

The classification of quasi-elliptic fibrations and unexpected plane cubics in characteristics 2 and 3

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Abstract

In this paper, we categorize all isomorphism classes of quasi-elliptic surfaces over a field k of characteristic 2 or 3. For every quasi-elliptic surface X , we classify all possible sequences of blow-downs from X to the projective plane $\mathbb{P}_k^2$. We then use these categorizations to identify all unexpected plane cubic curves in characteristic 2 and present a proof of the lack of unexpected cubics in characteristic 3. Before the work in this paper—based partly on the author’s thesis—the complete classification of unexpected plane cubic curves in characteristic 2 was unknown, as well as the question of the existence of unexpected plane cubic curves in characteristic 3.

1 Introduction to quasi-elliptic fibrations

This paper concerns the notion of a quasi-elliptic surface X . The main result of this paper is found in Theorem 1 and Corollary 3 in section 4, wherein we prove there are no unexpected plane cubics in $\mathbb{P}_k^2$ for $\text{char } k = 3$. In this section, we prove some general results about quasi-elliptic fibrations. In section 2, we describe in detail the Dynkin diagrams associated to each quasi-elliptic surface in characteristic 2. We use the Dynkin diagrams to investigate all possible ways of blowing down the surface X to a plane, up to automorphism of X , providing explicit examples of each type of fibration. All the ways of blowing X down to a plane then give us all possible labelings of the vertices of the Dynkin diagram of X , with labels as (-2) -curves of the Picard group, up to automorphism of X . In section 3, we will collect all the data acquired about the blow-downs seen in section 2 to determine which blow-downs yield unexpected plane cubics in characteristic 2.

In section 4, we describe the Dynkin diagrams associated to quasi-elliptic surfaces in characteristic 3, and perform the same analysis as in section 2 to classify all possible ways to blow up a plane to get a quasi-elliptic surface, up to automorphism of X . Finally, we investigate each possible manner of blowing up $\mathbb{P}^2$ to a quasi-elliptic surface and go on to show that none of these quasi-elliptic surfaces yield an unexpected plane cubic curve, and indeed that there are no unexpected plane cubics in characteristic 3.

This is related to work done in characteristic 0 by Zanardini [17], who had classified constructions of Halphen pencils of index 2; (quasi-)elliptic surfaces are examples of index-1 Halphen pencils [10].

Definition 1. Let k be an algebraically closed field, and let X be a smooth projective surface. A **quasi-elliptic fibration** $f : X \rightarrow \mathbb{P}_k^1$ is a morphism whose general fiber is isomorphic to a singular plane cubic curve.

Note that if X is a surface admitting such a morphism f , we will often simply refer to X as a quasi-elliptic fibration. It is known that quasi-elliptic fibrations can occur only in characteristics 2 and 3 [2, 15]. Also, any quasi-elliptic surface is also the consecutive blowing-up of the projective plane $\mathbb{P}^2$ at nine (possibly infinitely-near) points [11]. The nine points are in such a configuration that $\dim | -K_X| = 1$, where $K_X \in \text{Pic} X$ is the canonical divisor of X ; thus X is Jacobian [12].

We will show that every quasi-elliptic fibration X is extremal; that is, every quasi-elliptic fibration has only finitely many sections (equivalently, the Mordell-Weil group of sections $\text{MW}(X)$ is finite).

Each section of X corresponds to a linear combination of the base points under the group law of each fibre, with a fixed base point chosen as the group identity. We will show such a set of points is well-defined.

Lemma 1. *Let $A, B, C, D \in F \subset \mathbb{P}^2$, a general plane cubic curve. Let $\langle P_1, \dots, P_n \rangle_Q$ denote the subgroup of F generated by the points $P_1, \dots, P_n \in F$ with $Q \in F$ chosen as the zero point. Let $P_1 +_Q P_2$, $-_Q P$, and $n \cdot_Q P$ denote addition, negation, and $\mathbb{Z}$ -module multiplication with Q as the zero point, respectively. Then $A +_B C \in \langle A, B, C \rangle_D$, $A +_B C \in \langle A, B \rangle_C$, $-_A B \in \langle A \rangle_B$, and $-_A B \in \langle A, B \rangle_C$.*

Proof. This is immediate from the bijection $F \leftrightarrow \text{Pic}^1 F$. □

Thus we immediately have the following corollaries.

Corollary 1. *For any nine points $P_1, \dots, P_9 \in F$, we have $\langle P_1, \dots, P_9 \rangle_{P_i} = \langle P_1, \dots, P_9 \rangle_{P_j}$ for all $1 \leq i, j \leq 9$.*

Corollary 2. *Let X be a blow-up of $\mathbb{P}^2$ that gives a quasi-elliptic fibration. Then X is extremal.*

Proof. By Lemma 1, it is well-defined to say each section of X corresponds to a linear combination of the base points under the group law of each fibre, with some fixed base point chosen as the zero.

Let F_η be the fiber of $\eta \in \mathbb{P}_k^1$, and define G_η as $\langle P_1, \dots, P_9 \rangle_{P_i}$ as a subgroup of F_η . Since each fibre F_η is of additive type [11], and $\text{char } k \in \{2, 3\}$, the subgroup G_η is a set of linear combinations of finitely many $\mathbb{Z}/2\mathbb{Z}$ - or $\mathbb{Z}/3\mathbb{Z}$ -vectors and is thus finite. Since each (-1) -curve is a section, they correspond to a curve given by a point in G_η as η runs through $\mathbb{P}_k^1$, there must be finitely many (-1) -curves. Therefore X is extremal. □

The following lemma will be used to enumerate the number of (-1) -curves of every quasi-elliptic surface we will see in this paper.

Lemma 2. *Let X be a (quasi-)elliptic surface with canonical divisor $K \in \text{Pic}(X)$. Define one of the (-1) -curves s_0 of X as the zero-section of X in the Mordell-Weil group $\text{MW}(X)$. Then define $U = \langle K, s_0 \rangle$ and $V \leq \text{Pic}(X)$ to be generated by the (-2) -curves that do not intersect s_0 . Then $V \leq U^\perp$ and*

$$\text{MW}(X) \cong U^\perp / V.$$

Proof. Line VII.2.5 of Theorem VII.2.1 in [16]. $\square$

In particular, since X is extremal, we have that the number of (-1) -curves is equal to the size of the quotient group $U^\perp/V$.

Definition 2. Let $Z \subseteq \mathbb{P}_k^2$ be a set of points. Let $I_{Z+dP} \subseteq k[\mathbb{P}^2]$ be the ideal $\bigcap_{q \in Z} I(q) \cap I(P)^d$. We say Z admits an **unexpected curve** of degree $d+1$ if

$$h^0(\mathbb{P}_k^2, I_{Z+dP}(d+1)) > \max\left(0, I_Z(d+1) - \binom{d+1}{2}\right),$$

for a general point $P \in \mathbb{P}_k^2$.

For a finite set $Z \subseteq \mathbb{P}_k^n$ for $n \geq 3$, we consider whether Z admits an unexpected cone of degree d , which is when

$$h^0(\mathbb{P}_k^n, I_{Z+dP}(d)) > \max\left(0, I_Z(d) - \binom{d+n-1}{n}\right)$$

for a general point $P \in \mathbb{P}_k^n$. We will borrow notation from Chiantini and Migliore [4], who denote Z admitting an unexpected cone of degree d as Z having the $C(d)$ property, in $\mathbb{P}_k^n$ (or $C_Z(d)$ if it is not clear which Z is intended).

In 2017, Solomon Akesseh [1] proved that a finite set $Z \subseteq \mathbb{P}^2$ of distinct points has an unexpected plane cubic curve only when Z consists of the seven points of the Fano plane in characteristic 2. Farnik, Galuppi, Sodomaco, and Trok [8] recovered the result that there is no unexpected cubic in characteristic 0, and in addition showed in characteristic 0 that unexpected quartics arise for a unique (up to choice of coordinates) configuration of 9 points in $\mathbb{P}^2$. Configurations that admit unexpected varieties in higher dimensions are connected to an interesting geometric property known as the geproci property [3], which also has a home in the positive characteristic setting [14].

To begin discussing unexpected cubic curves, it is necessary to detail their connection to quasi-elliptic fibrations with the following lemma. Note that throughout this paper, we refer to a set Z of “not necessarily distinct” points; this refers to the fact that some points of Z may be infinitely-near other points of Z . Given an algebraic set X , a point Q is **infinitely-near** the point P if Q maps to P under the standard blowup map $\pi_P : \text{Bl}_P(X) \rightarrow X$, as in Chapter 4.2 of [9]. On the other hand, if $Q \in X \setminus P$, then Q and P are **distinct**. We will often denote an ordinary point of $\mathbb{P}_k^2$ as $P^{(1)}$ and denote a point infinitely-near $P^{(i)}$ as $P^{(i+1)}$, and the set of points $P^{(1)}, P^{(2)}, \dots, P^{(n)}$ as $P \times n$.

Lemma 3. Let Z be a set of seven (not necessarily distinct) points in $\mathbb{P}_k^2$ admitting an unexpected cubic. Then the points impose independent conditions on cubics, every cubic curve through Z is singular, the general cubic through Z is reduced and irreducible, and for a general point P there is a unique cubic C_P singular at P ; it is reduced and irreducible.

Proof. The proof of Lemma 3.2.9 in Akesseh [1] carries over to the case that Z contains infinitely near points. $\square$

Proposition 1. Let $Z = \{P_1, \dots, P_7\}$ be a set of seven (not necessarily distinct) points in $\mathbb{P}_k^2$ admitting an unexpected cubic. Let $P \in \mathbb{P}^2$ be general. Then the set $\mathcal{P}_P$ of cubic curves containing

$Z \cup \{P\}$ is a quasi-elliptic fibration. Furthermore, this fibration can be viewed as a surface that is isomorphic to the blowup of $\mathbb{P}_k^2$ at nine points: the seven points of Z , the point P and a point infinitely-near P .

