Recursive construction and enumeration of self-orthogonal and self-dual codes over finite commutative chain rings of even characteristic - I

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Abstract

Let $\mathscr{R}_{e,m}$ denote a finite commutative chain ring of even characteristic with maximal ideal $\langle u \rangle$ of nilpotency index $e \geq 3$, Teichmüller set $\mathcal{T}_m$, and residue field $\mathscr{R}_{e,m}/\langle u \rangle$ of order 2^m. Suppose that $2 \in \langle u^\kappa \rangle \setminus \langle u^{\kappa+1} \rangle$ for some odd integer κ with $3 \leq \kappa \leq e$. In this paper, we first develop a recursive method to construct a self-orthogonal code $\mathscr{D}_e$ of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ from a chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(\lceil \frac{e}{2} \rceil)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, and vice versa, subject to the following conditions:

(i) $\dim \mathcal{C}^{(i)} = \lambda_1 + \lambda_2 + \cdots + \lambda_i$ for $1 \leq i \leq \lceil \frac{e}{2} \rceil$;

(ii) the code $\mathcal{C}^{(\lfloor \frac{e}{2} \rfloor - \lfloor \frac{\kappa}{2} \rfloor)}$ is doubly even; and

(iii) the all-one vector $\mathbf{1} = (1, 1, \ldots, 1) \notin \mathcal{C}^{(\lceil \frac{e}{2} \rceil - \kappa)}$ provided that $2\kappa \leq e$, $n \equiv 4 \pmod 8$ and m is odd,

where $\lambda_1, \lambda_2, \ldots, \lambda_e$ are non-negative integers satisfying $2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{e-i+1} + \lambda_{e-i+2} + \lambda_{e-i+3} + \cdots + \lambda_i \leq n$ for $\lceil \frac{e+1}{2} \rceil \leq i \leq e$, and $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions, respectively. This construction ensures that $Tor_i(\mathscr{D}_e) = \mathcal{C}^{(i)}$ for $1 \leq i \leq \lceil \frac{e}{2} \rceil$. With the help of this recursive construction method and by applying results from group theory and finite geometry, we obtain explicit enumeration formulae for all self-orthogonal and self-dual codes of an arbitrary length over $\mathscr{R}_{e,m}$. We also illustrate these results with some examples. In a subsequent study [30], we will address the complementary case where κ is even.

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1 Introduction

Self-orthogonal and self-dual codes are among the most significant and widely studied classes of linear codes. These codes exhibit strong connections with design theory [2, 19] as well as with the theory of modular forms and unimodular lattices [3, 10, 13]. These codes are fundamental in constructing quantum error-correcting codes, which protect quantum information from decoherence and noise [1, 17]. These codes are also used in designing secret-sharing schemes with desirable access structures [6, 12]. This has inspired numerous coding theorists to investigate these codes and develop construction methods for these codes [6, 12, 17, 27].

The study of self-orthogonal and self-dual codes has evolved significantly over the past few decades. In the 1990s, it was discovered that many binary non-linear codes, such as Kerdock, Preparata, Goethals, and Delsarte-Goethals codes, can be represented as Gray images of linear codes over the ring $\mathbb{Z}_4$ of integers modulo 4 [7, 8]. Since then, there has been growing interest in studying self-orthogonal and self-dual codes over finite commutative chain rings [9, 14, 15, 22, 23, 24]. In particular, the explicit enumeration of self-orthogonal and self-dual codes over various finite commutative chain rings has gained significant attention, as these enumeration formulae are instrumental in classifying these codes up to monomial equivalence [9, 15, 31, 33]. Below, we summarize key results in this direction.

Let $GR(p^\mathfrak{s}, m)$ denote the Galois ring of characteristic $p^\mathfrak{s}$ and cardinality $p^{\mathfrak{s}m}$, where p is a prime and m, $\mathfrak{s}$ are positive integers. By Theorem XVII.5 of [21], any finite commutative chain ring is isomorphic to a quotient ring of the form $\mathscr{R}_{e,m} = \frac{GR(p^\mathfrak{s}, m)[y]}{\langle h(y), p^{\mathfrak{s}-1}y^\mathfrak{t} \rangle}$, where $h(y)$ is an Eisenstein polynomial over $GR(p^\mathfrak{s}, m)$ of degree κ, and

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the parameter $\mathtt{t}$ satisfies $1 \le \mathtt{t} \le \kappa$ when $\mathfrak{s} \ge 2$, while $\mathtt{t} = \kappa$ when $\mathfrak{s} = 1$. In the special case when $\mathfrak{s} = \kappa = \mathtt{t} = 1$, the ring $\mathscr{R}_{e,m}$ reduces to the finite field $\mathbb{F}_{p^m}$ of order p^m and the enumeration formulae for self-orthogonal and self-dual codes over $\mathbb{F}_{p^m}$ are obtained by Pless [27]. Further, for a detailed study on the enumeration of self-orthogonal and self-dual codes over finite commutative chain rings of odd characteristic (*i.e.,* when p is odd), the reader is referred to [4, 5, 14, 15, 23, 24, 33] and references therein.

On the other hand, when $p = 2$, $m = 1$, $\mathfrak{s} \ge 1$ and $\kappa = 1$, we have $\mathtt{t} = 1$ and $\mathscr{R}_{e,m} \simeq \mathbb{Z}_{2^{\mathfrak{s}}}$, and the enumeration formula for self-dual codes over $\mathbb{Z}_{2^{\mathfrak{s}}}$ are obtained by Nagata *et al.* [22]. When $p = 2$ and $\mathfrak{s} = 1$, we have $\mathtt{t} = \kappa$, $GR(2^{\mathfrak{s}}, m) = \mathbb{F}_{2^m}$ and $\mathscr{R}_{e,m} \simeq \mathbb{F}_{2^m}[y]/\langle y^\kappa \rangle$, and Yadav and Sharma [32] provided a recursive method to construct and enumerate self-orthogonal and self-dual codes over $\mathbb{F}_{2^m}[y]/\langle y^\kappa \rangle$ from self-orthogonal codes over $\mathbb{F}_{2^m}$. When $p = 2$, $\mathfrak{s} \ge 1$ and $\kappa = 1$, we have $\mathtt{t} = 1$ and $\mathscr{R}_{e,m} \simeq GR(2^{\mathfrak{s}}, m)$, and Yadav and Sharma [31] observed that the recursive construction methods and enumeration techniques employed in [22, 32, 33], do not directly extend to construct and enumerate self-orthogonal and self-dual codes over Galois rings of even characteristic. To address this limitation, they modified their recursive method, where they constructed self-orthogonal and self-dual codes over Galois rings of even characteristic from doubly even codes over the Teichmüller set $\mathcal{T}_m$ of $\mathscr{R}_{e,m}$. However, working as in Section 3 of [32], we observe that even these modified recursive methods are insufficient to construct and enumerate self-orthogonal and self-dual codes over finite commutative chain rings of even characteristic in the case when $p = 2$ and both $\kappa, s \ge 2$, as illustrated in Example 2.1. The main goal of this paper is to provide a modified recursive method for constructing and enumerating all self-orthogonal and self-dual codes over finite commutative chain rings of even characteristic when $p = 2$ and $\kappa \ge 3$ is odd. The complementary case, where κ is even, will be addressed in a subsequent work [30], in which we develop a recursive method for constructing self-orthogonal and self-dual codes over $\mathscr{R}_{e,m}$ from a nested sequence of self-orthogonal codes over finite fields that meet certain additional conditions. Together, the aforementioned works provide a complete solution to the problem of enumeration of self-orthogonal and self-dual codes over finite commutative chain rings. Notably, as shown in [18, 20], the resulting enumeration formulae have broader applications, including the enumeration of self-orthogonal and self-dual quasi-abelian codes and self-orthogonal and self-dual Galois additive cyclic codes over finite commutative chain rings, as well as various other special classes of linear codes that decompose into direct sums of self-orthogonal and self-dual linear codes over finite commutative chain rings. Additionally, these enumeration formulae are instrumental in classifying such codes up to monomial equivalence [5, 9, 15, 31, 33]. It is worth noting that all monomially equivalent linear codes over finite commutative chain rings have the same homogenous and overweight distances. Consequently, these enumeration formulae are also instrumental in the effective implementation of search algorithms aimed at identifying new codes with optimal parameters [16, 26].

This paper is structured as follows: In Section 2, we present the necessary preliminaries required for proving our main results. In Section 3, we begin by defining doubly even codes over the Teichmüller set $\mathcal{T}_m$ of the chain ring $\mathscr{R}_{e,m}$ (see Definition 3.1). We then provide a recursive method for constructing a self-orthogonal (*resp.* self-dual) code $\mathscr{D}_e$ of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ satisfying $Tor_i(\mathscr{D}_e) = \mathcal{C}^{(i)}$ from a chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(\lceil \frac{e}{2} \rceil)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, and vice versa, under the following conditions: (i) $\dim \mathcal{C}^{(i)} = \lambda_1 + \lambda_2 + \cdots + \lambda_i$ for $1 \le i \le \lceil \frac{e}{2} \rceil$, (ii) the code $\mathcal{C}^{(\lfloor \frac{e}{2} \rfloor - \lfloor \frac{\kappa}{2} \rfloor)}$ is doubly even, and (iii) the all-one vector $\mathbf{1} = (1, 1, \ldots, 1) \notin \mathcal{C}^{(\lceil \frac{e}{2} \rceil - \kappa)}$ provided that $2\kappa \le e$, $n \equiv 4 \pmod 8$ and m is odd, where $\lambda_1, \lambda_2, \ldots, \lambda_e$ are non-negative integers satisfying $2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{e-i+1} + \lambda_{e-i+2} + \lambda_{e-i+3} + \cdots + \lambda_i \le n$ for $\lceil \frac{e+1}{2} \rceil \le i \le e$ (see Propositions 3.1-3.8 and Construction methods (A) and (B)). In Section 4, we employ Construction methods (A) and (B) to derive explicit enumeration formulae for all self-orthogonal and self-dual codes of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ (Theorems 4.1 and 4.2), which, using Remark 2.1 and equation (4.1), further provide enumeration formulae for all self-orthogonal and self-dual codes of length n over $\mathscr{R}_{e,m}$.

2 Some preliminaries

A finite commutative ring with unity is called a chain ring if its ideals are totally ordered under the set-theoretic inclusion relation. The following theorem provides a complete characterization of all finite commutative chain rings.

Theorem 2.1. *[21] For a prime number p and positive integers m and $\mathfrak{s}$, let $GR(p^{\mathfrak{s}}, m)$ denote the Galois ring of characteristic $p^{\mathfrak{s}}$ and cardinality $p^{\mathfrak{s}m}$. The quotient ring $\mathcal{R} = \frac{GR(p^{\mathfrak{s}}, m)[y]}{\langle h(y), p^{\mathfrak{s}-1} y^{\mathtt{t}} \rangle}$ is a finite commutative chain ring, where $h(y)$ is an Eisenstein polynomial over $GR(p^{\mathfrak{s}}, m)$ of degree κ and the parameter $\mathtt{t}$ satisfies $1 \le \mathtt{t} \le \kappa$ when $\mathfrak{s} \ge 2$, while $\mathtt{t} = \kappa$ when $\mathfrak{s} = 1$. If $u := y + \langle h(y), p^{\mathfrak{s}-1} y^{\mathtt{t}} \rangle \in \mathcal{R}$, then the ideal $\langle u \rangle$ is the unique maximal ideal of $\mathcal{R}$ of nilpotency index $e = \kappa(\mathfrak{s} - 1) + \mathtt{t}$ and the residue field $\overline{\mathcal{R}} = \mathcal{R}/\langle u \rangle$ of $\mathcal{R}$ is of order p^m. Moreover, the chain consisting of all ideals of $\mathcal{R}$ is given by $\{0\} \subsetneq \langle u^{e-1} \rangle \subsetneq \langle u^{e-2} \rangle \subsetneq \cdots \subsetneq \langle u \rangle \subsetneq \langle 1 \rangle = \mathcal{R}$ and the element*

$p \in \langle u^\kappa\rangle \setminus \langle u^{\kappa+1}\rangle$. *Conversely, any finite commutative chain ring is isomorphic to a quotient ring of the form* $\mathcal{R}$ *for some prime number* p*, positive integers* m*,* $\mathfrak{s}$*,* κ *and* $\mathtt{t}$ *satisfying* $1 \le \mathtt{t} \le \kappa$ *if* $\mathfrak{s} \ge 2$*, while* $\mathtt{t} = \kappa$ *if* $\mathfrak{s} = 1$*, and for some Eisenstein polynomial* $h(y) \in GR(p^{\mathfrak{s}}, m)[y]$ *of degree* κ*, (note that the integers* p*,* m*,* $\mathfrak{s}$*,* κ *and* $\mathtt{t}$ *are called invariants of the chain ring* $\mathcal{R}$*).*

In particular, when $\mathfrak{s} = 1$, we see, by Theorem 2.1, that $\mathtt{t} = \kappa$, $GR(p^{\mathfrak{s}}, m) = \mathbb{F}_{p^m}$ and the ring $\mathcal{R}$ is the quasi-Galois ring $\mathbb{F}_{p^m}[y]/\langle y^\kappa\rangle$, where $\mathbb{F}_{p^m}$ is the finite field of order p^m. On the other hand, when $\kappa = 1$, we see, by Theorem 2.1 again, that $\mathtt{t} = 1$ and the ring $\mathcal{R}$ is the Galois ring $GR(p^{\mathfrak{s}}, m)$ of characteristic $p^{\mathfrak{s}}$ and cardinality $p^{\mathfrak{s}m}$. Thus, quasi-Galois rings and Galois rings constitute two special classes of finite commutative chain rings.

Throughout this paper, we assume that p is a prime number and m, $\mathfrak{s}$, κ and $\mathtt{t}$ are positive integers satisfying $1 \le \mathtt{t} \le \kappa$ if $\mathfrak{s} \ge 2$, whereas $\mathtt{t} = \kappa$ if $\mathfrak{s} = 1$. Let $\mathscr{R}_{e,m}$ denote a finite commutative chain ring with invariants p, m, $\mathfrak{s}$, κ and $\mathtt{t}$ and maximal ideal $\langle u\rangle$ of nilpotency index $e = \kappa(\mathfrak{s}-1)+\mathtt{t}$. Here, the residue field $\overline{\mathscr{R}}_{e,m} = \mathscr{R}_{e,m}/\langle u\rangle$ is the finite field of order p^m. By Theorem 2.1, we note that all ideals of $\mathscr{R}_{e,m}$ form a chain $\{0\} \subsetneq \langle u^{e-1}\rangle \subsetneq \langle u^{e-2}\rangle \subsetneq \ldots \subsetneq \langle u\rangle \subsetneq \langle 1\rangle = \mathscr{R}_{e,m}$ and that the element $p \in \langle u^\kappa\rangle \setminus \langle u^{\kappa+1}\rangle$. We further note, by Lemma XVII.4 of [21], that $|\langle u^i\rangle| = p^{m(e-i)}$ for $0 \le i \le e$, (throughout this paper, $|\cdot|$ denotes the cardinality function). By Theorem XVII.5 of [21], we see that there exists an element $\xi \in \mathscr{R}_{e,m}$ with multiplicative order $p^m - 1$. Moreover, the cyclic subgroup generated by ξ is the only subgroup of the unit group of $\mathscr{R}_{e,m}$, which is isomorphic to the multiplicative group of the residue field $\overline{\mathscr{R}}_{e,m}$. The set $\mathcal{T}_{e,m} = \{0, 1, \xi, \xi^2, \ldots, \xi^{p^m-2}\}$ is called the Teichmüller set of $\mathscr{R}_{e,m}$. By Lemma XVII.4 of [21], we see that each element $d \in \mathscr{R}_{e,m}$ can be uniquely expressed as $d = d_0 + ud_1 + \cdots + u^{e-1}d_{e-1}$, where $d_i \in \mathcal{T}_{e,m}$ for $0 \le i \le e-1$, (such a representation is called the Teichmüller representation of d). Furthermore, the element d is a unit in $\mathscr{R}_{e,m}$ if and only if $d_0 \ne 0$. Now, define a projection map $\pi_0 : \mathscr{R}_{e,m} \to \mathcal{T}_{e,m}$ as $\pi_0(d) = d_0$ for all $d = d_0 + ud_1 + \cdots + u^{e-1}d_{e-1}$, where $d_i \in \mathcal{T}_{e,m}$ for $0 \le i \le e-1$. Next, define a binary operation $\oplus$ on $\mathcal{T}_{e,m}$ as $a \oplus b = \pi_0(a+b)$ for all $a, b \in \mathcal{T}_{e,m}$. It is easy to see that the set $\mathcal{T}_{e,m}$ is the finite field of order p^m under the addition operation $\oplus$ and the usual multiplication operation of $\mathscr{R}_{e,m}$ and that the finite field $\mathcal{T}_{e,m}$ is isomorphic to the residue field $\overline{\mathscr{R}}_{e,m}$ of $\mathscr{R}_{e,m}$. There also exists a canonical epimorphism $^- : \mathscr{R}_{e,m} \to \overline{\mathscr{R}}_{e,m}$, defined as $a \mapsto \bar{a} = a + \langle u\rangle$ for all $a \in \mathscr{R}_{e,m}$. Note that the restriction map $^-\restriction_{\mathcal{T}_{e,m}} : \mathcal{T}_{e,m} \to \overline{\mathscr{R}}_{e,m}$ is a field isomorphism.

