RATIOS OF TWO POWERS OF VAN DER LAAN-PADOVAN NUMBERS

TOMOHIRO YAMADA*

ABSTRACT. The van der Laan-Padovan sequence P_n $(n = 0, 1, \ldots)$ is defined by $P_0 = 1, P_1 = P_2 = 0$, and $P_{n+3} = P_{n+1} + P_n$ for $n = 0, 1, \ldots$. We determine all pairs (P_m, P_n) satisfying $P_m^b = 2^{g_1} 3^{g_2} 5^{g_3} 7^{g_4} P_n^a$ for some integers g_1, g_2, g_3, g_4, a, and b. More generally, for a linear recurrence sequence u_n satisfying the dominant root condition and a given set of primes $p_1, \ldots, p_k$, there exist only finitely many pairs (u_m, u_n) satisfying $u_m^b = p_1^{g_1} \cdots p_k^{g_k} u_n^a$ for some integers $g_1, \ldots, g_k$, a, and b.

1. INTRODUCTION

An r-th order linear recurrence sequence (u_n) is a sequence determined by initial terms $u_0, \ldots, u_{r-1}$ and the recurrence relation

$$u_{n+r} = s_{r-1} u_{n+r-1} + \cdots + s_0 u_n \tag{1}$$

for $n \geq 0$ with some complex numbers $s_0, \ldots, s_{r-1}$ and its characteristic polynomial

$$P(X) = X^r - s_{r-1} X^{r-1} - \cdots - s_0 = (X - \alpha_1)^{d_1} \cdots (X - \alpha_t)^{d_t},$$

where we write distinct roots of the polynomial $P(X)$ for $\alpha_1, \ldots, \alpha_t$.

As is well known (see for example, 1.1.6 of [6]), u_n can be written in the form

$$u_n = \sum_{i=1}^{t} q_i(n) \alpha_i^n, q_i(n) = \sum_{j=1}^{d_i} \kappa_{w_i + j} n^{j-1}. \tag{2}$$

for certain real numbers $\kappa_1, \ldots, \kappa_r$, where $w_i = d_1 + \cdots + d_{i-1}$.

Throughout this paper, we assume that $s_0, \ldots, s_{r-1}, u_0, \ldots, u_{r-1}$ are integers.

Arithmetic properties and diophantine equations concerning linear recurrence sequences have been extensively studied. Among such sequences, the most extensively studied ones are Lucas sequences $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ with α and β distinct roots of a given quadratic equation $X^2 - s_1 X - s_0 = 0$ and the discriminant $\Delta = s_1^2 - 4s_0$, where s_0 and s_1 are relatively prime and α/β is not a root of unity. For such sequences, Bilu-Hanrot-Voutier theorem [3] states that, for $n \geq 30$, u_n always has a prime factor which does

2020 *Mathematics Subject Classification.* 11A05, 11B37, 11B83, 11D61, 11J86.

Key words and phrases. Exponential diophantine equation; Linear recurrence sequence; van der Laan-Padovan sequence.

not divide u_m with $0 \leq m < n$. It is well known that a prime p not dividing $s_0 s_1 \Delta$ always divides $u_{p-(\Delta/p)}$ for a given Lucas sequence (u_n) (such properties of Lucas sequences are concisely explained in [27, Section 2.IV]). Hence, for any given Lucas sequence (u_n) and prime numbers $p_1, \ldots, p_k$, we can easily determine all integers in this sequence (u_n) all of whose prime factors belong to $p_1, \ldots, p_k$. Moreover, Bugeaud, Mignotte, and Siksek [4] gives a practical way to determine all powers in a given Lucas sequences, although they apply an elaborate combination of modular techniques via Frey curves and Baker's method.

Less is known on general linear recurrence sequences. However, many results are known under some restrictions. For an overview of arithmetic and other properties of linear recurrence sequences and their applications, we refer to [6]. Earlier results are surveyed by Stewart [32].

Assume that α_i/α_j are not roots of unity for $1 \leq i < j \leq r$. Mahler proved that $\lim_{n\to\infty} |u_n| = \infty$. Later, van der Poorten and Schlickewei [26] proved that for any $\epsilon > 0$, the inequality

$$|u_n| > |\alpha_1|^{n(1-\epsilon)}$$

holds for sufficiently large n (a more accessible proof is given by Fuchs and Heintze [10]). They also proved that, writing $P(a/b)$ with $\gcd(a, b) = 1$ for the largest prime factor of an integer ab, $\lim_{n\to\infty} P(u_n) = \infty$. Evertse [7] proved that $\lim_{n\to\infty, n>s, u_s\neq 0} P(u_n/u_s) = \infty$.

We note that these results are ineffective. Indeed, no effective version of even an inequality $\lim_{n\to\infty} |u_n| = \infty$ has been known applicable for every non-degenerate sequence (u_n).

For special cases, effective results are known. Mignotte [20] proved that If $|\alpha_1| = \cdots = |\alpha_\ell| > |\alpha_{\ell+1}|$ with $\ell \leq 3$, then

$$|u_n| > c' |\alpha_1|^n / n^c$$

for $n \geq n_0$ whenever $q_1(n)\alpha_1^n + \cdots + q_\ell(n)\alpha_\ell^n \neq 0$, where c, c', and n_0 are positive constants effectively computable in terms of $\alpha_1, \ldots, \alpha_\ell$, $q_1, \ldots, q_\ell$. Shparlinskii [30, Theorem 1] proved that if $|\alpha_1| > |\alpha_2|$, then $P(u(n)) > C \log n$ for $n \geq n_0$, where $C > 0$ and n_0 are effectively computable in terms of the sequence (u_n).

Stewart [31, Theorem 4] proved that, for an algebraic field $\mathcal{K}$ of degree d over $\mathbb{Q}$, a real algebraic number α in $\mathcal{K}$ with $|\alpha| > 1$, if an integer $u(n)$ can be written in the form

$$u(n) = f(n)\alpha^n + h(n),$$

where $f(n)$ is a nonzero polynomial with coefficients from $\mathcal{K}$ and $|h(n)| < |\alpha|^{\gamma n}$ for some $\gamma < 1$, then

$$P(u(n)) > (1 - \epsilon) \log n$$

for $n > n_0$, where n_0 is an effectively computable constant depending on d, α, f, γ, and ϵ. Stewart [33] replaced this lower bound by $C \log n \log\log n / \log\log\log n$ with n_0 and C effectively computable depending on α, f, and γ.