Proof. Since Z imposes independent conditions on cubics by Lemma 3 and since P is general, $\mathcal{P}_P$ is a linear pencil of cubics. By Lemma 3, the cubics are singular but fixed component free, so blowing up the vase points gives a quasi-elliptic fibration. Since the general member of the pencil is a cubic, there are 9 base points. By Lemma 3, the cubics containing $Z \cup \{P\}$ contain C_P and the general one, C , is reduced and irreducible. Since $C.C_P = P_1 + \dots + P_7 + 2P$, the 9 base points are $P_1, \dots, P_7, P, P_9$, where P_9 is infinitely near P . $\square$

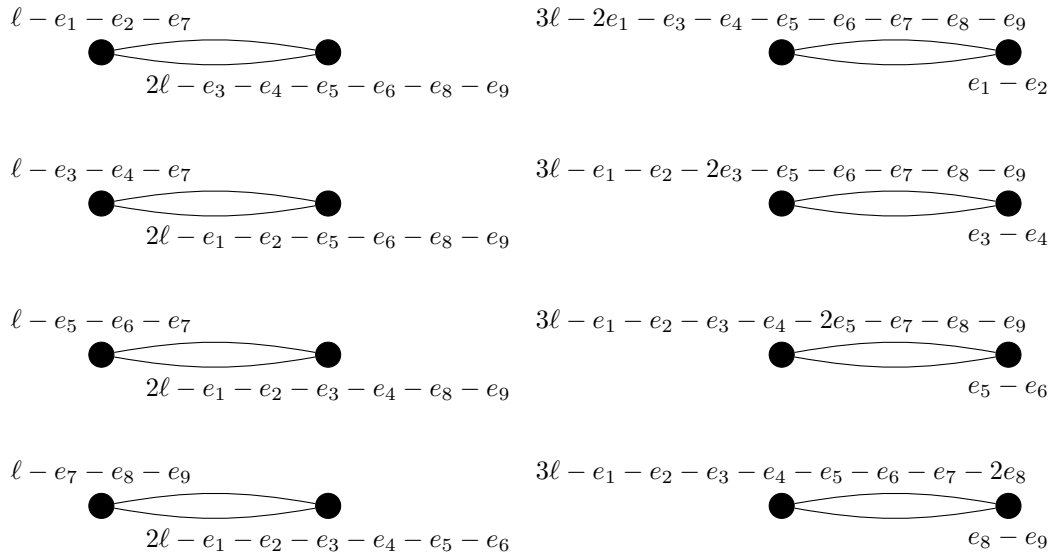
2 Quasi-Elliptic Fibrations in Characteristic 2

Given a quasi-elliptic fibration X , one may construct a multigraph D the following way: the vertex set is the set of curves on X with a self-intersection of -2 , and the number of edges connecting two vertices is equal to the multiplicity of the intersection of their respective curves. The multigraph D constructed this way is called the **Dynkin diagram** of X .

The possible Dynkin diagrams for quasi-elliptic fibrations in characteristic 2 are $\tilde{A}_1^{\oplus 8}$, $\tilde{A}_1^{\oplus 4} \oplus \tilde{D}_4$, $\tilde{A}_1^{\oplus 2} \oplus \tilde{D}_6$, $\tilde{D}_4^{\oplus 2}$, $\tilde{A}_1 \oplus \tilde{E}_7$, $\tilde{D}_8$, and $\tilde{E}_8$ [5], Proposition 5.6.3, or [6] Corollary 4.3.22. We will show that there are 2 ways to blow down $\tilde{A}_1^{\oplus 8}$ to $\mathbb{P}^2$, 4 ways to blow down $\tilde{A}_1^{\oplus 4} \oplus \tilde{D}_4$, 4 ways to blow down $\tilde{A}_1^{\oplus 2} \oplus \tilde{D}_6$, 2 ways to blow down $\tilde{D}_4^{\oplus 2}$, 2 ways to blow down $\tilde{A}_1 \oplus \tilde{E}_7$, 2 ways to blow down $\tilde{D}_8$, and 1 way to blow down $\tilde{E}_8$.

Case 1: $\tilde{A}_1^{\oplus 8}$

Let us start with $\tilde{A}_1^{\oplus 8}$. We can get this Dynkin diagram by blowing up $P_1^{(1)}, P_1^{(2)}, P_2^{(1)}, P_2^{(2)}, P_3^{(1)}, P_3^{(2)}, P_4^{(1)}, P_4^{(2)}, P_5$ where $P_n^{(2)}$ is infinitely-near $P_n^{(1)}$ in the direction of P_5 . Let us call this surface X . Note $\text{Pic}X = \langle \ell, e_1, \dots, e_9 \rangle$. We will draw the Dynkin diagram below:



Using Lemma 2, we know that there are 16 (-1) -curves. The 16 (-1) -curves are $e_2, e_4, e_6, e_7, e_9, \ell - e_1 - e_3, \ell - e_1 - e_5, \ell - e_1 - e_8, \ell - e_3 - e_5, \ell - e_5 - e_8, 2\ell - e_1 - e_2 - e_3 - e_5 - e_8, 2\ell - e_1 - e_3 - e_4 - e_5 - e_8, 2\ell - e_1 - e_3 - e_5 - e_6 - e_8, 2\ell - e_1 - e_3 - e_5 - e_7 - e_8$, and $2\ell - e_1 - e_3 - e_5 - e_8 - e_9$.

Taking the (-2) -curves and the (-1) -curves together, we can form a multigraph on 32 vertices that yields the following adjacency matrix.

$$\Gamma = \begin{pmatrix} \mathbf{B} & \mathbf{M} \\ \mathbf{M}^\top & \mathbf{R} \end{pmatrix},$$

where

$$\mathbf{B} = -2\mathbf{I}_{16} + 2\mathbf{J}_{16},$$

represents the intersection products between the (-2) -curves ($\mathbf{I}_{16}$ is the 16×16 identity matrix and $\mathbf{J}_{16}$ is the 16×16 exchange matrix),

$$\mathbf{R} = -\mathbf{I}_{16} + \mathbf{J}_{16}$$

represents the intersection products between the (-1) -curves, and

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

represents the intersection products between the (-1) - and (-2) -curves.

Proposition 2. *There are– up to isomorphism of the adjacency diagram of X – two ways to blow down X to the projective plane $\mathbb{P}^2$.*

Proof. By Castelnuovo's contraction theorem, one can blow down a curve C on a smooth projective surface X to get a smooth projective surface if and only if the self-intersection number of C is -1 [13] V.5.7.2. Let us denote by B the set of (-2) -curves and by R the set of (-1) -curves.

First choose one (-1) -curve R_1 to blow down. Note that since we are blowing down to $\mathbb{P}^2$, we will need to blow down exactly nine curves. Once we blow down a (-1) -curve, the single adjacent (-1) -curve as given by Γ cannot be blown down since its self-intersection number would increase. There are eight pairs of adjacent (-1) curves. Therefore, if we blow down eight disjoint (-1) -curves, we would still need to blow down one of the (-2) -curves as the ninth. Thus there must be at least one (-2) -vertex present in the blow-down. Choose B_1 adjacent to R_1 to blow down.

Now let us take an inventory of which curves cannot be blown down. The single (-2) -curve adjacent to B_1 cannot be blown down, because it intersects B_1 with multiplicity 2, so its self-intersection number has increased to 0. In addition, the other 7 (-1) -curves adjacent to B_1 cannot be blown down, because their self-intersection have also increased to 0. Finally, the other 7 (-2) -curves adjacent to R_1 cannot be blown down, because their self intersection increases to (-1) after R_1 is blown down, but since B_1 also intersects R_1 , their self intersection would then increase to 0 after B_1 is blown down.

What is left of Γ after deleting all the vertices whose self-intersection number has risen to a nonnegative value after blowing down R_1 and B_1 is a three-regular bipartite graph on 14 vertices containing 7 disjoint (-1) -vertices and 7 disjoint (-2) -vertices.

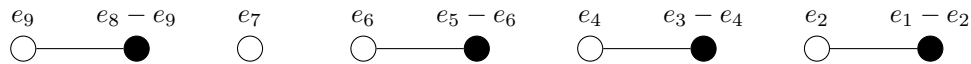
In this subgraph, the two possibilities for blow-downs are to blow down the 7 disjoint (-1) -vertices, or to blow down one (-1) -vertex and three (-1) - (-2) pairs.

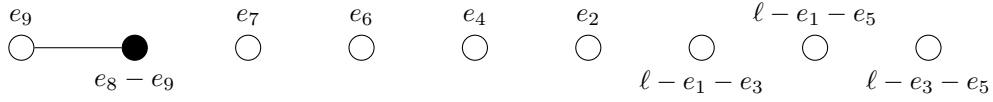
It is impossible to blow down three disjoint (-1) vertices and two (-1) - (-2) pairs because there do not exist three (-1) -vertices that no two (-2) -vertices are adjacent to.

It is impossible to blow down five disjoint (-1) -vertices and one (-1) - (-2) pair because every (-2) -vertex is adjacent to three (-1) -vertices: if one has already been blown down, then there are two of the remaining six connected to it, leaving four out of six not connected to it. This is less than the five required, so this blow down is impossible. Thus there are only two types of blow-downs. $\square$

Note that we can run this procedure algebraically by performing the following computations on the adjacency matrix Γ . We use the fact that blowing down the curve R_1 increases the self-intersection number of a curve C by $(C.R_1)^2$ and increases the intersection product of two general divisors C and D by $(C.R_1) \cdot (D.R_1)$, [13] Proposition V.3.2(d) and Proposition V.3.6. We can thus form a new graph Γ_{R_1} induced by blowing down R_1 by using the column $\mathbf{r}_1$ of Γ corresponding to the vertex representing R_1 and computing $\mathbf{r}_1 \mathbf{r}_1^T + \Gamma$. Then the row and column corresponding to R_1 are now completely zero, and the (i, j) -entry of Γ_{R_1} is the intersection product of the curves corresponding to Γ_i and Γ_j after having blown down R_1 . We may repeat this procedure until we blow down to $\mathbb{P}^2$, when Γ will have been transformed into a matrix with no negative entries.

The two blow-downs are given by the following diagrams. The (-1) -curves are illustrated by the hollow vertices, and the (-2) -curves are illustrated by the solid black vertices.