Now, let n be a positive integer, and let $\mathscr{R}^n_{e,m}$ be an $\mathscr{R}_{e,m}$-module consisting of all n-tuples over $\mathscr{R}_{e,m}$. A linear code $\mathscr{D}$ of length n over $\mathscr{R}_{e,m}$ is defined as an $\mathscr{R}_{e,m}$-submodule of $\mathscr{R}^n_{e,m}$, whose elements are referred to as codewords. A generator matrix of the code $\mathscr{D}$ is defined as a matrix over $\mathscr{R}_{e,m}$ whose rows form a minimal generating set of the code $\mathscr{D}$. Further, two linear codes of length n over $\mathscr{R}_{e,m}$ are said to be permutation equivalent if one code can be obtained from the other by permuting its coordinate positions. By Proposition 3.2 of Norton and Sălăgean [25], we note that every linear code $\mathscr{D}$ of length n over $\mathscr{R}_{e,m}$ is permutation equivalent to a code with a generator matrix in the standard form

$$\mathcal{G} = \begin{bmatrix} \mathtt{I}_{\lambda_1} & \mathtt{B}_{1,1} & \cdots & \mathtt{B}_{1,e-2} & \mathtt{B}_{1,e-1} & \mathtt{B}_{1,e} \\ 0 & u\mathtt{I}_{\lambda_2} & \cdots & u\mathtt{B}_{2,e-2} & u\mathtt{B}_{2,e-1} & u\mathtt{B}_{2,e} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & u^{e-2}\mathtt{I}_{\lambda_{e-1}} & u^{e-2}\mathtt{B}_{e-1,e-1} & u^{e-2}\mathtt{B}_{e-1,e} \\ 0 & 0 & \cdots & 0 & u^{e-1}\mathtt{I}_{\lambda_e} & u^{e-1}\mathtt{B}_{e,e} \end{bmatrix} = \begin{bmatrix} \mathtt{W}_1 \\ u\mathtt{W}_2 \\ \vdots \\ u^{e-2}\mathtt{W}_{e-1} \\ u^{e-1}\mathtt{W}_e \end{bmatrix} \tag{2.1}$$

whose columns are partitioned into blocks of sizes $\lambda_1, \lambda_2, \ldots, \lambda_e, \lambda_{e+1} = n - (\lambda_1 + \lambda_2 + \cdots + \lambda_e)$, the matrix $\mathtt{I}_{\lambda_i}$ is the $\lambda_i \times \lambda_i$ identity matrix over $\mathscr{R}_{e,m}$ and the matrix $\mathtt{B}_{i,j} \in \mathcal{M}_{\lambda_i \times \lambda_{j+1}}(\mathscr{R}_{e,m})$ is considered modulo u^{j-i+1} for $1 \le i \le j \le e$, *i.e.,* the matrix $\mathtt{B}_{i,j} \in \mathcal{M}_{\lambda_i \times \lambda_{j+1}}(\mathscr{R}_{e,m})$ is of the form $\mathtt{B}_{i,j} = \mathtt{B}^{(0)}_{i,j} + u\mathtt{B}^{(1)}_{i,j} + \cdots + u^{j-i}\mathtt{B}^{(j-i)}_{i,j}$ with $\mathtt{B}^{(0)}_{i,j}, \mathtt{B}^{(1)}_{i,j}, \ldots, \mathtt{B}^{(j-i)}_{i,j} \in \mathcal{M}_{\lambda_i \times \lambda_{j+1}}(\mathcal{T}_{e,m})$ for $1 \le i \le j \le e$, (throughout this paper, the set of all $f \times h$ matrices over a ring R is denoted by $\mathcal{M}_{f\times h}(R)$). A linear code $\mathscr{D}$ of length n over $\mathscr{R}_{e,m}$ is said to be of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ if it is permutation equivalent to a code with a generator matrix in the standard form (2.1). Further, a linear code $\mathscr{D}$ of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ over $\mathscr{R}_{e,m}$ contains $(p^m)^{\sum_{i=1}^{e}(e-i+1)\lambda_i}$ codewords.

Next, the Euclidean bilinear form on $\mathscr{R}^n_{e,m}$ is a map $\cdot : \mathscr{R}^n_{e,m} \times \mathscr{R}^n_{e,m} \to \mathscr{R}_{e,m}$, defined as $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} \mathrm{a}_i\mathrm{b}_i$ for all $\mathbf{a} = (\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_n)$, $\mathbf{b} = (\mathrm{b}_1, \mathrm{b}_2, \ldots, \mathrm{b}_n) \in \mathscr{R}^n_{e,m}$. Note that the map $\cdot$ is a non-degenerate and symmetric bilinear form on $\mathscr{R}^n_{e,m}$. The (Euclidean) dual code $\mathscr{D}^\perp$ of a linear code $\mathscr{D}$ of length n over $\mathscr{R}_{e,m}$ is defined as $\mathscr{D}^\perp = \{\mathbf{v}_1 \in \mathscr{R}^n_{e,m} : \mathbf{v}_2 \cdot \mathbf{v}_1 = 0 \text{ for all } \mathbf{v}_2 \in \mathscr{D}\}$. Note that the dual code $\mathscr{D}^\perp$ is also a linear code of length n over $\mathscr{R}_{e,m}$. Further, by Theorem 3.10 of Norton and Sălăgean [25], we see that the dual code $\mathscr{D}^\perp$ of $\mathscr{D}$ is of type $\{n - (\lambda_1 + \lambda_2 + \cdots + \lambda_e), \lambda_e, \lambda_{e-1}, \ldots, \lambda_2\}$ if the code $\mathscr{D}$ is of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$. The code $\mathscr{D}$ is said to be self-orthogonal if $\mathscr{D} \subseteq \mathscr{D}^{\perp}$, while the code $\mathscr{D}$ is said to be self-dual if $\mathscr{D} = \mathscr{D}^{\perp}$. Now, the following theorem provides a necessary and sufficient condition under which a linear code of length n over $\mathscr{R}_{e,m}$ is self-orthogonal.

Lemma 2.1. *[11, 33] For integers $e \geq 2$ and $n \geq 1$, let $\lambda_1, \lambda_2, \ldots, \lambda_{e+1}$ be non-negative integers satisfying $n = \lambda_1 + \lambda_2 + \cdots + \lambda_{e+1}$. Let $\mathscr{D}$ be a linear code of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ with a generator matrix $\mathcal{G}$ of the form (2.1). The code $\mathscr{D}$ is self-orthogonal if and only if*

$$\mathtt{W}_i \mathtt{W}_j^t \equiv \mathbf{0} \pmod{u^{e-i-j+2}} \quad \text{for } 1 \leq i \leq j \leq e \text{ and } i + j \leq e + 1.$$

Furthermore, the code $\mathscr{D}$ is self-dual if and only if it is self-orthogonal and $\lambda_i = \lambda_{e-i+2}$ for $1 \leq i \leq e$. (Throughout this paper, $(\cdot)^t$ denotes the matrix transpose.)

For an integer ℓ satisfying $1 \leq \ell < e$, one can easily see that the quotient ring $\mathscr{R}_{e,m}/\langle u^\ell \rangle$ is also a finite commutative chain ring with maximal ideal $\langle u + \langle u^\ell \rangle \rangle$ of nilpotency index ℓ. From now on, we shall denote the chain ring $\mathscr{R}_{e,m}/\langle u^\ell \rangle$ by $\mathscr{R}_{\ell,m}$ for our convenience. Furthermore, the element $\xi_\ell := \xi + \langle u^\ell \rangle \in \mathscr{R}_{\ell,m}$ has multiplicative order $p^m - 1$ and the set $\mathcal{T}_{\ell,m} = \{0, 1, \xi_\ell, \xi_\ell^2, \ldots, \xi_\ell^{p^m-2}\}$ is the Teichmüller set of $\mathscr{R}_{\ell,m}$. The canonical epimorphism from $\mathscr{R}_{e,m}$ onto $\mathscr{R}_{\ell,m}$ is defined as $a \mapsto a + \langle u^\ell \rangle$ for all $a \in \mathscr{R}_{e,m}$. In view of this, we will identify each element $a + \langle u^\ell \rangle \in \mathscr{R}_{\ell,m}$ with the element $a \in \mathscr{R}_{e,m}$, performing addition and multiplication in $\mathscr{R}_{\ell,m}$ modulo u^ℓ. Specifically, we will identify the element $\xi_\ell \in \mathcal{T}_{\ell,m}$ with the element $\xi \in \mathcal{T}_{e,m}$. Accordingly, we assume, throughout this paper, that

$$\mathcal{T}_{1,m} = \mathcal{T}_{2,m} = \cdots = \mathcal{T}_{e-1,m} = \mathcal{T}_{e,m} = \{0, 1, \xi, \xi^2, \ldots, \xi^{p^m-2}\} = \mathcal{T}_m \text{ (say)}$$

and

$$\mathscr{R}_{1,m} = \overline{\mathscr{R}}_{1,m} = \overline{\mathscr{R}}_{2,m} = \cdots = \overline{\mathscr{R}}_{e-1,m} = \overline{\mathscr{R}}_{e,m} = \mathcal{T}_m.$$

In view of this, we see, for $1 \leq \ell \leq e$, that each element $b \in \mathscr{R}_{\ell,m}$ can be uniquely written as $b = b_0 + ub_1 + u^2 b_2 + \cdots + u^{\ell-1} b_{\ell-1}$, where $b_0, b_1, b_2, \ldots, b_{\ell-1} \in \mathcal{T}_m$. Further, each matrix $Z \in \mathcal{M}_{f \times h}(\mathscr{R}_{\ell,m})$ can be uniquely written as $Z = Z_0 + uZ_1 + u^2 Z_2 + \cdots + u^{\ell-1} Z_{\ell-1}$, where $Z_0, Z_1, \ldots, Z_{\ell-1} \in \mathcal{M}_{f \times h}(\mathcal{T}_m)$.

The set $\mathcal{T}_m^n$ consisting of all n-tuples over $\mathcal{T}_m$ can be viewed as an n-dimensional vector space over the finite field $\mathcal{T}_m$ under the component-wise addition induced by $\oplus$ and the component-wise scalar multiplication. We further define a map $B_m : \mathcal{T}_m^n \times \mathcal{T}_m^n \to \mathcal{T}_m$ as $B_m(\mathbf{v}, \mathbf{w}) = \pi_0(\mathbf{v} \cdot \mathbf{w}) = \pi_0(\sum_{i=1}^{n} v_i w_i)$ for all $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathcal{T}_m^n$, (here $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i$ is viewed as an element of $\mathscr{R}_{e,m}$). Note that the map B_m is a symmetric and non-degenerate bilinear form on $\mathcal{T}_m^n$. A linear code $\mathcal{C}$ of length n over $\mathcal{T}_m$ is defined as a $\mathcal{T}_m$-linear subspace of $\mathcal{T}_m^n$. The dual code of $\mathcal{C}$ with respect to B_m is defined as $\mathcal{C}^{\perp_{B_m}} = \{\mathbf{w} \in \mathcal{T}_m^n : B_m(\mathbf{w}, \mathbf{c}) = 0 \text{ for all } \mathbf{c} \in \mathcal{C}\}$. Note that $\mathcal{C}^{\perp_{B_m}}$ is also a linear code of length n over $\mathcal{T}_m$. Further, a linear code $\mathcal{C}$ of length n over $\mathcal{T}_m$ is said to be (i) self-orthogonal if it satisfies $\mathcal{C} \subseteq \mathcal{C}^{\perp_{B_m}}$, and (ii) self-dual if it satisfies $\mathcal{C} = \mathcal{C}^{\perp_{B_m}}$.

For a linear code $\mathscr{D}$ of length n over $\mathscr{R}_{e,m}$, the i-th torsion code of $\mathscr{D}$ is defined as

$$Tor_i(\mathscr{D}) = \{\mathbf{w} \in \mathcal{T}_m^n : u^{i-1}(\mathbf{w} + u\mathbf{w}') \in \mathscr{D} \text{ for some } \mathbf{w}' \in \mathscr{R}_{e,m}^n\}$$

for $1 \leq i \leq e$. Note that $Tor_i(\mathscr{D})$ is a linear code of length n over $\mathcal{T}_m$ for each i. Additionally, if the code $\mathscr{D}$ has a generator matrix $\mathcal{G}$ of the form (2.1), then the i-th torsion code $Tor_i(\mathscr{D})$ has dimension $\lambda_1 + \lambda_2 + \cdots + \lambda_i$ over $\mathcal{T}_m$ and has a generator matrix

$$\begin{bmatrix} \mathtt{I}_{\lambda_1} & \mathtt{B}_{1,1}^{(0)} & \mathtt{B}_{1,2}^{(0)} & \cdots & \mathtt{B}_{1,i-1}^{(0)} & \cdots & \mathtt{B}_{1,e-1}^{(0)} & \mathtt{B}_{1,e}^{(0)} \\ 0 & \mathtt{I}_{\lambda_2} & \mathtt{B}_{2,2}^{(0)} & \cdots & \mathtt{B}_{2,i-1}^{(0)} & \cdots & \mathtt{B}_{2,e-1}^{(0)} & \mathtt{B}_{2,e}^{(0)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathtt{I}_{\lambda_i} & \cdots & \mathtt{B}_{i,e-1}^{(0)} & \mathtt{B}_{i,e}^{(0)} \end{bmatrix}. \tag{2.2}$$

Note that $Tor_i(\mathscr{D}) \subseteq Tor_{i+1}(\mathscr{D})$ for $1 \leq i \leq e-1$ and $|\mathscr{D}| = \prod_{i=1}^{e} |Tor_i(\mathscr{D})|$. Now, to count all self-orthogonal and self-dual codes over $\mathscr{R}_{e,m}$ of a given type, we need the following lemma.

Lemma 2.2. *[11, 33] For a self-orthogonal code $\mathscr{D}$ of length n over $\mathscr{R}_{e,m}$, the following hold.*

(a) $Tor_i(\mathscr{D}) \subseteq Tor_i(\mathscr{D})^{\perp_{B_m}}$ for $1 \leq i \leq \lfloor \frac{e+1}{2} \rfloor$.

(b) $Tor_i(\mathscr{D}) \subseteq Tor_{e-i+1}(\mathscr{D})^{\perp_{B_m}}$ for $\lfloor \frac{e+1}{2} \rfloor + 1 \leq i \leq e$.

Furthermore, if the code $\mathscr{D}$ is self-dual, then we have $Tor_i(\mathscr{D}) = Tor_{e-i+1}(\mathscr{D})^{\perp_{B_m}}$ for $\lceil \frac{e+1}{2} \rceil \leq i \leq e$. (Throughout this paper, $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions, respectively.)

Remark 2.1. *From the above lemma, it follows that if $\mathscr{D}$ is a self-orthogonal code of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$, then $2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{e-i+1} + \lambda_{e-i+2} + \cdots + \lambda_i \leq n$ for $\lceil \frac{e+1}{2} \rceil \leq i \leq e$, which implies that $n \geq 2(\lambda_1 + \lambda_2 + \cdots + \lambda_{\frac{e}{2}}) + \lambda_{\frac{e}{2}+1}$ if e is even, while $n \geq 2(\lambda_1 + \lambda_2 + \cdots + \lambda_{\frac{e+1}{2}})$ if e is odd.*

Yadav and Sharma [31, 32, 33] counted all self-orthogonal and self-dual codes of length n over $\mathscr{R}_{e,m}$ in the following three cases: (i) p is an odd prime (*i.e.,* the ring $\mathscr{R}_{e,m}$ is of odd characteristic), (ii) $p = 2$ and $\mathfrak{s} = 1$ (*i.e.,* $\mathscr{R}_{e,m}$ is the quasi-Galois ring $\mathbb{F}_{2^m}[u]/\langle u^e \rangle$ of even characteristic), and (iii) $p = 2$ and $\kappa = 1$ (*i.e.,* $\mathscr{R}_{e,m}$ is the Galois ring $GR(2^e, m)$ of characteristic 2^e and cardinality 2^{em}). When either p is odd or $p = 2$ and $\mathfrak{s} = 1$, they provided recursive methods for constructing self-orthogonal and self-dual codes of type $\{\lambda_1, \lambda_1, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ from a self-orthogonal code of length n and dimension $\lambda_1 + \lambda_2 + \cdots + \lambda_{\lceil \frac{e}{2} \rceil}$ over $\mathcal{T}_m$, where $\lambda_1, \lambda_2, \ldots, \lambda_e$ are non-negative integers satisfying $2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{e-i+1} + \lambda_{e-i+2} + \cdots + \lambda_i \leq n$ for $\lceil \frac{e+1}{2} \rceil \leq i \leq e$ (see [32, Sec. 4] and [33, Sec. 5]). On the other hand, when $p = 2$ and $\kappa = 1$, they provided a recursive method for constructing self-orthogonal and self-dual codes of type $\{\lambda_1, \lambda_1, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ from a doubly even code of length n and dimension $\lambda_1 + \lambda_2 + \cdots + \lambda_{\lceil \frac{e}{2} \rceil}$ over $\mathcal{T}_m$, where $\lambda_1, \lambda_2, \ldots, \lambda_e$ are non-negative integers satisfying $2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{e-i+1} + \lambda_{e-i+2} + \cdots + \lambda_i \leq n$ for $\lceil \frac{e+1}{2} \rceil \leq i \leq e$ (see [31, Sec. 4]). It is worth noting that when $p = 2$ and either $\mathfrak{s} = 1$ or $\kappa = 1$, the recursive method provided in [33, Sec. 5] does not always lift a self-orthogonal code over $\mathcal{T}_m$ to a self-orthogonal code over $\mathscr{R}_{e,m}$ (see [31, Sec. 1] and [32, Sec. 3]). When $p = 2$ and both $\mathfrak{s}, \kappa \geq 2$, the following example illustrates that the recursive methods provided in [31, Sec. 4] and [32, Sec. 4] do not always lift a self-orthogonal code over $\mathcal{T}_m$ and a doubly even code over $\mathcal{T}_m$ (see [31, Def. 2.1]) to a self-orthogonal code over $\mathscr{R}_{e,m}$, respectively.