Pethö [24] proved that, for a given second-order linear recurrence sequence (u_n) with integral parameters satisfying quite natural conditions and a given finite set S of primes, $u_n = wx^q$ has only finitely many solutions in integers n, w, x, q with $|x| > 1$, $q \geq 2$, and w composed of primes in S, which can be bounded by an effectively computable constant. Bugeaud and Kaneko [5] proved that if $|\alpha_1| > |\alpha_2|$ and $s_0 \neq 0$, then there are only finitely many perfect powers in (u_n) and their number can be bounded by an effectively computable constant.

On the other hand, Evertse [8] had proved that if no quotient α_i/α_j with $i \neq j$ is a root of unity, then $P(u_n) \to \infty$ together with n and $P(u_m/u_n) \to \infty$ when $m > n$, $u_n \neq 0$, and m tends to infinity. However, Evertse uses Schlickewei's p-adic subspace theorem [28], which makes these results ineffective.

Odjoumani and Ziegler [21] proved that, if $|\alpha_1| > |\alpha_2| > |\alpha_3|$ or $|\alpha_1| > |\alpha_2|$ and κ_1 and α_1 are multiplicatively independent, then for any prime p outside an effectively determinable finite set, (u_n) contains at most one prime power $\pm p^m$ with $m \neq 0$. They [22] extended this result by proving that under some additional conditions, (u_n) contains at most two integers $\pm p^a q^b$ with $p^a q^b > 1$ composed by two given primes p and q unless p or q belongs an effectively determinable finite set.

Results of Odjoumani and Ziegler imply that, for any given linear recurrence sequence u_n satisfying a certain condition, there are only finitely many triples (m, n, ℓ) of nonnegative integers satisfying

$$u_m^a u_n^b u_\ell^c = 1 \tag{3}$$

for some integers a, b, c. Gómez Ruiz and Luca [11] proved that for given k different binary recurrence sequences $u_n^{(1)}, \ldots, u_n^{(k)}$ satisfying certain conditions, there are only finitely many k-tuples $u_{n_1}^{(1)}, \ldots, u_{n_k}^{(k)}$ such that

$$u_n^{(1)a_1} \cdots u_n^{(k)a_k} = 1 \tag{4}$$

for some integers $a_1, \ldots, a_k$. Independently of them, we determined all integer solutions of

$$\sigma(2^m)^a \sigma(3^n)^b \sigma(5^\ell)^c = 1 \tag{5}$$

with $m, n, \ell > 0$ in [37].

In this paper, we prove the following finiteness result on multiplicative relations in a given linear recurrence sequence.

Theorem 1.1. *Let u_n be an r-th order linear recurrence sequence defined by (1). Assume that roots $\alpha_1, \ldots, \alpha_t$ of its characteristic polynomials satisfy $|\alpha_1| > |\alpha_2| \geq \cdots \geq |\alpha_t|$, $\kappa_1 \neq 0$, $\kappa_i \neq 0$ for some $i \geq 2$, and $d_1 = 1$ (we note that α_1 must be real and $q_1(n) = \kappa_1$ under this assumption).*

If $n > m$ and $u_n^a = p_1^{g_1} \cdots p_k^{g_k} u_m^b$ for some integers $g_1, \ldots, g_k, a, b$ with $a > 0$ and $b \geq 0$, then m and n can be bounded by an effectively computable constant as given in Theorem 4.1 explicitly.

Remark 1.2. The condition $b \geq 0$ is essentially not restrictive. Indeed, if $b < 0$, then u_n^a divides $p_1^{g_1} \cdots p_k^{g_k}$ and therefore $u_n = p_1^{h_1} \cdots p_k^{h_k}$ for some integers $h_1, \ldots, h_k$.

Although our bound given in Theorem 4.1 is considerably large, we can determine all solutions of (1) with the aid of the lattice reduction method, when given parameters are not extremely large. We would like to do it for the van der Laan-Padovan sequence

$$1, 0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, \ldots$$

defined by

$$P_0 = 1, P_1 = P_2 = 0$$

and

$$P_{n+3} = P_{n+1} + P_n \tag{6}$$

for $n \geq 0$. In literature, this sequence has been called the Padovan sequence after Richard Padovan. However, Padovan himself attributes this sequence to Dom Hans van der Laan (see for example, [23, Chapter 5]). Indeed, this sequence has been implicitly introduced in a work [14, Chapter 8] of van der Laan. So that, this sequence should be called the van der Laan-Padovan sequence.

We find all pairs (m, n) of nonnegative integers m, n satisfying

$$P_n^a = 2^{g_1} 3^{g_2} 5^{g_3} 7^{g_4} P_m^b. \tag{7}$$

Theorem 1.3. *If* (7) *holds for some integers* g_1, g_2, g_3, g_4, a, b *with* $a > 0$, *then we must have* $m, n \in \{1, 2, 4\}$, $m, n \in \{0, 3, 5, \ldots, 18, 20, 25, 36\}$, *or* $m, n \in \{21, 27, 49\}$.

Our PARI-GP script is available at `https://drive.google.com/file/d/1_IIBLRToi9jB3FeSmOR3Qx5FH1KFjYGD/view?usp=sharing`.

2. Preliminaries

In this section, we shall introduce some notations and lemmas.

Let $\mathcal{K}$ and $\mathcal{O}$ denote the number field $\mathbb{Q}(\alpha_1)$ and its ring of integers respectively.

Moreover, we define the absolute logarithmic height $\mathrm{h}(\alpha)$ of an algebraic number α in $\mathcal{K}$. For an algebraic number α in $\mathcal{K}$ and a prime ideal $\mathfrak{p}$ over $\mathcal{K}$ such that $\alpha = (\zeta_1/\zeta_2)\xi$ with $\xi \in \mathfrak{p}^k$ and ζ_1, ζ_2 in $\mathcal{O}\backslash\mathfrak{p}$, we define the absolute value $|\alpha|_{\mathfrak{p}}$ by

$$|\alpha|_{\mathfrak{p}} = N\mathfrak{p}^{-k}$$

as usual, where $N\mathfrak{p}$ denotes the norm of $\mathfrak{p}$, i.e., the rational prime lying over $\mathfrak{p}$. Now the absolute logarithmic height $\mathrm{h}(\alpha)$ is defined by

$$\mathrm{h}(\alpha) = \frac{1}{2}\left(\log^+ |\alpha| + \log^+ |\bar{\alpha}| + \sum_{\mathfrak{p}} \log^+ |\alpha|_{\mathfrak{p}}\right),$$

where $\log^+ t = \max\{0, \log t\}$ and $\mathfrak{p}$ in the sum runs over all prime ideals over $\mathcal{K}$.