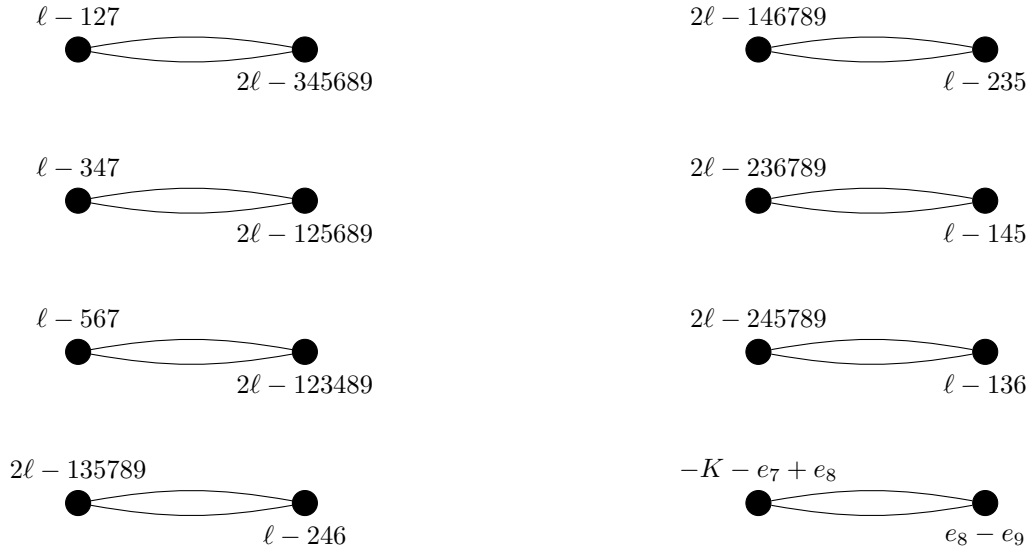
The first blow-down diagram corresponds to the exact points on $\mathbb{P}^2$ that were blown up to obtain the surface X in the first place.

The second diagram induces a map $A : \text{Pic}X \rightarrow \text{Pic}X$ given by the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The map A sends each (-1) -curve shown in the second blow-down diagram to a (-1) -curve of the form e_i and each (-2) -curve shown to a (-2) -curve of the form $e_i - e_j$. This allows us to express all curves in $\text{Pic}(X)$ as $\mathbb{Z}$ -linear combinations of the curves of the second blow-down diagram, thus allowing us to learn what configuration of points in $\mathbb{P}_k^2$ we are required to blow up to attain X . In section 3, we will determine which configurations yield unexpected cubics.

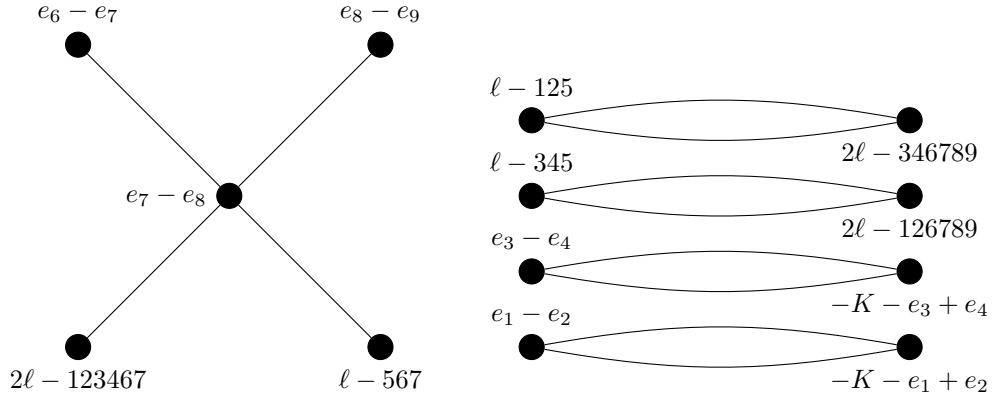
Applying A to the Dynkin diagram gives us a new Dynkin diagram shown below. To save space, we will denote $3\ell - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - 2e_8$ as $-K - e_7 + e_8$, where $K = -3\ell + e_1 + \dots + e_9$ is the canonical divisor and rename $2\ell - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6}$ as $2\ell - i_1 i_2 i_3 i_4 i_5 i_6$, and $\ell - e_{i_1} - e_{i_2} - e_{i_3}$ as $\ell - i_1 i_2 i_3$.



- The original Dynkin diagram corresponds to the blowup of $\mathbb{P}^2$ at five ordinary points $P_1^{(1)}$ through $P_5^{(1)}$ and four infinitely-near points $P_1^{(2)}, \dots, P_4^{(2)}$ where each infinitely-near point $P_i^{(2)}$ corresponds to the line containing the ordinary points $P_i^{(1)}$ and P_5 . An example of a pencil of cubics with base locus at such an arrangement is that spanned by $a^2(xy + xz + yz) + (ab + ac + bc)x^2$ and $b^2(xy + xz + yz) + (ab + ac + bc)y^2$, where $(a, b, c) \in \mathbb{P}_k^2$ may be a generic point.
- The second diagram corresponds to seven points blown up in a Fano plane configuration, and a general eighth point blown up twice. An example of a pencil of cubics whose base locus is such an arrangement is that spanned by $(ac + c^2)(x^2y + xy^2) + (ab + b^2)(x^2z + xz^2)$ and $(bc + c^2)(x^2y + xy^2) + (ab + b^2)(y^2z + yz^2)$, where $(a, b, c) \in \mathbb{P}_k^2$ may be a generic point.

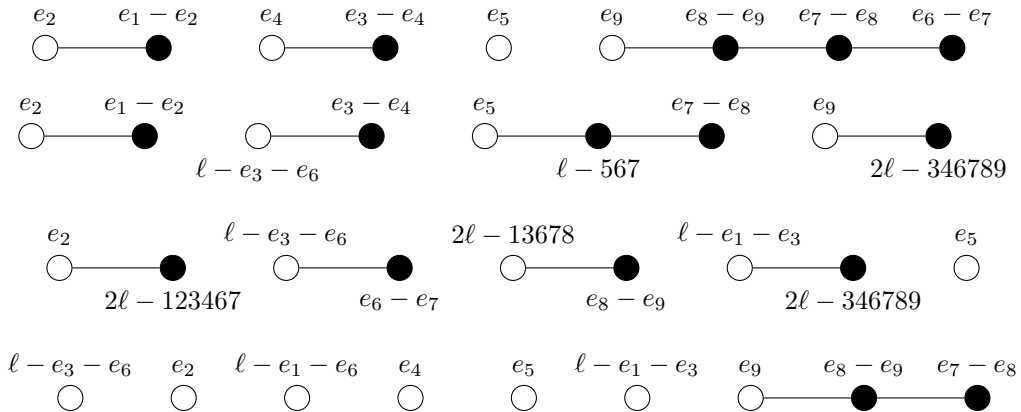
Case 2: $\tilde{A}_1^{\oplus 4} \oplus \tilde{D}_4$

Let's now look at the Dynkin diagram $\tilde{A}_1^{\oplus 4} \oplus \tilde{D}_4$, as depicted below.



Using Lemma 2, we know there are eight (-1) -curves; they are $e_2, e_4, e_5, e_9, \ell - e_1 - e_3, \ell - e_1 - e_6, \ell - e_3 - e_6$, and $2\ell - e_1 - e_3 - e_6 - e_7 - e_8$.

A similar analysis as in Proposition 2 reveals that there are— up to isomorphism of the graph— four ways to blow down the surface to $\mathbb{P}^2$. The details of this analysis have been omitted for the sake of brevity. The four ways to blow down this diagram (up to isomorphism) are given by the following diagrams.



The three bottom blow-downs give us the matrices

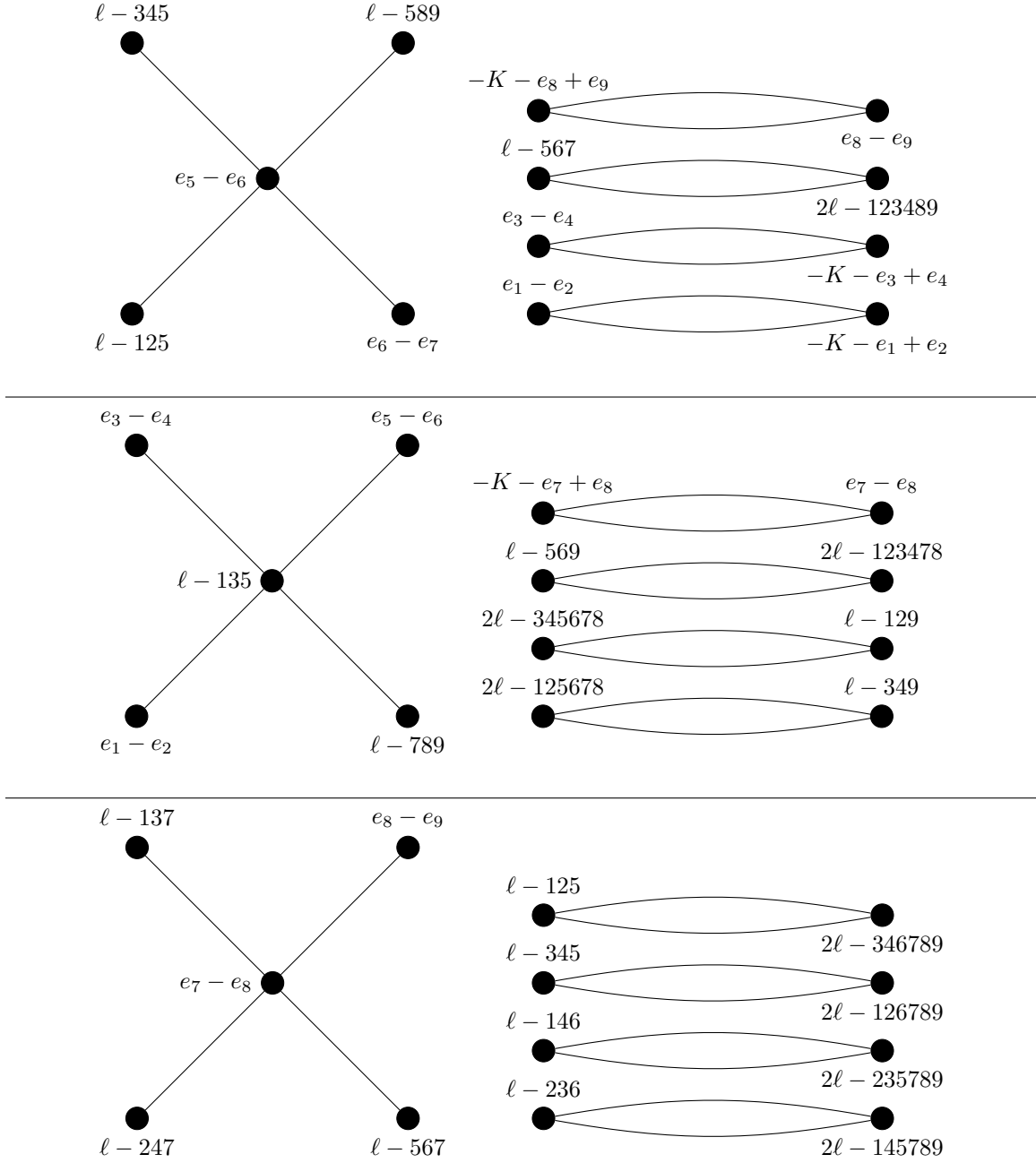
$$\begin{pmatrix} 3 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X,$$

$$\begin{pmatrix} 5 & 2 & 0 & 3 & 1 & 0 & 2 & 2 & 1 & 1 \\ -2 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ -2 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 \\ -3 & -1 & 0 & -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X,$$

and

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X$$

respectively. Applying these change-of-basis matrices to the original Dynkin diagram gives us the following diagrams.

