Sum of elements preceding records in set partitions

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Abstract

In this paper, we aim to derive an explicit formula for the total number of elements preceding records over all set partitions of $[n]$ with exactly k blocks, as well as an asymptotic estimate for the total sum of elements preceding records in all set partitions of $[n]$, expressed in terms of Bell numbers. To achieve this, we analyze the generating function that enumerates set partitions of $[n]$ according to this statistic, which we denote by sume.

Keywords: Records, Sum of elements preceding records, Set partitions, Generating functions, Bell numbers and Asymptotic estimate.

1 Introduction

Define $[k] = \{1, 2, \ldots, k\}$ as a totally ordered alphabet with k letters, and let $[k]^n$ be the set of words of length n over this alphabet. Let $\omega = \omega_1\omega_2 \cdots \omega_n \in [k]^n$, an element ω_i in ω is a *record* if $\omega_i > \omega_j$ for all $j = 1, 2, \cdots, i-1$, and i is called the *position* of the record ω_i. Researchers have studied the record statistics on various combinatorial structures. The statistic sumrec was defined and studied by Kortchemski [6] on permutations (words without repeated letters). For a permutation π, sumrec(π) is defined as the sum of the positions of all records in π. For example, the permutation $\pi = 122634$ has 3 records, 1, 2 and 6, occur at positions 1, 2 and 4, respectively, yielding sumrec$(\pi) = 1+2+4 = 7$. In the context of compositions (a composition $\sigma = \sigma_1\sigma_2 \cdots \sigma_m$ of an integer n is a word over alphabet $\mathbb{N}$ whose sum is n), Knopfmacher and Mansour [3] studied the total number of records in all compositions of a given integer n, as well as the sum of the positions of these records. Myers and Wilf [8] studied records on words over finite alphabet.

A *set partition* Π of $[n]$ of size k (or, *set partition of* $[n]$ *with exactly* k *blocks*) is a collection $\{B_1, B_2, \ldots, B_k\}$ of nonempty disjoint subsets of $[n]$, called *blocks*, whose union is equal to $[n]$. We assume that blocks are listed in increasing order of their minimal elements, that is, $minB_1 < minB_2 < \cdots < minB_k$. We denote the set of all set partitions of $[n]$ with exactly k blocks to be $P_{n,k}$, and we denote the set of all set partitions of $[n]$ to be P_n. It is well-known that the number of all set partitions of $[n]$ with exactly k blocks is given by the Stirling numbers of the second kind $S_{n,k}$, and the number of all set partitions of $[n]$ is the n-th Bell number B_n, see [7]. A partition Π can be written as a word $\pi = \pi_1\pi_2 \cdots \pi_n$, where $i \in B_{\pi_i}$ for all i, and this form is called the *canonical sequential form*, see [7]. For example, the partition $\Pi = \{\{1, 5\}, \{2, 3\}, \{4\}\} \in P_{5,3}$ has the canonical sequential form

$\pi = 12231$. Knopfmacher, Mansour, and Wagner [4] found the asymptotic mean values and variances for the number, and for the sum of positions of records in all partitions of $[n]$. Moreover, Asakly [2] studied the statistic *swrec*, where $swrec(\pi)$ is defined as the sum of the position of a record in π multiplied by the value of the record over all the records in P_n. For instance, if $\pi = 122313$, then $swrec(\pi) = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 4 = 17$.

Motivated by the aforementioned results, we define $\mathrm{sume_a}(\pi)$ to be the sum of all elements preceding a record a in π, and define $\mathrm{sume}(\pi)$ as the total sum of all such $\mathrm{sume_a}(\pi)$ over all records in π. In this paper, we study this statistic in set partitions in context of canonical sequential form. For instance, if $\pi = 121132$, then $\mathrm{sume}_2(\pi) = 1$, $\mathrm{sume}_3(\pi) = 1 + 2 + 1 + 1$ and $\mathrm{sume}(\pi) = 1 + (1 + 2 + 1 + 1) = 6$. In particular, we show that the total number of $\mathrm{sume}(\pi)$ taken over all set partitions of $[n]$ is given by

$$\frac{1}{3}B_{n+3} - \frac{1}{4}B_{n+2} - (\frac{1}{2}n + \frac{13}{12})B_{n+1} - (\frac{1}{12} + \frac{1}{2}n)B_n,$$

see Theorem 7.

