{I}$ in (5.5) is called *the associated common index jump interval.*

In fact, it is shown in [ÇGG24, Section 4.1.1] that the orbits from Γ_- must have a degree less than $d-n-1$, thus cannot have a degree in $\mathcal{I}$. (For the reader's convenience we will briefly recall the proof for this argument later. The construction of N in (5.4) is essentially used there.) Consequently, the only closed orbits with degree in $\mathcal{I}$ must be from Γ_0. Note that there are exactly n integer spots in $\mathcal{I}$ of parity $n+1$, which must all be filled by $\deg(x_i^{k_i})$ when d is sufficiently large; see (O1) in Theorem 5.1. This implies that $r+1 \geq n$.

Now we recall the sketch of the proof for

$$\deg(x_i^k) \leq d-n-1 \quad \textit{when } k < k_i \textit{ and } x_i^k \in \mathcal{P}. \tag{5.6}$$

(The terminology has been adjusted to maintain notational consistency throughout this article, which leads to a slight simplification of the original proof in [ÇGG24].)

Proof. Start with $x = x_0$ and $k = k_0 - 1$. Recall that x is prime with $\mu_-(x) = \deg(x) = n+1$. By construction of N we see x^k is an admissible iteration and k is odd, which implies $x^k \in \mathcal{P}$ by

$$\chi^{\mathrm{eq}}(x^k) = \chi^{\mathrm{eq}}(x) = (-1)^{n+1} \neq 0,$$

which follows from Lemma 2.6. Since x is prime, by (2.3) and (2.4) one can compute $\deg(x^k)$ through the local Floer homology of its Poincaré return map to get

$$\deg(x^k) = \deg(x) + \mu_+(x^k) - \mu_+(x), \tag{5.7}$$

see [ÇGG24, Lemma 4.2], Remark 4.3 and Remark 4.4. Applying (3.14) of Theorem 3.12 to (5.7) yields

$$\begin{aligned}\deg(x^k) &= \deg(x) + d - \mu_-(x) - 2S_x^+(1) + \nu(x) - \mu_+(x)\\ &= d - n - 1 - 2S_x^+(1)\\ &\leq d - n - 1,\end{aligned} \tag{5.8}$$

where the inequality holds by Lemma 3.8.

For $k' < k$ we have $\mathcal{A}(x^{k'}) < \mathcal{A}(x^k)$, which leads to $\deg(x^{k'}) < \deg(x^k) \leq d - n - 1$ by Theorem 5.1, provided $x^{k'} \in \mathcal{P}$.

Finally we turn to the case $x_i^{k_i - s}$ with $i \geq 1$ and $s > 0$. Fix i. Note that (5.6) is an immediately consequence of the following claim:

Claim: $\deg(x_i^{k_i-s}) > d - n$ *for some* $s > 0$ *implies* $x_i^{k_i-s} \notin \mathcal{P}$.

Indeed, by (O1) of Theorem 5.1 and the fact $\deg(x_i^{k_i-s}) > d - n - 1 \geq \deg x_0^{k_0-1}$, we have

$$\mathcal{A}(x_i^{k_i-s}) > \mathcal{A}(x_0^{k_0-1}). \tag{5.9}$$

Combining (3.12) of Theorem 3.12 with (O2) of Theorem 5.1 yields

$$|\mathcal{A}(x_i^{k_i}) - \mathcal{A}(x_0^{k_0})| < \mathrm{const} \cdot \eta. \tag{5.10}$$

Provied $\eta > 0$ sufficiently small, (5.9) together with (5.10) implies $\mathcal{A}(x_i^s) < \mathcal{A}(x_0)$. Then by the definition of x_0, $x_i^s \notin \mathcal{P}$. Consequently $\chi^{eq}(x_i^s) = 0$. Moreover, $s \leq s_i$ with s_i defined in (5.3), hence k_i is divisible by s by the construction of N. Set $k_i' = k_i/s$. Then $x_i^{k_i-s} = (x_i^s)^{k_i'-1}$ is an admissible odd iterate of x_i^s. Then by Lemma 2.6 we have

$$\chi^{\mathrm{eq}}(x_i^{k_i-s}) = \chi^{\mathrm{eq}}(x_i^s) = 0,$$

which implies $x_i^{k_i-s} \notin \mathcal{P}$ by (5.1). □

5.2. **Proof of two irrationally elliptic closed Reeb orbits.** From now on, we assume that **(DF'):** *the Reeb flow on* Σ^{2n-1} *is dynamically convex and has exactly* n *prime closed orbits.* Denote by $\{y_1, \ldots, y_n\}$ these n prime closed Reeb orbits. By our discussion in the previous subsection, for each common index jump event $\{k_i, d\}$ with d sufficiently large, the n integer spots of parity $n+1$ in the common index jump interval $\mathcal{I}$, must be filled by distinct values of $\deg(y_i^{k_i})$ for $1 \leq i \leq n$. (This implies that all the n prime closed Reeb orbits are eventually $\mathbb{Q}$-visible!)

