On the generating series of the degree sequence

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October 8, 2025

Abstract

We study the generating series associated with the degree sequence of a monomial self-map of a projective toric variety. We establish conditions under which this series has its circle of convergence as a natural boundary, and hence is a transcendental, non-holonomic function. In the case of toric surfaces, our results are sharp; moreover, we answer a question of Bell by proving that their reductions modulo p are transcendental for all but finitely many prime numbers p.

1 Introduction

Degree sequences and the associated dynamical degrees play a central role in algebraic dynamics. They are defined for arbitrary dominant rational maps on normal projective varieties over any field (see [DS05, Tru20, Dan20]). As fundamental birational invariants, dynamical degrees govern key dynamical features of the system and measure its complexity. Although their general behavior remains largely mysterious (cf. [BDJ20]), they can be explicitly computed and are known to be algebraic in certain situations, for instance, for dominant toric rational maps. In this paper, we restrict to this latter setting. Dynamical degrees are defined via the asymptotic behavior of degree sequences, which therefore encode finer information and are, in principle, more difficult to study. Our main objective is to show that, although dynamical degrees are comparatively simple and well understood in this case, the associated degree sequences may already display non-trivial complexity. We capture this complexity through the study of the corresponding generating series.

Throughout this paper, we work in the following setting. Let $X(\Sigma)$ be a projective toric variety of dimension $d \geq 2$ over an algebraically closed field of arbitrary characteristic. The variety $X(\Sigma)$ is determined by a lattice $N \cong \mathbb{Z}^d$, its dual lattice M, and a complete fan Σ in $N_{\mathbb{R}}$. Let $\varphi : X(\Sigma) \dashrightarrow X(\Sigma)$ be a toric dominant rational map. It corresponds to a homomorphism $M \to M$ given by a matrix $A \in M_d(\mathbb{Z})$ of non-zero determinant. Let $0 \leq k \leq d$ be an integer and D an ample Cartier divisor on $X(\Sigma)$, then the *k-degree sequence* is defined by the intersection numbers $\deg_{D,k}(\varphi^n) = ((\varphi^n)^* D^k \cdot D^{d-k})$ for all $n \geq 0$ (we use the convention $\varphi^0 = \mathrm{id}$). This sequence does not depend on the linear equivalence class of D. The *k-dynamical degree* of φ is defined as $\lambda_k(\varphi) := \lim \deg_{D,k}(\varphi^n)^{1/n}$, which is equal to the product of the k eigenvalues of largest modulus of A, cf. [HP07, FW12, Lin12]. We denote $\lambda_k(\varphi)$ by λ_k, unless otherwise specified. Our main object of interest is the *generating power series*

$$\Delta_{k,\varphi,D}(z) := \sum_{n \geq 0} \deg_{D,k}(\varphi^n) z^n \in \mathbb{Z}[[z]],$$

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which converges inside the open disk $\mathbb{D}(0, \lambda_k(\varphi)^{-1})$.

These series have been studied extensively in [AABM99, BHM03] when $k = 1$ and have been proven to be rational in many cases, most notably for regular maps and for birational surface maps. When the base field is $\mathbb{C}$, the dynamical degrees and the degree sequences are well-understood when the dynamical system $(X(\Sigma), \varphi)$ can be lifted to a simplicial projective toric variety $X(\tilde{\Sigma})$ and a *k-stable* self-map $\tilde{\varphi}$ in the sense that $\tilde{\varphi}^{n*} = \tilde{\varphi}^{*n}$ for the induced pullbacks on $H^{2k}(X(\tilde{\Sigma}), \mathbb{R})$ and for all $n \geq 0$, a property first introduced in [FS95]. A consequence of k-stability is that the degree sequences satisfy a *linear recurrence relation* (cf. [DF01, Corollary 2.2] and [LW14, Proposition 6.3]), which implies that $\Delta_{k,\varphi,D}$ is a *rational function*. On the other hand, under certain conditions on A, it is shown in [BK04, HP07, LW14] that the degree sequences do not satisfy any linear recurrence, so that $\Delta_{k,\varphi,D}$ is not rational.

A natural way to measure the complexity of power series is to place them within the following hierarchy:

$$\mathrm{Rat} \subset \mathrm{Alg} \subset \mathrm{Hol} \subset \mathrm{DiA} \subset \mathbb{C}[[z]]. \tag{1}$$

Here, $\mathrm{Rat} = \mathbb{C}(z) \cap \mathbb{C}[[z]]$ denotes the ring of rational power series, Alg the ring of power series that are *algebraic* over $\mathbb{C}(z)$ (i.e., those satisfying a polynomial equation with coefficients in $\mathbb{C}(z)$), Hol the ring of *holonomic* power series (that is, those satisfying a linear differential equation with coefficients in $\mathbb{C}(z)$), and DiA the ring of *differentially algebraic* power series, those satisfying an algebraic differential equation with coefficients in $\mathbb{C}(z)$. Power series that do not belong to Alg (resp. DiA) are called *transcendental* (resp. *hypertranscendental* or *differentially transcendental*). This classification also provides information about the corresponding coefficient sequences. Indeed, a power series is rational if and only if its coefficients satisfy a linear recurrence with constant coefficients, while it is holonomic if and only if its coefficients satisfy a linear recurrence relation with polynomial coefficients (see [Sta11, Chapter 4] and [SF99, Chapter 6]). Thus, more involved functional equations give rise to more complex coefficient sequences. When restricted to power series with algebraic coefficients, the rings Hol and DiA are countable, and they play for arithmetic holomorphic functions a role analogous to that of the ring of periods among complex numbers [KZ01]. The classification (1) is also central in enumerative combinatorics, where the nature of a generating series reflects structural properties of the objects being counted (cf. [BM06]). It has long been a classical problem to determine to which ring a given series arising from number theory, combinatorics, complex dynamics, or physics belongs. Hilbert [Hil02] already observed that many important number-theoretic functions, such as the Riemann zeta function, do not belong to DiA. For further results in this direction, we refer to [Sta80, BT94, SF99, ADH21] and the references therein.

A distinctive feature of holonomic convergent power series is that they admit analytic continuation to all but finitely many points of $\mathbb{C}$ (see, for instance, [Sta80, Theorem 2.1 and 4(a)]). A natural strategy to prove that a power series is not algebraic or holonomic is therefore to show that it has infinitely many singularities. In the case of the series $\Delta_{k,\varphi,D}(z)$, we investigate its analytic properties in greater detail and establish a stronger result. Namely, we will prove that it admits the circle $\mathcal{C}(0, {\lambda_k}^{-1})$ as a *natural boundary*, i.e., it cannot be analytically continued at any point on $\mathcal{C}(0, {\lambda_k}^{-1})$. Our main theorem, Theorem A, may be viewed as a generalization of [Fav03, Théorème principal], [BK04, Theorem 1.1], [HP07, Proposition 7.6], and [LW14, Theorem C]. The existence of a natural boundary shows that even in relatively simple cases where the dynamical degrees can be computed explicitly, the associated degree sequences may nevertheless exhibit some complexity.

Theorem A. *Let $\varphi : X(\Sigma) \dashrightarrow X(\Sigma)$ be a dominant toric rational self-map of a projective toric variety of dimension $d \geq 2$, which corresponds to a matrix $A \in M_d(\mathbb{Z})$ with $\det A \neq 0$. Let D be an ample Cartier divisor on $X(\Sigma)$. Assume that the eigenvalues of A are ordered as $|\mu_1| \geq |\mu_2| \geq |\mu_3| \geq \ldots \geq |\mu_d|$. Let $1 \leq k \leq d-1$, and assume further that*

- $|\mu_{k-1}| > |\mu_k| = |\mu_{k+1}| > |\mu_{k+2}|$,

- *μ_k and μ_{k+1} are complex conjugate, and μ_{k+1}/μ_k is not a root of unity.*

Then the series $\Delta_{k,\varphi,D}(z)$ has the circle $\mathcal{C}(0,{\lambda_k}^{-1})$ as a natural boundary. In particular, the series $\Delta_{k,\varphi,D}(z)$ is transcendental and non-holomonic over $\mathbb{C}(z)$.

Remark 1.1. There are several settings in which it has been shown that a power series is either rational or has its circle of convergence as a natural boundary. See, for instance, [BCR13] in the context of Mahler equations, and [BGNS23, BHN25] in the context of Artin-Mazur zeta functions arising from dynamics. It would be interesting to know whether a similar dichotomy also holds in our setting; in this direction, see Theorem D. We stress, however, that proving the existence of a natural boundary does not imply that the corresponding function is differentially transcendental. Establishing the latter property usually requires more sophisticated algebraic tools and is often related to the existence of a functional equation (not of differential type) satisfied by the function in question (see [ADH21]). It would therefore be interesting to determine whether the generating series in Theorem A are differentially transcendental.

Remark 1.2. One can consider other variants of the generating series, such as the Artin-Mazur style zeta function

$$\zeta_{k,\varphi,D}(z) := \exp\left(\sum_{n\geq 1}\frac{\deg_{D,k}(\varphi^n)}{n}z^n\right)$$

which also converges on $\mathbb{D}(0,\lambda_k^{-1})$. Observe that $z\dfrac{d}{dz}\log\zeta_{k,\varphi,D}(z) = \Delta_{k,\varphi,D}(z) - (D^d)$. Therefore $\zeta_{k,\varphi,D}(z)$ also has $\mathcal{C}(0,\lambda_k^{-1})$ as a natural boundary.

The Cesaro mean of the sequence $\deg_{D,k}(\varphi^n)$ is given by

$$\sum_{n\geq 1}\frac{\sum_{0\leq m\leq n-1}\deg_{D,k}(\varphi^m)}{n}z^n.$$

A computation similar to Theorem A implies that under the same assumption, it also admits $\mathcal{C}(0,\lambda_k^{-1})$ as a natural boundary.

The proof of Theorem A proceeds as follows. First, we use the standard dictionary relating intersection theory in toric varieties and mixed volumes to compute the degrees, following [HS95]. It expresses the degree sequence as the values of a piecewise linear function at the sequence of powers of an auxiliary matrix. Under our assumption on the eigenvalues of A, we are then able to reduce the whole problem to a two-dimensional situation where the auxiliary matrix becomes conformal. At this stage, we need to control a series of the form $\sum h(\chi^n)z^n$ where h is piecewise linear on the complex plane, and χ is a complex number whose argument is incommensurable with 2π. The key argument is to consider the Fourier series of the restriction of h to the unit circle, and to exploit the discontinuity of the derivative of h to infer that infinitely many of its Fourier coefficients are non-zero. To conclude that the generating series has a natural boundary, we use in a crucial way that these coefficients are non-zero along some arithmetic progression, which follows from the Skolem-Mahler-Lech theorem.

In [BDJ20], the authors construct the first example of a dominant rational self-map f_μ of $\mathbb{P}^2$ with transcendental dynamical degree $\lambda_1(f_\mu)$. The map f_μ is of the form $f_\mu = g\circ\varphi_\mu$, where

$$g:\mathbb{P}^2 \dashrightarrow \mathbb{P}^2, [x_0:x_1:x_2]\to[x_0(x_1+x_2-x_0):x_1(x_2+x_0-x_1):x_2(x_0+x_1-x_2)],$$

is a birational involution of $\mathbb{P}^2$ and φ_μ is the monomial map associated with $A_\mu = \begin{pmatrix}\operatorname{Re}\mu & -\operatorname{Im}\mu\\ \operatorname{Im}\mu & \operatorname{Re}\mu\end{pmatrix}$ where $\mu\in\mathbb{Z}[\mathrm{i}]$ is a Gaussian integer whose argument is incommensurable with 2π. Recall from *loc. cit.* that $\lambda_1(f_\mu) > \lambda_1(\varphi_\mu) = \lambda_2(\varphi_\mu)^{1/2} = \lambda_2(f_\mu)^{1/2}$ and we have the identity:

$$(1+\Delta_{1,f_\mu,\mathcal{O}(1)}(z))(2-\Delta_{1,\varphi_\mu,\mathcal{O}(1)}(z)) = 2$$

for $|z| < \lambda_1(f_\mu)^{-1}$, cf. Formula (2.2) and Proposition 2.8 in *loc. cit.*. It is straightforward to deduce the following corollary by applying Theorem A and [BFJ06, Main Theorem].