In order to obtain an upper bound for the size of solutions, we use an lower bound for linear forms of logarithms due to Matveev [19, Theorem 2.2].

Lemma 2.1. *Assume that $\mathcal{K}$ is an real algebraic field of degree D. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be algebraic integers in $\mathcal{K}$ which are multiplicatively independent and $b_1, b_2, \ldots, b_n$ be arbitrary integers with $b_n \neq 0$. Let $A(\alpha) = \max\{D\mathrm{h}(\alpha), |\log \alpha|\}$ and $A_j = A(\alpha_j)$.*

Put

$$\begin{aligned} B &= \max\{1, |b_1|\, A_1/A_n, |b_2|\, A_2/A_n, \ldots, |b_n|\}, \\ \Omega &= A_1 A_2 \ldots A_n, \\ C(n) &= \frac{16}{n!} e^n (2n+3)(n+2)(4(n+1))^{n+1} \\ &\quad \times \left(\frac{1}{2} en\right) (4.4n + 5.5 \log n + 7 + 2\log D + \log(1 + \log D)), \\ c_1 &= 1.5eD(1 + \log D) \end{aligned} \tag{8}$$

and

$$\Lambda = b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n. \tag{9}$$

Then we have $\Lambda = 0$ or

$$\log |\Lambda| > -C(n)\Omega \log(c_1 B). \tag{10}$$

Remark 2.2. When $\mathcal{K} = \mathbb{Q}$, Aleksentsev's result in [1] gives a better estimate. However, the upper bound for our problem derived by Aleksentsev's result would be still considerably large.

Let

$$\Lambda = x_1\theta_1 + \cdots x_n\theta_n$$

be a linear form with $n \geq 2$, $x_1, \ldots, x_n \in \mathbb{Z}$, and $\theta_1, \ldots, \theta_n \in \mathbb{R}$. Let $M = (m_{ij})$ be the n-th order square matrix defined by $m_{ij} = 0$ for $1 \leq i \leq n-1$, $1 \leq j \leq n$ with $i \neq j$, $m_{11} = \cdots = m_{n-1,n-1} = \gamma$, and $m_{ni} = \lfloor C\gamma\theta_i \rfloor$ for $i = 1, \ldots, n$, where C and γ are constants chosen later, and $\mathbf{v}_i$ be the i-th column vector M. Let $l(M)$ be the the shortest length of vectors in the lattice generated by the column vectors $\mathbf{v}_1$, ..., $\mathbf{v}_n$ of M.

We prove the following relation of a lower bound for $|\Lambda|$ and a lower bound for $l(M)$. This is implicit in the proof of Lemma 3.7 of [36] and used in our previous work [37]. However, we would like to make it explicit here and give its proof.

Lemma 2.3. *If X_1 be a real number such that $l(M) > X_1\sqrt{(n+1)^2 + (n-1)\gamma^2}$ and $X_1 \geq \max\{x_1, \ldots, x_n\}$, then $|\Lambda| > X_1/(C\gamma)$.*

Proof. We proceed as in the proof of Lemma 3.7 of [36]. We put

$$\mathbf{v} = \sum x_i \mathbf{v}_i = \begin{pmatrix} \gamma x_1 \\ \vdots \\ \gamma x_{n-1} \\ \tilde{\Lambda} \end{pmatrix},$$

where $\tilde{\Lambda} = x_1 \lfloor C\gamma\theta_1 \rfloor + \cdots + x_n \lfloor C\gamma\theta_n \rfloor$. Then

$$|\mathbf{v}|^2 = \gamma^2 \sum_{i=1}^{n-1} x_i^2 + \tilde{\Lambda}^2 \tag{11}$$

and

$$\left|\tilde{\Lambda} - \gamma C\Lambda\right| \leq \sum_{i=1}^{n} |x_i| \, |\lfloor C\gamma\theta_i \rfloor - C\gamma\theta_i| < \sum_{i=1}^{n} |x_i| \leq nX_1. \tag{12}$$

By the assumption, we have

$$|\gamma C\Lambda| \geq \left|\tilde{\Lambda}\right| - \left|\tilde{\Lambda} - \gamma C\Lambda\right| \geq \sqrt{l(M)^2 - (n-1)\gamma^2 X_1^2} - nX_1 \geq X_1, \tag{13}$$

which proves the lemma. □

Finally, we prove the following fact.

Lemma 2.4. $\kappa_1 \in \mathbb{Q}(\alpha_1)$.

Proof. We may assume that $\alpha_1, \ldots, \alpha_{r_0}$ are the conjugates of α_1 without the loss of generality. Let A be the extended vandermonde matrix of order r with its n-th row consisting of $n^k \alpha_i^n$ for $i = 1, \ldots, t$ and $k = 0, \ldots, d_i - 1$ and A_{ij} be its (i,j)-cofactor. Then, κ_1 can be written in a linear form $\kappa_1 = B_1 s_0 + \cdots + B_r s_{r-1}$ of $s_0, \ldots, s_{r-1}$, where each $B_i = \det A_{i1} / \det A$.

We see that $\det A$ can be represented as a polynomial of $n^k \alpha_\ell^n$'s with $\ell = 1, \ldots, t$, $k = 0, \ldots, d_i - 1$ alternating over $(k,l)'s$ and each $\det A_{i1}$ can be represented as a polynomial of $n^k \alpha_\ell^n$'s with $\ell = 2, \ldots, t$ and $k = 0, \ldots, d_i - 1$ alternating over $(k,l)'s$ (explicit constructions for $\det A$ are given in [2] and [9]).