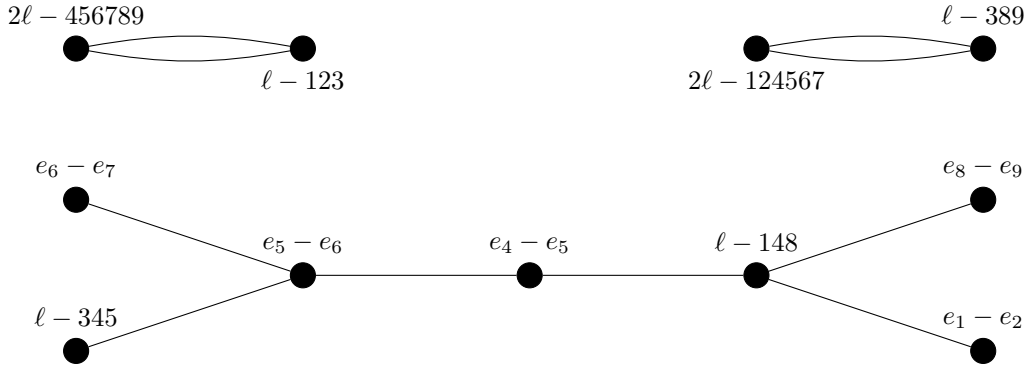


- The original diagram is a pencil defined by the union of a conic C_1 and a tangent line L_1 , and the union of a conic C_2 that intersects C_1 at two points with multiplicity 2 and a line L_2 that intersects C_1 at one point with multiplicity 2. An example of such a pencil is $x^2z + xy^2$ and $x^2z + \varphi xz^2 + \varphi^2 y^2z$ where φ satisfies $\varphi^2 + \varphi + 1 = 0$.
- The second diagram is a pencil of the union of a conic and a line with the union of three concurrent lines, each tangent to the conic. An example of such a pencil is that spanned by $(y^2 + xz)(x + \varphi z)$ and $xz(x + z)$.

- The third diagram is a pencil spanned by the union of a conic and a line, and the union of a line that is tangent to the conic and a double line. An example of such a pencil is that spanned by $(y^2 + xz)(x + \varphi z)$ and $y^2(x + z)$.
- The fourth diagram corresponds to a pencil spanned by the union of three concurrent lines, and the union of a line and a conic that contains that contains the point where the three lines intersect. The seven ordinary points in the intersection of these two cubics form a Fano plane. An example of such a pencil is that spanned by $(x + y)(x + z)(y + z)$ and $(x + y + z)(Ax(y + z) + By(x + z))$ where $(A, B) \in \mathbb{P}^1$.

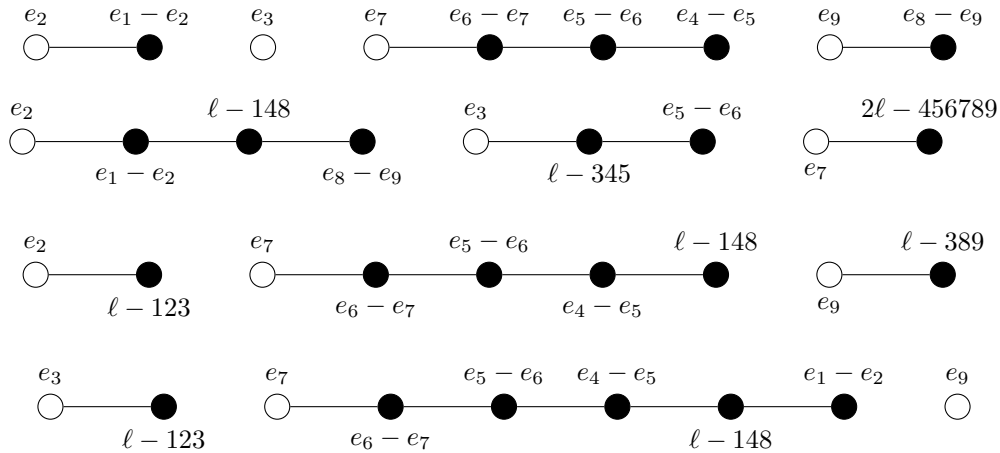
Case 3: $\tilde{A}_1^{\oplus 2} \oplus \tilde{D}_6$

Let's now look at the Dynkin diagram $\tilde{A}_1^{\oplus 2} \oplus \tilde{D}_6$, as depicted below.



Using Lemma 2, we know there are four (-1) -curves; they are e_2 , e_3 , e_7 , and e_9 .

A similar analysis as in Proposition 2 reveals that there are— up to isomorphism of the graph— four ways to blow down the surface to $\mathbb{P}^2$. The details of this analysis have been omitted for the sake of brevity. The four ways to blow down this diagram (up to isomorphism) are given by the following diagrams.



The bottom three blow-downs give us the change-of-basis matrices

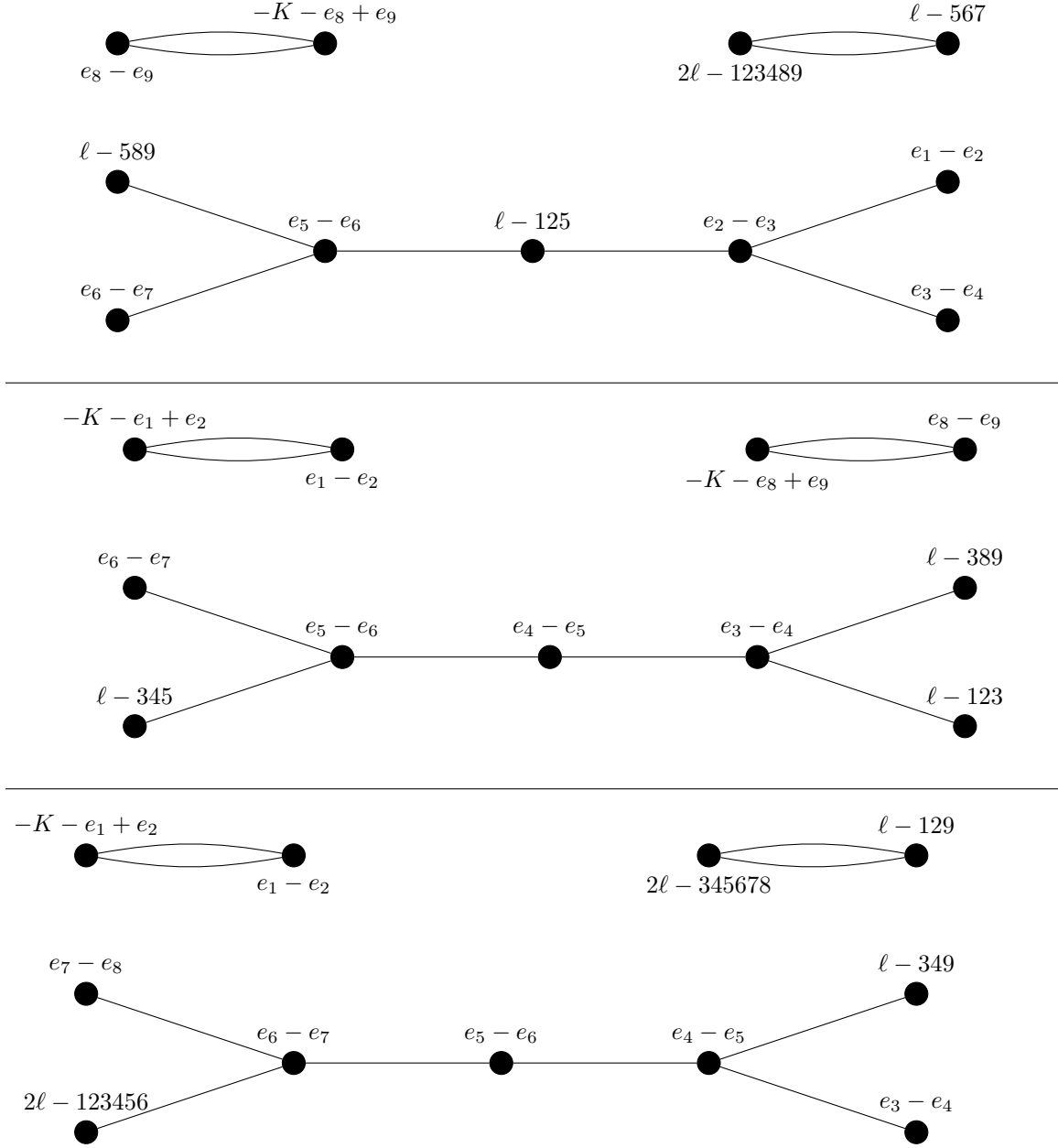
$$\begin{pmatrix} 3 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X,$$

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X,$$

and

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X$$

respectively. Applying these change-of-basis matrices to the original Dynkin diagram gives us the following diagrams.