Corollary B. *For any Gaussian integer μ whose argument is incommensurable with 2π, the generating series $\Delta_{1,f_\mu,\mathcal{O}(1)}(z)$ of the map $f_\mu = g\circ\varphi_\mu$ is meromorphic in the disk $\mathbb{D}(0,\lambda_2(f_\mu)^{-1/2})$ with a simple pole at $\lambda_1(f_\mu)^{-1}$, and admits the circle $\mathcal{C}(0,\lambda_2(f_\mu)^{-1/2})$ as a natural boundary.*

Let $\mathfrak{M}$ denote the category of all normal projective varieties X over arbitrary fields K, together with all ample Cartier divisors D, and all dominant rational self-maps f. It is shown in [Ure18] that the set of degree sequences attached to elements of $\mathfrak{M}$ is countable[1]. In view of the examples of transcendental dynamical degrees [BDJ20, BDJK24, Sug25], it is thus natural to ask whether

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\lambda_k(f) : (X,K,D,f) \in \mathfrak{M}) = +\infty \quad \forall k \geq 1.$$

However, even the weaker question $\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\lambda_k(f) : (X,K,D,f) \in \mathfrak{M}) \geq 2$ remains open. On the other hand, at the level of functions, Theorem A allows us to deduce the following.

Corollary C. *For every $k \geq 1$, we have $\mathrm{tr.deg}_{\mathbb{C}(z)}(\Delta_{k,f,D}(z) : (X,K,D,f) \in \mathfrak{M}) = +\infty$.*

There is a parallel story for the degree sequences and dynamical degrees in the context of *monomial correspondences* [DR21]. Combining our results with those in *loc. cit.*, it is straightforward to establish a version of Theorem A and its corollaries in this setting. To keep the paper reasonably short, we refrain from stating it.

In the two-dimensional case, Theorem A covers all cases, and we obtain:

Theorem D. *Let $\varphi : X(\Sigma) \dashrightarrow X(\Sigma)$ be a dominant toric rational self-map of a projective toric surface. Let $A \in M_2(\mathbb{Z})$ be the matrix associated with φ, and let μ_1, μ_2 be its eigenvalues. Let D be an ample Cartier divisor on $X(\Sigma)$.*

1. *If μ_1 and μ_2 are complex conjugate and μ_1/μ_2 is not a root of unity, then $\Delta_{1,\varphi,D}$ admits $\mathcal{C}(0,\lambda_1^{-1})$ as a natural boundary. Furthermore, for any prime number p sufficiently large, the reduction modulo p*

$$\Delta_{1,\varphi,D}(z) \bmod p := \sum_{n\geq 0} (\deg_D(\varphi^n) \bmod p) z^n \in \mathbb{F}_p[[z]]$$

is transcendental over $\mathbb{F}_p(z)$.

2. *Otherwise, $\Delta_{1,\varphi,D}$ is a rational function.*

The proof of the dichotomy in Theorem D relies on [Fav03, Théorème principal] (see also [JW10, Theorem C']), which provides a necessary and sufficient condition for the existence of algebraically stable models for monomial self-maps of toric surfaces. Moreover, the transcendence of the reductions modulo p of the generating series answers a question posed by Bell (see [Bel24]). Our argument in this direction relies on Christol's theorem from automata theory.

Remark 1.3. We stress that the study of reductions modulo p of power series in $\mathbb{Z}[[x]]$ is an active topic, another way to measure the complexity of the coefficient sequence. When such a power series is algebraic, all its reductions modulo p are necessarily algebraic. The converse, however, is false: there exist transcendental power series whose reductions are algebraic for every prime p. In particular, this intriguing phenomenon occurs for important families of holonomic power series arising from geometry (such as diagonals of algebraic power series [Del84] and some hypergeometric series [VM21]). It has proved useful in questions of functional transcendence and has also led to challenging open problems (see, for instance, [AB13, ABD19, ABC23, CFVM25]).

[1] Although the result in *loc. cit.* is stated only for smooth varieties, its proof extends verbatim to the normal case.

Remark 1.4. Note that by Remark 1.2, the same dichotomy holds for the Artin-Mazur zeta function $\zeta_{1,\varphi,D}(z)$ in the toric surface case. Similar results have been proved using the Pólya–Carlson theorem for other kinds of Artin-Mazur zeta functions, see [MT88, Corollary 10.4] and [BGNS23, Theorem 4.2] for examples. However, the latter papers put conditions on the coefficients of the series that are not satisfied in our case.

Finally, we prove an intriguing rigidity result for the generating series of monomial surface maps, which follows from Theorem D. We refer to [JX23] and the references therein for deep rigidity results in holomorphic dynamics in dimension 1.

To state our result, we need an auxiliary notion. A matrix $A \in M_2(\mathbb{Z})$ is said to be *absolutely irreducible over* $\mathbb{Q}$ if there exists no A^n-invariant line in $\mathbb{Q}^2$ for all $n \geq 1$, or equivalently, A^n has no rational eigenvalues. A toric rational self-map φ of a toric surface is said to be *absolutely irreducible* over $\mathbb{Q}$ if its associated matrix is absolutely irreducible over $\mathbb{Q}$.

Theorem E. *Let φ and φ' be dominant toric rational self-maps of a projective toric surface $X(\Sigma)$ that are absolutely irreducible over $\mathbb{Q}$. Let D and D' be ample Cartier divisors on $X(\Sigma)$. Assume that φ, with respect to D, and φ', with respect to D', have the same 1-degree sequence, i.e.,*

$$\Delta_{1,\varphi,D} = \Delta_{1,\varphi',D'}.$$

Then there exists $u \in \{1, 2, 3, 4, 6\}$ such that φ^u and φ'^u are semi-conjugate, that is, $f \circ \varphi^u = \varphi'^u \circ f$ for some dominant toric rational map $f : X(\Sigma) \dashrightarrow X(\Sigma)$.

The difficulty of Theorem E is when the series are not rational. In this case, we need to use an explicit expression of $\Delta_{1,\varphi,D}$ and $\Delta_{1,\varphi',D'}$ to compare the coefficients of both sides. One can ask whether $\Delta_{1,f,D} = \Delta_{1,f,D'}$ implies $D = D'$, but it is unclear at this stage how to recover the line bundle from the generating series.

The paper is organized as follows. In Section 2, we cover basic preliminaries on toric geometry. Section 3 is devoted to presenting a computation that plays a key role in the proof of Theorem A. The readers can skip Section 3 on a first reading. In Section 4, we aim to present the proof of Theorem A and Corollary C. Finally, we restrict our attention to the surface case in Section 5.

Notations

Throughout this paper, we use $\mathbb{N} = \{0, 1, 2, \ldots\}$. We denote $\arg z \in [0, 2\pi)$ for the argument of $z \in \mathbb{C}$, and $\{x\} \in [0, 1)$ for the fractional part of $x \in \mathbb{R}$. For two functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \asymp g$ when there exist constants $c_1, c_2 > 0$ such that $c_1 f(n) \leq g(n) \leq c_2 f(n)$ for all n sufficiently large.

Acknowledgment

I am grateful to my advisors Boris Adamcewski and Charles Favre for their suggestions, helpful discussions, and constant encouragement.

This project has received funding from the European Union's MSCA-Horizon Europe, grant agreement No. 101126554.

Disclaimer

2 Toric geometry

In this section, we recall some materials from toric geometry and aim to interpret the degree sequences using mixed volumes, following [Oda93].

A *toric variety* is a normal variety that contains an open dense torus, whose natural multiplicative action on itself extends to the whole variety.

2.1 Toric varieties

Let $d \geq 1$, N a free abelian group of rank d, and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual lattice. We denote $N_{\mathbb{R}} := N \otimes \mathbb{R} \cong \mathbb{R}^d$, $M_{\mathbb{R}} := M \otimes \mathbb{R}$ the associated vector spaces equipped with the natural pairing $\langle \cdot, \cdot \rangle$, and similarly for $N_{\mathbb{Q}}$, $M_{\mathbb{Q}}$. We fix a basis of N and the dual basis for M.

A *polyhedral convex cone* in $N_{\mathbb{R}}$ is a set of the form $\sigma = \left\{ \sum_{v \in S} \lambda_v v : \lambda_v \geq 0 \right\}$ for a finite subset $S \subset N_{\mathbb{R}}$. A polyhedral convex cone σ is said to be *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$, and said to be *rational* if σ is generated by a finite subset S of N. For the rest of this section, by cones we mean strongly convex rational polyhedral cones. The *dimension* $\dim \sigma$ of σ is the dimension of the vector space generated by σ. One-dimensional cones are also called *rays*. A cone is *simplicial* if it is generated by linearly independent vectors. A cone is *regular* if it is generated by a set which is part of a basis of N.

A *fan* Σ in $N_{\mathbb{R}}$ is a finite collection of cones such that every face of a cone in Σ is also a cone in Σ, and the intersection of two cones in Σ is a face of each. The *support* of Σ is defined as $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$. We denote by $\Sigma(1)$ the set of rays in Σ. A fan is said to be *simplicial* (resp. *regular*) if all of its cones are simplicial (resp. regular). A fan Σ is said to be *complete* if $|\Sigma| = N_{\mathbb{R}}$. A fan $\tilde{\Sigma}$ is a *refinement* of Σ if every cone in $\tilde{\Sigma}$ is included in a cone of Σ.

Each fan Σ determines a toric variety $X(\Sigma)$ of dimension d with a dense torus $\mathbb{G}_m^d$, and all toric varieties arise in this way. The variety $X(\Sigma)$ is complete (resp. regular) if and only if Σ is complete (resp. regular). We say that $X(\Sigma)$ is simplicial if Σ is simplicial.

We say that Σ is projective if $X(\Sigma)$ is projective.

2.2 Cartier divisors and piecewise linear functions

Let Σ be a complete fan in N. On the toric variety $X(\Sigma)$, the action of the dense torus extends to an action on $\mathbb{Q}$-Cartier divisors. Those invariant under such an action are called $\mathbb{G}_m^d$*-invariant* $\mathbb{Q}$*-Cartier divisors*. Every $\mathbb{Q}$-Cartier divisor is linear equivalent to a $\mathbb{G}_m^d$-invariant one.

It turns out that $\mathbb{G}_m^d$-invariant $\mathbb{Q}$-Cartier divisors on $X(\Sigma)$ can be described in terms of support functions as follows. Let ($\mathbb{Q}$-)SF(Σ) be the set of all *($\mathbb{Q}$-)support functions with respect to* Σ, that is, for every cone $\sigma \in \Sigma$, there exists $m(\sigma) \in M$ (resp., $m(\sigma) \in M_{\mathbb{Q}}$) such that $h|_\sigma = m(\sigma)$. For each ray $\rho \in \Sigma(1)$, the associated *ray generator* is the unique lattice point v_ρ generating $\rho = \mathbb{R}_{\geq 0} v_\rho$. Every $h \in$ ($\mathbb{Q}$-)SF(Σ) determines a $\mathbb{G}_m^d$-invariant ($\mathbb{Q}$-)Cartier divisor $D(h) := \sum_{\rho \in \Sigma(1)} h(v_\rho) D_\rho$. Conversely, every $\mathbb{G}_m^d$-invariant ($\mathbb{Q}$-)Cartier divisor D can be regarded as a ($\mathbb{Q}$)-Weil divisor of the form $D = \sum_{\rho \in \Sigma(1)} a_\rho D(\rho)$ with $a_\rho \in \mathbb{Z}$ (resp. $a_\rho \in \mathbb{Q}$), which defines a function $h_D \in$ ($\mathbb{Q}$-)SF(Σ) by $h_D(v_\rho) = a_\rho$ for all $\rho \in \Sigma(1)$. This gives a one-to-one correspondence between ($\mathbb{Q}$-)SF(Σ) and $\mathbb{G}_m^d$-invariant ($\mathbb{Q}$-)Cartier divisors. A $\mathbb{G}_m^d$-invariant $\mathbb{Q}$-Cartier divisor D is ample if and only if h_D is *strictly convex*, i.e., it is a convex function and defined by different $m(\sigma)$ for different d-dimensional cones σ in Σ. Thus Σ is projective if and only if there is a strictly convex function in SF(Σ).