Hence, each B_i can be written in a rational function of α_1, ..., α_t with rational coefficients which is symmetric on α_i's in other conjugate classes than the class of α_1 and symmetric on $\alpha_2, \ldots, \alpha_{r_0}$. In other words, each B_i can be written in a rational function of $\alpha_2, \ldots, \alpha_{r_0}$ whose coefficients belongs to $\mathbb{Q}(\alpha_1)$. This means that each B_i itself belongs to $\mathbb{Q}(\alpha_1)$ and so does κ_1. □

3. A preliminary upper bound

Let (u_n) be an r-th order linear recurrence sequence u_n satisfying (1) and assume that the characteristic roots $\alpha_1, \ldots, \alpha_t$ satisfy $|\alpha_1| > |\alpha_2| \geq \cdots |\alpha_t|$. We write u_n in the form (2).

Lemma 3.1. *Let $m_1, \ldots, m_k$ be rational integers such that $m_1, \ldots, m_k$, α_1, and κ_1 are multiplicatively independent and put $A = \max\{\log m_1, \ldots, \log m_k, \mathrm{h}(\alpha_1), \mathrm{h}(\kappa_1)\}$. Let $n_0 \geq 1$, $K > 0$, $\delta > 1$ be real numbers such that*

$$\left| \frac{u_n}{\kappa_1 \alpha_1^n} - 1 \right| < K\delta^{-n} \tag{14}$$

for $n \geq n_0$ (we can take such real numbers K and δ since $|\alpha_1| > |\alpha_2|$) and $n_0 > A/c_2$, where we put

$$c_2 = c_1 \max\{\mathrm{h}(\alpha_1), \mathrm{h}(\kappa_1)/n_0, \log \alpha_1 + \max\{0, \log(K\kappa_1(1+\epsilon))\}/n_0\}.$$

Then, we write $\epsilon = K\delta^{-n_0}$ and $\epsilon_1 = \epsilon/(1-\epsilon)$.

Moreover, we put D to be the degree of the field $\mathbb{Q}(\alpha_1)$ over $\mathbb{Q}$ and, for each $s = 1, \ldots, k$,

$$\begin{aligned} A^{(s)} &= \max\{\log m_1, \ldots, \log m_s, \mathrm{h}(\alpha_1), \mathrm{h}(\kappa_1)\}, \\ \Psi^{(s)} &= (\log m_1) \cdots (\log m_s)\mathrm{h}(\alpha_1)\mathrm{h}(\kappa_1), \Psi'^{(s)} = \Psi/A^{(s)}, \\ c_3^{(s)} &= \frac{\log((1+\epsilon_1)K)}{\Psi^{(s)} \log(c_2 n_0/A^{(s)})}, \\ C'^{(s)} &= \frac{C(s+2)D^{s+2} + c_3^{(s)}}{\log \delta}, \end{aligned}$$

and $C''^{(s)} = C'^{(s)}\eta_1$, where we put η_1 to be the constant satisfying

$$\frac{\eta_1 Y \log Y}{\log(\eta_1 Y \log Y)} = Y$$

with $Y = c_2 C'^{(s)} \Psi'^{(s)}$. We simply write $\Psi = \Psi^{(k)}$ and so on.

If $n \geq n_0$ and $u_n = m_1^{e_1} \cdots m_k^{e_k}$ for some integers $e_1, \ldots e_k$, then

$$n < C''\Psi \log(c_2 C' \Psi'). \tag{15}$$

Remark 3.2. If $P(X)$ has no double root, then we have $t = r$, $u_n = \sum_{i=1}^r \kappa_i \alpha_i^n$ for $n = 0, 1, \ldots$, and we can put $\delta = |\alpha_1/\alpha_2|$ and

$$K = \frac{|\kappa_2| + \cdots + |\kappa_r|}{|\kappa_1|}. \tag{16}$$

Proof. We put $\Lambda_0 = e_1 \log m_1 + \cdots + e_k \log m_k - \log \kappa_1 - n \log \alpha_1 = \log(u_n/\kappa_1 \alpha_1^n)$. It is clear that

$$\left| \frac{u_n}{\kappa_1 \alpha_1^n} - 1 \right| \leq \left| \frac{\kappa_2}{\kappa_1} \left(\frac{\alpha_2}{\alpha_1} \right)^n + \cdots + \frac{\kappa_r}{\kappa_1} \left(\frac{\alpha_r}{\alpha_1} \right)^n \right| \leq K\delta^{-n}. \tag{17}$$

We observe that

$$|\log(1+x)| < \frac{|x|}{1-|x|} \tag{18}$$

for $|x| < 1$ to obtain

$$|\Lambda_0| < \frac{K\delta^{-n}}{1 - K\delta^{-n}} \leq (1+\epsilon_1)K\delta^{-n}. \tag{19}$$

Since $u_n \neq \kappa_1 \alpha_1^n$ by assumption, we have $\Lambda_0 \neq 0$. We note that $D\mathrm{h}(\alpha_1) \geq \log|\alpha_1| = |\log \alpha_1|$ and $D\mathrm{h}(\kappa_1) \geq \log|\kappa_1| = |\log \kappa_1|$. Moreover, $n \geq n_0 > 0$ by assumption.

Now we put s to be the largest index i such that $e_i > 0$. Since $\mathbb{Q}(\alpha_1)$ is a real field and $\kappa \in \mathbb{Q}(\alpha_1)$ by Lemma 2.4, we can apply Lemma 2.1 for Λ_0 with $\mathcal{K} = \mathbb{Q}(\alpha_1)$, $n = s+2$, and $A_n = DA$ to obtain

$$-\log|\Lambda_0| < C(s+2)D^{s+2}\Psi \log(c_1 B^{(s)}), \tag{20}$$

where we observe that $\Omega = D^{s+2}\Psi$ and

$$\begin{aligned} B^{(s)} &= \frac{\max\{e_1 \log m_1, \ldots, e_k \log m_k, \mathrm{h}(\kappa_1), n\mathrm{h}(\alpha_1)\}}{A^{(s)}} \\ &\leq \frac{\max\{n\mathrm{h}(\alpha_1), \mathrm{h}(\kappa_1), \log u_n\}}{A^{(s)}} \leq \frac{c_2 n}{c_1 A^{(s)}}. \end{aligned} \tag{21}$$

From (19), (20), and (21), we obtain

$$n \log \delta < \log(2K_1) + C(s+2)D^{s+2}\Psi^{(s)} \log\left(\frac{c_2 n}{A^{(s)}}\right), \tag{22}$$

which immediately gives that

$$\frac{c_2 n}{A^{(s)}} < c_2 C'^{(s)} \Psi'^{(s)} \log\left(\frac{c_2 n}{A^{(s)}}\right). \tag{23}$$