Example 2.1. *Let $\mathscr{R}_{8,2} = \frac{GR(8,2)[y]}{\langle y^3+2, 4y^2 \rangle}$ be a finite commutative chain ring with maximal ideal $\langle u \rangle$ of nilpotency index 8 and $u^3 + u^6 = 2$ in $\mathscr{R}_{8,2}$, where $u := y + \langle y^3 + 2, 4y^2 \rangle$. Let $n = 4$, $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 0$. Let $\mathcal{C}_0$ be a self-orthogonal code of length 4 and dimension $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$ over $\mathcal{T}_2$ with a generator matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. Note that $\pi_1(\boldsymbol{d} \cdot \boldsymbol{d}) = \pi_3(\boldsymbol{d} \cdot \boldsymbol{d}) = 0$ for all $\boldsymbol{d} \in \mathcal{C}_0$, where each $\boldsymbol{d} \cdot \boldsymbol{d}$ is viewed as an element of $\mathscr{R}_{8,2}$.*

I. *Here, we will show that the code $\mathcal{C}_0$ can not be lifted to a self-orthogonal code of type $\{1, 0, 0, 0, 0, 0, 0, 0\}$ and length 4 over $\mathscr{R}_{8,2}$ using the recursive construction method provided in Section 4 of Yadav and Sharma [32]. Towards this, let us consider a linear code $\mathcal{C}_2$ of type $\{1, 0\}$ and length 4 over $\mathscr{R}_{2,2}$ with a generator matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + u \begin{bmatrix} 0 & a_1 & b_1 & c_1 \end{bmatrix}$, where $a_1, b_1, c_1 \in \mathcal{T}_2$. Using the recursive construction method provided in Section 4 of [32], we see that the code $\mathcal{C}_0$ can be lifted to a self-orthogonal code of type $\{1, 0, 0, 0, 0, 0, 0, 0\}$ over $\mathscr{R}_{8,2}$ if and only if $\mathcal{C}_2$ is a self-orthogonal code over $\mathscr{R}_{2,2}$ satisfying property $(*)$ (see [32, Sec. 4]), which holds if and only if $a_1^2 + b_1^2 + c_1^2 = 0$. In particular, for $a_1 = b_1 = c_1 = 0$, the code $\mathcal{C}_2$ of length 4 over $\mathscr{R}_{2,2}$ with a generator matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ is a self-orthogonal code satisfying property $(*)$. Corresponding to this particular choice of the code $\mathcal{C}_2$, consider a linear code $\mathcal{C}_4$ of type $\{1, 0, 0, 0\}$ and length 4 over $\mathscr{R}_{4,2}$ with a generator matrix*

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + u^2 \begin{bmatrix} 0 & a_2 & b_2 & c_2 \end{bmatrix} + u^3 \begin{bmatrix} 0 & a_3 & b_3 & c_3 \end{bmatrix},$$

where $a_2, b_2, c_2, a_3, b_3, c_3 \in \mathcal{T}_2$. Using again the recursive construction method provided in Section 4 of [32], we see that the code $\mathcal{C}_4$ can be lifted to a self-orthogonal code of type $\{1, 0, 0, 0, 0, 0, 0, 0\}$ over $\mathscr{R}_{8,2}$ if and only if $\mathcal{C}_4$ is a self-orthogonal code over $\mathscr{R}_{4,2}$ satisfying property $(*)$, which holds if and only if $a_2^2 + b_2^2 + c_2^2 = 0$ and $a_3^2 + b_3^2 + c_3^2 = 0$. In particular, for $a_2 = b_2 = c_2 = a_3 = b_3 = c_3 = 0$, we see that the code $\mathcal{C}_4$ of length 4 over $\mathscr{R}_{4,2}$ with a generator matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ is a self-orthogonal code satisfying property $(*)$. Further, corresponding to this particular choice of the code $\mathcal{C}_4$, consider a linear code $\mathcal{C}_6$ of type $\{1, 0, 0, 0, 0, 0\}$ and length 4 over $\mathscr{R}_{6,2}$ with a generator matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + u^4 \begin{bmatrix} 0 & a_4 & b_4 & c_4 \end{bmatrix} + u^5 \begin{bmatrix} 0 & a_5 & b_5 & c_5 \end{bmatrix},$$

where $a_4, b_4, c_4, a_5, b_5, c_5 \in \mathcal{T}_2$. By the recursive construction method provided in Section 4 of [32], we see that the code $\mathcal{C}_6$ can be lifted to a self-orthogonal code of type $\{1, 0, 0, 0, 0, 0, 0, 0\}$ over $\mathscr{R}_{8,2}$ if and only if $\mathcal{C}_6$ is a self-orthogonal code over $\mathscr{R}_{6,2}$ satisfying property $(*)$, which holds for all choices of $a_4, b_4, c_4, a_5, b_5, c_5 \in \mathcal{T}_2$. In particular, for $a_4 = b_4 = c_4 = a_5 = b_5 = c_5 = 0$, we see that the code $\mathcal{C}_6$ of length 4 over $\mathscr{R}_{6,2}$ with a generator matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ is a self-orthogonal code over $\mathscr{R}_{6,2}$ satisfying property $(*)$. Finally, consider a linear code $\mathcal{C}_8$ of type $\{1, 0, 0, 0, 0, 0, 0, 0\}$ and length 4 over $\mathscr{R}_{8,2}$ with a generator matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + u^6 \begin{bmatrix} 0 & a_6 & b_6 & c_6 \end{bmatrix} + u^7 \begin{bmatrix} 0 & a_7 & b_7 & c_7 \end{bmatrix},$$

where $a_6, b_6, c_6, a_7, b_7, c_7 \in \mathcal{T}_2$. *Note that the code* $\mathcal{C}_8$ *is not self-orthogonal for any choice of* $a_6, b_6, c_6, a_7, b_7, c_7 \in \mathcal{T}_2$. *This illustrates that the recursive construction method provided in Section 4 of Yadav and Sharma [32] does not always lift a self-orthogonal code over* $\mathcal{T}_m$ *to a self-orthogonal code over* $\mathscr{R}_{e,m}$.

II. *We will next show that the code* $\mathcal{C}_0$ *can not be lifted to a self-orthogonal code of type* $\{1,0,0,0,0,0,0,0\}$ *and length* 4 *over* $\mathscr{R}_{8,2}$ *using the recursive construction method provided in Section 4 of Yadav and Sharma [31]. For this, we first recall [31, Def. 2.1] that a self-orthogonal code* $\mathscr{D}$ *over* $\mathcal{T}_2$ *is said to be doubly even if it satisfies* $\pi_1(\boldsymbol{d}\cdot\boldsymbol{d}) = 0$ *for all* $\boldsymbol{d} \in \mathscr{D}$, *where each* $\boldsymbol{d}\cdot\boldsymbol{d}$ *is viewed as an element of* $\mathscr{R}_{8,2}$. *One can easily see that* $\pi_1(\boldsymbol{v}\cdot\boldsymbol{v}) = 0$ *for all* $\boldsymbol{v} \in \mathcal{C}_0$, *so the code* $\mathcal{C}_0$ *is doubly even. Now, let us consider a linear code* $\mathcal{D}_2$ *of type* $\{1,0\}$ *and length* 4 *over* $\mathscr{R}_{2,2}$ *with a generator matrix* $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + u\begin{bmatrix}0 & f_1 & g_1 & h_1\end{bmatrix}$, *where* $f_1, g_1, h_1 \in \mathcal{T}_2$. *By the recursive construction method provided in Section 4 of [31], we see that the code* $\mathcal{D}_2$ *can be lifted to a self-orthogonal code of type* $\{1,0,0,0,0,0,0,0\}$ *over* $\mathscr{R}_{8,2}$ *if and only if* $\mathcal{D}_2$ *is a* λ_1*-doubly even self-orthogonal code over* $\mathscr{R}_{2,2}$ *(see [31, Def. 2.1]), which holds if and only if* $f_1^2+g_1^2+h_1^2 \equiv 0 \pmod{u}$. *In particular, for* $f_1 = g_1 = h_1 = 0$, *we see that the code* $\mathcal{D}_2$ *of length* 4 *over* $\mathscr{R}_{2,2}$ *with a generator matrix* $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$ *is a* λ_1*-doubly even self-orthogonal code over* $\mathscr{R}_{2,2}$. *Corresponding to this particular choice of the code* $\mathcal{D}_2$, *let us consider a linear code* $\mathcal{D}_4$ *of type* $\{1,0,0,0\}$ *and length* 4 *over* $\mathscr{R}_{4,2}$ *with a generator matrix*

$$\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + u^2\begin{bmatrix}0 & f_2 & g_2 & h_2\end{bmatrix} + u^3\begin{bmatrix}0 & f_3 & g_3 & h_3\end{bmatrix},$$

where $f_2, g_2, h_2, f_3, g_3, h_3 \in \mathcal{T}_2$. *By the recursive construction method provided in Section 4 of [31], we see that the code* $\mathcal{D}_4$ *can be lifted to a self-orthogonal code of type* $\{1,0,0,0,0,0,0,0\}$ *over* $\mathscr{R}_{8,2}$ *if and only if* $\mathcal{C}_4$ *is a* λ_1*-doubly even self-orthogonal code over* $\mathscr{R}_{4,2}$, *which holds if and only if* $f_2^2+g_2^2+h_2^2 \equiv 0 \pmod{u}$. *In particular, for* $f_2 = g_2 = h_2 = f_3 = g_3 = h_3 = 0$, *we see that the code* $\mathcal{D}_4$ *of length* 4 *over* $\mathscr{R}_{4,2}$ *with a generator matrix* $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$ *is a* λ_1*-doubly even self-orthogonal code. Now, corresponding to this particular choice of the code* $\mathcal{D}_4$, *consider a linear code* $\mathcal{D}_6$ *of type* $\{1,0,0,0,0,0\}$ *and length* 4 *over* $\mathscr{R}_{6,2}$ *with a generator matrix*

$$\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + u^4\begin{bmatrix}0 & f_4 & g_4 & h_4\end{bmatrix} + u^5\begin{bmatrix}0 & f_5 & g_5 & h_5\end{bmatrix},$$

where $f_4, g_4, h_4, f_5, g_5, h_5 \in \mathcal{T}_2$. *By the recursive construction method provided in Section 4 of [31], we see that the code* $\mathcal{D}_6$ *can be lifted to a self-orthogonal code of type* $\{1,0,0,0,0,0,0,0\}$ *over* $\mathscr{R}_{8,2}$ *if and only if* $\mathcal{D}_6$ *is a* λ_1*-doubly even self-orthogonal code over* $\mathscr{R}_{6,2}$. *One can easily observe that* $\mathcal{D}_6$ *is not a* λ_1*-doubly even code for any choice of* $f_4, g_4, h_4, f_5, g_5, h_5 \in \mathcal{T}_2$. *This illustrates that the recursive construction method provided in Section 4 of Yadav and Sharma [31] does not always lift a doubly even code over* $\mathcal{T}_m$ *to a self-orthogonal code over* $\mathscr{R}_{e,m}$.

From now on, we assume, throughout this paper, that $p = 2$ and the integer $\kappa \geq 3$ is odd, *i.e.,* the element $2 \in \langle u^\kappa\rangle \setminus \langle u^{\kappa+1}\rangle$ with $\kappa \geq 3$ being odd. Accordingly, we assume, from now on, that $2 \equiv u^\kappa(\eta_0 + u\eta_1 + u^2\eta_2 + \cdots + u^{e-1-\kappa}\eta_{e-1-\kappa}) \pmod{u^e}$, where $\eta_0 \in \mathcal{T}_m \setminus \{0\}$ and $\eta_1, \eta_2, \ldots, \eta_{e-1-\kappa} \in \mathcal{T}_m$. In the following section, we will provide a recursive method for lifting a self-orthogonal code over $\mathcal{T}_m$ to a self-orthogonal code over $\mathscr{R}_{e,m}$ when $\kappa \geq 3$ is odd. We will consider the case when κ is even in a subsequent work [30]. To do this, let us define $\mathrm{f}_w = \lfloor \frac{w}{2} \rfloor$ and $\mathrm{c}_w = \lceil \frac{w}{2} \rceil$ for any positive integer w. Let us define $\theta_e = 0$ if e is even, while $\theta_e = 1$ if e is odd, and let us define the numbers $s = \mathrm{f}_e = \lfloor \frac{e}{2} \rfloor$ and $\kappa_1 = \mathrm{f}_\kappa = \lfloor \frac{\kappa}{2} \rfloor = \frac{\kappa-1}{2}$ for our convenience. Unless otherwise stated, we will adhere to the notations introduced in Section 2.

3 A recursive method for constructing self-orthogonal and self-dual codes over $\mathscr{R}_{e,m}$ from a chain of self-orthogonal codes over $\mathcal{T}_m$

Throughout this section, let n be a positive integer, and let $\lambda_1, \lambda_2, \ldots, \lambda_{e+1}$ be non-negative integers satisfying $n = \lambda_1 + \lambda_2 + \cdots + \lambda_{e+1}$ and $2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{e-i+1} + \lambda_{e-i+2} + \cdots + \lambda_i \leq n$ for $s+1 \leq i \leq e$. Let us define $\Lambda_0 = 0$ and $\Lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i$ for $1 \leq i \leq e+1$. Note that $\Lambda_{e+1} = n$. Further, for $1 \leq \ell \leq e$ and $0 \leq i \leq \ell - 1$, let us define a mapping $\pi_i : \mathscr{R}_{\ell,m} \to \mathcal{T}_m$ as $\pi_i(a) = a_i$ for all $a = a_0 + ua_1 + \cdots + u^{\ell-1}a_{\ell-1} \in \mathscr{R}_{\ell,m}$, where $a_0, a_1, \ldots, a_{\ell-1} \in \mathcal{T}_m$. We further recall, from Section 2, that $(\mathcal{T}_m, \oplus, \cdot)$ is the finite field of order 2^m. Now, if $\mathcal{C}$ is a self-orthogonal code of length n over $\mathcal{T}_m$, then for all $\mathbf{v} \in \mathcal{C}$, one can easily see that $\pi_0(\mathbf{v}\cdot\mathbf{v}) = 0$ and $\pi_{2j+1}(\mathbf{v}\cdot\mathbf{v}) = 0$ for $0 \leq j \leq \kappa_1 - 1$, where $\kappa_1 = \mathrm{f}_\kappa = \frac{\kappa-1}{2}$ and each $\mathbf{v}\cdot\mathbf{v}$ is viewed as an element of $\mathscr{R}_{e,m}$. We next define a special class of self-orthogonal codes over $\mathcal{T}_m$.

Definition 3.1. *A self-orthogonal code $\mathcal{D}$ of length n over $\mathcal{T}_m$ is said to be doubly even if it satisfies $\pi_\kappa(\boldsymbol{b}\cdot\boldsymbol{b})=0$ for all $\boldsymbol{b}\in\mathcal{D}$, where each $\boldsymbol{b}\cdot\boldsymbol{b}$ is viewed as an element of $\mathscr{R}_{e,m}$. Moreover, a self-orthogonal code $\mathcal{D}$ of length n over $\mathcal{T}_m$ is said to be $\mathfrak{d}$-doubly even if it has a $\mathfrak{d}$-dimensional doubly even subcode.*

For $e\geq 3$ and $2\leq \ell\leq e$, we further define a special class of linear codes of length n over $\mathscr{R}_{\ell,m}$ as follows:

Linear codes over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$: *Let e,ℓ be integers satisfying $e\geq 3$, $2\leq\ell\leq e$ and $\ell\equiv e \pmod 2$. Let us define $\gamma_\ell=s-\mathrm{f}_\ell=\left\lfloor\frac{e}{2}\right\rfloor-\left\lfloor\frac{\ell}{2}\right\rfloor$. Let $\mathscr{D}_\ell$ be a linear code of type $\{\Lambda_{\gamma_\ell+1},\lambda_{\gamma_\ell+2},\ldots,\lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ with a generator matrix*

$$\mathcal{G}_\ell=\begin{bmatrix}\mathtt{W}_1\\ \mathtt{W}_2\\ \vdots\\ \mathtt{W}_{\gamma_\ell+1}\\ u\mathtt{W}_{\gamma_\ell+2}\\ \vdots\\ u^{\ell-1}\mathtt{W}_{\gamma_\ell+\ell}\end{bmatrix},$$

where $\mathtt{W}_i\in\mathcal{M}_{\lambda_i\times n}(\mathscr{R}_{\ell,m})$ for $1\leq i\leq\gamma_\ell+1$ and the matrix $\mathtt{W}_{\gamma_\ell+j}\in\mathcal{M}_{\lambda_{\gamma_\ell+j}\times n}(\mathscr{R}_{\ell,m})$ is to be considered modulo $u^{\ell-j+1}$ for $2\leq j\leq\ell$.