A *rational polytope* in $M_{\mathbb{R}}$ is the convex hull of finitely many points in $M_{\mathbb{Q}}$. A function $h \in \mathbb{Q}\text{-SF}(\Sigma)$ determines a (non-empty) rational polytope $P(h) := \{m \in M_{\mathbb{R}} : m \leq h\} \subset M_{\mathbb{R}}$. It follows that a $\mathbb{G}_m^d$-invariant $\mathbb{Q}$-Cartier divisor D defines a rational polytope $P_D := P(h_D)$. Two $\mathbb{G}_m^d$-invariant $\mathbb{Q}$-Cartier divisors D_1 and D_2 are linear equivalent if and only if $h_{D_1} - h_{D_2}$ is linear on $N_{\mathbb{R}}$, or equivalently, P_{D_1} is a translate of P_{D_2}. We note that $h \in \mathbb{Q}\text{-SF}(\Sigma)$ is strictly convex if and only if $P(h)$ is a polytope with non-empty interior, cf. [Sch13, Lemma 1.7.12].

If $h \in \text{SF}(\Sigma)$ is convex, then $P(h)$ is a *lattice polytope*, that is, the convex hull of finitely many points in M.

2.3 Toric rational maps and the pull-back operator

For a homomorphism $A : M \to M$, we denote by A the induced map $M_{\mathbb{R}} \to M_{\mathbb{R}}$, and by $\check{A}$ the dual maps $N \to N$ and $N_{\mathbb{R}} \to N_{\mathbb{R}}$.

Let Σ be a complete fan. A rational map $\varphi : X(\Sigma) \dashrightarrow X(\Sigma)$ is said to be *toric* if it is equivariant under the action of the torus $\mathbb{G}_m^d$. Every group homomorphism $A : M \to M$ with $A = (a_{ij}) \in M_d(\mathbb{Z})$ gives rise to the group homomorphism $\check{A} : N \to N$, which in turn determines the monomial map $\varphi : \mathbb{G}_m^d \to \mathbb{G}_m^d, (z_1, \ldots, z_d) \mapsto (z_1^{a_{11}} \cdots z_d^{a_{d1}}, \ldots, z_1^{a_{1d}} \cdots z_d^{a_{dd}})$ where $z_1, \ldots, z_d$ are coordinates on $\mathbb{G}_m^d$, hence a toric rational map $X(\Sigma) \dashrightarrow X(\Sigma)$. Conversely, all toric rational self-maps arise in this way. The map $\varphi : X(\Sigma) \dashrightarrow X(\Sigma)$ is dominant if and only if $\det(A) \neq 0$. In this case, its topological degree is equal to $|\det(A)|$.

For a $\mathbb{Q}$-Cartier divisor D on $X(\Sigma)$, its *pullback* via the toric rational map φ is defined as follows. We can find a regular refinement $\tilde{\Sigma}$ of Σ so that the induced map $\tilde{\varphi}$ is regular. It gives rise to a regular modification $\pi : X(\tilde{\Sigma}) \to X(\Sigma)$, induced by the identity map $\text{id} : M \to M$, such that $\tilde{\varphi} = \varphi \circ \pi$. The pullback $\varphi^* D$ is defined as $\varphi^* D := \pi_* \tilde{\varphi}^* D$ which is only a $\mathbb{Q}$-Weil divisor in general. This definition does not depend on the choice of $\tilde{\Sigma}$.

If $D = \sum_{\rho \in \Sigma(1)} h_D(v_\rho) D(\rho)$ is a $\mathbb{G}_m^d$-invariant ($\mathbb{Q}$-)Cartier divisor which corresponds to $h_D \in$ ($\mathbb{Q}$-)SF(Σ) and the polytope $P_D \subset M_{\mathbb{R}}$, then we have $\varphi^* D = \sum_{\rho \in \Sigma(1)} h_D(\check{A}(v_\rho)) D(\rho)$, which is also a $\mathbb{G}_m^d$-invariant ($\mathbb{Q}$-)Cartier divisor. Further, $\varphi^* D$ corresponds to $h_{\varphi^* D} = h_D \circ \check{A} \in$ ($\mathbb{Q}$-)SF(Σ) and $P(h_D \circ \check{A}) = AP(h_D) \subset M_{\mathbb{R}}$.

2.4 Mixed volumes and intersection numbers

Let $K_1, \ldots, K_d$ be convex bodies in $M_{\mathbb{R}}$. A theorem of Minkowski and Steiner says that the map given by the *Minkowski sum*

$$\mathbb{R}_{\geq 0}^d \to \mathbb{R}_{\geq 0}, (r_1, \ldots, r_d) \mapsto \text{Vol}(r_1 K_1 + \cdots + r_d K_d)$$

is a homogeneous polynomial of total degree d. The *mixed volume* $\text{Vol}(K_1, \ldots, K_d)$ of $K_1, \ldots, K_d$ is the non-negative number defined as the coefficient of the monomial $r_1 \cdots r_d$, divided by $d!$.

By [Oda93, p. 79], if $D_1, \ldots, D_d$ are $\mathbb{G}_m^d$-invariant $\mathbb{Q}$-Cartier divisors on a complete toric variety $X(\Sigma)$, then the intersection product can be interpreted by mixed volumes as

$$(D_1 \cdots D_d) = d! \,\text{Vol}(P_{D_1}, \ldots, P_{D_d}) \in \mathbb{Q}.$$

Here, we recall that $P_{D_1}, \ldots, P_{D_d}$ are the polytopes associated with $D_1, \ldots, D_d$, respectively.

2.5 Degree sequences

Let $X(\Sigma)$ be a projective toric variety defined by a fan Σ in a lattice N of rank $d \geq 1$. Let $\varphi : X(\Sigma) \dashrightarrow X(\Sigma)$ be a dominant toric rational map associated with a group homomorphism $A : M \to M$, D be an ample Cartier divisor on $X(\Sigma)$. Since every Cartier divisor is linear equivalent to a $\mathbb{G}_m^d$-invariant one, we may assume that D is $\mathbb{G}_m^d$-invariant. For $1 \leq k \leq d$, the k-degree sequence of φ can be rewritten as

$$\deg_{D,k}(\varphi^n) = ((\varphi^n)^* D^k \cdot D^{d-k}) = d! \operatorname{Vol}(A^n P_D[k], P_D[d-k]) \in \mathbb{Z}.$$

Here, the notation $Q[\ell]$ stands for ℓ times repetition of Q.

Recall that the k-dynamical degree of φ is $\lambda_k = \lambda_k(\varphi) = \lim \deg_{k,D}(\varphi^n)^{1/n} \in [1, \infty)$. Thus $\deg_{D,k}(\varphi^n) \asymp \lambda_k^n$.

3 Some auxiliary results on piecewise linear functions

In this section, we present two auxiliary results for real-valued piecewise linear functions that will be useful later. The readers can skip this section on a first reading.

A piecewise linear function on a finite-dimensional real vector space V is a continuous 1-homogeneous function $h\colon V \to \mathbb{R}$ such that there exists a complete (not necessarily rational) fan Σ of V and for each cone $\sigma \in \Sigma$ a linear function $h_\sigma : V \to \mathbb{R}$ such that $h = h_\sigma$ on the cone σ.

Lemma 3.1. *Every piecewise linear function is Lipschitz.*

Proof. Let $h : V \to \mathbb{R}$ be a piecewise linear function with respect to some fan. We fix a norm $|\cdot|$ on V. We can find some $C > 0$ sufficiently large so that if $v, w \in V$ lie in the same cone, then $|h(v) - h(w)| \leq C|v - w|$. If v and w are not in the same cone, we can join them by a segment $v = v_0, v_1, \ldots, v_n = w$ where v_i, v_{i+1} stay in the same cone for every $i = 0, \ldots, n-1$. Therefore

$$|h(v) - h(w)| \leq \sum_{i=0}^{n-1} |h(v_i) - h(v_{i+1})| \leq C \sum_{i=0}^{n-1} |v_i - v_{i+1}| = C|v - w|$$

as desired. ∎

Proposition 3.2. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a piecewise linear function. Let χ be some non-zero complex number. We denote by θ its argument. Then the power series $\sum_{n\geq 0} g(\operatorname{Re}(\chi^n), \operatorname{Im}(\chi^n)) z^n$ converges inside the disk $\mathbb{D}(0, |\chi|^{-1})$ and can be written as*

$$\sum_{m\in\mathbb{Z}} \frac{a_m}{1 - e^{im\theta}|\chi| z}$$

where $a_m \in \mathbb{C}$ such that $\sum_{m\in\mathbb{Z}} |a_m| < \infty$. Further, there is a linear recurrence sequence $\{d_m : m \in \mathbb{Z}\}$, such that $a_m = \dfrac{d_m}{m^2 - 1}$ for all $m \neq \pm 1$.

Proof. Since g is piecewise linear, we have $|g(\operatorname{Re}(\chi^n), \operatorname{Im}(\chi^n))| = O(|\chi|^n)$ which yields that $\sum_{n\geq 0} g(\operatorname{Re}(\chi^n), \operatorname{Im}(\chi^n)) z^n$ converges inside the disk $\mathbb{D}(0, |\chi|^{-1})$.

It is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ to make use of the conformal structure on $\mathbb{C}$, and view g as a 1-homogeneous continuous function on $\mathbb{C}$. Then

$$g(\operatorname{Re}(\chi^n), \operatorname{Im}(\chi^n)) z^n = \sum_{n\geq 0} |\chi|^n g(e^{in\theta}) z^n$$

inside the disk $\mathbb{D}(0, |\chi|^{-1})$.

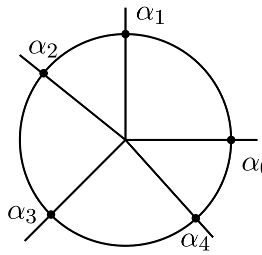
We decompose the function $g|_{\mathbb{S}^1}$ in the Hilbertian basis consisting of $\{e^{imx} : m \in \mathbb{Z}\}$ in $L^2(\mathbb{S}^1)$. Since $g|_{\mathbb{S}^1}$ is a piecewise linear function on $\mathbb{S}^1$, it is Lipschitz by Lemma 3.1. Therefore, its Fourier series converges uniformly to $g|_{\mathbb{S}^1}$. In addition, since $g|_{\mathbb{S}^1}$ is piecewise linear, $g|_{\mathbb{S}^1}$ belongs to the Sobolev space $W^{1,2}(\mathbb{S}^1) := \{f \in L^2(\mathbb{S}^1) : f' \in L^2(\mathbb{S}^1)\}$. Thus, for $x \in [0, 2\pi]$, we may write

$$g(e^{\mathrm{i}x}) = \sum_{m\in\mathbb{Z}} a_m e^{\mathrm{i}mx}$$

for $a_m \in \mathbb{C}$ and we have $\sum_{m\in\mathbb{Z}} m^2|a_m|^2 < \infty$ by Parseval's identity. It follows that $\sum_{m\in\mathbb{Z}} |a_m| < \infty$. Therefore, for $z \in \mathbb{D}(0, |\chi|^{-1})$, we obtain

$$\sum_{n\geq 0} |\chi|^n g(e^{\mathrm{i}n\theta})z^n = \sum_{m\in\mathbb{Z}} a_m \sum_{n\geq 0} |\chi|^n e^{\mathrm{i}mn\theta} z^n = \sum_{m\in\mathbb{Z}} a_m \frac{1}{1 - e^{\mathrm{i}m\theta}|\chi|z}.$$

Now we compute a_m. Since g is piecewise linear, we can write $g(z) = b_j z + c_j \overline{z}$ when $\arg z \in [\alpha_j, \alpha_{j+1})$, where $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_t = 2\pi$ are the break points on the unit circle $\mathbb{S}^1$ for some $t \in \mathbb{N}$, see the picture below as an illustration.