By the definition of $C''^{(s)}$, we have

$$\frac{c_2 n}{A^{(s)}} < c_2 C''^{(s)} \Psi'^{(s)} \log(c_2 C'^{(s)} \Psi'^{(s)}) \tag{24}$$

and therefore

$$n < C''^{(s)} \Psi^{(s)} \log(c_2 C'^{(s)} \Psi'^{(s)}). \tag{25}$$

We can easily see that the resulting constants $C''^{(s)}\Psi^{(s)}$ and $C'^{(s)}\Psi'^{(s)}$ grow as s becomes large to obtain (15) as desired. □

4. General upper bounds

Let $p_1, \ldots, p_k$ be distinct primes and x_0 be the smallest prime other than them. We take an arbitrary constant $\mu > 1$. We put p_s to be the largest prime among $p_1, \ldots, p_k$ such that $g_s \neq 0$, $A = \max\{\log p_s, \mathrm{h}(\alpha_1), \mathrm{h}(\kappa_1)\}$, and $C_0 = \mu/(c_1 \mathrm{h}(\alpha_1))$, and

$$\Psi = (\log p_1) \cdots (\log p_k)\mathrm{h}(\alpha_1)\mathrm{h}(\kappa_1).$$

Theorem 4.1. *Let $n_1 \geq 1$, $K > 0$, $\delta > 1$ be real numbers satisfying (14) for $n \geq n_1$ and $n_1 > C_0 A$. We write $\epsilon = K\delta^{-n_1}$ and assume that $\epsilon < 1$. Moreover, we write $\epsilon_1 = \epsilon/(1-\epsilon)$ and $\epsilon_2 = \max\{0, \log \kappa_1/(n_1 \log \alpha_1)\}$.*

We take $C_1' = c_2 C'$ with

$$C' = \frac{C(k+2)D^{k+2} + \log((1+\epsilon_1)K)/(\Psi \log \mu)}{\log \delta}$$

and $C_1 = C''$ obtained from C' as in Lemma 3.1.

Nextly, we put $C_2' = c_2 C'$ *with*

$$C' = \frac{C(k+3)D^{k+3} + \log((1+\epsilon_1)K)/(\Psi(\log x_0)(\log \mu))}{\log \delta},$$

$C_2 = C''$ *obtained from* C' *as in Lemma 3.1, and* $C_3 = C_2(1+\epsilon)$. *Moreover, We put*

$$\begin{aligned}
C_4 &= C(k+2)D^{k+2},\\
C_5' &= c_1 \frac{C_3 \Psi^2 (\log \alpha_1) \log^2(C_2' \Psi) \max\{C_2 \mathrm{h}(\alpha_1), C_3 \log \alpha_1\}}{A},\\
C_5 &= C_4 + \frac{\log(2(1+\epsilon_1)KC_3 \Psi \log \alpha \log(C_2' \Psi))}{2C_4 \Psi (\log C_5')(\log \delta)},\\
C_6 &= 2C_5(1+\epsilon)(1+\epsilon_2)\log \alpha_1,\\
C_7 &= \eta_2 C_6,
\end{aligned}$$

where we put η_2 *to be the constant satisfying*

$$\frac{\eta_2 Y \log Y}{\log(\eta_2 Y \log Y)} = Y$$

with $Y = C_6 \Psi$.

If $n > m \geq n_1$ *and* $u_n^a = p_1^{g_1} \cdots p_k^{g_k} u_m^b$ *for some integers* $g_1, \ldots, g_k, a, b$ *with* $a > 0$ *and* $b \geq 0$, *then we have either*

$$m \leq n < 2\max\{C_0, C_2 \Psi \log(C_2' \Psi)\} C_5', \tag{26}$$

or

$$m < 2C_5 \Psi \log(C_7 \Psi \log(C_6 \Psi)) \tag{27}$$

and

$$n < \max\{C_0, C_2 \Psi \log(C_2' \Psi)\} C_7 \Psi \log(C_6 \Psi). \tag{28}$$

We begin by observing that $u_m, u_n > (1-\epsilon)\kappa_1 \alpha_1^n > 0$ from the assumption that $n > m \geq n_1$ and $\epsilon < 1$. Moreover, we may assume that $\gcd(a, b) = 1$. Since $p_1, \ldots, p_k$ are distinct primes, we have

$$u_m = p_1^{e_1} \cdots p_k^{e_k} x^a \tag{29}$$

and

$$u_n = p_1^{f_1} \cdots p_k^{f_k} x^b \tag{30}$$

for some integer x not divisible by p_1, ..., or p_k. Indeed, we can write $u_m = p_1^{e_1} \cdots p_k^{e_k} U$ and $u_n = p_1^{f_1} \cdots p_k^{f_k} V$ with $\gcd(U, p_1 \cdots p_k) = \gcd(V, p_1 \cdots p_k) = 1$. Then, we observe that $U^b = V^a$ and therefore $U = x^a$ and $V = x^b$ for some integer x.

If $x = 1$ or $b = 0$, then Lemma 3.1 with $n_0 = n_1$ gives that $m < n < C_1 \Psi \log(C_1' \Psi)$, where we note that $n_0 \geq \mu A / c_2 > A/c_2$ and $c_3 \leq \log((1+\epsilon_1)K)/(\Psi \log \mu)$.

Hence, we may assume that $x \geq 2$ and $b > 0$. We put $A' = \max\{A, \log x\}$. Now, we would like to apply Lemma 3.1 with $k+1$, $\Psi \log x$, and A' in place of k, Ψ, and A respectively and with

$$n_0 = \max\{n_1, C_0 A'\}, m_i = p_i \ (i = 1, \dots, k), m_{k+1} = x.$$

Then, we see that $n_0 \geq \mu A'/c_2 > A'/c_2$ and, either $e_{k+1} = a > 0$ or $e_{k+1} = b > 0$ holds. Moreover, we have $c_3 \leq \log((1+\epsilon_1)K)/(\Psi(\log x_0)(\log \mu))$ in Lemma 3.1. Hence, we can apply Lemma 3.1 to obtain

$$\max\{m, n\} \leq \max\{n_0, C_2 \Psi (\log x) \log(C_2' \Psi)\}. \tag{31}$$

Since we have assumed that $n_1 > C_0 A$, we have

$$\max\{m, n\} \leq \max\{n_0, \max\{C_0, C_2 \Psi \log(C_2' \Psi)\} \log x\}. \tag{32}$$