Furthermore, by the same proof of [W22, Lemma 4.4], the mean indices of closed Reeb orbits in the common index jump interval have the following commutative property:

Proposition 5.2. *There exists two common index jump events* $\{k_i, d\}$ *and* $\{k_i', d'\}$ *such that for any integers* $1 \leq i, j \leq n$ *and* $i \neq j$, *we have*

$$\hat{\mu}(y_i^{k_i}) < \hat{\mu}(y_j^{k_j}), \ \textit{while} \ \hat{\mu}(y_i^{k_i'}) > \hat{\mu}(y_j^{k_j'}),$$

i.e., the ordering of mean indices of any two closed Reeb orbits with degree in the common index jump intervals gets interchanged.

Theorem 5.1 implies that the ordering of mean indices is exactly the ordering of degrees for any two closed Reeb orbits in $\mathcal{P}$. Combining this with Proposition 5.2, after reordering $\{y_1, \ldots, y_n\}$ if necessary, we can find common index jump events $\{k_i, d\}$ and $\{k_i', d'\}$ with $d < d'$ such that the following diagrams hold:

$$\begin{array}{ccccc} d-n+1 & d-n+3 & \cdots & d+n-3 & d+n-1 \\ \| & \| & & \| & \| \\ \deg(y_1^{k_1}) & \deg(y_2^{k_2}) & & \deg(y_{n-1}^{k_{n-1}}) & \deg(y_n^{k_n}) \end{array}$$

$$\begin{array}{ccccc} d'-n+1 & d'-n+3 & \cdots & d'+n-3 & d'+n-1 \\ \| & \| & & \| & \| \\ \deg(y_n^{k_n'}) & \deg(y_{n-1}^{k_{n-1}'}) & & \deg(y_2^{k_2'}) & \deg(y_1^{k_1'}) \end{array}$$

Proof of Theorem 1.7. In the following we will show that both y_1 and y_n are irrational elliptic. By the symmetric positions of y_1 and y_n, we only state the proof for y_1 here, which is divided into several steps:

Step 1. *We show that* $\deg(y_1^{k_1'}) = \mu_+(y_1^{k_1'})$ *and* $\deg(y_1^{k_1}) = \mu_+(y_1^{k_1})$.
Abbreviate $\mu_+(y_1^{k_1'})$ and $\mu_+(y_1^{k_1})$ by μ_+' and μ_+ respectively. By (5.2),

$$d' + n - 1 = \deg(y_1^{k_1'}) \le \mu_+' \le \hat{\mu}(y_1^{k_1'}) + n - 1. \tag{5.11}$$

In Theorem 3.12 we have $|d' - \hat{\mu}(y_1^{k_1'})| < \eta$ and we have assumed that η is sufficiently small (for instance, $0 < \eta < \frac{1}{2}$). Since μ_+' is an integer, (5.11) implies that

$$\mu_+' = d' + n - 1 = \deg(y_1^{k_1'}). \tag{5.12}$$

By the construction of N in (5.4), we conclude that both $y_1^{k_1'}$ and $y_1^{k_1}$ are admissible iteration of y_1^N. Then by (5.12), we can apply Theorem 1.8 to get

$$\mathrm{CH}_*(y_1^{k_1}; \mathbb{Q}) = 0 \quad \text{whenever } * \neq \mu_+,$$

which implies

$$\mu_+ = \deg(y_1^{k_1}) = d - n + 1. \tag{5.13}$$

Step 1 is finished by (5.12) and (5.13).

Step 2. *We show that* y_1 *is strongly non-degenerate, i.e. any iteration of* y_1 *is non-degenerate.*
Indeed, by (5.2) and (5.13),

$$\hat{\mu}(y_1^{k_1}) - n + 1 \le \mu_-(y_1^{k_1}) \le \deg(y_1^{k_1}) = d - n + 1,$$

which implies $\mu_-(y_1^{k_1}) = d - n + 1$ since $|d - \hat{\mu}(y_1^{k_1})| < \frac{1}{2}$. Combining this fact with (5.13) we have $\mu_-(y_1^{k_1}) = \mu_+(y_1^{k_1})$, which implies $y_1^{k_1}$ is non-degenerate by (3.4). Since k_1 is divisible by N, we conclude that y_1^N is also non-degenerate. Thus, by the construction of N in (5.4), we know that there is no root of unity in $\sigma(y_1)$, i.e., for each eigenvalue $\lambda = e^{\sqrt{-1}\theta} \in \sigma(y) \cap \mathbb{U}$, θ/π is irrational. Consequently, y_1 is strongly non-degenerate.