(i) When $\ell\leq\min\{\kappa-1,e-\kappa\}$, we say that the code $\mathscr{D}_\ell$ satisfies property $(\mathfrak{P})$ if

$$\begin{aligned}\mathtt{Diag}\left([\mathtt{W}]_{\gamma_\ell+1-\mathrm{f}_i}[\mathtt{W}]^t_{\gamma_\ell+1-\mathrm{f}_i}\right)&\equiv\mathbf{0}\pmod{u^{\ell+i}}\quad\text{for }\ i\in\{2,4,6,\ldots,\ell-\theta_e\}\ \text{ and}\\ \pi_{\ell+j}\big(\mathtt{Diag}\big([\mathtt{W}]_{\gamma_\ell+1-\mathrm{f}_{j+2}}[\mathtt{W}]^t_{\gamma_\ell+1-\mathrm{f}_{j+2}}\big)\big)&=\mathbf{0}\quad\text{for }\ j\in\{\ell-1,\ell+1,\ldots,\kappa-2+\theta_e\}.\end{aligned}$$

(ii) When $e-\kappa<\ell\leq\kappa-\mathrm{f}_{2\kappa-e}+1$, we say that the code $\mathscr{D}_\ell$ satisfies property $(\mathfrak{P})$ if

$$\begin{aligned}\mathtt{Diag}\left([\mathtt{W}]_{\gamma_\ell+1-\mathrm{f}_i}[\mathtt{W}]^t_{\gamma_\ell+1-\mathrm{f}_i}\right)&\equiv\mathbf{0}\pmod{u^{\ell+i}}\quad\text{for }\ i\in\{2,4,6,\ldots,\ell-\theta_e\}\ \text{and}\\ \pi_{\ell+j}\Big(\mathtt{Diag}\big([\mathtt{W}]_{\gamma_\ell+1-\mathrm{f}_{j+2}}[\mathtt{W}]^t_{\gamma_\ell+1-\mathrm{f}_{j+2}}\big)\Big)&=\mathbf{0}\quad\text{for }\ j\in\{\ell-1,\ell+1,\ell+3,\ldots,e-\ell-1-\theta_e\}.\end{aligned}$$

(iii) When $\kappa\leq\ell\leq e-\kappa$, we say that the code $\mathscr{D}_\ell$ satisfies property $(\mathfrak{P})$ if

$$\mathtt{Diag}\left([\mathtt{W}]_{\gamma_\ell+1-\mathrm{c}_i}[\mathtt{W}]^t_{\gamma_\ell+1-\mathrm{c}_i}\right)\equiv\mathbf{0}\pmod{u^{\ell+i}}\quad\text{for }\ i\in\{2,4,6,\ldots,\kappa-1\}\cup\{\kappa\}.$$

(iv) When $\ell>\max\{e-\kappa,\kappa-\mathrm{f}_{2\kappa-e}+1\}$, we say that the code $\mathscr{D}_\ell$ satisfies property $(\mathfrak{P})$ if

$$\mathtt{Diag}\left([\mathtt{W}]_{\gamma_\ell+1-\mathrm{f}_i}[\mathtt{W}]^t_{\gamma_\ell+1-\mathrm{f}_i}\right)\equiv\mathbf{0}\pmod{u^{\ell+i}}\quad\text{for }\ i\in\{2,4,6,\ldots,e-\ell\}.$$

(Here, all matrices are considered over $\mathscr{R}_{e,m}$ and all matrix multiplications are performed over $\mathscr{R}_{e,m}$. Throughout this paper, for any square matrix X, $\mathtt{Diag}(X)$ denotes the diagonal matrix whose principal diagonal is the same as that of the matrix X. Also, for any positive integer v, let $[Y]_v$ denote the column block matrix with z-th block Y_z for $1\leq z\leq v$.)

By carrying out direct computations, one can easily observe that linear codes over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ coincide with linear codes over $\mathscr{R}_{\ell,m}$ satisfying property $(*)$ when $\mathfrak{s}=1$ [32, Sec. 4]. Next, for positive integers v and $w\leq e$, let $(\mathtt{H})_{v,w}$ denote the block matrix whose (i,j)-th block is the matrix $\mathtt{H}_{i,j}\in\mathcal{M}_{\lambda_i\times\lambda_{j+1}}(\mathcal{T}_m)$ for $1\leq i\leq v$ and $w\leq j\leq e$. Further, if $\mathscr{D}$ is a self-orthogonal code of type $\{\lambda_1,\lambda_2,\ldots,\lambda_e\}$ and length n over $\mathscr{R}_{e,m}$, then we see, by Lemma 2.2, that the torsion code $Tor_{s+\theta_e}(\mathscr{D})$ is also a self-orthogonal code of length n and dimension $\Lambda_{s+\theta_e}$ over $\mathcal{T}_m(=\overline{\mathscr{R}_{e,m}})$. We now make the following observation.

Remark 3.1. *[31, 32] Every self-orthogonal code of length n and dimension $\Lambda_{s+\theta_e}$ over $\mathcal{T}_m$ is permutation equivalent to a self-orthogonal code with a generator matrix*

$$G=\begin{bmatrix}\mathtt{I}_{\lambda_1}&\mathtt{B}_{1,1}&\mathtt{B}_{1,2}&\cdots&\mathtt{B}_{1,s+\theta_e-1}&\mathtt{B}_{1,s+\theta_e}&\cdots&\mathtt{B}_{1,e}\\ \mathbf{0}&\mathtt{I}_{\lambda_2}&\mathtt{B}_{2,2}&\cdots&\mathtt{B}_{2,s+\theta_e-1}&\mathtt{B}_{2,s+\theta_e}&\cdots&\mathtt{B}_{2,e}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&\mathtt{I}_{\lambda_{s+\theta_e}}&\mathtt{B}_{s+\theta_e,s+\theta_e}&\cdots&\mathtt{B}_{s+\theta_e,e}\end{bmatrix}$$

whose columns are partitioned into blocks of sizes $\lambda_1,\lambda_2,\ldots,\lambda_{e+1}$, the matrix $\mathtt{I}_{\lambda_i}$ is the $\lambda_i\times\lambda_i$ identity matrix over $\mathcal{T}_m$, $\mathtt{B}_{i,j}\in\mathcal{M}_{\lambda_i\times\lambda_{j+1}}(\mathcal{T}_m)$ for $1\leq i\leq s+\theta_e$ and $i\leq j\leq e$, and the matrices $(\mathtt{B})_{s,s+\theta_e+1}$, $(\mathtt{B})_{s-1,s+\theta_e+2},\ldots,(\mathtt{B})_{2,e-1}$, $\mathtt{B}_{1,e}$ are of full row ranks.

Now, let $\mathscr{D}_e$ be a linear code of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ with a generator matrix

$$\mathcal{G}_e = \begin{bmatrix} \mathtt{W}_1^{(e)} \\ u\mathtt{W}_2^{(e)} \\ u^2\mathtt{W}_3^{(e)} \\ \vdots \\ u^{e-1}\mathtt{W}_e^{(e)} \end{bmatrix}, \tag{3.1}$$

where the matrix $\mathtt{W}_j^{(e)} \in \mathcal{M}_{\lambda_j \times n}(\mathscr{R}_{e,m})$ is of the form $\mathtt{W}_j^{(e)} = \mathtt{W}_j^{(0)} + u\mathtt{V}_j^{(1)} + u^2\mathtt{V}_j^{(2)} + \cdots + u^{e-j}\mathtt{V}_j^{(e-j)}$ with $\mathtt{W}_j^{(0)}, \mathtt{V}_j^{(1)}, \mathtt{V}_j^{(2)}, \ldots, \mathtt{V}_j^{(e-j)} \in \mathcal{M}_{\lambda_j \times n}(\mathcal{T}_m)$ for $1 \leq j \leq e$. Now, for each integer ℓ satisfying $2 \leq \ell \leq e-2$ and $\ell \equiv e \pmod 2$, let us define $\gamma_\ell = s - \mathrm{f}_\ell$, and let us consider a linear code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ with a generator matrix

$$\mathcal{G}_\ell = \begin{bmatrix} \mathtt{W}_1^{(\ell)} \\ \mathtt{W}_2^{(\ell)} \\ \vdots \\ \mathtt{W}_{\gamma_\ell+1}^{(\ell)} \\ u\mathtt{W}_{\gamma_\ell+2}^{(\ell)} \\ \vdots \\ u^{\ell-1}\mathtt{W}_{\gamma_\ell+\ell}^{(\ell)} \end{bmatrix}, \tag{3.2}$$

where $\mathtt{W}_h^{(\ell)} = \mathtt{W}_h^{(0)} + u\mathtt{V}_h^{(1)} + \cdots + u^{\ell-1}\mathtt{V}_h^{(\ell-1)}$ $\big($*i.e.*, $\mathtt{W}_h^{(\ell)} \equiv \mathtt{W}_h^{(e)} \pmod{u^\ell}\big)$ for $1 \leq h \leq \gamma_\ell + 1$ and $\mathtt{W}_{\gamma_\ell+j}^{(\ell)} = \mathtt{W}_{\gamma_\ell+j}^{(0)} + u\mathtt{V}_{\gamma_\ell+j}^{(1)} + \cdots + u^{\ell-j}\mathtt{V}_{\gamma_\ell+j}^{(\ell-j)}$ $\big($*i.e.*, $\mathtt{W}_{\gamma_\ell+j}^{(\ell)} \equiv \mathtt{W}_{\gamma_\ell+j}^{(e)} \pmod{u^{\ell-j+1}}\big)$ for $2 \leq j \leq \ell$. In the following lemma, we derive a necessary condition for a self-orthogonal (*resp.* self-dual) code over $\mathscr{R}_{\ell,m}$ to be lifted to a self-orthogonal (*resp.* self-dual) code over $\mathscr{R}_{e,m}$, where ℓ is an integer satisfying $2 \leq \ell \leq e-2$ and $\ell \equiv e \pmod 2$.

Lemma 3.1. *Let $\mathscr{D}_e$ be a linear code of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ with a generator matrix $\mathcal{G}_e$ (as defined in* (3.1)*). For each integer ℓ satisfying $2 \leq \ell \leq e-2$ and $\ell \equiv e \pmod 2$, let $\mathscr{D}_\ell$ be the corresponding linear code of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ with a generator matrix $\mathcal{G}_\ell$ (as defined in* (3.2)*), where $\gamma_\ell = s - \mathrm{f}_\ell$. If the code $\mathscr{D}_e$ is self-orthogonal (resp. self-dual), then $\mathscr{D}_\ell$ is also a self-orthogonal (resp. self-dual) code over $\mathscr{R}_{\ell,m}$ satisfying property* $(\mathfrak{P})$.

Proof. To prove the result, let ℓ be a fixed integer satisfying $2 \leq \ell \leq e-2$ and $\ell \equiv e \pmod 2$. Here, we see, working as in Theorem 5.1 of Yadav and Sharma [33], that if the code $\mathscr{D}_e$ is self-orthogonal (*resp.* self-dual), then the code $\mathscr{D}_i$ is also self-orthogonal (*resp.* self-dual) for any integer $i < e$ with $i \equiv e \pmod 2$, and in particular, the code $\mathscr{D}_\ell$ is self-orthogonal (*resp.* self-dual). Now, it remains to show that the code $\mathscr{D}_\ell$ satisfies property $(\mathfrak{P})$. For this, we will distinguish the following two cases: (I) $2\kappa \leq e$, and (II) $2\kappa > e$.

(I) Let $2\kappa \leq e$. Here, we will consider the following three cases separately: (i) $2 \leq \ell \leq \kappa - 1$, (ii) $\kappa \leq \ell \leq e - \kappa$, and (iii) $\ell \geq e - \kappa + 1$.

(i) Let $2 \leq \ell \leq \kappa - 1$. Since $\mathscr{D}_\ell$ is a self-orthogonal code over $\mathscr{R}_{\ell,m}$, we have $\mathtt{Diag}\big([\mathtt{W}^{(\ell)}]_{\gamma_\ell+1}[\mathtt{W}^{(\ell)}]^t_{\gamma_\ell+1}\big) \equiv \mathbf{0} \pmod{u^\ell}$. Let us write $\mathtt{Diag}\big([\mathtt{W}^{(\ell)}]_{\gamma_\ell+1}[\mathtt{W}^{(\ell)}]^t_{\gamma_\ell+1}\big) \equiv u^\ell \mathtt{E}^{(\ell)} + u^{\ell+1}\mathtt{E}^{(\ell+1)} + \cdots + u^{\ell+\kappa-1}\mathtt{E}^{(\ell+\kappa-1)} \pmod{u^{\ell+\kappa}}$, where $\mathtt{E}^{(\ell)}, \mathtt{E}^{(\ell+1)}, \ldots, \mathtt{E}^{(\ell+\kappa-1)} \in \mathcal{M}_{(\gamma_\ell+1)\times(\gamma_\ell+1)}(\mathcal{T}_m)$. Further, let $\mathtt{E}_{g,g}^{(b)}$ denote the (g,g)-th entry of the matrix $\mathtt{E}^{(b)}$ for $1 \leq g \leq \gamma_\ell + 1$ and $\ell \leq b \leq \ell + \kappa - 1$.

Now, for $i \in \{2, 4, \ldots, \ell - \theta_e\}$, as $\mathscr{D}_{\ell+i}$ is a self-orthogonal code over $\mathscr{R}_{\ell+i,m}$, we see, by Lemma 2.1, that $\mathtt{Diag}\big([\mathtt{W}^{(\ell+i)}]_{\gamma_\ell+1-\mathrm{f}_i}[\mathtt{W}^{(\ell+i)}]^t_{\gamma_\ell+1-\mathrm{f}_i}\big) \equiv \mathbf{0} \pmod{u^{\ell+i}}$, from which one can easily observe that

$$\mathtt{Diag}\big([\mathtt{W}^{(\ell)}]_{\gamma_\ell+1-\mathrm{f}_i}[\mathtt{W}^{(\ell)}]^t_{\gamma_\ell+1-\mathrm{f}_i}\big) \equiv \mathtt{Diag}\big([\mathtt{W}^{(\ell+i)}]_{\gamma_\ell+1-\mathrm{f}_i}[\mathtt{W}^{(\ell+i)}]^t_{\gamma_\ell+1-\mathrm{f}_i}\big) \equiv \mathbf{0} \pmod{u^{\ell+i}}.$$

Further, for $j \in \{\ell-1, \ell+1, \ldots, \kappa-2+\theta_e\}$, since $\mathscr{D}_{\ell+j+1+\theta_e}$ is a self-orthogonal code over $\mathscr{R}_{\ell+j+1+\theta_e,m}$, by Lemma 2.1 again, we get

$$\mathtt{Diag}\big([\mathtt{W}^{(\ell+j+1+\theta_e)}]_{\gamma_\ell+1-\mathrm{f}_{j+2}}[\mathtt{W}^{(\ell+j+1+\theta_e)}]^t_{\gamma_\ell+1-\mathrm{f}_{j+2}}\big) \equiv \mathbf{0} \pmod{u^{\ell+j+1+\theta_e}}. \tag{3.3}$$

We further observe, for $1 \leq a \leq \gamma_\ell + 1 - \mathtt{f}_{j+2}$, that the (a,a)-th entry, say L_a, of the matrix $\mathtt{Diag}\big([\mathtt{W}^{(\ell+j+1+\theta_e)}]_{\gamma_\ell+1-\mathrm{f}_{j+2}}[\mathtt{W}^{(\ell+j+1+\theta_e)}]^t_{\gamma_\ell+1-\mathrm{f}_{j+2}}\big)$ satisfies

$$L_a \equiv u^{2\ell}\big(\mathtt{E}^{(2\ell)}_{a,a} + \mathbf{v}^{(\ell)}_a \cdot \mathbf{v}^{(\ell)}_a\big) + u^{2\ell+1}\mathtt{E}^{(2\ell+1)}_{a,a} + \cdots + u^{\ell+j-2}\mathtt{E}^{(\ell+j-2)}_{a,a} + u^{\ell+j-1}\big(\mathtt{E}^{(\ell+j-1)}_{a,a} + \mathbf{v}^{(\mathtt{f}_{\ell+j-1})}_a \cdot \mathbf{v}^{(\mathtt{f}_{\ell+j-1})}_a\big) + u^{\ell+j}\mathtt{E}^{(\ell+j)}_{a,a} \pmod{u^{\ell+j+1}}$$

if ℓ is even, while it satisfies

$$L_a \equiv u^{2\ell-1}\mathtt{E}^{(2\ell-1)}_{a,a} + u^{2\ell}\big(\mathtt{E}^{(2\ell)}_{a,a} + \mathbf{v}^{(\ell)}_a \cdot \mathbf{v}^{(\ell)}_a\big) + u^{2\ell+1}\mathtt{E}^{(2\ell+1)}_{a,a} + \cdots + u^{\ell+j}\mathtt{E}^{(\ell+j)}_{a,a} + u^{\ell+j+1}\big(\mathtt{E}^{(\ell+j+1)}_{a,a} + \mathbf{v}^{(\mathtt{f}_{\ell+j+1})}_a \cdot \mathbf{v}^{(\mathtt{f}_{\ell+j+1})}_a\big) \pmod{u^{\ell+j+2}}$$

if ℓ is odd, where $\mathbf{v}^{(b)}_a$ denotes the a-th row of the matrix $[\mathtt{V}^{(b)}]_{\gamma_\ell+1}$ for $1 \leq a \leq \gamma_\ell + 1 - \mathtt{f}_{j+2}$ and $\ell \leq b \leq \mathtt{f}_{\ell+j-1+2\theta_e}$. We next observe, by (3.3), that $\mathtt{E}^{(\ell+j)}_{a,a} = 0$ for $1 \leq a \leq \gamma_\ell + 1 - \mathtt{f}_{j+2}$, which implies that

$$\pi_{\ell+j}\big(\mathtt{Diag}\big([\mathtt{W}^{(\ell)}]_{\gamma_\ell+1-\mathrm{f}_{j+2}}[\mathtt{W}^{(\ell)}]^t_{\gamma_\ell+1-\mathrm{f}_{j+2}}\big)\big) = \mathbf{0}$$

for $j \in \{\ell-1, \ell+1, \ldots, \kappa-2+\theta_e\}$.