2 Main Results

2.1 The ordinary generating function for the number of set partitions according to the statistic sume

Let $P_{k,a}(x, q)$ be the generating function for the number of set partitions of $[n]$ with exactly k blocks according to the statistic sume_a, that is

$$P_{k,a}(x, q) = \sum_{n \geq k} \sum_{\pi \in P_{n,k}} x^n q^{\mathrm{sume}_a(\pi)}.$$

And Let $P_k(x, q)$ be the generating function for the number of set partitions of $[n]$ with exactly k blocks according to the statistic sume, that is

$$P_k(x, q) = \sum_{n \geq k} \sum_{\pi \in P_{n,k}} x^n q^{\mathrm{sume}(\pi)}.$$

We aim to find an explicit formula for the ordinary generating function $P_k(x, q)$, and using the same methods as in [4], also obtain an explicit formula for the corresponding exponential generating function.

Theorem 1 *The generating function for the number of set partitions of $[n]$ with exactly k blocks according to the statistic sume_a is given by*

$$P_{k,a}(x, q) = x^k q^{\frac{a(a-1)}{2}} \left(\prod_{j=1}^{a-1} \left(\frac{1}{1 - xq(\frac{1-q^j}{1-q})} \prod_{i=a}^{k} \frac{1}{1 - ax} \right) \right). \quad (1)$$

Proof Let Π be a set partition of $[n]$ with exactly k blocks, and let π denote its canonical sequential form, defined as follows: $\pi = 1\pi^{(1)}2\pi^{(2)}\cdots k\pi^{(k)}$, where $\pi^{(j)}$ denotes an arbitrary word over an alphabet $[j]$ including the empty word. Thus, for a fixed $1 \leq a \leq k-1$ the contribution of $\pi' = 1\pi^{(1)}2\pi^{(2)}\cdots(a-1)\pi^{(a-1)}$ to the generating function $P_k(x,q)$ is $q^{1+2+\cdots+a-1}\prod_{j=1}^{a-1}\frac{x}{1-xq(\frac{1-q^j}{1-q})}$ and the contribution of $a\pi^{(a+1)}\cdots(k-1)\pi^{(k-1)}k\pi^{(k)}$ to the generating function $P_k(x,q)$ is $\prod_{i=a}^{k}\frac{x}{1-ax}$, as required. □

Corollary 2 *The generating function for the number of set partitions of* $[n]$ *with exactly* k *blocks according to the statistic* sume *is given by*

$$P_k(x,q) = \sum_{a=1}^{k} x^k q^{\frac{a(a-1)}{2}} \left(\prod_{j=1}^{a-1} \left(\frac{1}{1-xq(\frac{1-q^j}{1-q})} \prod_{i=a}^{k} \frac{1}{1-ax} \right) \right). \tag{2}$$

Proof By summing over all $1 \leq a \leq k$ in (1) , we obtain the required result. □

2.2 Exact expression for $\sum_{\pi\in P_{n,k}} \mathrm{sume}(\pi)$

In this section we aim to prove that the total number of the sume over all set partitions of $[n]$ with exactly k blocks is

$$S_{n,k}\sum_{a=1}^{k}\frac{a(a-1)}{2} + \sum_{i=1}^{k-1}\left(\frac{(k-i)i(i+1)}{2}\sum_{j=1}^{n-k}S_{n-j,k}i^{j-1}\right).$$

For that we need the following Lemma:

Lemma 3 *For all* $k \geq 1$,

$$\frac{d}{dq}P_k(x,q)\mid_{q=1} = \frac{x^k}{(1-x)\ldots(1-kx)}\sum_{a=1}^{k}\frac{a(a-1)}{2} + \frac{x^{k+1}}{(1-x)\ldots(1-kx)}\sum_{i=1}^{k-1}\frac{(k-i)i(i+1)}{2(1-ix)}. \tag{3}$$

