

Corrigendum to “Degree-Based Approximations for Network Reliability Polynomials”. Comment on J. Complex Networks 2025, 13, cnaf001

XINHAN LIU *

Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

AND

PIET VAN MIEGHEM

Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands
Email: P.F.A.VanMieghem@tudelft.nl

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Our paper [1] described the stochastic approximation $\overline{rel}_G(p) = [1 - \phi_D(1 - p)]^N$ in [1, eq. (2.2)] and the first-order approximation $(R_1)_G(p) = \prod_{i=1}^N [1 - (1 - p)^{d_i}]$ in [1, eq. (4.1)] as upper bounds for the all-terminal reliability polynomial $rel_G(p)$. The present corrigendum clarifies that the unique upper bound is $\Pr[\hat{D}_{\min} \geq 1]$, which is difficult to compute exactly, because we must account for correlated node-isolation events. Both the stochastic approximation $\overline{rel}_G$ and the first-order approximation $(R_1)_G$ ignore those correlations, assume independence and, consequently, do not always upperbound $rel_G(p)$ as stated previously. The complete graph K_3 is a counterexample, where both approximations lie below the exact reliability polynomial $rel_{K_3}(p)$, illustrating that they are not upper bounds. Moreover, as claimed in [1], the first-order approximation $(R_1)_G$ is not always more accurate than the stochastic approximation $\overline{rel}_G$. We show by an example that the relative accuracy of the stochastic approximation $\overline{rel}_G$ and the first-order approximation $(R_1)_G$ varies with the graph G and the link operational probability p .

Keywords: network robustness, node failure, probabilistic graph, reliability polynomial

1. Brief summary of our original statement

In the original paper [1], we investigated two *degree-based* formulas for the all-terminal reliability polynomial $rel_G(p)$: (i) our *stochastic approximation* $\overline{rel}_G$ and (ii) the *first-order approximation* $(R_1)_G$ due to Jason Brown *et al.* in [2]. We have argued that both *degree-based* approximations are upper bounds for $rel_G(p)$. Here, we first recall the relation that underlies these formulas and then explain the arguments in [1].

Let $\hat{G}$ denote the companion random graph, obtained by retaining each link of G independently with probability p , so that the reliability polynomial is $rel_G(p) = \Pr[\hat{G} \text{ connected}]$. For any simple graph on $N \geq 2$ nodes, connectivity of a graph implies the absence of isolated nodes, which means that the minimum degree $D_{\min}$ in the graph should be at larger than 1. The opposite implication is not always true, because a network can consist of separate, disconnected clusters containing nodes each with minimum degree larger than 1. The event $\{\hat{D}_{\min} \geq 1\}$ should hold in the graph $\hat{G}$ to be connected,

*Corresponding author. Email: x.liu-22@tudelft.nl

which leads to the equivalence

$$\{\hat{G} \text{ connected}\} \subseteq \{\hat{D}_{\min} \geq 1\} \quad \Rightarrow \quad \text{rel}_G(p) \leq \Pr[\hat{D}_{\min} \geq 1]$$

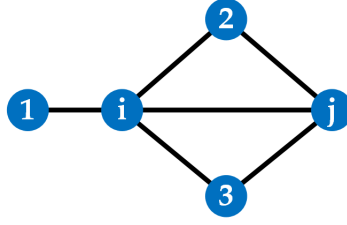


FIG. 1. Illustration of dependence between events $\{\text{Node } i \text{ is isolated}\}$ and $\{\text{Node } j \text{ is isolated}\}$. Each link is independently operational with probability p . The probability of event $\{\text{Node } i \text{ is isolated}\}$ is $\Pr[\text{Node } i \text{ is isolated}] = 1 - p^4$. For node j , the probability is $\Pr[\text{Node } j \text{ is isolated}] = 1 - p^3$. The joint probability is $\Pr[\{\text{Node } i \text{ is isolated}\} \cap \{\text{Node } j \text{ is isolated}\}] = 1 - p^6$. If the two events were independent, the probability is $\Pr[\{\text{Node } i \text{ is isolated}\} \cap \{\text{Node } j \text{ is isolated}\}] = \Pr[\{\text{Node } i \text{ is isolated}\}] \Pr[\{\text{Node } j \text{ is isolated}\}] = 1 - p^3 - p^4 + p^7$.