(ii) Next, let $\kappa \leq \ell \leq e-\kappa$. Here, for $i \in \{2,4,\ldots,\kappa-1\} \cup \{\kappa\}$, as $\mathscr{D}_{\ell+i}$ is a self-orthogonal code over $\mathscr{R}_{\ell+i,m}$, by Lemma 2.1, we get $\mathtt{Diag}\big([\mathtt{W}^{(\ell+i)}]_{\gamma_\ell+1-\mathrm{c}_i}[\mathtt{W}^{(\ell+i)}]^t_{\gamma_\ell+1-\mathrm{c}_i}\big) \equiv \mathbf{0} \pmod{u^{\ell+i}}$, which implies that

$$\mathtt{Diag}\big([\mathtt{W}^{(\ell)}]_{\gamma_\ell+1-\mathrm{c}_i}[\mathtt{W}^{(\ell)}]^t_{\gamma_\ell+1-\mathrm{c}_i}\big) \equiv \mathbf{0} \pmod{u^{\ell+i}}.$$

(iii) Finally, let $\ell \geq e-\kappa+1$. Here, for $i \in \{2,4,\ldots,e-\ell\}$, since $\mathscr{D}_{\ell+i}$ is a self-orthogonal code over $\mathscr{R}_{\ell+i,m}$, by Lemma 2.1, we get $\mathtt{Diag}\big([\mathtt{W}^{(\ell+i)}]_{\gamma_\ell+1-\mathrm{f}_i}[\mathtt{W}^{(\ell+i)}]^t_{\gamma_\ell+1-\mathrm{f}_i}\big) \equiv \mathbf{0} \pmod{u^{\ell+i}}$, which implies that

$$\mathtt{Diag}\big([\mathtt{W}^{(\ell)}]_{\gamma_\ell+1-\mathrm{f}_i}[\mathtt{W}^{(\ell)}]^t_{\gamma_\ell+1-\mathrm{f}_i}\big) \equiv \mathbf{0} \pmod{u^{\ell+i}}.$$

This shows that the code $\mathscr{D}_\ell$ satisfies property $(\mathfrak{P})$.

(II) When $2\kappa > e$, the desired result follows by working as in case (I). ☐

From the above lemma, we see, for $2 \leq \ell \leq e-2$ and $\ell \equiv e \pmod 2$, that a necessary condition for lifting a linear code $\mathscr{D}_\ell$ over $\mathscr{R}_{\ell,m}$ to a self-orthogonal (*resp.* self-dual) code $\mathscr{D}_e$ over $\mathscr{R}_{e,m}$ is that $\mathscr{D}_\ell$ must be a self-orthogonal (*resp.* self-dual) code over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$. We next prove the following lemma.

Lemma 3.2. *Let ℓ be a fixed integer satisfying $\ell \equiv e \pmod 2$ and $4 \leq \ell \leq \kappa+2$ if $2\kappa \leq e$, while the integer ℓ satisfies $\ell \equiv e \pmod 2$ and $4 \leq \ell \leq e-\kappa+1-2\theta_e$ if $2\kappa > e$. Let us define $\gamma_\ell = s - \mathrm{f}_\ell$. If there exists a self-orthogonal code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$, then there exists a $\Lambda_{s-\kappa_1}$-dimensional doubly even subcode of the torsion code $Tor_1(\mathscr{D}_\ell)$.*

Proof. To prove the result, we assume, without any loss of generality, that the code $\mathscr{D}_\ell$ has a generator matrix $\mathcal{G}_\ell$ as defined in (3.2). Note that the torsion code $Tor_1(\mathscr{D}_\ell)$ is a $\Lambda_{\gamma_\ell+1}$-dimensional linear code over $\mathcal{T}_m$ with a generator matrix $[\mathtt{W}^{(0)}]_{\gamma_\ell+1}$. Now, let $\mathcal{C}^{(s-\kappa_1)}$ be a $\Lambda_{s-\kappa_1}$-dimensional linear subcode of $Tor_1(\mathscr{D}_\ell)$ with a generator matrix $[\mathtt{W}^{(0)}]_{s-\kappa_1}$. We assert that $\mathcal{C}^{(s-\kappa_1)}$ is a doubly even code over $\mathcal{T}_m$.

To prove this assertion, we see, by Lemma 2.2, that $\mathcal{C}^{(s-\kappa_1)}$ is a self-orthogonal code over $\mathcal{T}_m$. We further note, by Lemma 2.1, that if $\mathscr{D}_\ell$ is a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$, then

$$\begin{aligned} \pi_i\Big(\mathtt{Diag}\big([\mathtt{W}^{(\ell)}]_{\gamma_\ell+1}[\mathtt{W}^{(\ell)}]^t_{\gamma_\ell+1}\big)\Big) &= \mathbf{0} \text{ for } i \in \{1,3,\ldots,\ell-1-\theta_e\} \text{ and} \\ \pi_{\ell+j-\theta_e}\Big(\mathtt{Diag}\big([\mathtt{W}^{(\ell)}]_{\gamma_\ell+1-\mathrm{f}_{j+2-\theta_e}}[\mathtt{W}^{(\ell)}]^t_{\gamma_\ell+1-\mathrm{f}_{j+2-\theta_e}}\big)\Big) &= \mathbf{0} \text{ for } j \in \{1,3,\ldots,\kappa-2\}. \end{aligned}$$

We further note that $\pi_g\big(\mathtt{Diag}\big([\mathtt{W}^{(0)}]_{s-\kappa_1}[\mathtt{W}^{(0)}]^t_{s-\kappa_1}\big)\big) = \pi_g\big(\mathtt{Diag}\big([\mathtt{W}^{(\ell)}]_{s-\kappa_1}[\mathtt{W}^{(\ell)}]^t_{s-\kappa_1}\big)\big) = \mathbf{0}$ for all $g \in \{1,3,\ldots,\kappa\}$. From this, it follows that $\mathcal{C}^{(s-\kappa_1)}$ is a doubly even code over $\mathcal{T}_m$. ☐

In the following lemma, we show that if $\mathscr{D}_e$ is a self-orthogonal code of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$, then $Tor_{s+\theta_e}(\mathscr{D}_e)$ is a $\Lambda_{s-\kappa_1}$-doubly even code of length n and dimension $\Lambda_{s+\theta_e}$ over $\mathcal{T}_m$ and has $Tor_{s-\kappa_1}(\mathscr{D}_e)$ as its $\Lambda_{s-\kappa_1}$-dimensional doubly even subcode.

Lemma 3.3. *Let $\mathscr{D}_e$ be a self-orthogonal code of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ with a generator matrix $\mathcal{G}_e$ (as defined in (3.1)). The torsion code $Tor_{s+\theta_e}(\mathscr{D}_e)$ is a $\Lambda_{s-\kappa_1}$-doubly even code of length n and dimension $\Lambda_{s+\theta_e}$ over $\mathcal{T}_m$ with its $\Lambda_{s-\kappa_1}$-dimensional doubly even subcode as $Tor_{s-\kappa_1}(\mathscr{D}_e)$.*

Proof. To prove the result, we note, by Lemma 2.2, that the torsion code $Tor_{s+\theta_e}(\mathscr{D}_e)$ is a self-orthogonal code of length n and dimension $\Lambda_{s+\theta_e}$ over $\mathcal{T}_m$. Here, we assert that the torsion code $Tor_{s-\kappa_1}(\mathscr{D}_e)$ is a doubly even code, *i.e.,* $\pi_\kappa(v \cdot v) = 0$ for all $v \in Tor_{s-\kappa_1}(\mathscr{D}_e)$, where each $v \cdot v$ is viewed as an element of $\mathscr{R}_{e,m}$. To prove this assertion, let $a \in Tor_{s-\kappa_1}(\mathscr{D}_e)$ be fixed. Thus, there exist $a_1, a_2, \ldots, a_{s+\kappa_1+\theta_e} \in \mathcal{T}_m^n$ such that $u^{s-\kappa_1-1}(a + ua_1 + u^2 a_2 + \cdots + u^{s+\kappa_1+\theta_e} a_{s+\kappa_1+\theta_e}) \in \mathscr{D}_e$. Now, since $\mathscr{D}_e$ is a self-orthogonal code over $\mathscr{R}_{e,m}$, we have $u^{2s-2\kappa_1-2}(a + ua_1 + \cdots + u^{s+\kappa_1+\theta_e} a_{s+\kappa_1+\theta_e}) \cdot (a + ua_1 + \cdots + u^{s+\kappa_1+\theta_e} a_{s+\kappa_1+\theta_e}) \equiv 0 \pmod{u^e}$. This implies that $(a+ua_1+\cdots+u^{s+\kappa_1+\theta_e} a_{s+\kappa_1+\theta_e}) \cdot (a+ua_1+\cdots+u^{s+\kappa_1+\theta_e} a_{s+\kappa_1+\theta_e}) \equiv 0 \pmod{u^{\kappa+1+\theta_e}}$, which further implies that $a \cdot a + u^2(a_1 \cdot a_1) + \cdots + u^{\kappa-1}(a_{\kappa_1} \cdot a_{\kappa_1}) + \theta_e u^{\kappa+1}(a_{\kappa_1+1} \cdot a_{\kappa_1+1} + \eta_0 a \cdot a_1) \equiv 0 \pmod{u^{\kappa+1+\theta_e}}$. From this, one can easily see that $\pi_\kappa(a \cdot a) = \pi_\kappa\left(a \cdot a + u^2(a_1 \cdot a_1) + \cdots + u^{\kappa-1}(a_{\kappa_1} \cdot a_{\kappa_1}) + \theta_e u^{\kappa+1}(a_{\kappa_1+1} \cdot a_{\kappa_1+1} + \eta_0 a \cdot a_1)\right) = 0$. This shows that $Tor_{s-\kappa_1}(\mathscr{D}_e)$ is a doubly even code over $\mathcal{T}_m$. From this, the desired result follows. $\square$

From the above lemma, we see that a necessary condition for a linear code $\mathscr{D}_0$ of length n and dimension $\Lambda_{s+\theta_e}$ over $\mathcal{T}_m$ to be lifted to a self-orthogonal code over $\mathscr{R}_{e,m}$ is that the code $\mathscr{D}_0$ must be a $\Lambda_{s-\kappa_1}$-doubly even code over $\mathcal{T}_m$. Next, let $\mathcal{T}$ be the Teichmüller set of the Galois ring $GR(2^e, m)$ of characteristic 2^e and rank m. Now, a linear code C of length n over $\mathcal{T}$ is said to be doubly even if it satisfies $\mathbf{c} \cdot \mathbf{c} \equiv 0 \pmod 4$ for all $\mathbf{c} \in C$, where each $\mathbf{c} \cdot \mathbf{c}$ is viewed as an element of $GR(2^e, m)$ (see [31, Def. 2.1]). Yadav and Sharma [31, Th. 3.2 and 3.3] obtained enumeration formulae for the number $\widehat{\sigma}_m(n; \mathfrak{d})$ of doubly even codes of length n and dimension $\mathfrak{d}$ over $\mathcal{T}$ containing the all-one vector and the number $\sigma_m(n; \mathfrak{d})$ of doubly even codes of length n and dimension $\mathfrak{d}$ over $\mathcal{T}$ that do not contain the all-one vector. In the following lemma, we establish a 1-1 correspondence between doubly even codes over $\mathcal{T}_m$ and doubly even codes over the Teichmüller set $\mathcal{T}$ of $GR(2^e, m)$.

Lemma 3.4. *There is a 1-1 correspondence between doubly even codes of length n over $\mathcal{T}_m$ and doubly even codes of the same length n over the Teichmüller set $\mathcal{T}$ of $GR(2^e, m)$.*

Proof. To prove the result, we first note that $\mathcal{T} \simeq \mathbb{F}_{2^m} \simeq \mathcal{T}_m$, and let ϕ be the field isomorphism from $\mathcal{T}$ onto $\mathcal{T}_m$. The isomorphism ϕ can be extended component-wise to an isomorphism from $\mathcal{T}^n$ onto $\mathcal{T}_m^n$, which we shall denote by ϕ itself for the sake of simplicity. The isomorphism ϕ induces a 1-1 correspondence between linear codes of length n over $\mathcal{T}$ and linear codes of length n over $\mathcal{T}_m$.

Now, let C be a doubly even code of length n over $\mathcal{T}$. We assert that $\phi(C)$ is a doubly even code over $\mathcal{T}_m$. To prove this assertion, it is easy to see that the code $\phi(C)$ is self-orthogonal. Next, let $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in C$. Since the code C is doubly even over $\mathcal{T}$, we have $\mathbf{c} \cdot \mathbf{c} \equiv 0 \pmod 4$, where $\mathbf{c} \cdot \mathbf{c}$ is viewed as an element of $GR(2^e, m)$. This implies that $(\sum\limits_{i=1}^{n} c_i)^2 - 2 \sum\limits_{1 \le i < j \le n} c_i c_j \equiv 0 \pmod 4$, which further implies that $\pi_0(-\sum\limits_{1 \le i < j \le n} c_i c_j) = 0$. As $\phi(\mathbf{c}) \cdot \phi(\mathbf{c})$ can be viewed as an element of $\mathscr{R}_{e,m}$, we observe that $\pi_\kappa\big(\phi(\mathbf{c}) \cdot \phi(\mathbf{c})\big) = \eta_0 \pi_0\big(-\sum\limits_{1 \le i < j \le n} \phi(c_i)\phi(c_j)\big) = \eta_0 \phi\big(\pi_0(-\sum\limits_{1 \le i < j \le n} c_i c_j)\big) = \phi(0) = 0$, which implies that the code $\phi(C)$ is doubly even over $\mathcal{T}_m$.

Conversely, let D be a doubly even code of length n over $\mathcal{T}_m$. There exists a linear code D' of length n over $\mathcal{T}$ such that $\phi(D') = D$. We claim that the code D' is doubly even. It is easy to see that the code D' is self-orthogonal. Now, let $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in D'$ be fixed. Since D is doubly even, we have $\pi_\kappa\big(\phi(\mathbf{d}) \cdot \phi(\mathbf{d})\big) = \eta_0 \pi_0\big(-\sum\limits_{1 \le i < j \le n} \phi(d_i)\phi(d_j)\big) = 0$. This implies that $\pi_0\big(-\sum\limits_{1 \le i < j \le n} d_i d_j\big) = 0$, which further implies that $\mathbf{d} \cdot \mathbf{d} \equiv 0 \pmod 4$, where $\mathbf{d} \cdot \mathbf{d}$ is viewed as an element of $GR(2^e, m)$. This implies that the code D' is doubly even. $\square$

Further, let $\mathtt{Sym}_j(\mathcal{T}_m)$ and $\mathtt{Alt}_j(\mathcal{T}_m)$ denote the sets consisting of all $j \times j$ symmetric and alternating matrices over $\mathcal{T}_m$, respectively. We further recall, from Section 2, that $2 \equiv u^\kappa(\eta_0 + u\eta_1 + u^2\eta_2 + \cdots + u^{e-1-\kappa}\eta_{e-1-\kappa}) \pmod{u^e}$, where $\eta_0 \in \mathcal{T}_m \setminus \{0\}$ and $\eta_1, \eta_2, \ldots, \eta_{e-1-\kappa} \in \mathcal{T}_m$. Now, the following proposition considers the case where $e \ge 4$ is an even integer and κ is an odd integer satisfying $3 \le \kappa \le e-1$. It provides a method for lifting a $\Lambda_{s-\kappa_1}$-doubly even code of length n and dimension Λ_s over $\mathcal{T}_m$ to a self-orthogonal code of type $\{\Lambda_s, \lambda_{s+1}\}$ and the same length n over $\mathscr{R}_{2,m}$ satisfying property $(\mathfrak{P})$ with also enumerates the number of possible ways to perform this lifting.