Now, the theorem immediately follows when $\log x < C_5'$. Thus, we may assume that $\log x \geq C_5'$. We put

$$\begin{aligned} \Lambda = & \log \left(\frac{u_m^b u_n^{-a}}{\kappa_1^{b-a} \alpha_1^{bm-an}} \right) = \log \left(\frac{p_1^{g_1} \cdots p_k^{g_k}}{\kappa_1^{b-a} \alpha_1^{bm-an}} \right) \\ = & \sum_{i=1}^{k} g_i \log p_i - (b-a) \log \kappa_1 - (bm - an) \log \alpha_1. \end{aligned} \tag{33}$$

We can easily see that

$$|\Lambda| \leq \frac{bK\delta^{-m}}{1 - K\delta^{-m}} + \frac{aK\delta^{-n}}{1 - K\delta^{-n}} < (1+\epsilon_1)(a+b)K\delta^{-m}. \tag{34}$$

If $\Lambda = 0$, then, from the assumption that $p_1, \dots, p_k, \kappa_1, \alpha_1$ are multiplicatively independent, we see that $b - a = bm - an = 0$. Since $m \neq n$, we must have $a = b = 0$ while we have assumed that $a > 0$.

Thus, we have $\Lambda \neq 0$. Hence, we can apply Lemma 2.1 for Λ with $n = k+2$, $A_n = A$ noting that $g_s \neq 0$, and

$$B = \frac{\max\{g_1 \log p_1, \dots, g_k \log p_k, |b-a| \, \mathrm{h}(\kappa_1), |bm - an| \, \mathrm{h}(\alpha_1)\}}{A} \tag{35}$$

to obtain

$$-\log |\Lambda| \leq C_4 \Psi \log(c_1 B). \tag{36}$$

Combining this bound with (34), we have

$$m \log \delta < \log((a+b)(1+\epsilon_1)K) + C_4 \Psi \log(c_1 B). \tag{37}$$

From (32), we have

$$\max\{a, b\} \leq C_3 \Psi (\log \alpha_1) \log(C_2' \Psi) \tag{38}$$

and the same upper bound also applies to $|a - b|$ since a and b are both positive. We can easily see that $g_i = be_i - af_i$ from (29) and (30) and

$$|g_i| \leq C_3^2 \Psi_i \Psi (\log^2 \alpha_1)(\log x) \log^2(C_2 \Psi) \tag{39}$$

for $i = 1, \dots, k$, where $\Psi_i = \Psi / \log p_i$. Moreover, combining (32) and (38), we immediately see that

$$|bm - an| \le C_2 C_3 \Psi^2 (\log \alpha_1)(\log x) \log^2(C_2 \Psi). \tag{40}$$

Hence, we obtain

$$B \le \frac{C_3 \Psi^2 (\log \alpha_1)(\log x) \log^2(C_2' \Psi) \max\{C_2 \mathrm{h}(\alpha_1), C_3 \log \alpha_1\}}{A} \tag{41}$$

and therefore $c_1 B \le C_5' \log x$. From (38), we see that $a+b < 2C_3 \Psi (\log \alpha_1) \log(C_2' \Psi)$. Now (37) yields that

$$m \log \delta < \log(2C_3(1+\epsilon_1) K \Psi (\log \alpha_1) \log(C_2' \Psi)) + C_4 \Psi \log(C_5' \log x) \tag{42}$$

and we conclude that

$$m < C_5 \Psi \log(C_5' \log x) < 2C_5 \Psi \log\log x. \tag{43}$$

Now, observing that $u_m \ge x^a$ and $a \log x \le \log u_m < (1+\epsilon)(\log \kappa_1 + m \log \alpha_1)$, we have

$$a \log x < C_6 \Psi \log\log x < C_6 \Psi \log(a \log x) \tag{44}$$

and

$$a \log x < C_7 \Psi \log(C_6 \Psi). \tag{45}$$

Now, recalling the assumption that $a > 0$, the theorem follows from (43) and (32) again.

5. Proof of Theorem 1.3

We apply our method to the van der Laan-Padovan sequence. We put $\alpha_1 = 1.324717\cdots$ be the plastic ratio, the real solution of the equation $X^3 - X - 1 = 0$, $\alpha_2, \alpha_3 = -0.662358\cdots \pm 0.562279\cdots i$ be the remaining solutions, which are mutually conjugate, and $\kappa_i = 1/(2\alpha_i + 3)$ for $i = 1, 2, 3$. Now we can write $P_n = \kappa_1 \alpha_1^n + \kappa_2 \alpha_2^n + \kappa_3 \alpha_3^n$. We note that $K = 5.599815\cdots$ and $\delta = 1.524702\cdots$.

Now we assume that m and n are two nonnegative integers satisfying (7) for some integers g_1, g_2, g_3, g_4, a, b with $a \ge 0$. Moreover, we may assume that $b > 0$ without the loss of generality.

We begin by settling the special case $P_n = 2^{f_1} 3^{f_2} 5^{f_3} 7^{f_4}$.

Lemma 5.1. *If $P_n = 2^{f_1} 3^{f_2} 5^{f_3} 7^{f_4}$, then $n = 0, \dots, 18$, 20, 25, or 36.*

Proof. We proceed as in the proof of Theorem 4.1 taking $n_1 = 27$ and $\mu = 10$. Indeed, we have $A = \max\{\log 7, \mathrm{h}(\alpha_1), \mathrm{h}(\kappa_1)\} = \log 7$ and therefore $n_1 > 9 > C_0 A$. Thus we see that $\epsilon < 6.3413 \times 10^{-5}$ and $\epsilon_1 < 6.3417 \times 10^{-5}$.

Now we apply Lemma 3.1 with $A = \max\{\log 7, \mathrm{h}(\alpha_1), \mathrm{h}(\kappa_1)\}$ to obtain

$$n < C_1 \Psi \log(C_1' \Psi) < 2.456 \times 10^{24}$$

with $C_1 < 2.066058 \times 10^{23}$ and $C_1' < 5.381845 \times 10^{22}$.

We can easily see that $P_n = 2^{f_1}3^{f_2}5^{f_3}7^{f_4} < (1+\epsilon)\kappa_1\alpha^n$ and therefore $f_i < 2.456 \times 10^{24}$ for each i. Now, putting

$$\Lambda_0 = f_1 \log 2 + \cdots + f_4 \log 7 - \log \kappa_1 - n\alpha_1,$$

we have $|\Lambda_0| < (1+\epsilon_1)K\delta^{-n}$ for $n \geq n_1$.