In general, the computation of the probability $\Pr[\hat{D}_{\min} \geq 1]$ is difficult. Due to the existence of common links, the node-isolation events are correlated, as exemplified in Fig 1. Both degree-based formulas assume independence and should thus be viewed as approximations of $\Pr[\hat{D}_{\min} \geq 1]$:

- **The stochastic approximation in [1].** Let $\varphi_D(z) = \mathbb{E}[z^D]$ be the probability generating function of the degree D of a node in the graph G . Approximating the possible N node isolation events by independent events gives

$$\overline{\text{rel}}_G(p) = \left(1 - \varphi_D(1 - p)\right)^N \approx \Pr[\hat{D}_{\min} \geq 1].$$

- **The first-order approximation ([2]).** If d_i denotes the degree of node i , then

$$(R_1)_G(p) = \prod_{i=1}^N (1 - (1 - p)^{d_i}) = \prod_{i=1}^N \Pr[\hat{D}_i \geq 1] \approx \Pr[\hat{D}_{\min} \geq 1],$$

with equality to $\Pr[\bigcap_i \{\hat{D}_i \geq 1\}]$ only under *mutual independence* of the events $\{\hat{D}_i \geq 1\}$.

Among these two degree-based approximations, the inequality

$$(R_1)_G(p) \leq \overline{\text{rel}}_G(p),$$

with equality only in regular graphs, was established in our original paper (see [1, Sec. 4.1]).

CLARIFICATION. In [1], we have implicitly assumed that good approximations of the upper bound $\Pr[\hat{D}_{\min} \geq 1]$ also upperbound the reliability polynomial $\text{rel}_G(p)$. Here, we show that this implicit assumption is not always correct. In addition, in the original paper [1], we have claimed that $(R_1)_G(p)$ is always more accurate than $\overline{\text{rel}}_G(p)$, which is also not generally true. Since both the stochastic approximation $\overline{\text{rel}}_G(p)$ and the first-order approximation $(R_1)_G(p)$ are independence-based and degree-based approximations of the upper bound $\Pr[\hat{D}_{\min} \geq 1]$, either approximation can be closer to the reliability polynomial $\text{rel}_G(p)$ depending on the graph G and the operational link probability p , as demonstrated by a counterexample in Fig. 2 below.

2. Corrigendum

The inclusion of the events $\{\hat{G} \text{ connected}\} \subseteq \{\hat{D}_{\min} \geq 1\}$ implies that the corresponding probabilities of the events obey

$$\text{rel}_G(p) \leq \Pr[\hat{D}_{\min} \geq 1].$$

Both degree-based approximations $\overline{\text{rel}}_G(p)$ and $(R_1)_G(p)$ approximate the *joint* event $\Pr[\bigcap_i \{\hat{D}_i \geq 1\}]$ by assuming independence of the events $\{\hat{D}_i \geq 1\}$ for each node i . The events $\{\hat{D}_i \geq 1\}$ are dependent as follows from the basic law of the degree, stating that $\sum_{i=1}^N \hat{D}_i = 2L$, where L is the number of links in the graph G . In general, the joint probability $\Pr[\bigcap_i \{\hat{D}_i \geq 1\}]$ depends upon the graph's degree correlation structure, which is related to the assortativity [3] of the graph G .

In summary, there is no universal inequality for the degree-based approximations: neither lower bound $\overline{\text{rel}}_G(p) \leq \Pr[\hat{D}_{\min} \geq 1]$ nor upper bound $\overline{\text{rel}}_G(p) \geq \Pr[\hat{D}_{\min} \geq 1]$ holds for all graphs G and all $p \in (0, 1)$. A similar property applies to the first-order approximation $(R_1)_G(p)$. Depending on the graph G and the probability $p \in (0, 1)$, the stochastic approximation $\overline{\text{rel}}_G(p)$ can satisfy either $\overline{\text{rel}}_G(p) > \Pr[\hat{D}_{\min} \geq 1]$ or $\overline{\text{rel}}_G(p) < \Pr[\hat{D}_{\min} \geq 1]$. Consequently, neither degree-based approximation is always an upper bound for $\text{rel}_G(p)$.