Proposition 3.1. *Let $e \ge 4$ be an even integer, and let κ be an odd integer satisfying $3 \le \kappa \le e-1$. Let $\mathcal{C}^{(s)}$ be a $\Lambda_{s-\kappa_1}$-doubly even code of length n and dimension Λ_s over $\mathcal{T}_m$. The following hold.*

(a) There exists a self-orthogonal code $\mathscr{D}_2$ of type $\{\Lambda_s, \lambda_{s+1}\}$ and length n over $\mathscr{R}_{2,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_2) = \mathcal{C}^{(s)}$.

(b) Moreover, each $\Lambda_{s-\kappa_1}$-doubly even code $\mathcal{C}^{(s)}$ of length n and dimension Λ_s over $\mathcal{T}_m$ gives rise to precisely

$$(2^m)^{\sum_{i=3}^{s+1}\lambda_i\Lambda_{i-2}+\Lambda_s(n-\Lambda_{s+1})-\Lambda_{s-1}-\frac{\Lambda_s(\Lambda_s-1)}{2}}\begin{bmatrix}\lambda_{s+1}+n-\Lambda_{s+1}-\Lambda_s\\ \lambda_{s+1}\end{bmatrix}_{2^m}$$

distinct self-orthogonal codes $\mathscr{D}_2$ of type $\{\Lambda_s,\lambda_{s+1}\}$ and length n over $\mathscr{R}_{2,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_2)=\mathcal{C}^{(s)}$.

Proof. Working as in Proposition 4.1 of Yadav and Sharma [31] and by applying Lemma 4.1 of Yadav and Sharma [32], the desired result follows. □

The following proposition considers the case when both e and κ are odd integers satisfying $3\leq\kappa\leq e-1$. It provides a method for lifting a $\Lambda_{s-\kappa_1}$-doubly even code of length n and dimension Λ_{s+1} over $\mathcal{T}_m$ to a self-orthogonal code of type $\{\Lambda_s,\lambda_{s+1},\lambda_{s+2}\}$ and the same length n over $\mathscr{R}_{3,m}$ satisfying property $(\mathfrak{P})$ with also enumerates the number of possible ways to perform this lifting.

Proposition 3.2. *Suppose that e and κ are both odd integers satisfying $3\leq\kappa\leq e-1$. Let $\mathcal{C}^{(s+1)}\supseteq\mathcal{C}^{(s)}\supseteq\mathcal{C}^{(s-\kappa_1)}\supseteq\mathcal{C}^{(s-\kappa_1-1)}$ be a chain of self-orthogonal codes of length n over $\mathcal{T}_m$, where $\dim\mathcal{C}^{(s+1)}=\Lambda_{s+1}$, $\dim\mathcal{C}^{(s)}=\Lambda_s$, $\dim\mathcal{C}^{(s-\kappa_1)}=\Lambda_{s-\kappa_1}$, $\dim\mathcal{C}^{(s-\kappa_1-1)}=\Lambda_{s-\kappa_1-1}$, and the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even. The following hold.*

(a) There exists a self-orthogonal code $\mathscr{D}_3$ of type $\{\Lambda_s,\lambda_{s+1},\lambda_{s+2}\}$ and length n over $\mathscr{R}_{3,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_3)=\mathcal{C}^{(s)}$ and $Tor_2(\mathscr{D}_3)=\mathcal{C}^{(s+1)}$.

(b) Moreover, for each such chain $\mathcal{C}^{(s+1)}\supseteq\mathcal{C}^{(s)}\supseteq\mathcal{C}^{(s-\kappa_1)}\supseteq\mathcal{C}^{(s-\kappa_1-1)}$ of self-orthogonal codes over $\mathcal{T}_m$, there are precisely

$$(2^m)^{\sum_{i=3}^{s+2}\lambda_i\Lambda_{i-2}+\sum_{j=4}^{s+2}\lambda_j\Lambda_{j-3}+(\Lambda_s+\Lambda_{s+1})(n-\Lambda_{s+2}-\Lambda_s)+\Lambda_s^2-\Lambda_{s-1}-\Lambda_{s-\kappa_1-1}+\omega_3}\begin{bmatrix}\lambda_{s+2}+n-\Lambda_{s+2}-\Lambda_s\\ \lambda_{s+2}\end{bmatrix}_{2^m}$$

distinct self-orthogonal codes $\mathscr{D}_3$ of type $\{\Lambda_s,\lambda_{s+1},\lambda_{s+2}\}$ and length n over $\mathscr{R}_{3,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_3)=\mathcal{C}^{(s)}$ and $Tor_2(\mathscr{D}_3)=\mathcal{C}^{(s+1)}$, where $\omega_3=1$ if $\boldsymbol{1}\in\mathcal{C}^{(s-\kappa_1-1)}$, while $\omega_3=0$ otherwise.

Proof. Here, by Remark 3.1, we assume, without any loss of generality, that the code $\mathcal{C}^{(s+1)}$ has a generator matrix

$$\mathtt{G}_0=[\mathtt{W}^{(0)}]_{s+1}=\begin{bmatrix}\mathtt{W}_1^{(0)}\\ \mathtt{W}_2^{(0)}\\ \vdots\\ \mathtt{W}_{s+1}^{(0)}\end{bmatrix}=\begin{bmatrix}\mathtt{I}_{\lambda_1} & \mathtt{B}_{1,1}^{(0)} & \mathtt{B}_{1,2}^{(0)} & \cdots & \mathtt{B}_{1,s}^{(0)} & \cdots & \mathtt{B}_{1,e-1}^{(0)} & \mathtt{B}_{1,e}^{(0)}\\ \mathbf{0} & \mathtt{I}_{\lambda_2} & \mathtt{B}_{2,2}^{(0)} & \cdots & \mathtt{B}_{2,s}^{(0)} & \cdots & \mathtt{B}_{2,e-1}^{(0)} & \mathtt{B}_{2,e}^{(0)}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathtt{I}_{\lambda_{s+1}} & \cdots & \mathtt{B}_{s+1,e-1}^{(0)} & \mathtt{B}_{s+1,e}^{(0)}\end{bmatrix},$$

where columns of the matrix $\mathtt{G}_0$ are partitioned into blocks of sizes $\lambda_1,\lambda_2,\ldots,\lambda_{e+1}$, the matrix $\mathtt{I}_{\lambda_i}$ is the $\lambda_i\times\lambda_i$ identity matrix over $\mathcal{T}_m$, the matrix $\mathtt{B}_{i,j}^{(0)}\in\mathcal{M}_{\lambda_i\times\lambda_{j+1}}(\mathcal{T}_m)$ for $1\leq i\leq s+1$ and $i\leq j\leq e$, and each of the matrices $(\mathtt{B}^{(0)})_{s,s+2}$, $(\mathtt{B}^{(0)})_{s-1,s+3},\ldots,(\mathtt{B}^{(0)})_{2,e-1}$, $\mathtt{B}_{1,e}^{(0)}$ are of full row-ranks. We further assume, without any loss of generality, that the code $\mathcal{C}^{(s)}$ has a generator matrix $[\mathtt{W}^{(0)}]_s$, the code $\mathcal{C}^{(s-\kappa_1)}$ has a generator matrix $[\mathtt{W}^{(0)}]_{s-\kappa_1}$, and the code $\mathcal{C}^{(s-\kappa_1-1)}$ has a generator matrix $[\mathtt{W}^{(0)}]_{s-\kappa_1-1}$. Under these assumptions, we have

$$\begin{aligned}[\mathtt{W}^{(0)}]_s[\mathtt{W}^{(0)}]_s^t &\equiv u\mathtt{F}^{(1)}+u^2\mathtt{F}^{(2)}+\cdots+u^{\kappa+2}\mathtt{F}^{(\kappa+2)}\pmod{u^{\kappa+3}},\\ [\mathtt{W}^{(0)}]_s\mathtt{W}_{s+1}^{(0)t} &\equiv uP\pmod{u^2},\end{aligned}$$

where $\mathtt{F}^{(j)}\in\mathtt{Sym}_{\Lambda_s}(\mathcal{T}_m)$ for $1\leq j\leq\kappa+2$ and $P\in\mathcal{M}_{\Lambda_s\times\lambda_{s+1}}(\mathcal{T}_m)$. Further, one can easily see that $\mathtt{F}_{i,i}^{(1)}=\mathtt{F}_{i,i}^{(3)}=\cdots=\mathtt{F}_{i,i}^{(\kappa-2)}=0$ for $1\leq i\leq\Lambda_s$ and $\mathtt{F}_{r,r}^{(\kappa)}=0$ for $1\leq r\leq\Lambda_{s-\kappa_1}$.

Now, to prove the result, let us define a matrix $\mathtt{G}_3$ over $\mathscr{R}_{3,m}$ as

$$\mathtt{G}_3=\begin{bmatrix}\mathtt{W}_1^{(3)}\\ \mathtt{W}_2^{(3)}\\ \vdots\\ \mathtt{W}_s^{(3)}\\ u\mathtt{W}_{s+1}^{(3)}\\ u^2\mathtt{W}_{s+2}^{(3)}\end{bmatrix}=\begin{bmatrix}\mathtt{W}_1^{(0)}+u\mathtt{V}_1^{(1)}+u^2\mathtt{V}_1^{(2)}\\ \mathtt{W}_2^{(0)}+u\mathtt{V}_2^{(1)}+u^2\mathtt{V}_2^{(2)}\\ \vdots\\ \mathtt{W}_s^{(0)}+u\mathtt{V}_s^{(1)}+u^2\mathtt{V}_s^{(2)}\\ u\mathtt{W}_{s+1}^{(3)}\\ u^2\mathtt{W}_{s+2}^{(3)}\end{bmatrix},$$

where the matrices $[\mathtt{V}^{(\mu)}]_s \in \mathcal{M}_{\Lambda_s\times n}(\mathcal{T}_m)$ for $\mu\in\{1,2\}$, $\mathtt{W}^{(3)}_{s+1}\in\mathcal{M}_{\lambda_{s+1}\times n}(\mathscr{R}_{3,m})$ and $\mathtt{W}^{(3)}_{s+2}\in\mathcal{M}_{\lambda_{s+2}\times n}(\mathcal{T}_m)$ are of the forms

$$[\mathtt{V}^{(\mu)}]_s=\begin{bmatrix}\mathtt{V}^{(\mu)}_1\\ \mathtt{V}^{(\mu)}_2\\ \vdots\\ \mathtt{V}^{(\mu)}_s\end{bmatrix}=\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}&\mathtt{B}^{(\mu)}_{1,\mu+1}&\mathtt{B}^{(\mu)}_{1,\mu+2}&\cdots&\mathtt{B}^{(\mu)}_{1,\mu+s}&\cdots&\mathtt{B}^{(\mu)}_{1,e}\\ \mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\mathtt{B}^{(\mu)}_{2,\mu+2}&\cdots&\mathtt{B}^{(\mu)}_{2,\mu+s}&\cdots&\mathtt{B}^{(\mu)}_{2,e}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&\mathtt{B}^{(\mu)}_{s,\mu+s}&\cdots&\mathtt{B}^{(\mu)}_{s,e}\end{bmatrix},$$

$$\mathtt{W}^{(3)}_{s+1}=\mathtt{W}^{(0)}_{s+1}+u\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}&\mathtt{B}^{(1)}_{s+1,s+2}&\mathtt{B}^{(1)}_{s+1,s+3}&\cdots&\mathtt{B}^{(1)}_{s+1,e}\end{bmatrix}\text{ and}$$

$$\mathtt{W}^{(3)}_{s+2}=\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}&\mathtt{I}_{\lambda_{s+2}}&\mathtt{B}^{(0)}_{s+2,s+2}&\mathtt{B}^{(0)}_{s+2,s+3}\cdots&\mathtt{B}^{(0)}_{s+2,e}\end{bmatrix}$$

with $\mathtt{B}^{(\mu)}_{i,j}\in\mathcal{M}_{\lambda_i\times\lambda_{j+1}}(\mathcal{T}_m)$ for $1\le i\le s$ and $i+\mu\le j\le e$, $\mathtt{B}^{(1)}_{s+1,g}\in\mathcal{M}_{\lambda_{s+1}\times\lambda_{g+1}}(\mathcal{T}_m)$ for $s+2\le g\le e$, and $\mathtt{B}^{(0)}_{s+2,h}\in\mathcal{M}_{\lambda_{s+2}\times\lambda_{h+1}}(\mathcal{T}_m)$ for $s+2\le h\le e$. Let $\mathscr{D}_3$ be a linear code of length n over $\mathscr{R}_{3,m}$ with a generator matrix $\mathtt{G}_3$. Clearly, the code $\mathscr{D}_3$ is of type $\{\Lambda_s,\lambda_{s+1},\lambda_{s+2}\}$ and satisfies $Tor_1(\mathscr{D}_3)=\mathcal{C}^{(s)}$ and $Tor_2(\mathscr{D}_3)=\mathcal{C}^{(s+1)}$. We further see, by Lemma 2.1, that the code $\mathscr{D}_3$ is a self-orthogonal code over $\mathscr{R}_{3,m}$ satisfying property $(\mathfrak{P})$ if and only if there exist matrices $[\mathtt{V}^{(1)}]_s,[\mathtt{V}^{(2)}]_s$, $[\mathbf{0}\ \cdots\ \mathbf{0}\ \mathtt{B}^{(1)}_{s+1,s+2}\ \cdots\ \mathtt{B}^{(1)}_{s+1,e}]$ and $\mathtt{W}^{(3)}_{s+2}$ satisfying the following system of matrix equations:

$$\begin{aligned}\mathtt{F}^{(1)}+[\mathtt{W}^{(0)}]_s[\mathtt{V}^{(1)}]^t_s+[\mathtt{V}^{(1)}]_s[\mathtt{W}^{(0)}]^t_s+u\Big(\mathtt{F}^{(2)}+[\mathtt{V}^{(1)}]_s[\mathtt{V}^{(1)}]^t_s&\\ +[\mathtt{W}^{(0)}]_s[\mathtt{V}^{(2)}]^t_s+[\mathtt{V}^{(2)}]_s[\mathtt{W}^{(0)}]^t_s\Big)&\equiv\mathbf{0}\pmod{u^2}, &(3.4)\\ \mathtt{Diag}\Big([\mathtt{W}^{(3)}]_{s-1}[\mathtt{W}^{(3)}]^t_{s-1}\Big)&\equiv\mathbf{0}\pmod{u^5}, &(3.5)\\ \pi_{j+3}\Big(\mathtt{Diag}\big([\mathtt{W}^{(3)}]_{s-\mathrm{f}_{j+2}}[\mathtt{W}^{(3)}]^t_{s-\mathrm{f}_{j+2}}\big)\Big)&=\mathbf{0}\quad\text{for }\ j\in\{2,4,\ldots,\kappa-1\}, &(3.6)\\ P+[\mathtt{W}^{(0)}]_s\Big[0\ \cdots\ 0\ \mathtt{B}^{(1)}_{s+1,s+2}\ \cdots\ \mathtt{B}^{(1)}_{s+1,e}\Big]^t&\equiv\mathbf{0}\pmod{u}, &(3.7)\\ [\mathtt{W}^{(0)}]_s\mathtt{W}^{(3)t}_{s+2}&\equiv\mathbf{0}\pmod{u}. &(3.8)\end{aligned}$$

First of all, we will establish the existence of the matrices $[\mathtt{V}^{(1)}]_s$ and $[\mathtt{V}^{(2)}]_s$ satisfying the system of matrix equations (3.4)-(3.6) and count their choices. To do this, let us suppose that $[\mathtt{W}^{(0)}]_s=(\mathbf{b}_i)$, $[\mathtt{V}^{(1)}]_s=(\mathbf{x}_j)$ and $[\mathtt{V}^{(2)}]_s=(\mathbf{y}_g)$, where $\mathbf{b}_i$'s, $\mathbf{x}_j$'s, and $\mathbf{y}_g$'s denote the rows of the matrices $[\mathtt{W}^{(0)}]_s$, $[\mathtt{V}^{(1)}]_s$, and $[\mathtt{V}^{(2)}]_s$, respectively. We further observe, for $1\le i\le\Lambda_s$, that the (i,i)-th entry, say J_i, of the matrix $\mathtt{Diag}\big([\mathtt{W}^{(3)}]_s[\mathtt{W}^{(3)}]^t_s\big)$ satisfies

$$\begin{aligned}J_i\equiv{}&u^2\Big(\mathtt{F}^{(2)}_{i,i}+\mathbf{x}_i\cdot\mathbf{x}_i\Big)+u^3\mathtt{F}^{(3)}_{i,i}+u^4\Big(\mathtt{F}^{(4)}_{i,i}+\mathbf{y}_i\cdot\mathbf{y}_i\Big)+u^5\mathtt{F}^{(5)}_{i,i}+\cdots+u^{\kappa}\mathtt{F}^{(\kappa)}_{i,i}+u^{\kappa+1}\Big(\mathtt{F}^{(\kappa+1)}_{i,i}+\eta_0\mathbf{b}_i\cdot\mathbf{x}_i\Big)\\ &+u^{\kappa+2}\Big(\mathtt{F}^{(\kappa+2)}_{i,i}+\eta_0\mathbf{b}_i\cdot\mathbf{y}_i+\eta_1\mathbf{b}_i\cdot\mathbf{x}_i\Big)\pmod{u^{\kappa+3}}.\end{aligned}$$