Step 3. *We show that* y_1 *is elliptic.*
Denoted by $\Phi \in \mathrm{Sp}(2n-2)$ the restricted linearized Poincaré return map of y_1. In Step 2 we have shown that y_1 is strongly non-degenerate, then by the abstract precise iteration formula

(3.9), we have

$$\begin{aligned}\mu(y_1^{k_1+1})-\mu(y_1^{k_1}) &= \mu(y_1)+\sum_{\theta\in(0,2\pi),\frac{\theta}{\pi}\notin\mathbb{Q}}(2\lceil\frac{(k_1+1)\theta}{2\pi}\rceil-2\lceil\frac{k_1\theta}{2\pi}\rceil-1)S^-_\Phi(e^{\sqrt{-1}\theta})\\ &= \mu(y_1)+\sum_{\theta\in(0,\pi),\frac{\theta}{\pi}\notin\mathbb{Q}}(2\lceil\frac{(k_1+1)\theta}{2\pi}\rceil-2\lceil\frac{k_1\theta}{2\pi}\rceil-1)(S^-_\Phi(e^{\sqrt{-1}\theta})-S^+_\Phi(e^{\sqrt{-1}\theta})),\end{aligned} \tag{5.14}$$

where the second equality holds by the fact that $S^-_\Phi(e^{\sqrt{-1}(2\pi-\theta)})=S^+_\Phi(e^{\sqrt{-1}\theta})$ and

$$\lceil\frac{(k_1+1)(2\pi-\theta)}{2\pi}\rceil-\lceil\frac{k_1(2\pi-\theta)}{2\pi}\rceil=1-\left(\lceil\frac{(k_1+1)\theta}{2\pi}\rceil-\lceil\frac{k_1\theta}{2\pi}\rceil\right). \tag{5.15}$$

By (3.13) and (5.13), (5.14) is equivalent to

$$n-1=\sum_{\theta\in(0,\pi),\frac{\theta}{\pi}\notin\mathbb{Q}}(2\lceil\frac{(k_1+1)\theta}{2\pi}\rceil-2\lceil\frac{k_1\theta}{2\pi}\rceil-1)(S^-_\Phi(e^{\sqrt{-1}\theta})-S^+_\Phi(e^{\sqrt{-1}\theta})). \tag{5.16}$$

From (S4) of Lemma 3.8, we have

$$\left|S^-_\Phi(e^{\sqrt{-1}\theta})-S^+_\Phi(e^{\sqrt{-1}\theta})\right|\le\nu_{e^{\sqrt{-1}\theta}}(\Phi), \tag{5.17}$$

Note that

$$\lceil\frac{k_1\theta}{2\pi}\rceil\le\lceil\frac{(k_1+1)\theta}{2\pi}\rceil=\lceil\frac{k_1\theta}{2\pi}+\frac{\theta}{2\pi}\rceil\le\lceil\frac{k_1\theta}{2\pi}\rceil+1. \tag{5.18}$$

Substituting (5.17) and (5.18) into (5.16), we see that

$$n-1\le\sum_{\theta\in(0,\pi),\frac{\theta}{\pi}\notin\mathbb{Q}}\nu_{e^{\sqrt{-1}\theta}}(\Phi)\le n-1,$$

which implies y_1 is elliptic.

Step 4. *We show that Φ is symplectic similar to $R(\theta_1)\diamond R(\theta_2)\diamond\cdots\diamond R(\theta_{n-1})$ with $\frac{\theta_i}{\pi}\notin\mathbb{Q}$, which finishes the proof.*