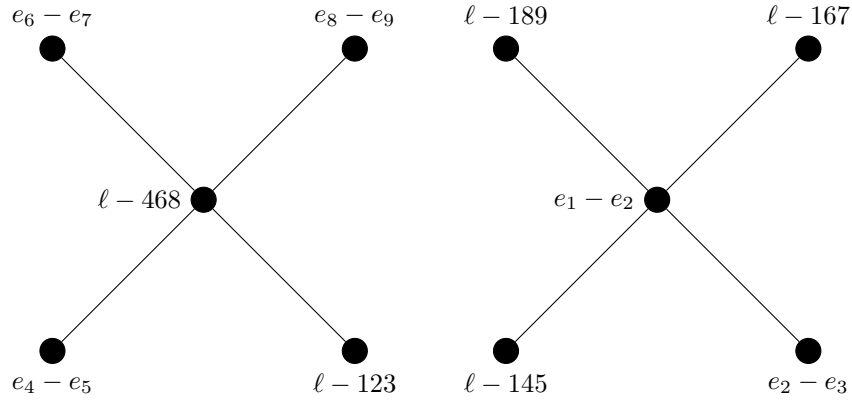
- The original diagram corresponds to a pencil spanned by the union of a conic and a line, and the union of a line tangent to the conic and a double line. An example of such a pencil is that spanned by $(y^2 + xz)(x + z)$ and xy^2 .
- The second diagram corresponds to a pencil spanned by the union of a conic and a line, and the union of two lines tangent to the conic and intersect on the first line, one of which is double. An example of such a pencil is $(y^2 + xz)(x + z)$ and x^2z .
- The third diagram corresponds to a pencil spanned by the union of three concurrent lines, and a cuspidal cubic that has a cusp on one of the lines, is tangent to another line at the concurrent

intersection point, and is tangent to the third line elsewhere. An example of such a pencil is that spanned by $x^3 + y^2z$ and $xz(x + z)$.

- The fourth diagram corresponds to a pencil spanned by the union of a conic and a tangent line, and the union of a line that is tangent to the first conic and a conic that intersects the first conic with multiplicity 4 at the original tangent point, creating a multiplicity-6 intersection of the two cubics at that point. An example of such a pencil is that spanned by $(y^2 + xz)x$ and $z(y^2 + x^2 + xz)$. The modulus is 1 because there is a pencil of conics that meet each other at a point of multiplicity 4.

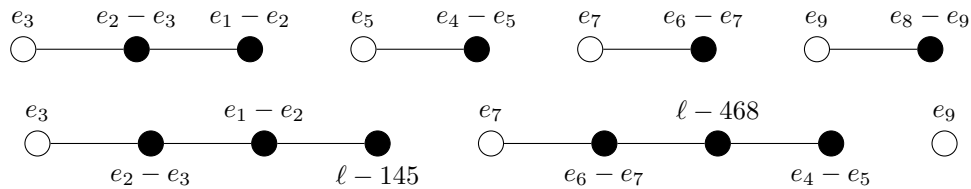
Case 4: $\tilde{D}_4^{\oplus 2}$

Let's now look at the Dynkin diagram $\tilde{D}_4^{\oplus 2}$, as depicted below.



Using Lemma 2, we know there are four (-1) -curves of this diagram; they are e_3 , e_5 , e_7 , and e_9 .

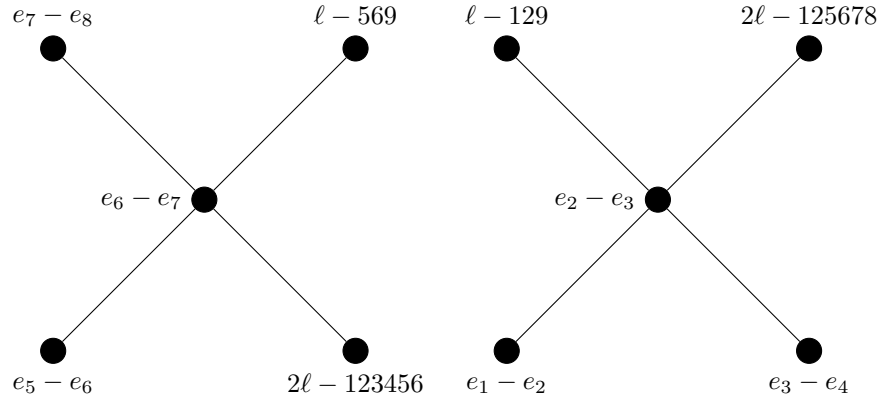
A similar analysis as in Proposition 2 reveals that there are—up to isomorphism of the graph—two ways to blow down the surface to $\mathbb{P}^2$. The details of this analysis have been omitted for the sake of brevity. The two blow-downs of the diagram (up to isomorphism) are represented by the diagrams below.



The second blow-down gives us the following change-of-basis matrix.

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X.$$

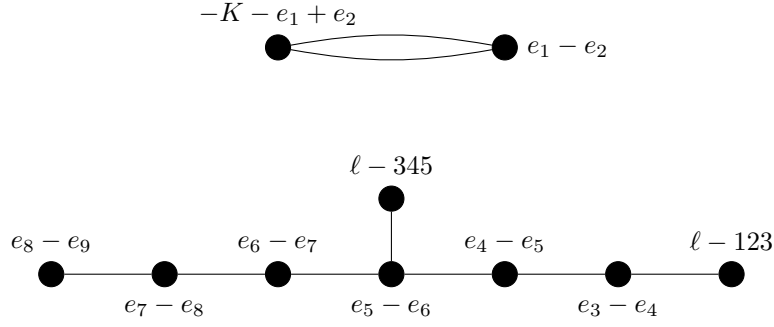
Applying this matrix to the Dynkin diagram gives us the following relabeling.



- The original diagram corresponds to a pencil spanned by the union of a line and a double line, and the union of three concurrent lines which all intersect on the single line. An example of such a pencil is that spanned by $x(x+z)z$ and $y^2(x+\varphi z)$.
- The second diagram corresponds to a pencil spanned by the union of a conic C_1 and a tangent line L_1 , and the union of a conic C_2 which intersects C_1 at two points with multiplicity 2, one of which is the intersection point of C_1 with L_1 , and a line L_2 which is tangent to C_1 and C_2 at the other point where they meet with multiplicity 2. An example of such a pencil is that spanned by $x(y^2+xz)$ and $z(\varphi y^2+xz)$.

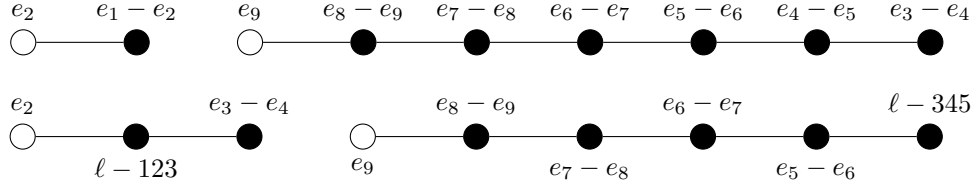
Case 5: $\tilde{A}_1 \oplus \tilde{E}_7$

Let's now look at the Dynkin diagram $\tilde{A}_1 \oplus \tilde{E}_7$, as depicted below.



Using Lemma 2, we know there are two (-1) -curves of this diagram; they are e_2 and e_9 .

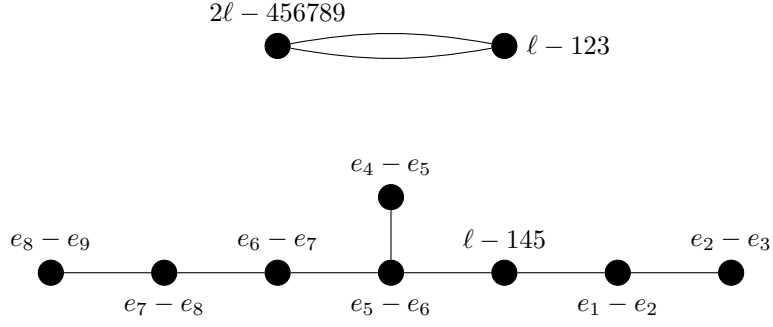
A similar analysis as in Proposition 2 reveals that there are— up to isomorphism of the graph— four ways to blow down the surface to $\mathbb{P}^2$. The details of this analysis have been omitted for the sake of brevity. The two blow-downs of the diagram (up to isomorphism) are represented by the diagrams below.



The second blow-down gives us the following change-of-basis matrix.

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X.$$

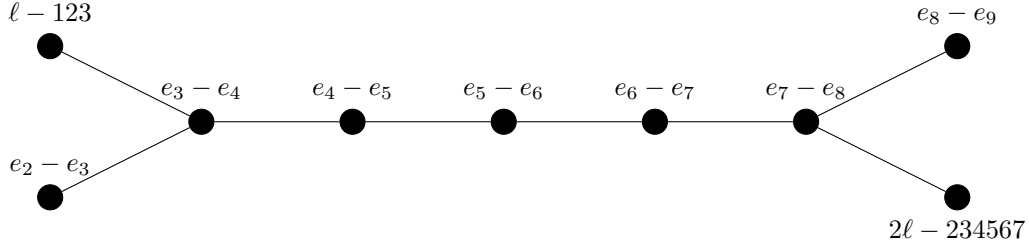
Applying this matrix to the Dynkin diagram gives us the following relabeling.



- The first diagram gives us a pencil spanned by a cuspidal cubic, and the union of a line and a double line, where the double line is tangent to the flex point of the cuspidal cubic and the line goes through the cusp and the flex point. An example of such a pencil is that spanned by $x^2 + y^2z$ and xz^2 .
- The second diagram corresponds to a pencil spanned by the union of a conic and a line, and a triple line that is tangent to the conic. An example of such a pencil is that spanned by $(y^2 + xz)z$ and x^3 .

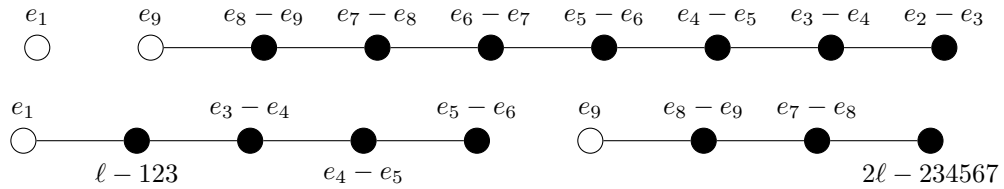
Case 6: $\tilde{D}_8$

Let's now look at the Dynkin diagram $\tilde{D}_8$, as depicted below.



Using Lemma 2, we know there are two (-1) -curves of this diagram; they are e_1 and e_9 .