Since g is real-valued, we have $a_{-m} = \overline{a_m}$. For $m \neq \pm 1$, we have

$$\begin{aligned} a_m &= \int_0^{2\pi} g(e^{\mathrm{i}x})e^{-\mathrm{i}mx}dx = \sum_{j=0}^{t-1} \int_{\alpha_j}^{\alpha_{j+1}} b_j e^{\mathrm{i}(-m+1)x} + c_j e^{\mathrm{i}(-m-1)x}dx \\ &= \sum_{j=0}^{t-1} b_j \frac{e^{\mathrm{i}(-m+1)\alpha_{j+1}} - e^{\mathrm{i}(-m+1)\alpha_j}}{\mathrm{i}(-m+1)} + \sum_{j=0}^{t-1} c_j \frac{e^{\mathrm{i}(-m-1)\alpha_{j+1}} - e^{\mathrm{i}(-m-1)\alpha_j}}{\mathrm{i}(-m-1)}. \end{aligned}$$

For each $j = 0, \ldots, t$, we denote $A_j = e^{\mathrm{i}\alpha_j}$, then

$$a_m = \frac{\mathrm{i}(-m-1)\sum_{j=0}^{t-1} b_j(A_{j+1}^{-m+1} - A_j^{-m+1}) + \mathrm{i}(-m+1)\sum_{j=0}^{t-1} c_j(A_{j+1}^{-m-1} - A_j^{-m-1})}{m^2 - 1}.$$

Finally, set $d_m := \mathrm{i}(-m-1)\sum_{j=0}^{t-1} b_j(A_{j+1}^{-m+1} - A_j^{-m+1}) + \mathrm{i}(-m+1)\sum_{j=0}^{t-1} c_j(A_{j+1}^{-m-1} - A_j^{-m-1})$ for $m \in \mathbb{Z}$, which is a power sum; thus, the sequence $\{d_m : m \in \mathbb{Z}\}$ satisfy a linear recurrence relation as wanted. □

Remark 3.3. It is known that every linear recurrence sequence can be represented as values of a power sum $\sum_{j=1}^{h} \rho_j^m P_j(m)$ for some distinct non-zero complex numbers ρ_j and some non-zero polynomials P_j. Such a representation is unique. The *degree* of such a power sum is defined as the maximal degree between P_j.

In Proposition 3.2, the sequence d_m is represented by a power sum of degree at most 1.

4 Proof of Theorem A and its corollaries

We have seen how to derive Corollary B from Theorem A in the introduction. In this section, we prove Theorem A and, subsequently, Corollary C.

4.1 Proof of Theorem A

The proof of Theorem A is divided into three steps.

- Step 1: Expressing the k-degree sequence as values at $\Lambda^d A^n$ of a piecewise linear function $h : \mathrm{End}(\Lambda^k M_{\mathbb{R}}) \to \mathbb{R}$.
- Step 2: We cut the series into two parts: the dominant part $\Delta_{\Pi}(z)$ whose radius of convergence is λ_k^{-1}, and the remaining part $H(z)$ whose convergence radius is strictly larger than λ_k^{-1}. We show that $\Delta_{\Pi}(z)$ is not rational.
- Step 3: We use Proposition 3.2 to express $\Delta_{\Pi}(z)$ as wanted. We then apply a result on the existence of the natural boundary of a power series to conclude, see Proposition 4.5.

Step 1: Our computations, following [LW14], rely on the formulas for computing mixed volumes of polytopes that were developed in [HS95].

Recall that the k-th exterior power $\Lambda^k M_{\mathbb{R}}$ is a real vector space of dimension $\binom{d}{k}$.

Proposition 4.1. *For any polytope P in $M_{\mathbb{R}}$ and for any $k \in \{1, \dots, d\}$, there exists a piecewise linear function $h_P \colon \mathrm{End}(\Lambda^k M_{\mathbb{R}}) \to \mathbb{R}$, depending on P, such that the following holds.*

For any endomorphism $u : M_{\mathbb{R}} \to M_{\mathbb{R}}$, one has

$$\mathrm{Vol}(u(P)[k], P[d-k]) = h_P(\Lambda^k u).$$

Proof. We need some notation.

Given $\mathcal{P} = (P_1, P_2)$ a pair of polytopes in $M_{\mathbb{R}}$ such that $P_1 + P_2$ has dimension d. A *cell* of $\mathcal{P}$ is a pair $\mathcal{C} = (C_1, C_2)$ of polytopes $C_i \subset P_i$, $i = 1, 2$. We denote by $\#C_i$ the number of vertices of C_i, and by C the Minkowski sum $C_1 + C_2$. A *fine mixed subdivision* of $\mathcal{P}$ is a collection of cells $\mathcal{S} = \{\mathcal{C}^{(1)}, \dots, \mathcal{C}^{(r)}\}$ such that $\dim C^{(j)} = d$, $\dim C_1^{(j)} + \dim C_2^{(j)} = d$, $\#(C_1^{(j)}) + \#(C_2^{(j)}) - 2 = d$ for all $1 \le j \le r$, $C^{(j)} \cap C^{(j')}$ is a face of $C^{(j)}$ and $C^{(j')}$ for all j, j', $\bigcup_j C^{(j)} = P$.

If $\mathcal{S}$ is a fine mixed subdivision of $\mathcal{P}$, then [HS95, Theorem 2.4] says that

$$\mathrm{Vol}(P_1[k], P_2[d-k]) = k!(d-k)! \sum_{\mathcal{C}^{(j)} \in \mathcal{S}, \dim C_i^{(j)} = k, i=1,2} \mathrm{Vol}(C^{(j)}). \tag{2}$$

Now, assume that P has vertices $v_1, \dots, v_s$ in $M_{\mathbb{R}}$ with $s = \#P$, then $u(P)$ has vertices $u(v_1), \dots, u(v_s)$. We can apply (2) to the fine mixed subdivision $\mathcal{S}$ of $(u(P), P)$ with cells of the form

$$\mathcal{C}_{\mathcal{I}\mathcal{J}} := (\mathrm{conv}(u(v_{i_0}), \dots, u(v_{i_k})), \mathrm{conv}(v_{j_0}, \dots, v_{j_{d-k}}))$$

for some subsets of indices $\mathcal{I} = \{i_0, \dots, i_k\}$ and $\mathcal{J} = \{j_0, \dots, j_{d-k}\}$ in $\{1, \dots, s\}$. Here, $\mathrm{conv}(\dots)$ denotes the convex hull. Let $\mathcal{S}_k$ be the set of all such cells. It follows from (2) that

$$\mathrm{Vol}(u(P)[k], P[d-k]) = k!(d-k)! \sum_{\mathcal{C}_{\mathcal{I}\mathcal{J}} \in \mathcal{S}_k} \mathrm{Vol}(C_{\mathcal{I}\mathcal{J}}).$$

Since $C_{\mathcal{I}\mathcal{J}} = \mathrm{conv}(u(v_{i_0}), \dots, u(v_{i_k})) + \mathrm{conv}(v_{j_0}, \dots, v_{j_{d-k}})$, we have $k!(d-k)! \mathrm{Vol}(C_{\mathcal{I}\mathcal{J}}) = |\det(D_{\mathcal{I}\mathcal{J}})|$ where $D_{\mathcal{I}\mathcal{J}}$ is the matrix with vectors $u(v_{i_1} - v_{i_0}), \dots, u(v_{i_k} - v_{i_0})$, $v_{j_1} - v_{j_0}, \dots, v_{j_{d-k}} - v_{j_0}$ as columns, cf. [HS95, Lemma 2.5]. Observe that $\det(D_{\mathcal{I}\mathcal{J}})$ can be written as a $\mathbb{Z}$-linear combination of $k \times k$-minors of u, hence a $\mathbb{Z}$-linear combination of entries of $\Lambda^k u$. Let $\mathcal{C}_{\mathcal{I}\mathcal{J}}$ run over $\mathcal{S}_k$, the equations $\det(D_{\mathcal{I}\mathcal{J}}) = 0$ thus give us hyperplanes in $\mathrm{End}(\Lambda^k M_{\mathbb{R}})$, hence decompose $\mathrm{End}(\Lambda^k M_{\mathbb{R}})$ into a union of $\binom{d}{k}$-dimensional cones. Further, for any u so that $\Lambda^k u$ lies on one of those cones (depending on, for each $\mathcal{C}_{\mathcal{I}\mathcal{J}} \in \mathcal{S}_k$, whether $\det(D_{\mathcal{I}\mathcal{J}})$ is non-negative or non-positive), the expression

$$\mathrm{Vol}(u(P)[k], P[d-k]) = \sum_{\mathcal{C}_{\mathcal{I}\mathcal{J}} \in \mathcal{S}_k} |\det(D_{\mathcal{I}\mathcal{J}})|$$

gives us a linear function in terms of $\Lambda^k u$. Therefore, we obtain a piecewise linear function $h_P : \mathrm{End}(\Lambda^k M_\mathbb{R}) \to \mathbb{R}$ with respect to some fan in $\mathrm{End}(\Lambda^k M_\mathbb{R})$ such that $\mathrm{Vol}(u(P)[k], P[d-k]) = h_P(\Lambda^k u)$ as desired. □

Step 2: We may assume that D is $\mathbb{G}_m^d$-invariant, in which case D gives rise to a lattice polytope P_D in $M_\mathbb{R}$. By Proposition 2, there is a piecewise linear function $h = d! h_{P_D} : \mathrm{End}(\Lambda^k M_\mathbb{R}) \to \mathbb{R}$ such that

$$\deg_{D,k}(\varphi^n) = d! \,\mathrm{Vol}((A^n P_D)[k], P_D[d-k]) = h(\Lambda^k A^n) \text{ for all } n \geq 0.$$

Recall that $1 \leq k \leq d-1$ and the eigenvalues of $A : M_\mathbb{R} \to M_\mathbb{R}$ satisfy $|\mu_{k-1}| > |\mu_k| = |\mu_{k+1}| > |\mu_{k+2}|$ and μ_k, μ_{k+1} are complex conjugate numbers whose arguments are incommensurable with 2π. Therefore, $\lambda_{k-1} = |\mu_1\mu_2 \cdots \mu_{k-1}|$, $\lambda_k = |\mu_1\mu_2 \cdots \mu_k|$ by [FW12, Corollary B], and the matrix $\Lambda^k A$ has two dominant eigenvalues $\chi_1 := \mu_1 \cdots \mu_{k-1}\mu_k$ and $\chi_2 := \mu_1 \cdots \mu_{k-1}\mu_{k+1}$. The other eigenvalues of $\Lambda^k A$ have moduli less than or equal to $\lambda := \lambda_{k-1}|\mu_{k+2}| < \lambda_k$. Observe that the product $\mu_1\mu_2 \cdots \mu_{k-1}$ is real; thus, χ_1 and χ_2 are complex conjugate numbers whose arguments, say θ and $-\theta$ respectively, are incommensurable with 2π. We obtain:

Lemma 4.2. *There exists a $\Lambda^k A$-invariant splitting $\Lambda^k M_\mathbb{R} = V \oplus W$ with $\dim(V) = 2$, and*

- *the eigenvalues of the restriction of $\Lambda^k A$ to V are χ_1 and χ_2,*
- *the spectral norm of the restriction on W is λ.*

For $n \geq 0$, we write $\Lambda^k A^n|_V$ for the restriction of $\Lambda^k A^n$ to V. Observe that the subspace Π of endomorphisms of V that commute with $\Lambda^k A|_V$ is equal to $\mathbb{R}\mathrm{Id} \oplus \mathbb{R}\Lambda^k A|_V$, hence is two-dimensional. We remark that $\Lambda^k A^n|_V \in \Pi$ for all $n \geq 0$.