Proof By differentiating (2) with respect to q, we obtain

$$\frac{d}{dq}P_k(x,q)\mid_{q=1} = x^k\sum_{a=1}^{k}\lim_{q\to 1}\left(\frac{d}{dq}L_{j,a}(q)\prod_{i=a}^{k}\frac{1}{1-ix}\right), \tag{4}$$

where $L_{j,a}(q) = q^{\frac{a(a-1)}{2}} \prod_{j=1}^{a-1} \frac{1}{1-xq(\frac{1-q^j}{1-q})}$. Observe that

$$\frac{d}{dq}L_{j,a}(q) = \frac{a(a-1)}{2} q^{\frac{a(a-1)}{2}-1} \prod_{j=1}^{a-1} \frac{1}{1-xq(\frac{1-q^j}{1-q})}$$
$$+ q^{\frac{a(a-1)}{2}} \sum_{m=1}^{a-1} \left(\frac{x(\frac{1-q^j}{1-q}) + xq(\frac{-jq^{j-1}(1-q)+(1-q^j)}{(1-q)^2})}{(1-xq(\frac{1-q^j}{1-q}))^2} \right) \left(\prod_{j=1,j\neq m}^{a-1} \frac{1}{1-xq(\frac{1-q^j}{1-q})} \right).$$

By substituting the following facts

$$\lim_{q\to 1} x(\frac{1-q^j}{1-q}) = xj,$$
$$\lim_{q\to 1} (1-xq(\frac{1-q^j}{1-q}))^2 = (1-xj)^2,$$
$$\lim_{q\to 1} xq(\frac{-jq^{j-1}(1-q)+(1-q^j)}{(1-q)^2}) = \frac{xj(j-1)}{2},$$

into (4), we obtain the required result. □

Theorem 4 *The total number of the* sume *over all set partitions of* $[n]$ *with exactly k blocks is*

$$S_{n,k} \sum_{a=1}^{k} \frac{a(a-1)}{2} + \sum_{i=1}^{k-1} \left(\frac{(k-i)i(i+1)}{2} \sum_{j=1}^{n-k} S_{n-j,k} i^{j-1} \right).$$

Proof In order to enumerate the total number of all partitions of $[n]$ with exactly k blocks according to the sume statistic , we need to find the coefficients of x^n in (3). We start with the fact that

$$\frac{x^k}{\prod_{j=1}^{k}(1-jx)} = \sum_{n\geq k} S_{n,k} x^n,$$

where $S_{n,k}$ denotes the Stirling numbers of the second kind.

Moreover, for any integer $1 \leq i \leq k-1$, we have

$$\frac{x^{k+1}}{\prod_{j=1}^{k}(1-jx)} \cdot \frac{1}{1-ix} \cdot \frac{(k-i)i(i+1)}{2} = \frac{(k-i)i(i+1)}{2} \sum_{n\geq k} S_{n,k} x^n \sum_{n\geq 1} i^{n-1} x^n$$

which is equal to

$$\frac{(k-i)i(i+1)}{2} \sum_{n\geq 0} \sum_{j=1}^{n-k} S_{n-j,k} i^{j-1} x^n.$$

Which completes the proof. □

2.3 Exact expression for $\sum_{\pi\in P_n}\mathrm{sume}(\pi)$

In this section we aim to prove that the total number of the sume over all set partitions of $[n]$ is

$$\frac{1}{3}B_{n+3}-\frac{1}{4}B_{n+2}-(\frac{1}{2}n+\frac{13}{12})B_{n+1}-(\frac{1}{12}+\frac{1}{2}n)B_n.$$

For that we pass from $\frac{d}{dq}P_k(x,q)\mid_{q=1}$ to $\frac{d}{dq}\widetilde{P}(x,u,q)\mid_{u=q=1}$, where $\widetilde{P}(x,u,q)$ is the exponential generating function for the number of set partitions of $[n]$ with exactly k blocks according to the statistic sume, where x,u,q mark $n,k,$ sume. We derive the total number of sume over all set partitions of $[n]$, by finding the coefficients x^n in $\frac{d}{dq}\widetilde{P}(x,u,q)\mid_{u=q=1}$. Define $\widetilde{P}(x,u,q)$ by

$$\widetilde{P}(x,u,q)=\sum_{k\geq 0}\sum_{n\geq k}\sum_{\pi\in P_{n,k}}\frac{x^n u^k q^{\mathrm{sume}(\pi)}}{n!}.$$

In order to find $\widetilde{P}(x,u,q)$, we need the following result.