Taking $C = 10^{150}$, $\gamma = 11$, and $X_1 = 2.456 \times 10^{24}$, we have a reduced matrix M such that $l(M) > X_1\sqrt{49+5\gamma^2}$. Thus, Lemma 2.3 gives that

$$|\Lambda| > \frac{X_1}{C\gamma} > 2.233 \times 10^{-127}$$

and $n \leq 695$. Checking each n, we can prove the lemma. □

Now we proceed to the remaining case. Then (7) holds with $b > 0$ and we may assume that $\gcd(a, b) = 1$.

Lemma 5.2. *If $P_n^a = 2^{g_1}3^{g_2}5^{g_3}7^{g_4}P_m^b$ with $\gcd(a,b) = 1$, then $m \leq 988$. Moreover, we must have $bm \leq 3.850562 \times 10^{29}$ and $an < 5.5553 \times 10^{29}$.*

Proof. We take $n_1 = 27$ and $\mu = 10$ again. Thus, we have $\epsilon < 6.3413 \times 10^{-5}$ and $\epsilon_1 < 6.3417 \times 10^{-5}$ We begin by observing that $P_m = 2^{e_1} \cdots 7^{e_4}x^a$ and $P_n = 2^{f_1} \cdots 7^{f_4}x^b$ for some integers $e_1, \ldots, e_4$, $f_1, \ldots, f_4$, and $x \geq 11$.

We put

$$\Lambda = g_1 \log 2 + \cdots + g_4 \log 7 + (a-b)\log \kappa_1 + (an - bm)\kappa\alpha_1.$$

Instead of (45), we obtain

$$a \log x < C_5(1+\epsilon)(1+\epsilon_2)\Psi \log \alpha_1 \log(C_5' a \log x) \tag{46}$$

with $\epsilon_2 = 0$, $C_2 < 1.062372 \times 10^{26}$, $C_2' < 7.587587 \times 10^{24}$, $C_3 < 1.062439 \times 10^{26}$, $C_5 < 8.072767 \times 10^{22}$, and $C_5' < 2.003782 \times 10^{54}$. Then, (46) implies that $a \log x < 9.561 \times 10^{23}$.

Hence, we must have $\log x \leq a \log x < 9.561 \times 10^{23}$. From (32), we obtain

$$an < \max\{an_0, C_2\Psi \log(C_2'\Psi)(a \log x)\} < 1.325038 \times 10^{51}.$$

Moreover, from (43), we obtain $m < 3.399751 \times 10^{24}$ and (38) gives $\max\{a, b\} < 3.897329 \times 10^{26}$. Hence, we have $bm < 1.325038 \times 10^{51}$.

Now we can apply Lemma 2.3 with $X_1 = 1.325038 \times 10^{51}$. Taking $C = 10^{310}$ and $\gamma = 2.6$, we have a reduced matrix M such that $l(M) > X_1\sqrt{49+5\gamma^2}$. Hence, Lemma 2.3 yields that $|\Lambda| > 5.0963 \times 10^{-260}$.

We observe that

$$\delta^m < 1.962208(a + 3.897329 \times 10^{24})(1+\epsilon_1)K \times 10^{259}.$$

Since $a \log x < (1+\epsilon)m \log \alpha_1$, we have $m \leq 1564$, $bm \leq 6.0955 \times 10^{29}$, $a \log x < 634.54$ and, using (32) again, $an < 8.794 \times 10^{29}$.

Now we use Lemma 2.3 again but with $C = 10^{183}$, $\gamma = 4$, and $X_1 = 8.794 \times 10^{29}$ to obtain $|\Lambda| > 2.1985 \times 10^{-154}$ and, proceeding like above, we have $m \leq 988$, $bm \leq 3.850562 \times 10^{29}$, $a \log x < 400.85$, and $an < 5.5553 \times 10^{29}$. This completes the proof of the lemma. □

Now we check each $m \le 988$. If $m \in \{0, \dots, 18, 20, 25, 36\}$, then so must n. Hence, we may assume that m is equal to none of $0, \dots, 18$, 20, 25, and 36. Now we play with the logarithmic form

$$\Lambda_1 = a \log \frac{P_n}{\kappa_1 \alpha_1^n} = b \log P_m + g_1 \log 2 + \cdots g_4 \log 7 - an \log \alpha_1 - a \log \kappa_1$$

instead of Λ for each $m \le 988$ not equal to $0, \dots, 18$, 20, 25, or 36. Indeed, for any such m, we confirmed that Lemma 2.3 works for some C and γ with $\gamma C \le 3 \times 10^{214}$ and therefore $|\Lambda_1| > 1.85176 \times 10^{-185}$. Moreover, putting $H_m = P_m/(2^{e_1} \cdots 7^{e_4})$ with $H_m = 0$ or $\gcd(H_m, 210) = 1$, we confirmed that H_m can never be a perfect power for $0 \le m \le 1012$ unless $H_m \in \{0, 1\}$. Hence, we must have $a = 1$.

Now we assume that $n > \max\{m, 100\}$. Since

$$|\Lambda_1| = |\log n - \log \kappa - n \log \alpha_1| < \frac{K(1+\epsilon_3)}{\delta^n} \tag{47}$$

with $\epsilon_3 < 3 \times 10^{-18}$, we have $n \le 1012$. We observe that $H_m = H_n$. Indeed, if $H_m = 1$, then $H_n = 1$ noting that $H_m^b = H_n^a$. Otherwise, H_m can never be a perfect power for $0 \le m \le 1012$ as mentioned above and we must have $a = b = 1$, which yields that $H_m = H_n$ again. We confirmed that this can occur only when $m, n \in \{1, 2, 4\}$ with $H_m = H_n = 0$, $m, n \in \{0, 3, 5, 6, \dots, 18, 20, 25, 36\}$ with $H_m = H_n = 1$, or $m, n \in \{21, 27, 49\}$ with $H_m = H_n = 13$. This completes the proof of Theorem 1.3.

References

[1] Y. M. Aleksentsev, The Hilbert polynomial and linear forms in the logarithms of algebraic numbers, *Izv. Ross. Akad. Nauk Ser. Mat.* **72** (6) (2008), 5–52, Eng. trans., *Izv. Math.* **72** (2008), 1063–1110.