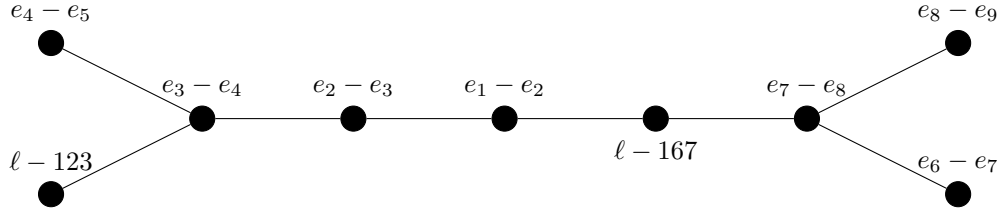
A similar analysis as in Proposition 2 reveals that there are— up to isomorphism of the graph— two ways to blow down the surface to $\mathbb{P}^2$. The details of this analysis have been omitted for the sake of brevity. The two blow-downs of the diagram (up to isomorphism) are represented by the diagrams below.



The second blow-down yields the following change-of-basis matrix.

$$\begin{pmatrix} 3 & 0 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X.$$

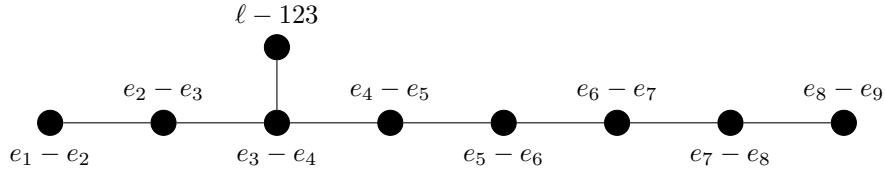
Applying the matrix to the Dynkin diagram gives us the following relabeling.



- The first diagram corresponds to a pencil spanned by a cuspidal cubic and the union of a line tangent to the cubic at somewhere other than the cusp or flex point, and a conic that meets the cubic at the same point with multiplicity 6. (Note: in characteristic 2, all points other than the flex and cusp on a cuspidal cubic are sextactic.) An example of such a conic is that spanned by $(y^2 + x^2 + xz + z^2)(x + z)$ and $x^3 + y^2z$.
- The second diagram corresponds to a pencil spanned by a cuspidal cubic and the union of a line and double line, where the line is tangent to the cubic at the flex point and the double line is tangent to a sextactic point. An example of such a pencil is that spanned by $x^3 + y^2z$ and $(x + z)^2z$.

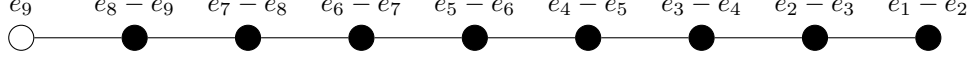
Case 7: $\tilde{E}_8$

Let's now look at the Dynkin diagram $\tilde{E}_8$, as depicted below.



The only (-1) -curve of this diagram is e_9 .

A similar analysis as in Proposition 2 reveals that there is— up to isomorphism of the graph— one way to blow down the surface to $\mathbb{P}^2$. The details of this analysis have been omitted for the sake of brevity. The only way to blow down this diagram is depicted in the following diagram.



Since there is only one way to blow down this surface, there are no change of basis matrices.

- This diagram corresponds to a pencil spanned by a cuspidal cubic and a triple line that meets the cubic at its cusp with multiplicity 3. An example of such a pencil is that spanned by $x^3 + y^2z$ and y^3 .

3 Unexpected Cubics in Characteristic 2

Let k be an algebraically closed field of characteristic 2, and let $\varphi \in k$ satisfy $\varphi^2 + \varphi + 1 = 0$.

We found in total eight types of configurations of points in $\mathbb{P}_k^2$ that yield unexpected cubics. Two come from the two blow-downs of $\tilde{A}_1^{\oplus 8}$, three come from the four blow-downs of $\tilde{A}_1^{\oplus 4} \oplus \tilde{D}_4$, and three come from the four blow-downs of $\tilde{A}_1^{\oplus 2} \oplus \tilde{D}_6$. We itemize them below.

1. The two $\tilde{A}_1^{\oplus 8}$ cases:

- In the first $\tilde{A}_1^{\oplus 8}$ case, the sheet of cubics spanned by $x^2(y+z)$, $y^2(x+z)$, and $z^2(x+y)$ produces cubic curves that contain the seven points $(1,0,0) \times 2$, $(0,1,0) \times 2$, $(0,0,1) \times 2$, and $(1,1,1)$, and have cusps at generic points.
- In the second $\tilde{A}_1^{\oplus 8}$ case (outlined in Akeseh's thesis), the sheet of cubics spanned by $xy(x+y)$, $xz(x+z)$, and $yz(y+z)$ produce cubic curves that contain the seven points of the Fano plane and have cusps at generic points.

2. The three $\tilde{A}_1^{\oplus 4} \oplus \tilde{D}_4$ cases:

- In the first $\tilde{A}_1^{\oplus 4} \oplus \tilde{D}_4$ blowdown, the sheet of cubics spanned by $x^2z + xy^2$, $x^2z + \varphi xz^2 + \varphi^2 y^2z$, and $x^2z + xz^2 + \varphi y^2z$ produce cubic curves containing the seven points $(0,0,1) \times 4$, $(1,0,0) \times 2$, and $(0,1,0)$, and have cusps at generic points.
- In the second $\tilde{A}_1^{\oplus 4} \oplus \tilde{D}_4$ blowdown, the sheet of cubics spanned by $(y^2 + xz)(x + \varphi z)$, $xz(x+z)$, and $xz(x + \varphi^2 z)$ produces cubic curves that contain the seven points $(1,0,0) \times 2$, $(0,0,1) \times 2$, and $(0,1,0) \times 3$, and have cusps at generic points.
- In the third $\tilde{A}_1^{\oplus 4} \oplus \tilde{D}_4$ blowdown, the sheet of cubics spanned by $(y^2 + xz)(x + \varphi z)$, $y^2(x+z)$, and $y^2(x + \varphi^2 z)$ produces cubic curves that contain the seven points $(0,0,1) \times 2$, $(\varphi, 0, 1) \times 2$, $(1,0,0) \times 2$, and $(0,1,0)$, and have cusps at generic points.

3. The three $\tilde{A}_1^{\oplus 2} \oplus \tilde{D}_6$ cases:

- In the first $\tilde{A}_1^{\oplus 2} \oplus \tilde{D}_6$ blowdown, the sheet of cubics spanned by $(y^2 + xz)(x + z)$, x^2z , and $x^2(x+z)$ produces cubic curves that contain the seven points $(0,0,1) \times 4$ and $(0,1,0) \times 2$, and have cusps at generic points.
- In the third $\tilde{A}_1^{\oplus 2} \oplus \tilde{D}_6$ blowdown, the sheet of cubics spanned by $x^3 + y^2z$, $xz(x+z)$, and $xz(x + \varphi z)$ produces a cubic curve that contains the seven points $(0,0,1) \times 2$, and $(0,1,0) \times 5$, and has a cusp at a generic point.

- (c) In the fourth $\tilde{A}_1^{\oplus 2} \oplus \tilde{D}_6$ blowdown, the sheet of cubics spanned by $x(y^2+xz)$, $z(y^2+x^2+xz)$, and $(x+z)(y^2+\varphi x^2+xz)$ produces a cubic curve that contains the seven points $(0,0,1) \times 6$ and $(0,1,0)$, and has a cusp at a generic point.

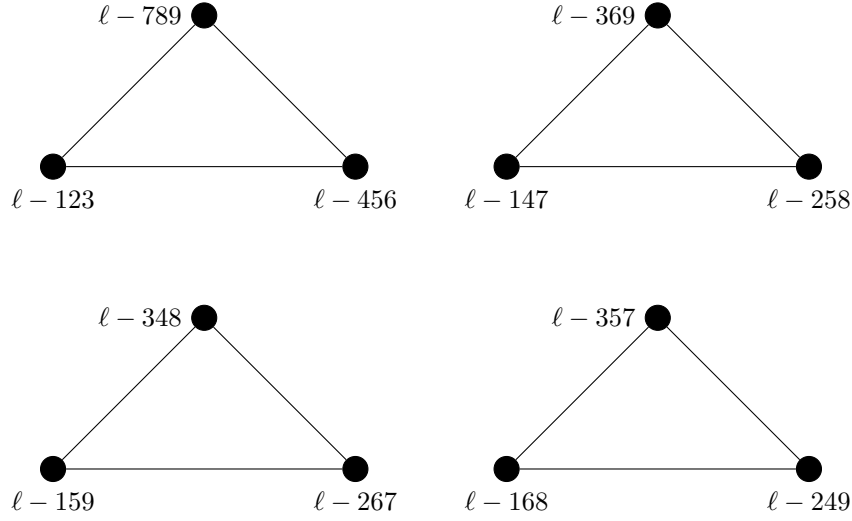
The cuspidal curves produced by each sheet above is an unexpected cubic for a set Z of seven points. The examples where some of the points of Z are infinitely near are new.

4 Quasi-Elliptic Fibrations in Characteristic 3

Now let us consider quasi-elliptic fibrations in characteristic 3. According to Cossec and Dolgachev, a quasi-elliptic fibration in characteristic 3 will have a Dynkin diagram of $\tilde{A}_2^{\oplus 4}$, $\tilde{A}_2 \oplus \tilde{E}_6$, or $\tilde{E}_8$, [5], Proposition 5.5.9, or [6] Corollary 4.3.22.

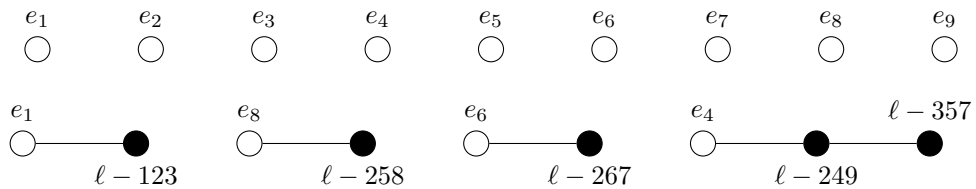
Case 1: $\tilde{A}_2^{\oplus 4}$

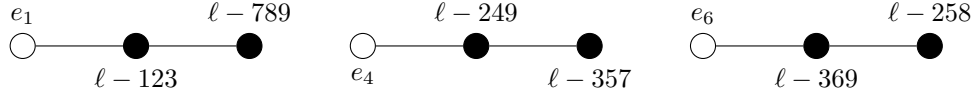
Let us first consider the diagram $\tilde{A}_2^{\oplus 4}$. We can label this diagram as follows.



Using Lemma 2, we know there are nine (-1) -curves; according to this labeling, they are $e_1, \dots, e_9$.