We can identify Π with its image in $\mathrm{End}(\Lambda^k M_\mathbb{R})$ under the map $u \mapsto u \oplus 0_W$. To simplify notation, we denote by $h \colon \Pi \to \mathbb{R}$ the piecewise linear function obtained by restricting h to Π, so that $h(\Lambda^k A^n|_V)$ is the value of h at the endomorphism $\Lambda^k A^n|_V \oplus 0_W$.

Since h is Lipschitz by Lemma 3.1, we have $\deg_{D,k}(\varphi^n) = h(\Lambda^k A^n) = h(\Lambda^k A^n|_V) + O(\lambda^n)$; hence, $h(\Lambda^k A^n|_V) \asymp \lambda_k^n$. It follows that

$$\Delta_{k,\varphi,D}(z) = \Delta_\Pi(z) + H(z)$$

where $\Delta_\Pi(z) = \sum_{n \geq 0} h(\Lambda^k A^n|_V) z^n$ converges inside $\mathbb{D}(0, \lambda_k^{-1})$ and $H(z)$ is a power series whose convergence radius is larger or equal to $\lambda^{-1} > \lambda_k^{-1}$.

Lemma 4.3. *The sequence $h(\Lambda^k A^n|_V), n \geq 0$, does not satisfy any linear recurrence relation.*

Proof. Assume the contrary. Since $h : \Pi \to \mathbb{R}$ is piecewise linear with respect to some fan, we can find a cone σ in Π so that $\Lambda^k A^n|_V$ lies in σ for infinitely many n. In other words, there is some linear function $h_\sigma : \Pi \to \mathbb{R}$ so that $h(\Lambda^k A^n|_V) = h_\sigma(\Lambda^k A^n|_V)$ for infinitely many n. We denote $\beta_n = h_\sigma(\Lambda^k A^n|_V)$ for $n \geq 0$.

The Cayley-Hamilton theorem implies that the sequence of matrices $\Lambda^k A^n|_V$, $n \geq 0$, satisfies the linear recurrence relation given by the characteristic polynomial of $\Lambda^k A|_V$. Therefore, the sequence β_n satisfies the same linear recurrence relation. Since the sequence $h(\Lambda^k A^n|_V)$ also satisfies a linear recurrence relation and coincides with the sequence β_n for infinitely many n, these two sequences must coincide on some arithmetic progression $a + b\mathbb{N}$ with $b > 0$ by the

Skolem-Mahler-Lech theorem, see, e.g., [Sta11, Chapter 4 Exercise 4]. Since $h(\Lambda^k A^n|_V) \asymp \lambda_k^n$, we have $\beta_n \asymp \lambda_k^n$ for $n \in a + b\mathbb{N}$. In particular, $\beta_n > 0$ for $n \in a + b\mathbb{N}$ sufficiently large.

Recall that $\theta = \arg\chi_1 \notin 2\pi\mathbb{Q}$ and $b \neq 0$. We may write

$$\beta_{a+bn} = h_\sigma(\Lambda^k A^{a+bn}|_V) = C\mathrm{Re}(\chi_1^{a+bn}) + D\mathrm{Im}\,(\chi_1^{a+bn}) = \lambda_k^{a+bn}\mathrm{Re}((C - \mathrm{i}D)e^{\mathrm{i}(a+bn)\theta})$$

for some $C, D \in \mathbb{R}$, not both equal to zero. Since $\{e^{\mathrm{i}(a+bn)\theta} : n \in \mathbb{N}\}$ is dense in the unit circle $\mathbb{S}^1$, there are infinitely many $n \in \mathbb{N}$ such that $\mathrm{Re}((C - \mathrm{i}D)e^{\mathrm{i}(a+bn)\theta}) < 0$, which is absurd. $\square$

It follows from [Sta11, Theorem 4.1.1] that $\Delta_\Pi(z)$ is not a rational function.

Step 3: Recall that θ is the argument of $\chi_1 = \mu_1 \cdots \mu_{k-1}\mu_k$ and $\lambda = |\mu_1 \cdots \mu_{k-1}\mu_{k+2}| < \lambda_k$.

Lemma 4.4. *Under the assumption of Theorem A, there are complex numbers $\{a_m\}_{m\in\mathbb{Z}}$ such that $\sum_{m\in\mathbb{Z}} |a_m| < \infty$ and*

$$\Delta_\Pi(z) = \sum_{m\in\mathbb{Z}} \frac{a_m}{1 - e^{\mathrm{i}m\theta}\lambda_k z}$$

for all $z \in \mathbb{D}(0, \lambda_k^{-1})$. Moreover, there exists a linear recurrence sequence $\{d_m : m \in \mathbb{Z}\}$ such that $a_m = \dfrac{d_m}{m^2 - 1}$ for all $m \neq \pm 1$. In addition, there exists an arithmetic progression $a + b\mathbb{N}$ with $b \neq 0$ such that $d_m \neq 0$.

Proof. We endow the plane V with a conformal structure so that it gets identified with the complex plane $\mathbb{C}$, and $\Lambda^k A|_V$ becomes the multiplication by χ_1. Via such identifications, the piecewise linear function $h : \Pi \to \mathbb{R}$ can be viewed as a piecewise linear function $h : \mathbb{R}^2 \to \mathbb{R}$ for which $h(\Lambda^k A^n|_V) = h(\mathrm{Re}(\chi_1^n), \mathrm{Im}\,(\chi_1^n))$ for all $n \geq 0$. By Proposition 3.2, we can write

$$\Delta_\Pi(z) = \sum_{m\in\mathbb{Z}} \frac{a_m}{1 - e^{\mathrm{i}m\theta}\lambda_k z}$$

for $z \in \mathbb{D}(0, \lambda_k^{-1})$. In addition, for $m \neq \pm 1$ we have $a_m = \dfrac{d_m}{m^2 - 1}$ for some linear recurrence sequence $\{d_m : m \in \mathbb{Z}\}$ as desired.

Since $\Delta_\Pi(z)$ is not rational, there are infinitely many $m \in \mathbb{Z}$ so that $d_m \neq 0$. The last assertion then follows from the Skolem-Mahler-Lech theorem [Sta11, Chapter 4 Exercise 4]. $\square$

Since $H(z)$ converges inside a disk properly containing the disk $\mathbb{D}(0, \lambda_k^{-1})$, it remains to show that $\Delta_\Pi(z)$ has $\mathcal{C}(0, \lambda_k^{-1})$ as a natural boundary. Since $\{e^{\mathrm{i}(a+bn)\theta}\lambda_k^{-1} : n \in \mathbb{N}\}$ is dense in $\mathcal{C}(0, \lambda_k^{-1})$, we complete the proof of Theorem A by applying the following result.

Lemma 4.5. *Let $\{p_m : m \in \mathbb{Z}\} \subset \mathbb{C}$ be a subset of $\mathcal{C}(0, \lambda_k)$. Let $\{b_m\}_{m\in\mathbb{Z}}$ be complex numbers such that $\sum_{m\in\mathbb{Z}} |b_m| < \infty$. Consider the power series $\Delta(z) := \sum_{m\in\mathbb{Z}} \dfrac{b_m}{1 - p_m z}$. Then we have*

$$\lim_{\rho\to 1^-} (1 - \rho)\Delta(p_m^{-1}\rho) = b_m \quad \textit{for all } m \in \mathbb{Z}.$$

If we assume further that the set of points $p_m \in \mathcal{C}(0, \lambda_k)$ satisfying $b_m \neq 0$ is dense in $\mathcal{C}(0, \lambda_k)$, then $\Delta(z)$ has $\mathcal{C}(0, \lambda_k^{-1})$ as a natural boundary.

Proof of Lemma 4.5. The hypothesis yields that $\Delta(z)$ converges in $\mathbb{D}(0, \lambda_k^{-1})$. Now, we fix some $m_0 \in \mathbb{Z}$, and take $z = p_{m_0}^{-1}\rho$ with $\rho \in \mathbb{R}$, $0 < \rho < 1$. Then

$$\Delta(p_{m_0}^{-1}\rho) = \sum_{m\in\mathbb{Z}\setminus\{m_0\}} \frac{b_m}{1 - p_m p_{m_0}^{-1}\rho} + \frac{b_{m_0}}{1 - \rho}.$$

Multiplying both sides by $(1-\rho)$ yields

$$(1-\rho)\Delta(p_{m_0}^{-1}\rho) = \sum_{m\in\mathbb{Z}\setminus\{m_0\}} b_m \frac{1-\rho}{1-p_m p_{m_0}^{-1}\rho} + b_{m_0}.$$

For every $m \neq m_0$, by triangle inequality we have

$$\left|\frac{1-\rho}{1-p_m p_{m_0}^{-1}\rho}\right| \leq \frac{1-\rho}{1-|p_m p_{m_0}^{-1}\rho|} \leq \frac{1-\rho}{1-\rho} \leq 1 \quad \text{for } 0<\rho<1.$$

Thus, it follows from $\sum_{m\in\mathbb{Z}} |b_m| < \infty$ that

$$\sum_{m\in\mathbb{Z}\setminus\{m_0\}} |b_m| \left|\frac{1-\rho}{1-p_m p_{m_0}^{-1}\rho}\right| \leq \sum_{m\in\mathbb{Z}\setminus\{m_0\}} |b_m| < \infty.$$

The Lebesgue dominated convergence theorem implies that

$$\lim_{\rho\to 1^-} \sum_{m\in\mathbb{Z}\setminus\{m_0\}} b_m \frac{1-\rho}{1-p_m p_{m_0}^{-1}\rho} = \sum_{m\in\mathbb{Z}\setminus\{m_0\}} \lim_{\rho\to 1^-} b_m \frac{1-\rho}{1-p_m p_{m_0}^{-1}\rho} = 0.$$

Therefore $\lim_{\rho\to 1^-} (1-\rho)\Delta(p_{m_0}^{-1}\rho) = b_{m_0}$.

In addition, if $b_{m_0} \neq 0$, then $\lim_{\rho\to 1^-} \Delta(p_{m_0}^{-1}\rho) = \infty$. Thus, if the set of points p_m satisfying $b_m \neq 0$ is dense in $\mathcal{C}(0,\lambda_k)$, then $\Delta(z)$ cannot be analytically continued at any point on $\mathcal{C}(0,{\lambda_k}^{-1})$ as wanted.

$\square$

4.2 Proof of Corollary C

Recall that we want to prove $\operatorname{tr.deg}_{\mathbb{C}(z)}(\Delta_{k,f,D}(z) : (X,K,D,f) \in \mathfrak{M}) = +\infty$ for all $k \geq 1$. We fix $k \geq 1$. Let $X(\Sigma)$ be any projective toric variety of dimension $k+1$ over some algebraically closed field, and D an ample Cartier divisor on $X(\Sigma)$. For instance, we can take $\mathbb{P}^{k+1}$ with $\mathcal{O}(1)$. It suffices to construct infinitely many dominant toric rational self-maps of $X(\Sigma)$ whose associated generating series of their k-degree sequences are algebraically independent over $\mathbb{C}(z)$.

Lemma 4.6. *Let $\varphi_1,\dots,\varphi_s$ be dominant toric rational self-maps of $X(\Sigma)$. Suppose that all φ_i satisfy the conditions in Theorem A. Suppose further that their k-dynamical degrees are pairwise distinct. Then the series $\Delta_{k,\varphi_1,D},\dots,\Delta_{k,\varphi_s,D}$ are algebraically independent over $\mathbb{C}(z)$.*

Granted this lemma, we can construct our desired toric maps as follows. By [Niv56, Corollary 3.12], we can choose infinitely many Gaussian integers $\mu_\ell \in \mathbb{Z}[i]$ of distinct moduli such that $\arg\mu_\ell \notin 2\pi\mathbb{Q}$ for all $\ell \geq 1$. We then consider for every ℓ the monomial maps φ_ℓ associated with a matrix $A_\ell \in M_{k+1}(\mathbb{Z})$ whose first $k-1$ dominant eigenvalues are $|\mu_\ell|^2$, and the k-th and $(k+1)$-th are μ_ℓ and $\overline{\mu}_\ell$, respectively. It follows that $\lambda_k(\varphi_\ell) = |\mu_\ell|^{2k}$. Lemma 4.6 yields that $\operatorname{tr.deg}_{\mathbb{C}(z)}(\Delta_{k,\varphi_\ell,D}(z) : \ell \geq 1) = +\infty$ as wanted.