[2] S. Barbero, Generalized Vandermonde determinants and characterization of divisibility sequences, *J. Number Theory* **173** (2017), 371–377.

[3] Y. Bilu, G. Hanrot, and P. M. Voutier, Existence of primitive divisors of Lucas and Lehmer numbers, *J. reine Angew. Math.* **539** (2001), 75–122.

[4] Y. Bugeaud, M. Mignotte, and S. Siksek, Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers, *Ann. Math.* **163** (2006), 969–1018.

[5] Y. Bugeaud and H. Kaneko, On perfect powers in linear recurrence sequences of integers, *Kyushu J. Math.* **73** (2019), 221–227.

[6] G. Everest, A. van der Poorten, I. Shparlinski, and T. Ward, *Recurrence Sequences*, Mathematical Surveys and Monographs, **104**, Amer. Math. Soc., Providence, RI, 2003.

[7] J.-H. Evertse, On sums of S-units and linear recurrences, *Compos. Math.* **53** (1984), 225–244.

[8] J.-H. Evertse, On equations in S-units and the Thue-Mahler equation, *Inv. Math.* **75** (1984), 561–584.

[9] R. P. Flowe and G. A. Harris, A note on generalized vandermonde determinants, *SIAM J. Matrix Anal. Appl.* **14** (1993), 1146–1151.

[10] C. Fuchs and S. Heintze, On the growth of multi-recurrences, *Arch. Math.* (Basel) **119** (2022), 489–494.

[11] C. A. Gómez Ruiz and F. Luca, Multiplicative diophantine equations with factors from different Lucas sequences, *J. Number Theory* **170** (2017), 282–301.

[12] P. Kiss, Pure powers and power classes in recurrence sequences, *Math. Slov.* **44** (1994), 525–529.
[13] P. Kiss and F. Mátyás, Perfect powers from the sums of terms of linear recurrences, *Period. Math. Hungar.* **42** (2001), 163–168.
[14] Dom H. van der Laan, *Het plastische getal, XV lesson over de grondslagen van de architectonische ordonnante*, E. J. Brill, Leiden, 1967. Eng. trans.: *Architectonic Space, Fifteen lessons on the disposition of the human habitat*, translated by Richard Padovan, E. J. Brill, Leiden, 1983.
[15] M. Laurent, Linear forms in two logarithms and interpolation determinants II, *Acta Arith.* **133** (2008), 325–348.
[16] M. Laurent, M. Mignotte, and Y. Nesterenko, Formes linéaires en deux logarithmes et déterminants d'interpolation, *J. Number Theory* **55** (1995), 285–321.
[17] A. K. Lenstra, H. W. Lenstra Jr. and L. Lovász, *Factoring polynomials with rational coefficients*, Math. Ann. **261** (1982), 515–534.
[18] F. Luca, Arithmetic properties of members of a binary recurrent sequence, *Acta Arith.* **109** (2003), 81–107.
[19] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (6) (2000), 125–180. Eng. trans.: *Izv. Math.* **64** (2000), 1127–1169.
[20] M. Mignotte, A note on linear recursive sequences, *J. Aust. Math. Soc. (Ser. A)* **20** (1975), 242–244.
[21] J. Odjoumani and V. Ziegler, On prime powers in linear recurrence sequences, *Ann. Math. Québec* **47** (2023), 349–366.
[22] J. Odjoumani and V. Ziegler, On products of prime powers in linear recurrence sequences, *Acta Arith.* **215** (2024), 355–384.
[23] Richard Padovan, *Dom Hans van der Laan: Modern Primitive*, Architectura & Natura Press, Amsterdam, 1994.
[24] A. Pethö, Perfect powers in second order linear recurrences, *J. Number Theory* **15** (1982), 5–13.
[25] A. Pethö and B. M. M. de Weger, Products of prime powers in binary recurrence sequences. I. The hyperbolic case, with an application to the generalized Ramanujan-Nagell equation, *Math. Comp.* **47** (1986), 713–727.
[26] A. J. van der Poorten and H. P. Schlickewei, The growth conditions for recurrence sequences, *Macquarie Math. Reports* **82-0041** (1982).
[27] P. Ribenboim, The New Book of Prime Number Records, 3rd edition, Springer, 1996.
[28] H. P. Schlickewei, The p-adic Thue-Siegel-Roth-Schmidt theorem, *Arch. Math. (Basel)*, **29** (1977), 267–270.
[29] T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge Tracts in Math., **87**, Cambridge University Press, 1986.
[30] I. E. Shparlinskii, Prime divisors of recurrence sequences, *Izv. Vyssh. Uchebn. Zaved. Mat.* **1980** (4), 100–103.
[31] C. L. Stewart, On divisors of terms of linear recurrence sequences, *J. reine Angew. Math.* **333** (1982), 12–31.
[32] C. L. Stewart, On the greatest prime factor of terms of a linear recurrence sequence, *Rocky Mountain J. Math.* **15** (1985), 599–608.
[33] C. L. Stewart, On the greatest square-free factor of terms of a linear recurrence sequence, *Diophantine Equations*, edited by N. Saradha, Narosa Publishing House, New Delhi, 2008, 257–264.
[34] László Szalay, A note on the products of the terms of linear recurrences, *Acta Acad. Paedagog. Agriensis Sect. Math.* **24** (1997), 47–53.
[35] B. M. M. de Weger, Products of prime powers in binary recurrence sequences. II. The elliptic case, with an application to a mixed quadratic-exponential equation, *Math. Comp.* **47** (1986), 729–739.

[36] B. M. M. de Weger, *Algorithms for diophantine equations*, CWI Tract 65, Stichting Mathematisch Centrum, Amsterdam, 1989, now available at `https://ir.cwi.nl/pub/13190`.
[37] T. Yamada, On the simultaneous equations $\sigma(2^a) = p^{f_1}q^{g_1}$, $\sigma(3^b) = p^{f_2}q^{g_2}$, $\sigma(5^c) = p^{f_3}q^{g_3}$, Publ. Math. Debrecen **93** (2018), 57–71.

* Center for Japanese language and culture
Osaka University
562-8678
3-5-10, Semba-Higashi, Minoo, Osaka
JAPAN
Email address: `tyamada1093@gmail.com`