Proposition 5 *The partial fraction decomposition of* $\frac{d}{dq}P_k(1/y,q)\mid_{q=1}$ *can be expressed as*

$$\sum_{m=1}^{k}\left(\frac{a_{k,m}}{(y-m)^2}+\frac{b_{k,m}}{y-m}\right), \tag{5}$$

where

$$\begin{aligned}a_{k,m}&=\frac{(-1)^{k-m}(k-m)m(m+1)}{2(m-1)!(k-m)!},\\ b_{k,m}&=\frac{(-1)^{k-m}\left(\frac{k^3}{12}-\frac{k^2(m+1)}{4}+\frac{k(6m^2+21m+10)}{12}-\frac{3m^2}{2}-m+\sum_{i=1}^{k}\frac{i(i-1)}{2}\right)}{(m-1)!(k-m)!}.\end{aligned}$$

Proof Equation (2) can be written as

$$\frac{d}{dq}P_k(x,q)\mid_{q=1}=x^k\left(\sum_{a=1}^{k}\frac{a(a-1)}{2}+\sum_{i=1}^{k-1}\frac{(k-i)i(i+1)x}{2(1-ix)}\right)\prod_{i=1}^{k}\frac{1}{1-ix}.$$

By substituting $x^{-1}=y$ in the above equation, we obtain

$$\frac{d}{dq}P_k(1/y,q)\mid_{q=1}=\left(\sum_{a=1}^{k}\frac{a(a-1)}{2}+\sum_{i=1}^{k}\frac{(k-i)i(i+1)}{2(y-i)}\right)\prod_{i=1}^{k}\frac{1}{y-i}. \tag{6}$$

The expression above can be decomposed in the form

$$\frac{d}{dq}P_k(1/y,q)\mid_{q=1}=\sum_{m=1}^{k}\left(\frac{a_{k,m}}{(y-m)^2}+\frac{b_{k,m}}{y-m}\right).$$

To determine the coefficients $a_{k,m}$ and $b_{k,m}$, we consider Laurent expansion of (4) around $y = m$:

$$\frac{d}{dq}P_k(1/y,q)\mid_{q=1} = \frac{\sum_{a=1}^{k}\frac{a(a-1)}{2} + \frac{(k-m)m(m+1)}{2(y-m)} + \sum_{\substack{i=1\\ i\neq m}}^{k}\frac{(k-i)i(i+1)}{2(y-i)}}{(y-m)\prod_{\substack{i=1\\ i\neq m}}^{k}(y-m+m-i)}$$

$$= \frac{\sum_{a=1}^{k}\frac{a(a-1)}{2} + \frac{(k-m)m(m+1)}{2(y-m)} + \sum_{\substack{i=1\\ i\neq m}}^{k}\frac{(k-i)i(i+1)}{2(y-i)}}{(y-m)\prod_{\substack{i=1\\ i\neq m}}^{k}\Big((m-i)(1+\frac{y-m}{m-i})\Big)}.$$

By using Laurent expansion of $\frac{1}{(1+\frac{y-m}{m-i})}$ and $\frac{(k-i)i(i+1)}{2(y-i)}$ with $i \neq m$ around $y = m$, we obtain

$$\frac{d}{dq}P_k(1/y,q)\mid_{q=1}$$
$$= \frac{(-1)^{k-m}}{(y-m)(m-1)!(k-m)!}\prod_{\substack{i=1\\ i\neq m}}^{k}\left(1-\frac{y-m}{m-i}+O((y-m)^2)\right)$$
$$\cdot\left(\sum_{a=1}^{k}\frac{a(a-1)}{2} + \frac{(k-m)m(m+1)}{2(y-m)} + \sum_{\substack{i=1\\ i\neq m}}^{k}\frac{(k-i)i(i+1)}{2(m-i)} + O(y-m)\right),$$

which equals

$$\frac{d}{dq}P_k(1/y,q)\mid_{q=1}$$
$$= \frac{(-1)^{k-m}}{(y-m)(m-1)!(k-m)!}\left(1-\sum_{\substack{i=1\\ i\neq m}}^{k}\frac{y-m}{m-i}+O((y-m)^2)\right)$$
$$\cdot\left(\sum_{a=1}^{k}\frac{a(a-1)}{2} + \frac{(k-m)m(m+1)}{2(y-m)} + \sum_{\substack{i=1\\ i\neq m}}^{k}\frac{(k-i)i(i+1)}{2(m-i)} + O(y-m)\right).$$