It remains to prove Lemma 4.6. By Theorem A, the series $\Delta_{k,\varphi_1,D_1},\dots,\Delta_{k,\varphi_s,D_s}$ have circles of convergence as natural boundaries. The lemma then follows from the following observation.

Lemma 4.7. *Let $\Delta_1,\dots,\Delta_s$ be power series in $\mathbb{C}[[z]]$ that have circles of pairwise distinct positive radii as natural boundaries. Then they are algebraically independent over $\mathbb{C}(z)$.*

Proof. Let $\rho_1, \ldots, \rho_s$ be their radii of convergence, ordered so that $0 < \rho_1 < \rho_2 < \ldots < \rho_s$. Suppose by contradiction that there is a polynomial relation between these functions. We may suppose that this relation is minimal in terms of the number of power series Δ_i that appear. We see that there are polynomials $P_0, \ldots, P_r \in \mathbb{C}[z][X_2, \ldots, X_\ell]$ for some $1 \leq \ell \leq s$ and non-zero P_r such that $P_r(\Delta_2, \ldots, \Delta_\ell) \neq 0$ and

$$\Delta_1^r P_r(\Delta_2, \ldots, \Delta_\ell) + \cdots + P_0(\Delta_2, \ldots, \Delta_\ell) = 0. \tag{3}$$

The left-hand side of (3) can be viewed as a polynomial in Δ_1 with coefficients in the ring of holomorphic functions inside the domain $\mathbb{D}(0, \rho_2)$. We denote by Δ its discriminant when considering Δ_1 as a variable. It is a holomorphic function inside $\mathbb{D}(0, \rho_2)$.

Recall that Δ_1 has the circle $\mathcal{C}(0, \rho_1)$ as a natural boundary. By the identity theorem, we can find a point $t \in \mathcal{C}(0, \rho_1)$ such that $P_r(\Delta_2(t), \ldots, \Delta_\ell(t)) \neq 0$ and $\Delta(t) \neq 0$. Then $P_r(\Delta_2(z), \ldots, \Delta_\ell(z)) \neq 0$ and $\Delta(z) \neq 0$ around some disk $\mathbb{D}$ centered at t. Now, there exists exactly r complex numbers c satisfying the algebraic equation

$$c^r P_r(\Delta_2(t), \ldots, \Delta_\ell(t)) + \cdots + P_0(\Delta_2(t), \ldots, \Delta_\ell(t)) = 0.$$

By the Cauchy-Kovalevskaya theorem [Hör83, Theorem 9.4.8], for each c, there is an open disk $\mathbb{D}_0 \subseteq \mathbb{D}$ centered at t and a holomorphic function y on $\mathbb{D}_0$ such that $y(t) = c$. By the uniqueness of solutions of the previous initial value problem, we deduce that Δ_1 must equal some y on $\mathbb{D}_0 \cap \mathbb{D}(0, \rho_1)$. In other words, there exists an analytic extension of Δ_1 to $\mathbb{D}_0 \cap \mathbb{D}(0, \rho_1)$, which is absurd. $\square$

5 The surface case

In this section, we restrict our attention to monomial surface maps. The advantage is that the eigenvalues of the associated 2×2 matrices are easier to control than in the general case.

5.1 Proof of the dichotomy part of Theorem D

There are two possibilities for the eigenvalues of $A \in M_2(\mathbb{Z})$ of non-zero determinant.

1. If $\overline{\mu_2} = \mu_1$ with μ_1/μ_2 is not a root of unity, then it follows from Theorem A that $\Delta_{1,\varphi,D}$ has the desired form. In addition, $\Delta_{1,\varphi,D}$ has the circle $\mathcal{C}(0, \lambda_1^{-1})$ as a natural boundary.
2. In the remaining case, by [Fav03, Théorème principal], φ can be lifted to a 1-stable toric rational map. Here, we note that toric surfaces are always simplicial. Therefore, $\Delta_{1,\varphi,D}$ is rational thanks to [DF01, Corollary 2.2][2].

This finishes the proof of the dichotomy.

Remark 5.1. In the first case, we have

$$\Delta_{1,\varphi,D}(z) = \sum_{m \in \mathbb{Z}} \frac{a_m}{1 - e^{im\theta} \lambda_1 z}$$

where $\theta = \arg \mu_1$ and $a_m = \dfrac{d_m}{m^2 - 1}$ for some linear recurrence sequence $\{d_m : m \in \mathbb{Z}\}$ with $\sum_{m \in \mathbb{Z}} |a_m| < \infty$. Further, the linear recurrence sequence d_m is represented by a power sum of degree at most 1, see Remark 3.3. This observation will be useful in the proof of Theorem E.

[2]Although the article was written for surfaces over $\mathbb{C}$, the result works with a proof unchanged over any algebraically closed field.

Remark 5.2. We propose a direct proof of Theorem D without relying on [Fav03]. When $\mu_1 > \mu_2 > 0$, since $d = 2$, we have

$$\Delta_{1,\varphi,D}(z) = \Delta_\Pi(z) = \sum_{n\geq 0} h(\mu_1^n, \mu_2^n) = \sum_{n\geq 0} \mu_1^n h\Big(1, \Big(\frac{\mu_2}{\mu_1}\Big)^n\Big) z^n$$

for some piecewise linear function $h : \mathbb{R}^2 \to \mathbb{R}$ with respect to some (not necessarily rational) fan. Since $(\mu_2/\mu_1)^n$ tends to 0 as n goes to infinity, there is some $N_0 \in \mathbb{N}$ so that the points $(1, (\mu_2/\mu_1)^n)$ belong to some two-dimensional cone of the fan for all $n \geq N_0$. Thus, there are some $p, q \in \mathbb{R}$ such that for $n \geq N_0$, we have $h(1, (\mu_2/\mu_1)^n) = p + q(\mu_2/\mu_1)^n$. Therefore

$$\Delta_{1,\varphi,D}(z) = \sum_{n=0}^{N_0} h(\mu_1^n, \mu_2^n) z^n + \sum_{n\geq N_0} (p\mu_1^n + q\mu_2^n) z^n \in \mathbb{C}(z).$$

Other cases for the eigenvalues μ_1, μ_2 can be treated similarly.

Remark 5.3. In light of Remark 1.1, we can ask whether, when $\overline{\mu_2} = \mu_1$ with μ_2/μ_1 is not a root of unity, $\Delta_{1,\varphi,D}$ is differentially transcendental over $\mathbb{C}(z)$. Although we are not able to prove it for $\Delta_{1,\varphi,D}$, we can prove it for

$$\Delta_0(z) := \sum_{m\in\mathbb{Z}} \frac{(m^2-1)a_m}{1 - e^{\mathrm{i}m\theta}\lambda_1 z} = \frac{-a_0}{1-\lambda_1 z} + \sum_{|m|>1} \frac{d_m}{1 - e^{\mathrm{i}m\theta}\lambda_1 z}.$$

Indeed, since both $\{d_m\}_{m>1}$ and $\{d_m\}_{m<-1}$ are linear recurrence sequences, we deduce that the series $\Delta_0(z)$ satisfies a linear $e^{\mathrm{i}\theta}$-difference equation with coefficients in $\mathbb{C}(z)$. Since $e^{\mathrm{i}\theta}$ is not a root of unity, it follows from [ADH21, Theorem 1.2] that either $\Delta_0 \in \bigcup_{j\geq 1} \mathbb{C}(z^{1/j})$, or it is differentially transcendental. Similar to $\Delta_{1,\varphi,D}(z)$, we observe that $\Delta_0(z)$ also admits the circle $\mathcal{C}(0, \lambda_1^{-1})$ as a natural boundary; in particular, $\Delta_0(z) \notin \bigcup_{j\geq 1} \mathbb{C}(z^{1/j})$. Therefore $\Delta_0(z)$ is differentially transcendental as wanted. We expect that there is a functional equation between $\Delta_{1,k,D}$ and Δ_0 from which one can deduce the differential transcendence for $\Delta_{1,k,D}$.

5.2 Proof of the reduction part of Theorem D

Our goal is to prove that when the eigenvalues μ_1 and μ_2 of the matrix A are complex conjugate and μ_1/μ_2 is not a root of unity, then for all prime numbers p sufficiently large, the reduction $\Delta_{1,\varphi,D}(z) \bmod p$ is transcendental over $\mathbb{F}_p(z)$. We need the following result from automata theory.

Lemma 5.4. *Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$. If $\sum_{n\geq 0} s_n z^n \in \mathbb{F}_q((z))$ is algebraic over $\mathbb{F}_q(z)$, then the sequence $(s_n)_{n\geq 0}$ is weakly periodic in the sense that for any integers $a > 0$ and $b \geq 0$, there exist $k > 0$ and $r > r' \geq 0$ such that $s_{a(kn+r)+b} = s_{a(kn+r')+b}$ for all $n \geq 0$.*

Proof. It follows from Christol's theorem [Chr79, Théorème 1][3], and [BK20, Lemma 2.1]. □

Therefore, we need to show that for primes p sufficiently large, the reduction modulo p of the degree sequence is not weakly periodic. In the proof of Theorem A, we have interpreted the degree sequence as values of a piecewise linear function. However, we need an additional property for such a function, namely, its convexity.

We may assume that D is an ample $\mathbb{G}_m^2$-Cartier divisor. Recall that $P_D \subset M_\mathbb{R}$ is the polyhedron associated to D, and $h_D : N_\mathbb{R} \to \mathbb{R}$ is its associated strictly convex piecewise linear

[3] Although the article stated the result only for the field $\mathbb{F}_p$, it holds verbatim for $\mathbb{F}_q$, cf. [AB19, Theorem 6.1].

function with respect to the fan Σ. We note that when restricting to every cone of Σ, h_D is a linear function with coefficients in $\mathbb{Z}$. We recall also that the subspace Π of endomorphisms of $M_{\mathbb{R}}$ that commutes with A is equal to $\mathbb{R}\mathrm{Id} \oplus \mathbb{R}A$, which is two-dimensional.

Lemma 5.5. *There exists a convex piecewise linear function $h : \Pi \to \mathbb{R}$ with respect to some complete fan in Π, which is not linear, such that for any endomorphism $u \in \Pi$, one has $2\mathrm{Vol}(u(P_D), P_D) = h(u)$. In particular, we have*

$$\deg_{D,1}(\varphi^n) = h(A^n) \text{ for all } n \geq 0.$$

Further, for each cone σ over which h is linear, the coefficients of h_σ are in $\mathbb{Z}$.

Proof. We will use the *surface area measure* to compute the mixed volume, following [Sch13]. We fix a scalar product for which the given basis of N forms an orthogonal basis, which gives rise to a Riemannian metric on $\mathbb{S}^1 \subset N_{\mathbb{R}}$. If Q is a compact polyhedron in $M_{\mathbb{R}}$, the surface area measure $\mathcal{S}(Q)$ on $\mathbb{S}^1 \subset N_{\mathbb{R}}$ is defined as $\sum_v \mathrm{Vol}(F_v)\delta_v$ where v runs over the outer unit normal vectors of the facets F_v of Q, and δ_v denotes the Dirac measure at v. Here, the volume of each facet is its one-dimensional volume, in other words, its length. By [Sch13, Formulae (5.19) and (5.23)], we have

$$\mathrm{Vol}(u(P_D), P_D) = \frac{1}{2}\int_{\mathbb{S}^1} (h_D \circ \check{u})\mathcal{S}(P_D) = \frac{1}{2}\sum_v \mathrm{Vol}(F_v)h_D(\check{u}(v))$$

where $\check{u} : N_{\mathbb{R}} \to N_{\mathbb{R}}$ is the dual linear map of u, and v runs over the outer unit normal vectors of the facets F_v of P_D. We set $h(u) := \sum_v \mathrm{Vol}(F_v)h_D(\check{u}(v))$ for $u \in \Pi$.