If w is the smallest positive integer satisfying $\kappa+1\le 2^w$, then we observe, for $1\le i\le\Lambda_s$, that

$$\begin{aligned}J_i\equiv{}&u^2\Big((\mathtt{F}^{(2)}_{i,i})^{2^{-w}}+(\mathbf{1}\cdot\mathbf{x}_i)^{2^{-(w-1)}}\Big)^{2^w}+u^3\mathtt{F}^{(3)}_{i,i}+u^4\Big((\mathtt{F}^{(4)}_{i,i})^{2^{-w}}+(\mathbf{1}\cdot\mathbf{y}_i)^{2^{-(w-1)}}\Big)^{2^w}+u^5\mathtt{F}^{(5)}_{i,i}+\cdots\\ &+u^{\kappa}\mathtt{F}^{(\kappa)}_{i,i}+u^{\kappa+1}\Big(\mathtt{F}^{(\kappa+1)}_{i,i}+\eta_0\mathbf{b}_i\cdot\mathbf{x}_i\Big)+u^{\kappa+2}\Big(\mathtt{F}^{(\kappa+2)}_{i,i}+\eta_0\mathbf{b}_i\cdot\mathbf{y}_i+\eta_1\mathbf{b}_i\cdot\mathbf{x}_i\\ &-\eta_0\Big((\mathbf{1}\cdot\mathbf{x}_i)(\mathtt{F}^{(2)}_{i,i})^{2^{-1}}+\sum_{1\le h<g\le n}\mathrm{x}_{i,h}\mathrm{x}_{i,g}\Big)\Big)\pmod{u^{\kappa+3}},\end{aligned}$$

where $\mathbf{x}_i=(\mathrm{x}_{i,1},\mathrm{x}_{i,2},\ldots,\mathrm{x}_{i,n})\in\mathcal{T}^n_m$. It is easy to see that $\big((\mathtt{F}^{(2)}_{i,i})^{2^{-w}}+(\mathbf{1}\cdot\mathbf{x}_i)^{2^{-(w-1)}}\big)^{2^w}\in\mathcal{T}_m$ and $\big((\mathtt{F}^{(4)}_{i,i})^{2^{-w}}+(\mathbf{1}\cdot\mathbf{y}_i)^{2^{-(w-1)}}\big)^{2^w}\in\mathcal{T}_m$ for $1\le i\le\Lambda_s$. In view of this, we see that the system of matrix equations (3.4)-(3.6) is equivalent to the following system of equations in unknowns $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_{\Lambda_s}$ and $\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_{\Lambda_s}$ over $\mathcal{T}_m$:

$$\begin{aligned}\mathtt{F}^{(1)}_{i,j}+\mathbf{b}_i\cdot\mathbf{x}_j+\mathbf{b}_j\cdot\mathbf{x}_i+u\Big(\mathtt{F}^{(2)}_{i,j}+\mathbf{x}_i\cdot\mathbf{x}_j+\mathbf{b}_i\cdot\mathbf{y}_j+\mathbf{b}_j\cdot\mathbf{y}_i\Big)&\equiv 0\pmod{u^2}\text{ for }1\le i<j\le\Lambda_s, &(3.9)\\ (\mathtt{F}^{(2)}_{i,i})^{2^{-1}}+\mathbf{1}\cdot\mathbf{x}_i&\equiv 0\pmod{u}\quad\text{for }1\le i\le\Lambda_s, &(3.10)\end{aligned}$$

$$(\mathtt{F}_{i,i}^{(4)})^{2^{-1}} + \mathbf{1}\cdot\mathbf{y}_i \equiv 0 \pmod{u} \quad \text{for } 1 \le i \le \Lambda_{s-1}, \tag{3.11}$$

$$\mathtt{F}_{i,i}^{(\kappa+2)} + \eta_0\mathbf{b}_i\cdot\mathbf{y}_i + \eta_1\mathbf{b}_i\cdot\mathbf{x}_i - \eta_0\Big((\mathbf{1}\cdot\mathbf{x}_i)(\mathtt{F}_{i,i}^{(2)})^{2^{-1}} + \sum_{1\le h<g\le n}\mathtt{x}_{i,h}\mathtt{x}_{i,g}\Big) \equiv 0 \pmod{u} \text{ for } 1 \le i \le \Lambda_{s-\kappa_1-1}. \tag{3.12}$$

Working as in Lemma 4.1 of Yadav and Sharma [32], we see that there exist vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{\Lambda_s}$ satisfying the following system of equations:

$$\mathtt{F}_{i,j}^{(1)} + \mathbf{b}_i\cdot\mathbf{x}_j + \mathbf{b}_j\cdot\mathbf{x}_i \equiv 0 \pmod{u} \quad \text{for } 1 \le i < j \le \Lambda_s, \tag{3.13}$$

$$(\mathtt{F}_{i,i}^{(2)})^{2^{-1}} + \mathbf{1}\cdot\mathbf{x}_i \equiv 0 \pmod{u} \quad \text{for } 1 \le i \le \Lambda_s, \tag{3.14}$$

and hence there exists a matrix $[\mathtt{V}^{(1)}]_s \in \mathcal{M}_{\Lambda_s\times n}(\mathcal{T}_m)$ satisfying (3.13) and (3.14). Moreover, such a matrix $[\mathtt{V}^{(1)}]_s \in \mathcal{M}_{\Lambda_s\times n}(\mathcal{T}_m)$ has precisely

$$(2^m)^{\sum\limits_{i=3}^{s+2}\lambda_i\Lambda_{i-2}+\Lambda_s(n-\Lambda_{s+2})-\Lambda_s-\frac{\Lambda_s(\Lambda_s-1)}{2}}$$

distinct choices. Next, for a given choice of the matrix $[\mathtt{V}^{(1)}]_s$ satisfying (3.13) and (3.14), we can write

$$\mathtt{F}^{(1)} + [\mathtt{W}^{(0)}]_s[\mathtt{V}^{(1)}]_s^t + [\mathtt{V}^{(1)}]_s[\mathtt{W}^{(0)}]_s^t \equiv uQ \pmod{u^2} \tag{3.15}$$

for some $Q \in \mathtt{Sym}_{\Lambda_s}(\mathcal{T}_m)$. This, by (3.9), implies that

$$Q + \mathtt{F}^{(2)} + [\mathtt{W}^{(0)}]_s[\mathtt{V}^{(2)}]_s^t + [\mathtt{V}^{(2)}]_s[\mathtt{W}^{(0)}]_s^t + [\mathtt{V}^{(1)}]_s[\mathtt{V}^{(1)}]_s^t \equiv \mathbf{0} \pmod{u}. \tag{3.16}$$

Note that $\mathcal{C}^{(s-\kappa_1-1)}$ is a doubly even code over $\mathcal{T}_m$. Further, if $\mathbf{1} \in \mathcal{C}^{(s-\kappa_1-1)}$, then by Theorem 3.2 of Yadav and Sharma [31], we see that $n \equiv 0, 4 \pmod 8$. Now, working as in Lemma 4.1 of Yadav and Sharma [31], we see that the matrix $[\mathtt{V}^{(2)}]_s$ satisfying (3.11), (3.12) and (3.16) has precisely

$$(2^m)^{\sum\limits_{i=4}^{s+2}\lambda_i\Lambda_{i-3}+\Lambda_s(n-\Lambda_{s+2})-\Lambda_{s-1}-\Lambda_{s-\kappa_1-1}-\frac{\Lambda_s(\Lambda_s-1)}{2}+\omega_3}$$

distinct choices, where $\omega_3 = 1$ if $\mathbf{1} \in \mathcal{C}^{(s-\kappa_1-1)}$, while $\omega_3 = 0$ otherwise. Further, working as in Proposition 4.2 of Yadav and Sharma [31], we see that there exist matrices $[\mathbf{0}\ \cdots\ \mathbf{0}\ \mathtt{B}_{s+1,s+2}^{(1)}\ \cdots\ \mathtt{B}_{s+1,e}^{(1)}]$ and $\mathtt{W}_{s+2}^{(3)}$ satisfying (3.7) and (3.8) and that such matrices $[\mathbf{0}\ \cdots\ \mathbf{0}\ \mathtt{B}_{s+1,s+2}^{(1)}\ \cdots\ \mathtt{B}_{s+1,e}^{(1)}]$ and $\mathtt{W}_{s+2}^{(3)}$ have precisely

$$(2^m)^{\lambda_{s+1}(n-\Lambda_{s+2}-\Lambda_s)}\begin{bmatrix}\lambda_{s+2}+n-\Lambda_{s+2}-\Lambda_s\\ \lambda_{s+2}\end{bmatrix}_{2^m}$$

distinct choices. From this, the desired result follows immediately. □

From this point on, we shall distinguish the following two cases: (i) $2\kappa \le e$, and (ii) $2\kappa > e$. The following proposition lifts a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$ to a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and the same length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$, where the integer ℓ satisfies $\ell \equiv e \pmod 2$ and $4 \le \ell \le \kappa$ if $2\kappa \le e$, while $4 \le \ell \le e-\kappa+1-2\theta_e$ if $2\kappa > e$. Additionally, it enumerates the number of possible ways to perform this lifting.

Proposition 3.3. *Let ℓ be a fixed integer satisfying $\ell \equiv e \pmod 2$ and $4 \le \ell \le \kappa$ if $2\kappa \le e$, while $4 \le \ell \le e-\kappa+1-2\theta_e$ if $2\kappa > e$. Let us define $\gamma_\ell = s-\mathrm{f}_\ell$. Let $\mathscr{D}_{\ell-2}$ be a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$. Let $\mathcal{C}^{(\gamma_\ell+1)}$ and $\mathcal{C}^{(\gamma_\ell+1-\kappa_1-\theta_e)}$ be linear subcodes of $Tor_1(\mathscr{D}_{\ell-2})$ such that $\mathcal{C}^{(\gamma_\ell+1-\kappa_1-\theta_e)} \subseteq \mathcal{C}^{(\gamma_\ell+1)}$, $\dim\mathcal{C}^{(\gamma_\ell+1)} = \Lambda_{\gamma_\ell+1}$, $\dim\mathcal{C}^{(\gamma_\ell+1-\kappa_1-\theta_e)} = \Lambda_{\gamma_\ell+1-\kappa_1-\theta_e}$ and the code $\mathcal{C}^{(\gamma_\ell+1-\kappa_1-\theta_e)}$ is doubly even, (note that such a pair of codes $\mathcal{C}^{(\gamma_\ell+1)}$ and $\mathcal{C}^{(\gamma_\ell+1-\kappa_1-\theta_e)}$ exists for each choice of $\mathscr{D}_{\ell-2}$, by Lemma 3.2). The following hold.*

- *(a) There exists a self-orthogonal code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{j+1}(\mathscr{D}_\ell) = Tor_j(\mathscr{D}_{\ell-2})$ for $1 \le j \le \ell-2$.*
- *(b) Moreover, each such choice of the codes $\mathscr{D}_{\ell-2}$, $\mathcal{C}^{(\gamma_\ell+1)}$ and $\mathcal{C}^{(\gamma_\ell+1-\kappa_1-\theta_e)}$) gives rise to precisely*

$$(2^m)^{\sum\limits_{i=\ell}^{\gamma_\ell+\ell}\lambda_i\Lambda_{i-\ell+1}+\sum\limits_{j=\ell+1}^{\gamma_\ell+\ell}\lambda_j\Lambda_{j-\ell}+Y_\ell}\begin{bmatrix}\lambda_{\gamma_\ell+\ell}+n-\Lambda_{\gamma_\ell+\ell}-\Lambda_{\gamma_\ell+1}\\ \lambda_{\gamma_\ell+\ell}\end{bmatrix}_{2^m}$$

distinct self-orthogonal codes $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{j+1}(\mathscr{D}_\ell) = Tor_j(\mathscr{D}_{\ell-2})$ for $1 \leq j \leq \ell-2$, where $Y_\ell = (\Lambda_{\gamma_\ell+\ell-1} + \Lambda_{\gamma_\ell+1})(n - \Lambda_{\gamma_\ell+\ell} - \Lambda_{\gamma_\ell+1}) + \Lambda^2_{\gamma_\ell+1} + \Lambda_{\gamma_\ell+1} - \Lambda_{s-2\mathrm{f}_\ell+2} - \Lambda_{s-2\mathrm{f}_\ell+1} - \Lambda_{\gamma_\ell+1-\kappa_1-\theta_e} + \omega_\ell$ with $\omega_\ell = 1$ if $\mathbf{1} \in \mathcal{C}^{(\gamma_\ell+1-\kappa_1-\theta_e)}$, while $\omega_\ell = 0$ otherwise.

Proof. To prove the result, we assume, without any loss of generality, that the code $\mathscr{D}_{\ell-2}$ has a generator matrix

$$\mathtt{G}_{\ell-2} = \begin{bmatrix} \mathtt{W}_1^{(\ell-2)} \\ \mathtt{W}_2^{(\ell-2)} \\ \vdots \\ \mathtt{W}_{\gamma_\ell+2}^{(\ell-2)} \\ u\mathtt{W}_{\gamma_\ell+3}^{(\ell-2)} \\ \vdots \\ u^{\ell-3}\mathtt{W}_{\gamma_\ell+\ell-1}^{(\ell-2)} \end{bmatrix} = \begin{bmatrix} \mathtt{W}_1^{(0)} + u\mathtt{V}_1^{(1)} + u^2\mathtt{V}_1^{(2)} + \cdots + u^{\ell-3}\mathtt{V}_1^{(\ell-3)} \\ \mathtt{W}_2^{(0)} + u\mathtt{V}_2^{(1)} + u^2\mathtt{V}_2^{(2)} + \cdots + u^{\ell-3}\mathtt{V}_2^{(\ell-3)} \\ \vdots \\ \mathtt{W}_{\gamma_\ell+2}^{(0)} + u\mathtt{V}_{\gamma_\ell+2}^{(1)} + u^2\mathtt{V}_{\gamma_\ell+2}^{(2)} + \cdots + u^{\ell-3}\mathtt{V}_{\gamma_\ell+2}^{(\ell-3)} \\ u\mathtt{W}_{\gamma_\ell+3}^{(\ell-2)} \\ \vdots \\ u^{\ell-3}\mathtt{W}_{\gamma_\ell+\ell-1}^{(\ell-2)} \end{bmatrix},$$

where

$$[\mathtt{W}^{(0)}]_{\gamma_\ell+2} = \begin{bmatrix} \mathtt{W}_1^{(0)} \\ \mathtt{W}_2^{(0)} \\ \vdots \\ \mathtt{W}_{\gamma_\ell+2}^{(0)} \end{bmatrix} = \begin{bmatrix} \mathtt{I}_{\lambda_1} & \mathtt{B}_{1,1}^{(0)} & \cdots & \mathtt{B}_{1,\gamma_\ell+1}^{(0)} & \cdots & \mathtt{B}_{1,e-1}^{(0)} & \mathtt{B}_{1,e}^{(0)} \\ \mathbf{0} & \mathtt{I}_{\lambda_2} & \cdots & \mathtt{B}_{2,\gamma_\ell+1}^{(0)} & \cdots & \mathtt{B}_{2,e-1}^{(0)} & \mathtt{B}_{2,e}^{(0)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathtt{I}_{\lambda_{\gamma_\ell+2}} & \cdots & \mathtt{B}_{\gamma_\ell+2,e-1}^{(0)} & \mathtt{B}_{\gamma_\ell+2,e}^{(0)} \end{bmatrix}$$

with $\mathtt{I}_{\lambda_i}$ as the $\lambda_i \times \lambda_i$ identity matrix over $\mathcal{T}_m$, $\mathtt{B}_{i,j}^{(0)} \in \mathcal{M}_{\lambda_i \times \lambda_{j+1}}(\mathcal{T}_m)$ for $1 \leq i \leq \gamma_\ell + 2$ and $i \leq j \leq e$, $[\mathtt{V}^{(a)}]_{\gamma_\ell+2} \in \mathcal{M}_{\Lambda_{\gamma_\ell+2}\times n}(\mathcal{T}_m)$ for $1 \leq a \leq \ell-3$, and the matrix $\mathtt{W}_{\gamma_\ell+g}^{(\ell-2)} \in \mathcal{M}_{\lambda_{\gamma_\ell+g}\times n}(\mathscr{R}_{\ell-2,m})$ is to be considered modulo $u^{\ell-g}$ for $3 \leq g \leq \ell-1$. Here, we note that the torsion code $Tor_1(\mathscr{D}_{\ell-2})$ is a $\Lambda_{\gamma_\ell+2}$-dimensional code over $\mathcal{T}_m$ with a generator matrix $[\mathtt{W}^{(0)}]_{\gamma_\ell+2}$. We further assume, without any loss of generality, that the code $\mathcal{C}^{(\gamma_\ell+1)}$ has a generator matrix $[\mathtt{W}^{(0)}]_{\gamma_\ell+1}$ and the code $\mathcal{C}^{(\gamma_\ell+1-\kappa_1-\theta_e)}$ has a generator matrix $[\mathtt{W}^{(0)}]_{\gamma_\ell+1-\kappa_1-\theta_e}$. Additionally, by Remark 3.1, we assume that the matrix $(\mathtt{B}^{(0)})_{\gamma_\ell+1,\gamma_\ell+\ell}$ is of full row-rank.