By (i) of Lemma 3.11, we assume that Φ can be connected within $\Omega^0(\Phi)$ to

$$\begin{aligned}&R(\vartheta_1)\diamond\cdots\diamond R(\vartheta_r)\diamond N_2(e^{\sqrt{-1}\rho_1},u_1)\diamond\cdots\diamond N_2(e^{\sqrt{-1}\rho_{r_*}},u_{r_*})\\ &\diamond N_2(e^{\sqrt{-1}\lambda_1},v_1)\diamond\cdots\diamond N_2(e^{\sqrt{-1}\lambda_{r_0}}.v_{r_0}),\end{aligned} \tag{5.19}$$

where $N_2(e^{\sqrt{-1}\rho_j},u_j)$s are non-trivial and $N_2(e^{\sqrt{-1}\lambda_{r_j}},v_j)$s are trivial basic normal forms in (3.11); ϑ_j, ρ_j, $\lambda_j\in(0,2\pi)$ with $\frac{\vartheta_j}{\pi}$, $\frac{\rho_j}{\pi}$, $\frac{\lambda_j}{\pi}\notin\mathbb{Q}$; and $r+2(r_*+r_0)=n-1$. Then by Lemma 3.11, (5.16) becomes

$$\begin{aligned}n-1&=\sum_{1\le j\le r}(2\lceil\frac{(k_1+1)\vartheta_j}{2\pi}\rceil-2\lceil\frac{k_1\vartheta_j}{2\pi}\rceil-1)\\ &\le r\le n-1,\end{aligned}$$

where the inequality holds by (5.18). Consequently, $r=n-1$, $r_*=0=r_0$, and the basic normal form decomposition (5.19) is precisely

$$R(\vartheta_1)\diamond\cdots\diamond R(\vartheta_{n-1}), \tag{5.20}$$

with $\vartheta_j\in(0,2\pi)$, $\frac{\vartheta_j}{\pi}\notin\mathbb{Q}$ and

$$\lceil\frac{(k_1+1)\vartheta_j}{2\pi}\rceil-\lceil\frac{k_1\vartheta_j}{2\pi}\rceil=1 \tag{5.21}$$

for each $1\le j\le n-1$.

As a direct consequence of (5.15) and (5.21), we have the following claim:

Claim 1. *If $R(\theta)$ appears in the basic normal form decomposition (5.20), then $R(2\pi-\theta)$ does not appear.*

Finally, we finish the proof by the following assertion:

Claim 2. *There exists $Q_1 \in \mathrm{Sp}(2n-2)$ such that*

$$Q_1^{-1}\Phi Q_1 = R(\theta_1)\diamond\cdots\diamond R(\theta_{n-1}) \tag{5.22}$$

with $\theta_i = \vartheta_i$ or $2\pi-\vartheta_i$ for $1\le i\le n-1$.

In fact, suppose $\omega_1^{\pm1},\dots,\omega_{n-1}^{\pm1}$ with $\omega_j = e^{\sqrt{-1}\vartheta_j}$ are the eigenvalues of Φ. By [L02, Theorem 1.6.11], there exists $X_1 \in \mathrm{Sp}(2n-2)$ such that

$$X_1\Phi X_1^{-1} = S_1\diamond\cdots\diamond S_{m_1}\diamond S_0, \tag{5.23}$$

where $S_0 \in \mathrm{Sp}(2k_0)$ with $k_0\ge 0$ and $\omega_1\notin\sigma(S_0)$, while for $1\le i\le m_1$, $k_i\ge 1$ and $S_i\in\mathrm{Sp}(2k_i)$ is in the normal form $N_{k_i}(\lambda_i,b_i)$ with $\lambda_i=\omega_1$ or ω_1^{-1} as defined in [L02, Section 1.6].

Applying Cases 3 and 4 from [L02, Section 1.8], if $k_i\ge 3$ for some $1\le i\le m_1$, then S_i can be connected within $\Omega^0(S_i)$ to some $\widetilde{S}_i$ whose total multiplicity of elliptic eigenvalues is strictly less than that of S_i. This contradicts (5.20). Hence $S_i = N_{l_i}(\lambda_i,b_i)$ with $l_i=1$ or 2 for $1\le i\le m_1$.

We now show $l_i=1$ for all $1\le i\le m_1$. Suppose $l_i=2$ for some i. Then by (5.20), S_i is not a basic normal form, implying

$$S_i\in\mathcal{M}^2_{\omega_1}(4) := \{M\in\mathrm{Sp}(4) : \dim_{\mathbb{C}}\ker_{\mathbb{C}}(M-\omega_1\mathrm{Id}) = 2\}.$$

By Case 4 in [L02, Section 1.8], S_i can be connected within $\Omega^0(S_i)$ to $R(\omega_1)\diamond R(2\pi-\omega_1)$, contradicting Claim 1. Thus (5.23) simplifies to

$$X_1\Phi X_1^{-1} = R(\theta_1)^{\diamond m_1}\diamond S_0, \tag{5.24}$$

with $\theta_1=\vartheta_1$ or $2\pi-\vartheta_1$. Iterating this argument at most $n-1$ times yields (5.22). □

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