We need to simplify the product, and consider the coefficients of $(y-m)^-1$ and $(y-m)^-2$ as follows:

$$\frac{(-1)^{k-m}}{(y-m)(m-1)!(k-m)!}$$
$$\cdot\left(\sum_{a=1}^{k}\frac{a(a-1)}{2} + \frac{(k-m)m(m+1)}{2(y-m)} + \sum_{\substack{i=1\\ i\neq m}}^{k}\frac{(k-i)i(i+1)-(k-m)m(m+1)}{2(m-i)} + O(y-m)\right).$$

With Maple, we can compute the summand, which shows that

$$\frac{(-1)^{k-m}}{(y-m)(m-1)!(k-m)!}$$
$$\cdot\left(\frac{m(k-m)(m+1)}{2(y-m)}+\sum_{a=1}^{k}\frac{a(a-1)}{2}+\frac{1}{2}\sum_{\substack{i=1\\ i\neq m}}^{k} i^2+(m-k+1)i+(m+1)(k-m)+O(y-m)\right)$$
$$=\frac{(-1)^{k-m}}{(y-m)(m-1)!(k-m)!}$$
$$\cdot\left(\frac{m(k-m)(m+1)}{2(y-m)}-\frac{k^3}{12}-\frac{k^2(m+1)}{4}+\frac{k(6m^2+21m+10)}{12}-\frac{3m^2}{2}-m+\sum_{a=1}^{k}\frac{a(a-1)}{2}\right).$$

Consequently, finding the coefficients of $\frac{1}{(y-m)}$ and $\frac{1}{(y-m)^2}$ completes the proof. □

Now, we proceed to find an explicit formula for the generating function $\frac{d}{dq}\widetilde{P}(x,1,q)\mid_{q=1}$.

Theorem 6 *We have*

$$\frac{d}{dq}\widetilde{P}(x,u,q)\mid_{u=q=1}=e^{e^x-1}\left(\frac{1}{3}e^{3x}-\frac{1}{2}xe^{2x}+\frac{3}{4}e^{2x}-xe^x-e^x-\frac{1}{12}\right). \tag{7}$$

Proof By replacing $\frac{1}{y-m}=\frac{x}{1-mx}=\sum_{\ell\geq 0}m^\ell x^{\ell+1}$ with $\frac{e^{mx}-1}{m}=\sum_{\ell\geq 0}\frac{m^\ell x^{\ell+1}}{(\ell+1)!}$ and $\frac{1}{(y-m)^2}$ with $\frac{e^{mx}(mx-1)+1}{m^2}$ in (5), we pass to exponential generating function. Moreover, by multiplying the result by u^k and summing over all k, we obtain

$$\frac{d}{dq}\widetilde{P}(x,u,q)\mid_{q=1}=\sum_{k\geq 1}u^k\sum_{m=1}^{k}\frac{(-1)^{k-m}(k-m)m(m+1)}{2(m-1)!(k-m)!}\cdot\frac{e^{mx}(mx-1)+1}{m^2}$$
$$+\sum_{k\geq 1}u^k\sum_{m=1}^{k}\frac{(-1)^{k-m}\left(-\frac{k^3}{12}-\frac{k^2(m+1)}{4}+\frac{k(6m^2+21m+10)}{12}-\frac{3m^2}{2}-m+\sum_{i=1}^{k}\frac{i(i-1)}{2}\right)}{(m-1)!(k-m)!}\cdot\frac{e^{mx}-1}{m}.$$

By changing the order of the summation, we get

$$\frac{d}{dq}\widetilde{P}(x,u,q)\mid_{q=1}=\sum_{m\geq 1}\frac{e^{mx}(mx-1)+1}{m!}\sum_{k\geq m}\frac{(-1)^{k-m}(k-m)(m+1)u^k}{2(k-m)!}$$
$$+\sum_{m\geq 1}\frac{e^{mx}-1}{m!}\sum_{k\geq m}\frac{(-1)^{k-m}\left(-\frac{k^3}{12}-\frac{k^2(m+1)}{4}+\frac{k(6m^2+21m+10)}{12}-\frac{3m^2}{2}-m+\sum_{i=1}^{k}\frac{i(i-1)}{2}\right)}{(k-m)!}u^k.$$