A similar analysis as in Proposition 2 reveals that there are— up to isomorphism of the graph— three ways to blow down the surface to $\mathbb{P}^2$. The details of this analysis have been omitted for the sake of brevity. The three blow-downs of the diagram (up to isomorphism) are represented by the diagrams below.





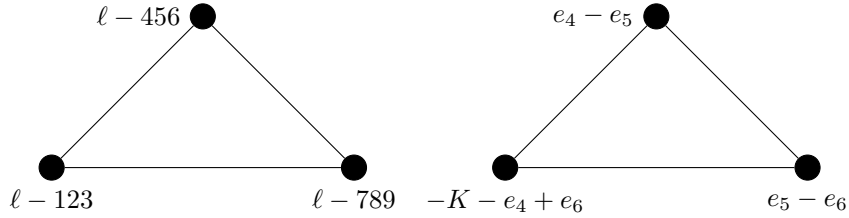
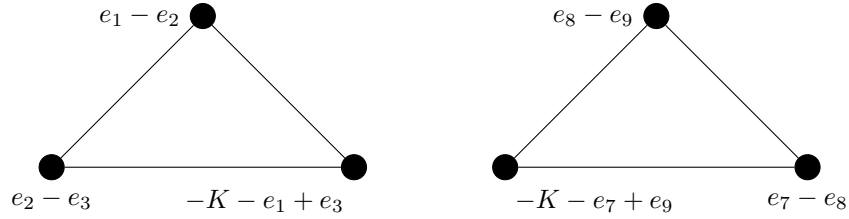
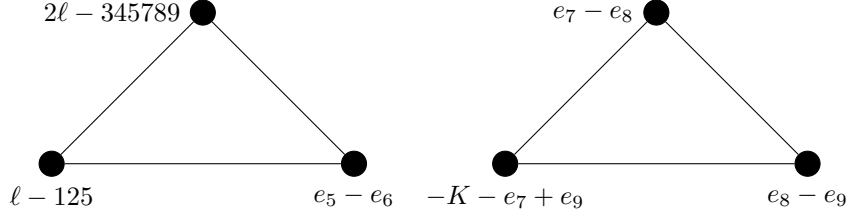
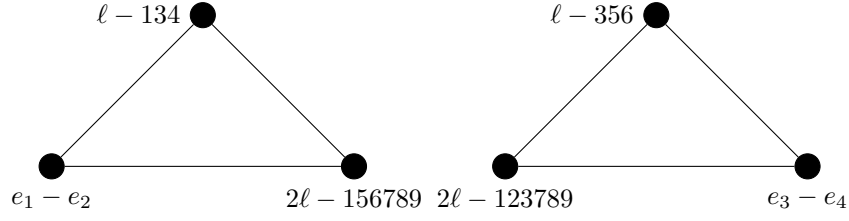
The second two blow-downs give us the change-of-basis matrices

$$\begin{pmatrix} 3 & 0 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X$$

and

$$\begin{pmatrix} 4 & 0 & 2 & 2 & 0 & 1 & 0 & 1 & 1 & 2 \\ -2 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X$$

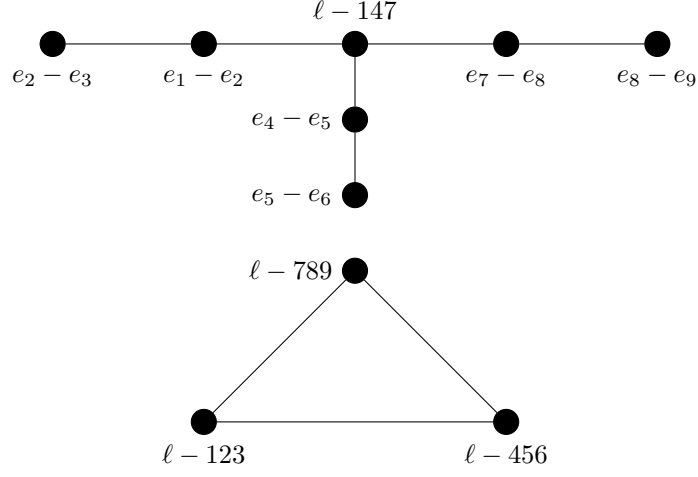
respectively. Applying these matrices to the Dynkin diagram above gives us the following relabelings.



- The original Dynkin diagram corresponds to a pencil spanned by the union of three concurrent lines and the union of three different concurrent lines. An example of such a pencil is that spanned by $x(x+z)(x-z)$ and $y(y+z)(y-z)$.
- The second diagram corresponds to a pencil spanned by the union of a conic and a line, and the union of a line tangent to the first conic and a conic that contains the intersection of the two lines and the intersection of the first conic and the first line. An example of such a pencil is that spanned by $2x^2z + yz^2$ and $xy^2 + 2yz^2$.
- The third diagram corresponds to a pencil spanned by a flexible cuspidal cubic and the union of three lines, two of which are tangent to flex points on the cubic and one which intersects the cusp with multiplicity 3. An example of such a pencil is that spanned by $x^3 - y^2z$ and $yz(y-z)$.

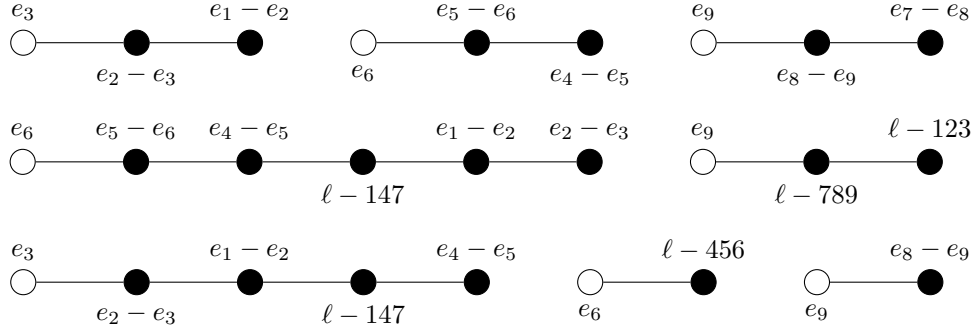
Case 2: $\tilde{A}_2 \oplus \tilde{E}_6$

Let us now consider the diagram $\tilde{A}_2 \oplus \tilde{E}_6$. A labeling is pictured below.



Using Lemma 2, we know this surface has three (-1) -curves; they are e_3 , e_6 , and e_9 .

A similar analysis as in Proposition 2 reveals that there are—up to isomorphism of the graph—three ways to blow down the surface to $\mathbb{P}^2$. The details of this analysis have been omitted for the sake of brevity. There are three ways (up to isomorphism) of blowing this surface down to $\mathbb{P}^2$. They are depicted below.



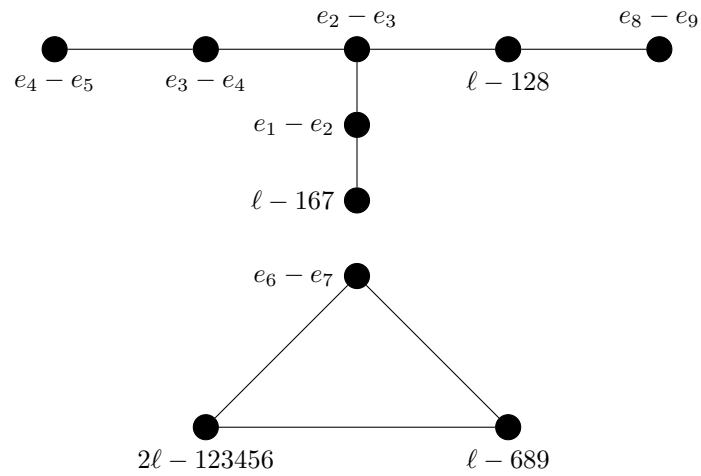
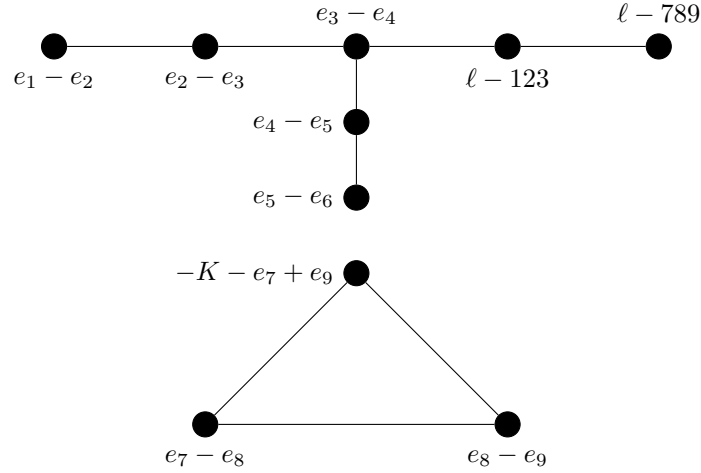
The second two blow-downs give us the change-of-basis matrices

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X$$

and

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \text{Pic}X \rightarrow \text{Pic}X$$

respectively. Applying these matrices to the Dynkin diagram gives us the relabelings depicted below.

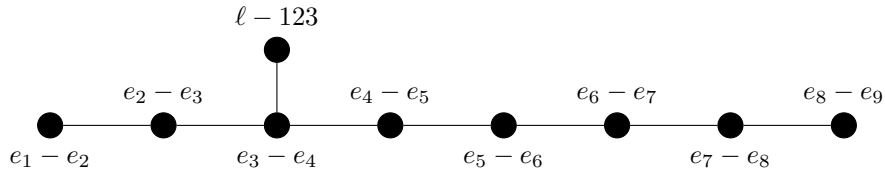


- The original diagram corresponds to a pencil spanned by the union of three lines and a triple line. An example of such a pencil is that spanned by xyz and $(x + y + z)^3$.

- The second diagram corresponds to a pencil spanned by a flexible cuspidal cubic and the union of a line and a double line, with the line meeting the cubic at a flex point with multiplicity three, and the double line intersecting the cusp of the cubic with multiplicity three.
- The third diagram corresponds to a pencil spanned by the union of a conic and a line and the union of a line and a double line, with the line containing one of the intersection points of the conic and the first line, and the double line is tangent to the conic at the other point it meets the second line. An example of such a pencil is that spanned by $x^2(x-y)$ and $(2xy+2xz+y^2)z$.

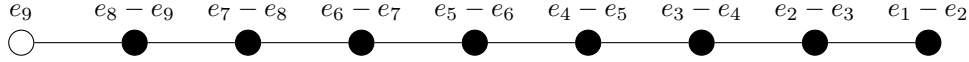
Case 3: $\tilde{E}_8$

Let us now consider the Dynkin diagram $\tilde{E}_8$, as depicted below.