Since all coefficients $\mathrm{Vol}(F_v)$ are positive, it is sufficient to show that for any v, the function $h_v(u) := h_D(\check{u}(v))$ is piecewise linear and convex. Recall the fact that every convex piecewise linear function can be represented as the maximum of linear forms; thus, $h_D = \max\{l_1, \ldots, l_n\}$ for some linear forms l_i with coefficients in $\mathbb{Z}$. It follows that

$$h_v(u) = \max\{L_1(u), \ldots, L_n(u)\}$$

where $L_i(u) = l_i(\check{u}(v))$ is a linear form on the space Π. Thus h_v is a convex piecewise linear function on Π for every v. In addition, since the vector $\mathrm{Vol}(F_v)v$ have coefficients in $\mathbb{Z}$ (as P_D is a lattice polytope), the linear functions $\mathrm{Vol}(F_v)L_i(u) = l_i(\check{u}(\mathrm{Vol}(F_v)v))$ have coefficients in $\mathbb{Z}$. Therefore, the piecewise linear functions $\mathrm{Vol}(F_v)h_v = \max\{\mathrm{Vol}(F_v)L_i\}$ also have coefficients in $\mathbb{Z}$, and so does h.

Since $A^n \in \Pi$ for all $n \geq 0$, we have $\deg_{D,1}(\varphi^n) = 2!\mathrm{Vol}(A^n P_D, P_D) = h(A^n)$ for all $n \geq 0$. Finally, if h is linear, then $\Delta_{1,\varphi,D}(z) = \sum_{n \geq 0} h(A^n)z^n$ would be rational which contradicts our result in §5.1. □

Remark 5.6. This map h is the same as the map h constructed in Step 3 of the proof of Theorem A when $d = 2$ and $k = 1$, but now we have an additional convex property.

Since A has complex conjugate eigenvalues μ_1 and μ_2, we can find $Q \in \mathrm{GL}_2(\mathbb{Q}(\mu_1))$ such that $QAQ^{-1} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$. By making a linear change by Q, we can endow $M_{\mathbb{R}}$ with a conformal structure and A becomes the multiplication by μ_1. Via this identification, we can view $h : \Pi \to \mathbb{R}$ as a convex piecewise linear function $h : \mathbb{C} \to \mathbb{R}$, which is not linear, such that $\deg_{D,1}(\varphi^n) = h(\mu_1^n)$. We note that h now has coefficients in $\mathbb{Q}(\mu_1)$ on each cone. Namely, there is an integer $s \geq 1$ together with a decomposition of $[0, 1] = \bigcup_{i=1}^{s} I_i$ into closed subsets, and there exist complex numbers $c_1, \dots, c_s$ in $\mathbb{Q}(\mu_1)$ such that $h(z) = c_i z + \overline{c_i z}$ whenever $\arg z \in 2\pi I_i$. Here, the closed sets I_i are subsets of the form $[\alpha, \beta]$ or of the form $[0,1]\backslash(\alpha,\beta)$ for some $0 \le \alpha < \beta \le 1$ with $\beta - \alpha < 1$. Via the identification $[0,1] \to \mathbb{S}^1, x \mapsto e^{\mathrm{i}2\pi x}$, we may assume that the sets I_i, for $i = 1, \dots, s$, are organized so that they correspond to consecutive closed connected subsets covering the unit circle $\mathbb{S}^1$ in clockwise order. For each i, we introduce the cone $\sigma_i := \{\lambda e^{\mathrm{i}2\pi x} : \lambda \ge 0, x \in I_i\}$ in $\mathbb{C}$, then $h(z) = c_i z + \overline{c_i z}$ whenever $z \in \sigma_i$.

Lemma 5.7. *If there are $1 \le i < j \le s$ such that $c_i = c_j$, then either $c_\ell = c_i$ for all integers $\ell \in (i,j)$ or $c_\ell = c_i$ for all integers $\ell \notin [i,j]$.*

Proof. Replacing $h(z)$ by $h(z) - (c_i z + \overline{c_i z})$, we may assume that $c_i = c_j = 0$. Thus $h(z) = 0$ for all $z \in C_i \cup C_j$.

We set σ to be the convex hull of σ_i and σ_j. Then either $\bigcup_{\ell\in(i,j)} \sigma_\ell \subset \sigma$ or $\bigcup_{\ell\notin[i,j]} \sigma_\ell \subset \sigma$ (or both). We assume the former case and aim to prove that $c_\ell = c_i$ for all $\ell \in (i,j)$; the latter case is n similarly.

Under our assumptions, we have $h(z) \le 0$ for all $z \in \sigma$ thanks to the convexity of h. Let z_i be any point in the interior of σ_i. For any $z \in \sigma$, we then can find some $0 < t < 1$ such that $tz_i + (1-t)z \in \sigma_i$, whence

$$0 = h(tz_i + (1-t)z) \le th(z_i) + (1-t)h(z) = (1-t)h(z),$$

which implies that $h(z) \ge 0$. We deduce that $h(z) = 0$ for all $z \in \sigma$. It follows that for any integer ℓ in (i,j), we have $h(z) = 0$ for $z \in \sigma_\ell$, hence $c_\ell = 0$ as wanted. $\square$

We can merge consecutive closed subsets I_i on which h has the same coefficients. Thanks to Lemma 5.7, we may then assume that the coefficients $c_1, \dots, c_s$ are pairwise distinct. Since h is not linear, we have $s > 1$.

Lemma 5.8. *For every $1 \le i < j \le s$, there is at most one $n \ge 0$ such that $c_i\mu_1^n + \overline{c_i}\mu_2^n = c_j\mu_1^n + \overline{c_j}\mu_2^n$.*

Proof. For every $0 \le n < m$, we note that two vectors (μ_1^n, μ_2^n) and (μ_1^m, μ_2^m) are linearly independent over $\mathbb{C}$, otherwise $\mu_1^{m-n} = \mu_2^{m-n}$, which is absurd since μ_1/μ_2 is not a root of unity. Since the kernel of the non-trivial linear form $(c_i - c_j)z_1 + (\overline{c_i} - \overline{c_j})z_2$ has dimension one over $\mathbb{C}$, the lemma follows. $\square$

We conclude that there is some positive integer b such that $c_1\mu_1^b + \overline{c_1}\mu_2^b \ne c_j\mu_1^b + \overline{c_j}\mu_2^b$ for all $1 < j \le s$. To simplify notation, let $K = \mathbb{Q}(\mu_1)$, consider the finite subset of non-zero elements in K

$$\mathcal{F} = \{\mu_1, c_i \text{ for } 1 \le i \le s, \text{ and } c_1\mu_1^b + \overline{c_1}\mu_2^b - c_i\mu_1^b - \overline{c_i}\mu_2^b \text{ for } 1 < j \le s\}.$$

The set of places (i.e., of equivalence classes of non-trivial norms) $|\cdot|_v$ on K such that $|x|_v \ne 1$ for some $x \in \mathcal{F}$ is finite. We infer that for all prime integers p sufficiently large, and for any place $|\cdot|_v$ extending the p-adic norm to K, then $|x|_v = 1$ for all $x \in \mathcal{F}$. Fix any such prime p and any place $|\cdot|_v$ extending the p-adic norm. We denote $K_v^\circ = \{x \in K : |x|_v \le 1\}$ and $K_v^{\circ\circ} = \{x \in K : |x|_v < 1\}$, and observe that the residue field $\tilde{K}_v = K_v^\circ / K_v^{\circ\circ}$ is a finite extension of $\mathbb{F}_p$.

Assume by contradiction that $\Delta_{1,\varphi,D}$ modulo p is algebraic over $\mathbb{F}_p(z)$. Thanks to the natural embedding $\mathbb{F}_p \hookrightarrow \tilde{K}_v$, the reduction $\Delta_{1,\varphi,D}(z) \bmod K_v^{\circ\circ} = \sum_{n\ge0} (h(\mu_1^n) \bmod K_v^{\circ\circ}) z^n \in \tilde{K}_v[[z]]$ is also algebraic over $\mathbb{F}_p(z)$, hence over $\tilde{K}_v(z)$. It follows that the sequence $\{h(\mu_1^n) \bmod K_v^{\circ\circ}\}_{n\ge0}$ is weakly periodic by Lemma 5.4. In particular, looking at the arithmetic progression $(q-1)\mathbb{N} + b$, there exist integers $k > 0$ and $r > r' \ge 0$ such that

$$h(\mu_1^{(q-1)(kn+r)+b}) = h(\mu_1^{(q-1)(kn+r')+b}) \bmod K_v^{\circ\circ} \text{ for all } n \ge 0. \tag{4}$$

We denote $r_n = (q-1)(kn+r)+b$ and $r'_n = (q-1)(kn+r')+b$ for convenience. Observe that $r_n - r'_n = (q-1)(r-r')$ for all $n \geq 0$. Recall that $\theta = \arg \mu_1 \notin 2\pi\mathbb{Q}$. Thus, the sets $\{\{r_n\theta/2\pi\} : n \in \mathbb{N}\}$ and $\{\{r'_n\theta/2\pi\} : n \in \mathbb{N}\}$ are both dense in $(0,1)$. We may assume that $I_1 = [\alpha, \beta]$ for some $0 \leq \alpha < \beta \leq 1$ with $\beta - \alpha < 1$, the case $I_1 = [0,1] \setminus (\alpha, \beta)$ is proven similarly. We have two cases.

- If $\{(q-1)(r-r')\theta/2\pi\} > \beta - \alpha$, then we can find $n \geq 0$ such that $\{r_n\theta/2\pi\} \in I_1$ (which is close to the right of α) and $\{r'_n\theta/2\pi\} \notin I_1$. In other words, $\{r_n\theta/2\pi\} \in I_1$ and $\{r'_n\theta/2\pi\} \in I_j$ for some $j \neq 1$. It follows that
$$h(\mu_1^{r_n}) = c_1\mu_1^{r_n} + \overline{c_1}\mu_2^{r_n} = c_1\mu_1^b + \overline{c_1}\mu_2^b \bmod K_v^{\circ\circ},$$
and
$$h(\mu_1^{r'_n}) = c_j\mu_1^{r'_n} + \overline{c_j}\mu_2^{r'_n} = c_j\mu_1^b + \overline{c_j}\mu_2^b \bmod K_v^{\circ\circ}.$$
Here, we use the fact that $x^{q-1} = 1 \bmod K_v^{\circ\circ}$ for every $x \in K_v^\circ \setminus K_v^{\circ\circ}$. Since $c_1\mu_1^b + \overline{c_1}\mu_2^b \neq c_j\mu_1^b + \overline{c_j}\mu_2^b \bmod K_v^{\circ\circ}$ by the assumption on p, the condition (4) cannot be fulfilled for such n, a contradiction.

- If $\{(q-1)(r-r')\theta/2\pi\} \leq \beta - \alpha$, then we can find $n \geq 0$ such that $\{r_n\theta/2\pi\} \notin I_1$ (which is close to the left of α) and $\{r'_n\theta/2\pi\} \in I_1$. Therefore, $\{r_n\theta/2\pi\} \in I_j$ for some $j \neq 1$ and $\{r'_n\theta/2\pi\} \in I_1$. An argument similar to the previous case also leads to a contradiction.

The proof of Theorem D is complete.