By Substituting $\ell = k - m$ and rewriting the above equation, we obtain

$$\frac{d}{dq}\widetilde{P}(x,u,q)\mid_{q=1} = \sum_{m\geq 1}\frac{e^{mx}(mx-1)+1}{m!}\sum_{\ell\geq 0}\frac{(-1)^\ell(m+1)\ell u^{m+\ell}}{2\ell!}$$
$$+\sum_{m\geq 1}\frac{e^{mx}-1}{m!}\sum_{\ell\geq 0}\frac{(-1)^\ell\left(-\frac{(\ell+m)^3}{12}-\frac{(\ell+m)^2(m+1)}{4}\right)}{\ell!}u^{m+\ell}$$
$$+\sum_{m\geq 1}\frac{e^{mx}-1}{m!}\sum_{\ell\geq 0}\frac{(-1)^\ell\left(\frac{(\ell+m)(6m^2+21m+10)}{12}-\frac{3m^2}{2}-m+\sum_{i=1}^{\ell+m}\frac{i(i-1)}{2}\right)}{\ell!}u^{m+\ell}.$$

By substituting $u=1$ into the previous terms, we complete the proof. $\square$

Theorem 7 *The total number of* sume *taken over all set partitions of* $[n]$ *is given by*

$$\frac{1}{3}B_{n+3}-\frac{1}{4}B_{n+2}-(\frac{1}{2}n+\frac{13}{12})B_{n+1}-(\frac{1}{12}+\frac{1}{2}n)B_n.$$

Proof According to Theorem 6, we have

$$\frac{d}{dq}\widetilde{P}(x,u,q)\mid_{u=q=1}= e^{e^x-1}\left(\frac{1}{3}e^{3x}-\frac{1}{2}xe^{2x}+\frac{3}{4}e^{2x}-xe^x-e^x-\frac{1}{12}\right).$$

Differentiating the generating function

$$e^{e^x-1}=\sum_{n\geq 0}B_n\frac{x^n}{n!}$$

three times, we get

$$e^xe^{e^x-1}=\sum_{n\geq 0}B_{n+1}\frac{x^n}{n!},$$

$$e^{2x}e^{e^x-1}=\sum_{n\geq 0}B_{n+2}\frac{x^n}{n!}-\sum_{n\geq 0}B_{n+1}\frac{x^n}{n!},$$

and

$$e^{3x}e^{e^x-1}=\sum_{n\geq 0}B_{n+3}\frac{x^n}{n!}-3\sum_{n\geq 0}B_{n+2}\frac{x^n}{n!}+2\sum_{n\geq 0}B_{n+1}\frac{x^n}{n!}.$$

From the above equations, it follows that

$$xe^xe^{e^x-1}=\sum_{n\geq 0}nB_n\frac{x^n}{n!},$$

and

$$xe^{2x}e^{e^x-1}=\sum_{n\geq 0}nB_{n+1}\frac{x^n}{n!}-\sum_{n\geq 0}nB_n\frac{x^n}{n!}.$$

Using all these facts together, we obtain

$$\frac{d}{dq}\widetilde{P}(x,u,q)\mid_{u=q=1}=\sum_{n\geq 0}\left(\frac{1}{3}B_{n+3}-\frac{1}{4}B_{n+2}-(\frac{1}{2}n+\frac{13}{12})B_{n+1}-(\frac{1}{12}+\frac{1}{2}n)B_n\right)\frac{x^n}{n!},$$

which completes the proof. □

In order to obtain asymptotic estimate for the moment as well as limiting distribution, we need the fact

$$B_{n+h}=B_n\frac{(n+h)!}{n!r^h}\left(1+O(\frac{\log n}{n})\right)$$

uniformly for $h=O(\log n)$, where r is the positive root of $re^r=n+1$. For more details about the asymptotic expansion of Bell numbers, we refer the reader to [1]. Therefore, Theorem 7 gives the following corollary.

Corollary 8 *Asymptotically, as $n\to\infty$, the total number of* swrec *taken over all set partitions of* $[n]$ *is given by* $B_n\frac{n^3}{r^3}\left(1+\frac{r}{n}\right)\left(1+O(\frac{\log n}{n})\right)$, *where r is the positive root of $re^r=n+1$.*

In addition, we show that asymptotically the total number of the sume over all set partitions of $[n]$ is given by

$$B_n\frac{n^3}{r^3}\left(1+\frac{r}{n}\right)\left(1+O(\frac{\log n}{n})\right),$$

where r is the positive root of $re^r=n+1$.

Acknowledgments

Walaa Asakly would like to dedicate this work to the memory of her father, whose love and encouragement continue to inspire her.

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