3. Illustrative counter examples

3.1 The complete-graph K_3

In the complete graph K_3 on three nodes, every node has degree $d_i = 2$. In regular graphs, the stochastic approximation (Eq. 2.2) and the first-order approximation (Eq. 4.1) are the same as shown in [1] and both yield for K_3

$$\overline{\text{rel}}_{K_3}(p) = (R_1)_{K_3}(p) = [1 - (1 - p)^2]^3 = (2p - p^2)^3.$$

However, the exact reliability polynomial in [1] for K_3 , with the number of nodes $N = 3$ and the number of links $L = 3$, is

$$\text{rel}_{K_3}(p) = \sum_{j=0}^1 F_j (1 - p)^j p^{3-j}.$$

where F_j counts the sets of j links whose removal leaves G connected. Clearly, if no node is removed, then $F_0 = 1$ and $F_1 = 3$, because deleting any one link of K_3 leaves a 3-node path, which is connected. After substituting these values of F_j , we obtain

$$\text{rel}_{K_3}(p) = 1 \cdot p^3 + 3 \cdot (1 - p)p^2 = 3p^2 - 2p^3.$$

If we define the difference $V(p) = (2p - p^2)^3 - (3p^2 - 2p^3)$, then a straightforward expansion shows $V(p) = p^2(-3 + 10p - 12p^2 + 6p^3 - p^4) < 0$ for every $0 < p < 1$ and $V(0) = V(1) = 0$. Hence, it holds that

$$(2p - p^2)^3 < 3p^2 - 2p^3 \quad \text{for } 0 < p < 1,$$

or, equivalently,

$$\overline{\text{rel}}_{K_3}(p) = (R_1)_{K_3}(p) < \text{rel}_{K_3}(p).$$

This example of K_3 shows that both the stochastic approximation $\overline{\text{rel}}_G(p)$ and the first-order approximation $(R_1)_G(p)$ *lower* bound and thus *not upper* bound the reliability polynomial $\text{rel}_G(p)$.

3.2 The modified circulant on $N = 15$ nodes

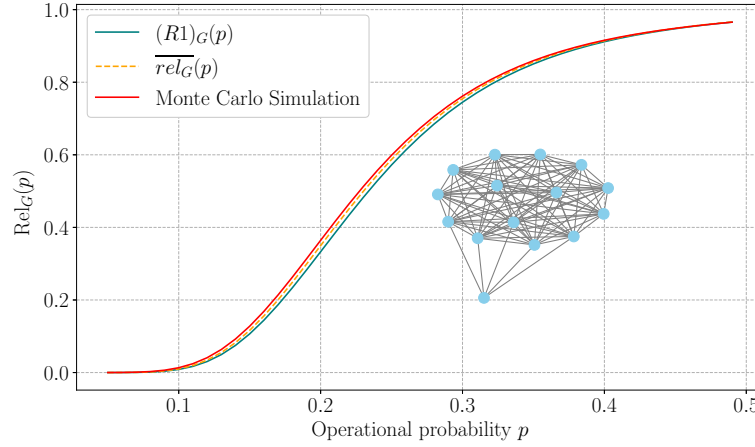


FIG. 2. The stochastic approximation, first-order approximation and Monte Carlo simulations for a modified circulant on 15 nodes (nodes numbered 1–15): start from the complete graph K_{15} and delete the nine links 1–2, 1–3, 1–4, 1–6, 1–7, 1–8, 1–13, 1–14, 1–15. In the resulting graph, node 1 has a degree of 5 (node 1 is linked only to nodes 5, 9, 10, 11, 12), while all other nodes have degree 13 or 14. Properties of circulant matrices of small-world graphs are deduced in [4, p. 194–200].

In the original paper [1], we also claimed that $(R_1)_G(p)$ is always more accurate than $\overline{\text{rel}}_G(p)$. This claim is not generally true, since there is no universal ordering between $(R_1)_G$, $\overline{\text{rel}}_G$ and $\text{rel}_G(p)$ among all graphs G and link operational probability p .

Consider the modified circulant graph in Fig. 2, where node 1 is *partially disconnected*, i.e., we delete $r \geq 1$ links incident to node 1 from the original circulant matrix, so that the degree of node 1 decreases by r , but node 1 remains non-isolated. Fig. 2 shows, over a broad intermediate range of the link operational probability p , that the stochastic approximation $\overline{\text{rel}}_G(p)$ (dashed) stays closer to the Monte Carlo evaluation of $\text{rel}_G(p)$ (red), which is very accurate and here regarded as benchmark, than the first-order product $(R_1)_G(p)$ (solid). In summary, in this example,

$$|\overline{\text{rel}}_G(p) - \text{rel}_G(p)| < |(R_1)_G(p) - \text{rel}_G(p)|$$

which contradicts the claim that $(R_1)_G$ is always more accurate than the stochastic approximation.

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