Remark 5.9. We cannot avoid the possibility of finitely many exceptional prime numbers p in Theorem D. Indeed, $\Delta_{1,\varphi,pD} \bmod p$ is always zero for all prime numbers p. A less trivial obstruction comes from the following example. Consider the series $\Delta_{1,\varphi_\mu,\mathcal{O}(1)}(z)$ as in Corollary B. In this case, the degree sequence is given by the values at $A_\mu^n = \begin{pmatrix} \operatorname{Re}\mu^n & -\operatorname{Im}\mu^n \\ \operatorname{Im}\mu^n & \operatorname{Re}\mu^n \end{pmatrix}$ of a piecewise linear function $h : \Pi \to \mathbb{R}$ with coefficients in $\mathbb{Z}$, cf. [BDJ20, Formula (1.1)]. Therefore, $\Delta_{1,\varphi_\mu,\mathcal{O}(1)}(z) \bmod p$ is equal to 1 for every common prime divisor p of $\operatorname{Re}\mu_1$ and $\operatorname{Im}\mu_1$.

5.3 Proof of Theorem E

Recall that φ and φ' are two monomial surface maps which are absolutely irreducible over $\mathbb{Q}$ with the same degree sequence. We need to show that they are semi-conjugate.

Let A and A' be their corresponding matrices in $M_2(\mathbb{Z})$. Let μ_1, μ'_2 be the eigenvalues of A and μ'_1, μ'_2 be the eigenvalues of A' ordered so that $|\mu_1| \geq |\mu_2|$ and $|\mu'_1| \geq |\mu'_2|$. Then $\lambda_1(\varphi) = |\mu_1|$ and $\lambda_1(\varphi') = |\mu'_1|$. The hypothesis yields $|\mu_1| = \lambda_1(\varphi) = \lambda_2(\varphi) = |\mu'_1|$ and we denote these numbers by λ_1 for short. We need to show that there exists some $u \in \{1,2,3,4,6\}$ such that A^u and A'^u are semi-conjugate via some $P \in M_2(\mathbb{Z})$ with $\det P \neq 0$.

Note that since A is absolutely $\mathbb{Q}$-irreducible, no eigenvalue of A is of the form $x^{1/n}$ for some $x \in \mathbb{Q}$ and $n \in \mathbb{N}$. Therefore, either $|\mu_1| > |\mu_2|$ and both eigenvalues are real; or they are complex conjugate with μ_1/μ_2 is not a root of unity.

Suppose first that μ_1 and μ_2 are real, then so are μ'_1 and μ'_2. In particular, A and A' are diagonalizable over $\mathbb{R}$. We note that μ_1 and μ_2 (respectively, μ'_1 and μ'_2) are Galois conjugate. Thus, it follows from $|\mu_1| = |\mu'_1|$ that $\mu_1^2 = \mu_1'^2$ and $\mu_2^2 = \mu_2'^2$. In other words, A^2 and A'^2 are matrices with coefficients in $\mathbb{Z}$ with the same eigenvalues. It then follows from [Bou90, A.VII.33 Corollary 3] that they are semi-conjugate by some $P \in M_2(\mathbb{Z})$ with $\det P \neq 0$.

Next, if μ_1 and μ_2 are complex conjugated, then so are μ'_1 and μ'_2 thanks to Theorem D. We set $\theta = \arg \mu_1 \notin 2\pi\mathbb{Q}$ and $\theta' = \arg \mu' \notin 2\pi\mathbb{Q}$. Recall that for $z \in \mathbb{D}(0, \lambda_1^{-1})$, we have
$$\Delta_{1,\varphi,D}(z) = \sum_{m \in \mathbb{Z}} \frac{a_m}{1 - e^{im\theta}\lambda_1 z} \text{ and } \Delta_{1,\varphi',D'}(z) = \sum_{m \in \mathbb{Z}} \frac{a'_m}{1 - e^{im\theta'}\lambda_1 z}.$$

Here, for $m \neq \pm 1$, $a_m = \dfrac{d_m}{m^2-1}$ and $a'_m = \dfrac{d'_m}{m^2-1}$ for some linear recurrence sequences $\{d_m : m \in \mathbb{Z}\}$ and $\{d'_m : m \in \mathbb{Z}\}$. Furthermore, both sequences $\{d_m : m \in \mathbb{Z}\}$ and $\{d'_m : m \in \mathbb{Z}\}$ are non-zero on some arithmetic progressions.

Consider the lattice $\Lambda_{\theta,\theta'} := \{(m, m') : m\theta = m'\theta' \text{ modulo } 2\pi\mathbb{Z}\} \subset \mathbb{Z}^2$. Its rank is at most 1 since θ and θ' are incommensurable to 2π. Thus it is generated by some $(u_0, u'_0) \in \mathbb{Z}^2$, i.e., $e^{\mathrm{i}mu_0\theta} = e^{\mathrm{i}mu'_0\theta'}$ for all $m \in \mathbb{Z}$. Lemma 4.5 yields that for $m \in \mathbb{Z}$, we have

$$\lim_{\rho\to 1^-}(1-\rho)\Delta_{1,\varphi,D}(\rho e^{-\mathrm{i}mu'_0\theta'}) = a_{mu_0}, \text{ and } \lim_{\rho\to 1^-}(1-\rho)\Delta_{1,\varphi',D'}(\rho e^{-\mathrm{i}mu_0\theta}) = a'_{mu'_0}.$$

On the other hand, for every $m_0 \in \mathbb{Z}$ such that $e^{\mathrm{i}m_0\theta} \notin \{e^{\mathrm{i}m'\theta'} : m' \in \mathbb{Z}\}$, we have

$$\lim_{\rho\to 1^-}(1-\rho)\Delta_{1,\varphi',D'}(\rho e^{-\mathrm{i}m_0\theta}\lambda_1) = 0.$$

Similarly for $m'_0 \in \mathbb{Z}$ such that $e^{\mathrm{i}m'_0\theta'} \notin \{e^{\mathrm{i}m\theta} : m \in \mathbb{Z}\}$.

It then follows from $\Delta_{1,\varphi,D}(z) = \Delta_{1,\varphi',D'}(z)$ that $\Lambda_{\theta,\theta'}$ is a non-trivial lattice. Therefore, $(u_0, u'_0) \neq (0,0)$ and $a_{mu_0} = a'_{mu'_0}$ for all $m \in \mathbb{Z}$ and they are not zero for infinitely many m. It follows that $\dfrac{d_{mu_0}}{(mu_0)^2-1} = \dfrac{d'_{mu'_0}}{(mu'_0)^2-1}$ for all $m \neq \pm 1$; hence,

$$m^2(u_0'^2 d_{mu_0} - u_0^2 d'_{mu'_0}) = d_{mu_0} - d'_{mu'_0}\ . \tag{5}$$

We will show that $u_0^2 = u_0'^2$. Assume the contrary, then $d_{mu_0} \neq d'_{mu'_0}$ for infinitely many m, hence $u_0'^2 d_{mu_0} \neq u_0^2 d'_{mu'_0}$ for infinitely many m. It follows that the linear recurrence sequence $\{d_{mu_0} - d'_{mu'_0} : m \in \mathbb{Z}\}$ is represented by a power sum of degree at most 1 by Remark 5.1 while the linear recurrence sequence $\{m^2(u_0'^2 d_{mu_0} - u_0^2 d'_{mu'_0}) : m \in \mathbb{Z}\}$ is represented by a power sum of degree at least 2, the equality (5) is absurd.

Therefore $u_0^2 = u_0'^2$. We then set $u = |u_0|$, whence either $u(\theta_1-\theta_2) \in 2\pi\mathbb{Z}$ or $u(\theta_1+\theta_2) \in 2\pi\mathbb{Z}$. This means that $e^{\mathrm{i}u\theta} = e^{\pm\mathrm{i}u'\theta}$, hence Λ_1^u and Λ_2^u are matrices with coefficients in $\mathbb{Z}$ and with the same eigenvalues. Therefore, they are semi-conjugate by some $P \in M_2(\mathbb{Z})$ with $\det P \neq 0$ by [Bou90, A.VII.33 Corollary 3].

It remains to show $u \in \{1,2,3,4,6\}$. Observe that $\xi := e^{\mathrm{i}(\theta-\theta')}$ is a primitive u-th root of unity, hence $\mu_1 = \xi\mu'_1$. Since μ_1 and μ'_1 are quadratic integers, $\mathbb{Q}(\mu_1,\mu'_1)$ is a Galois extension $\mathbb{Q}$ of degree $[\mathbb{Q}(\mu_1,\mu'_1) : \mathbb{Q}] \leq 4$. Since $\mathbb{Q}(\xi) \subseteq \mathbb{Q}(\mu_1,\mu'_1)$ and $[\mathbb{Q}(\xi) : \mathbb{Q}] = \phi(u)$, we have $\phi(u)|4$. Here, ϕ is Euler's totient function. Thus $u \in \{1,2,3,4,5,6,8,12\}$.

Next, assume that $u \in \{5,8,12\}$, so that $\phi(u) = 4$. Thus $[\mathbb{Q}(\mu_1,\mu'_1) : \mathbb{Q}] = [\mathbb{Q}(\xi) : \mathbb{Q}] = 4$, whence $\mathrm{Gal}(\mathbb{Q}(\mu_1,\mu'_1)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore $u \neq 5, 8$ since both $\mathrm{Gal}(\mathbb{Q}(e^{\mathrm{i}2\pi/5})/\mathbb{Q})$ and $\mathrm{Gal}(\mathbb{Q}(e^{\mathrm{i}2\pi/8})/\mathbb{Q})$ are cyclic of order 4. Thus, $u = 12$ and ξ is a 12th primitive root of unity. It is known that the minimal polynomial of ξ is $X^4 - X^2 + 1$, and $\mathbb{Q}(\xi)$ contains only three quadratic subfields: $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\mathrm{i})$, and $\mathbb{Q}(\sqrt{-3})$. Since $\mathbb{Q}(\mu_1)$ and $\mathbb{Q}(\mu'_1)$ are two complex quadratic subfield of $\mathbb{Q}(\xi)$ which are disjoint over $\mathbb{Q}$, one of them must be $\mathbb{Q}(\mathrm{i})$, and the other is $\mathbb{Q}(\sqrt{-3})$. We may assume that $\mathbb{Q}(\mu_1) = \mathbb{Q}(\mathrm{i}) = \mathbb{Q}(\xi^3)$ and $\mathbb{Q}(\mu'_1) = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\xi^4)$. Thus there are $a, b, c, d \in \mathbb{Q}$ such that $\mu_1 = a + b\xi^3$ and $\mu'_1 = c + d\xi^4 = c - d + d\xi^2$.

Since $\lambda_1^2 = \mu_1\overline{\mu_1} \in \mathbb{N}$, $\mathbb{Q}(\lambda_1)$ is a real subfield of $\mathbb{Q}(\xi)$. Therefore, there are two cases.

- $\mathbb{Q}(\lambda_1) = \mathbb{Q}$, i.e., $\lambda_1 \in \mathbb{N}$. From $\mu_1 = \xi\mu'_1$, we obtain

  $$a + b\xi^3 = \xi(c - d + d\xi^2) = (c-d)\xi + d\xi^3.$$

  Thus $a = 0$ and $b = c = d$; hence, $\mu_1 = \pm b\mathrm{i}$, which contradicts the hypothesis $\arg\mu_1 \notin 2\pi\mathbb{Q}$.

- $\mathbb{Q}(\lambda_1) = \mathbb{Q}(\sqrt{3})$, i.e., $\lambda_1 = \lambda\sqrt{3}$ for some $\lambda \in \mathbb{N}$. We have $3\lambda^2 = \lambda_1^2 = \mu_1\overline{\mu_1} = a^2 + b^2$. Since -1 is not a square modulo 3, it follows that $a = b = \lambda = 0$, which is absurd.

We conclude that $u \notin \{5, 8, 12\}$. The theorem is proven.

Remark 5.10. Observe that the irreducibility assumption is necessary. Indeed, consider monomial self-maps φ and φ' of $\mathbb{P}^2$ associated with matrices $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $A' = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ where a, b, c are positive integers and $a > b > c$. Let $D = D' = \mathcal{O}(1)$. Then $\Delta_{1,\varphi,\mathcal{O}(1)}(z) = \Delta_{1,\varphi',\mathcal{O}(1)}(z) = \sum_{n\geq 0} a^n z^n = \dfrac{1}{1-az}$. However, A^u and A'^u are not conjugate for all $u > 0$.

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