Using Lemma 2, we know there is only one (-1) -curve of this diagram; it is e_9 .

A similar analysis as in Proposition 2 reveals that there is—up to isomorphism of the graph—one way to blow down the surface to $\mathbb{P}^2$. The details of this analysis have been omitted for the sake of brevity. The only way to blow down this diagram is depicted in the following diagram.

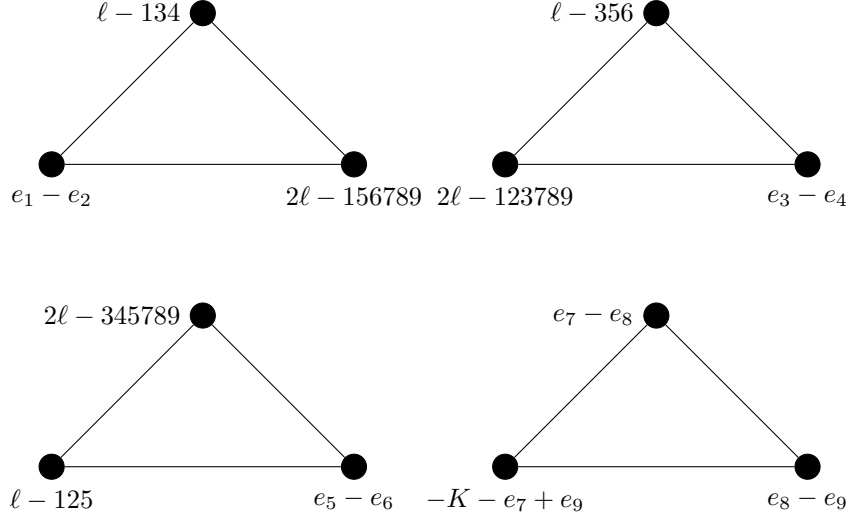


Since there is only one way to blow down this surface, there are no change of basis matrices.

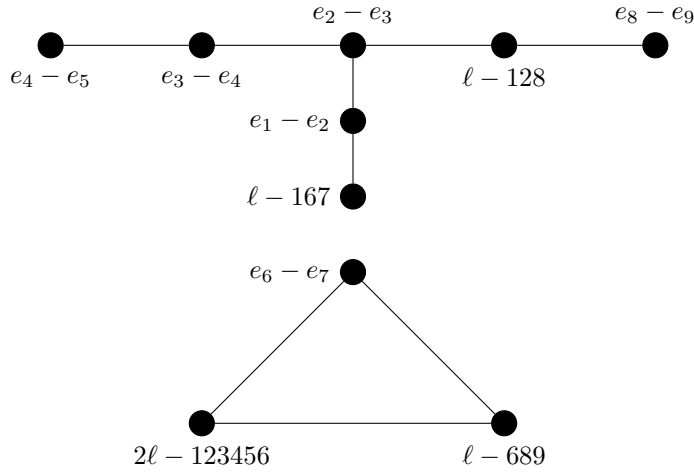
- This diagram corresponds to a pencil spanned by a cuspidal cubic and a triple line that meets the cubic at its cusp with multiplicity 3. An example of such a pencil is that spanned by $x^3 + y^2z$ and y^3 .

Theorem 1. *Let k be a field of characteristic 3. There are no unexpected cubic curves in $\mathbb{P}_k^2$ coming from the quasi-elliptic fibrations listed above.*

Proof. Recall that an unexpected cubic curve comes from choosing seven points $P_1, P_2, \dots, P_7 \in \mathbb{P}^2$ such that there is a cubic with a double point at the general point P . By Proposition 1, an unexpected cubic yields a quasi-elliptic fibration that is isomorphic to the blowup of $\mathbb{P}_k^2$ at the following nine points: the seven points $P_1, \dots, P_7$, a general point P and a point infinitely-near P . Since P is general, it will not be equal to any of the P_i . Therefore we need only consider the quasi-elliptic fibrations that are attained by blowing up $\mathbb{P}_k^2$ at some point exactly twice, with the other seven blowup points elsewhere. Out of all of the quasi-elliptic fibrations we've seen in this chapter, there are only two that come from blowing up $\mathbb{P}_k^2$ at some point exactly twice. We need only check that quasi-elliptic fibrations with the Dynkin diagrams



and



will not yield an unexpected cubic.

In the $\tilde{A}_2^{\oplus 4}$ diagram, we have an example of a pencil spanned by $2x^2z + yz^2$ and $xy^2 + 2yz^2$. This pencil has a base points of multiplicity 2 at $(0, 0, 1)$, $(0, 1, 0)$, and $(1, 0, 0)$ and has a base point of multiplicity 3 at $(1, 1, 1)$. We can create a two-dimensional linear system of cubics by introducing the third polynomial $(xz + 2y^2)(x + 2y)$. This linear system has two base points of multiplicity 2: one at $(0, 0, 1)$ and one at $(1, 0, 0)$. It also has a base point of multiplicity 3 at $(1, 1, 1)$. These seven points are all the base points of the linear system. Since unexpected cubics have a general double point, we must now determine whether this linear system has a general double point.

Let us examine the linear combination

$$F(xz + 2y^2)(x + 2y) + G(2x^2z + yz^2) + H(xy^2 + 2yz^2)$$

and the generic point $P = (a, b, c)$. It is sufficient to reduce to the affine case $z = 1$, and we want to

determine whether there are values for F , G , and H such that

$$F(x + 2y^2)(x + 2y) + G(2x^2 + y) + H(xy^2 + 2y)$$

has a double point at $(A, B) = (a/c, b/c)$. By shifting $x \mapsto x + A$ and $y \mapsto y + B$, we get that (F, G, H) annihilates the constant and linear components of the resulting polynomial if and only if

$$(F, G, H) \in \ker \begin{pmatrix} -A^2B + B^3 & -A^2 + B & AB^2 - B \\ -B^2 - A - B & A & B^2 \\ AB - A & 1 & -AB - 1 \end{pmatrix}.$$

But this matrix has a determinant

$$A^4B^2 - A^3B^3 - A^2B^4 + A^4B + A^2B^3 - AB^4 - B^5 + A^3B - AB^3 + A^3 \neq 0$$

and so this two-dimensional linear system of cubics does not have a generic double point. Therefore these seven base points do not admit an unexpected cubic. We can repeat this process by instead creating a two-dimensional linear system that omits the double point at $(0, 0, 1)$ or at $(1, 0, 0)$, and we will similarly find that no set of seven base points will admit an unexpected cubic.

Next we will look at the $\tilde{A}_2 \oplus \tilde{E}_6$ configuration. This is a pencil of cubics with base points α of multiplicity 2, β of multiplicity 2, and γ of multiplicity 5. We will denote the nine points as $\alpha^1, \alpha^2, \beta^1, \beta^2, \gamma^1, \gamma^2, \gamma^3, \gamma^4$, and γ^5 , where α^1, α^2 , and γ^1 are collinear, γ^1, γ^2 , and β^1 are collinear, and β^1, β^2 , and α^1 are collinear.

In order to get a two-dimensional linear system of cubics, we can choose to either take the seven base points $\alpha^1, \alpha^2, \gamma^1, \dots, \gamma^5$ or $\beta^1, \beta^2, \gamma^1, \dots, \gamma^5$.

In the first scenario, we want a cubic with a generic double point P given seven base points $\alpha^1, \alpha^2, \gamma^1, \dots, \gamma^5$. In order for the resulting pencil to still be quasi elliptic, P needs to be on the line determined by γ^1 and γ^2 , and so it turns out P cannot be generic after all. If P is not on this line, then we will not have the $\tilde{A}_2 \oplus \tilde{E}_6$ diagram, and there is no quasi-elliptic Dynkin diagram for the configuration we would get.

Similarly, in the second scenario, we want a cubic with a generic double point P given seven base points $\beta^1, \beta^2, \gamma^1, \dots, \gamma^5$. In order for the resulting pencil to still be quasi elliptic, P needs to be on the line determined by β^1 and β^2 , and so it turns out P cannot be generic after all. Therefore the quasi-elliptic fibration $\tilde{A}_2 \oplus \tilde{E}_6$ does not yield an unexpected cubic.

Therefore these $\tilde{A}_2^{\oplus 4}$ and $\tilde{A}_2 \oplus \tilde{E}_6$ diagrams do not yield unexpected cubics. No other diagram of a blow-down of a quasi-elliptic fibration could yield an unexpected cubic because no other diagram involves a point being blown up exactly twice, which is necessary for the generic double point. Therefore there are no unexpected cubics in characteristic 3 coming from the quasi-elliptic fibrations enumerated above.

□

Corollary 3. *Let k be a field of characteristic 3. There are no unexpected cubic curves in $\mathbb{P}_k^2$.*

Proof. Suppose there exists an unexpected cubic in $\mathbb{P}_k^2$. Then there are seven (possibly infinitely-near) points $P_1, \dots, P_7 \in \mathbb{P}_k^2$ such that there exists a cubic curve C containing the seven points and is singular at a general point P_8 . Then consider a cubic curve C' containing $P_1, \dots, P_7, P_8$ and is singular at some point Q' , so then $C.C' = P_1 + \dots + P_7 + 2P_8$. Then we may form a pencil $\mathcal{P}$ of cubic curves spanned by C and C' whose base locus is $P_1 + \dots + P_7 + P_8^{(1)} + P_8^{(2)}$, where $P_8^{(2)}$ is an infinitely-near point to $P_8^{(1)}$ corresponding to the tangent line of C' at $P_8 \in \mathbb{P}_k^2$.

The pencil $\mathcal{P}$ thus admits a quasi-elliptic surface X by blowing up the nine points of the base locus. Then X must admit a Dynkin diagram of $\tilde{A}_2^{\oplus 4}$, $\tilde{A}_2 \oplus \tilde{E}_6$, or $\tilde{E}_8$. By Corollary 2, X must be extremal and so has finitely many (-1) -curves. Furthermore, up to reflections across (-2) -classes (i.e. conjugation), each of the diagrams embed in the (-2) -curves of X in a unique way by Theorem 5.5 of [7]. Therefore X must be isomorphic to one of the seven surfaces explored in this chapter. As we have already seen in Theorem 1, there are no unexpected cubics coming from the seven surfaces of this chapter, giving us a contradiction. $\square$

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