Now, as $\mathscr{D}_{\ell-2}$ is a self-orthogonal code over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$, we have

$$\begin{aligned} [\mathtt{W}^{(\ell-2)}]_{\gamma_\ell+1}[\mathtt{W}^{(\ell-2)}]^t_{\gamma_\ell+1} &\equiv \mathbf{0} \pmod{u^{\ell-2}}, \\ \mathtt{Diag}\Big([\mathtt{W}^{(\ell-2)}]_{\gamma_\ell+2-\mathrm{f}_i}[\mathtt{W}^{(\ell-2)}]^t_{\gamma_\ell+2-\mathrm{f}_i}\Big) &\equiv \mathbf{0} \pmod{u^{\ell-2+i}} \quad \text{for} \quad i \in \{2,4,6,\ldots,\ell-2-\theta_e\}, \\ \pi_{\ell-2+j}\Big(\mathtt{Diag}\big([\mathtt{W}^{(\ell-2)}]_{\gamma_\ell+2-\mathrm{f}_{j+2}}[\mathtt{W}^{(\ell-2)}]^t_{\gamma_\ell+2-\mathrm{f}_{j+2}}\big)\Big) &= \mathbf{0} \quad \text{for} \quad j \in \{\ell-3,\ell-1,\ldots,\kappa-2+\theta_e\}, \\ [\mathtt{W}^{(\ell-2)}]_{\gamma_\ell+1}\mathtt{W}^{(\ell-2)t}_{\gamma_\ell+g} &\equiv \mathbf{0} \pmod{u^{\ell-g}} \;\; \text{for } 2 \leq g \leq \ell-1, \\ \mathtt{W}^{(\ell-2)}_{\gamma_\ell+i}\mathtt{W}^{(\ell-2)t}_{\gamma_\ell+j} &\equiv \mathbf{0} \pmod{u^{\ell+2-i-j}} \;\; \text{for } 2 \leq i,j \leq \ell-1 \text{ and } i+j \leq \ell+1. \end{aligned}$$

This implies that

$$\begin{aligned} [\mathtt{W}^{(\ell-2)}]_{\gamma_\ell+1}[\mathtt{W}^{(\ell-2)}]^t_{\gamma_\ell+1} &\equiv u^{\ell-2}\mathtt{F}^{(\ell-2)} + u^{\ell-1}\mathtt{F}^{(\ell-1)} + \cdots + u^{\ell+\kappa-1}\mathtt{F}^{(\ell+\kappa-1)} \pmod{u^{\ell+\kappa}}, \\ [\mathtt{W}^{(\ell-2)}]_{\gamma_\ell+1}\mathtt{W}^{(\ell-2)t}_{\gamma_\ell+g} &\equiv u^{\ell-g}P_g \pmod{u^{\ell+1-g}} \;\; \text{for } 2 \leq g \leq \ell-1, \\ \mathtt{W}^{(\ell-2)}_{\gamma_\ell+i}\mathtt{W}^{(\ell-2)t}_{\gamma_\ell+j} &\equiv \mathbf{0} \pmod{u^{\ell+2-i-j}} \;\; \text{for } 2 \leq i,j \leq \ell-1 \text{ and } i+j \leq \ell+1, \end{aligned}$$

where $\mathtt{F}^{(\ell-2)}, \mathtt{F}^{(\ell-1)}, \ldots, \mathtt{F}^{(\ell+\kappa-1)} \in \mathtt{Sym}_{\Lambda_{\gamma_\ell+1}}(\mathcal{T}_m)$ with $\mathtt{F}_{h,h}^{(\ell-4+i)} = \mathtt{F}_{h,h}^{(\ell-3+i)} = 0$ for $1 \leq h \leq \Lambda_{\gamma_\ell+2-\mathrm{f}_i}$ and $i \in \{2,4,\ldots,\ell-2-\theta_e\}$ and $\mathtt{F}_{a,a}^{(\ell-2+j)} = 0$ for $1 \leq a \leq \Lambda_{\gamma_\ell+2-\mathrm{f}_{j+2}}$ and $j \in \{\ell-3,\ell-1,\ell+1,\ldots,\kappa-2+\theta_e\}$, and the matrix $P_g \in \mathcal{M}_{\Lambda_{\gamma_\ell+1}\times\lambda_{\gamma_\ell+g}}(\mathcal{T}_m)$ for $2 \leq g \leq \ell-1$.

Now, to establish the result, we define a matrix $\mathtt{G}_\ell$ over $\mathscr{R}_{\ell,m}$ as follows:

$$\mathtt{G}_\ell=\begin{bmatrix}\mathtt{W}_1^{(\ell)}\\ \mathtt{W}_2^{(\ell)}\\ \vdots\\ \mathtt{W}_{\gamma_\ell+1}^{(\ell)}\\ u\mathtt{W}_{\gamma_\ell+2}^{(\ell)}\\ \vdots\\ u^{\ell-1}\mathtt{W}_{\gamma_\ell+\ell}^{(\ell)}\end{bmatrix}=\begin{bmatrix}\mathtt{W}_1^{(\ell-2)}+u^{\ell-2}\mathtt{V}_1^{(\ell-2)}+u^{\ell-1}\mathtt{V}_1^{(\ell-1)}\\ \mathtt{W}_2^{(\ell-2)}+u^{\ell-2}\mathtt{V}_2^{(\ell-2)}+u^{\ell-1}\mathtt{V}_2^{(\ell-1)}\\ \vdots\\ \mathtt{W}_{\gamma_\ell+1}^{(\ell-2)}+u^{\ell-2}\mathtt{V}_{\gamma_\ell+1}^{(\ell-2)}+u^{\ell-1}\mathtt{V}_{\gamma_\ell+1}^{(\ell-1)}\\ u\mathtt{W}_{\gamma_\ell+2}^{(\ell)}\\ \vdots\\ u^{\ell-1}\mathtt{W}_{\gamma_\ell+\ell}^{(\ell)}\end{bmatrix},$$

where the matrices $[\mathtt{V}^{(\mu)}]_{\gamma_\ell+1}\in\mathcal{M}_{\Lambda_{\gamma_\ell+1}\times n}(\mathcal{T}_m)$ for $\mu\in\{\ell-2,\ell-1\}$, $\mathtt{W}_{\gamma_\ell+g}^{(\ell)}\in\mathcal{M}_{\lambda_{\gamma_\ell+g}\times n}(\mathscr{R}_{\ell,m})$ for $2\le g\le\ell-1$ and $\mathtt{W}_{\gamma_\ell+\ell}^{(\ell)}\in\mathcal{M}_{\lambda_{\gamma_\ell+\ell}\times n}(\mathcal{T}_m)$ are of the forms

$$\begin{bmatrix}\mathtt{V}_1^{(\mu)}\\ \mathtt{V}_2^{(\mu)}\\ \vdots\\ \mathtt{V}_{\gamma_\ell+1}^{(\mu)}\end{bmatrix}=\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}&\mathtt{B}_{1,\mu+1}^{(\mu)}&\mathtt{B}_{1,\mu+2}^{(\mu)}&\cdots&\mathtt{B}_{1,\gamma_\ell+1+\mu}^{(\mu)}&\cdots&\mathtt{B}_{1,e}^{(\mu)}\\ \mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\mathtt{B}_{2,\mu+2}^{(\mu)}&\cdots&\mathtt{B}_{2,\gamma_\ell+1+\mu}^{(\mu)}&\cdots&\mathtt{B}_{2,e}^{(\mu)}\\ \vdots&\cdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&\mathtt{B}_{\gamma_\ell+1,\gamma_\ell+1+\mu}^{(\mu)}&\cdots&\mathtt{B}_{\gamma_\ell+1,e}^{(\mu)}\end{bmatrix},$$

$$\mathtt{W}_{\gamma_\ell+g}^{(\ell)}=\mathtt{W}_{\gamma_\ell+g}^{(\ell-2)}+u^{\ell-g}\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}&\mathtt{B}_{\gamma_\ell+g,\gamma_\ell+\ell}^{(\ell-g)}&\cdots&\mathtt{B}_{\gamma_\ell+g,e}^{(\ell-g)}\end{bmatrix}\text{ and}$$

$$\mathtt{W}_{\gamma_\ell+\ell}^{(\ell)}=\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}&\mathtt{I}_{\lambda_{\gamma_\ell+\ell}}&\mathtt{B}_{\gamma_\ell+\ell,\gamma_\ell+\ell}^{(0)}&\cdots&\mathtt{B}_{\gamma_\ell+\ell,e}^{(0)}\end{bmatrix}$$

with $\mathtt{B}_{i,j}^{(\mu)}\in\mathcal{M}_{\lambda_i\times\lambda_{j+1}}(\mathcal{T}_m)$ for $1\le i\le\gamma_\ell+1$ and $i+\mu\le j\le e$, $\mathtt{B}_{\gamma_\ell+g,h}^{(\ell-g)}\in\mathcal{M}_{\lambda_{\gamma_\ell+g}\times\lambda_{h+1}}(\mathcal{T}_m)$ for $\gamma_\ell+\ell\le h\le e$ and $\mathtt{B}_{\gamma_\ell+\ell,a}^{(0)}\in\mathcal{M}_{\lambda_{\gamma_\ell+\ell}\times\lambda_{a+1}}(\mathcal{T}_m)$ for $\gamma_\ell+\ell\le a\le e$. Next, let $\mathscr{D}_\ell$ be a linear code of length n over $\mathscr{R}_{\ell,m}$ with a generator matrix $\mathtt{G}_\ell$. Note that the code $\mathscr{D}_\ell$ is of type $\{\Lambda_{\gamma_\ell+1},\lambda_{\gamma_\ell+2},\ldots,\lambda_{\gamma_\ell+\ell}\}$ over $\mathscr{R}_{\ell,m}$ and satisfies $Tor_1(\mathscr{D}_\ell)=\mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{j+1}(\mathscr{D}_\ell)=Tor_j(\mathscr{D}_{\ell-2})$ for $1\le j\le\ell-2$. By Lemma 2.1, we see that the code $\mathscr{D}_\ell$ is a self-orthogonal code over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ if and only if there exist matrices $[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1}$, $[\mathtt{V}^{(\ell-1)}]_{\gamma_\ell+1}$, $\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}\ \mathtt{B}_{\gamma_\ell+g,\gamma_\ell+\ell}^{(\ell-g)}&\cdots&\mathtt{B}_{\gamma_\ell+g,e}^{(\ell-g)}\end{bmatrix}$ for $2\le g\le\ell-1$ and $\mathtt{W}_{\gamma_\ell+\ell}^{(\ell)}$ satisfying the following system of matrix equations:

$$\begin{aligned}\mathtt{F}^{(\ell-2)}+[\mathtt{W}^{(0)}]_{\gamma_\ell+1}[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1}^t+[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1}[\mathtt{W}^{(0)}]_{\gamma_\ell+1}^t+u\Big([\mathtt{W}^{(0)}]_{\gamma_\ell+1}[\mathtt{V}^{(\ell-1)}]_{\gamma_\ell+1}^t&\\ +[\mathtt{V}^{(\ell-1)}]_{\gamma_\ell+1}[\mathtt{W}^{(0)}]_{\gamma_\ell+1}^t+[\mathtt{V}^{(1)}]_{\gamma_\ell+1}[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1}^t+[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1}[\mathtt{V}^{(1)}]_{\gamma_\ell+1}^t+\mathtt{F}^{(\ell-1)}\Big)&\equiv\mathbf{0}\pmod{u^2}, &(3.17)\\ \mathtt{Diag}\Big(\mathtt{E}^{(2\ell-4)}+[\mathtt{V}^{(\ell-2)}]_{s-2\mathrm{f}_\ell+2}[\mathtt{V}^{(\ell-2)}]_{s-2\mathrm{f}_\ell+2}^t\Big)&\equiv\mathbf{0}\pmod{u}, &(3.18)\\ \mathtt{Diag}\Big(\mathtt{E}^{(2\ell-2)}+[\mathtt{V}^{(\ell-1)}]_{s-2\mathrm{f}_\ell+1}[\mathtt{V}^{(\ell-1)}]_{s-2\mathrm{f}_\ell+1}^t\Big)&\equiv\mathbf{0}\pmod{u}, &(3.19)\\ \mathtt{Diag}\Big(\mathtt{E}^{(\ell+\kappa-1)}+\eta_1[\mathtt{W}^{(0)}]_{\gamma_\ell-\kappa_1}[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell-\kappa_1}^t&\\ +\eta_0\Big([\mathtt{W}^{(0)}]_{\gamma_\ell-\kappa_1}[\mathtt{V}^{(\ell-1)}]_{\gamma_\ell-\kappa_1}^t+[\mathtt{V}^{(1)}]_{\gamma_\ell-\kappa_1}[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell-\kappa_1}^t\Big)\Big)&\equiv\mathbf{0}\pmod{u}\text{ if }\theta_e=1, &(3.20)\\ \mathtt{Diag}\Big(\mathtt{E}^{(\ell+\kappa-2)}+\eta_0[\mathtt{W}^{(0)}]_{\gamma_\ell+1-\kappa_1}[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1-\kappa_1}^t\Big)&\equiv\mathbf{0}\pmod{u}\text{ if }\theta_e=0, &(3.21)\\ P_g+[\mathtt{W}^{(0)}]_{\gamma_\ell+1}\begin{bmatrix}\mathbf{0}&\cdots&\mathbf{0}\ \mathtt{B}_{\gamma_\ell+g,\gamma_\ell+\ell}^{(\ell-g)}&\cdots&\mathtt{B}_{\gamma_\ell+g,e}^{(\ell-g)}\end{bmatrix}^t&\equiv\mathbf{0}\pmod{u}, &(3.22)\\ [\mathtt{W}^{(0)}]_{\gamma_\ell+1}\mathtt{W}_{\gamma_\ell+\ell}^{(\ell)t}&\equiv\mathbf{0}\pmod{u}, &(3.23)\end{aligned}$$

where the matrices $\mathtt{E}^{(2\ell-4)}$, $\mathtt{E}^{(2\ell-2)}$ and $\mathtt{E}^{(\ell+\kappa-2+\theta_e)}$ are of orders $\Lambda_{s-2\mathrm{f}_\ell+2}\times\Lambda_{s-2\mathrm{f}_\ell+2}$, $\Lambda_{s-2\mathrm{f}_\ell+1}\times\Lambda_{s-2\mathrm{f}_\ell+1}$ and $\Lambda_{\gamma_\ell+1-\kappa_1-\theta_e}\times\Lambda_{\gamma_\ell+1-\kappa_1-\theta_e}$ over $\mathcal{T}_m$ whose rows are the first $\Lambda_{s-2\mathrm{f}_\ell+2}$, $\Lambda_{s-2\mathrm{f}_\ell+1}$ and $\Lambda_{\gamma_\ell+1-\kappa_1-\theta_e}$ rows of the matrices $\mathtt{F}^{(2\ell-4)}$, $\mathtt{F}^{(2\ell-2)}$ and $\mathtt{F}^{(\ell+\kappa-2+\theta_e)}$, respectively.

First of all, we will establish the existence of the matrices $[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1}$ and $[\mathtt{V}^{(\ell-1)}]_{\gamma_\ell+1}$ satisfying the system of matrix equations (3.17)-(3.21). To do this, we will distinguish the following two cases: (I) ℓ is even, and (II) ℓ is odd.

(I) Let ℓ be even. In this case, we will first show that there exists a matrix $[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1} \in \mathcal{M}_{\Lambda_{\gamma_\ell+1}\times n}(\mathcal{T}_m)$ satisfying the following system of matrix equations:

$$\mathtt{F}^{(\ell-2)} + [\mathtt{W}^{(0)}]_{\gamma_\ell+1}[\mathtt{V}^{(\ell-2)}]^t_{\gamma_\ell+1} + [\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1}[\mathtt{W}^{(0)}]^t_{\gamma_\ell+1} \equiv \mathbf{0} \pmod{u}, \tag{3.24}$$

$$\mathtt{Diag}\left(\mathtt{E}^{(2\ell-4)} + [\mathtt{V}^{(\ell-2)}]_{s-2\mathrm{f}_\ell+2}[\mathtt{V}^{(\ell-2)}]^t_{s-2\mathrm{f}_\ell+2}\right) \equiv \mathbf{0} \pmod{u}, \tag{3.25}$$

$$\mathtt{Diag}\left(\mathtt{E}^{(\ell+\kappa-2)} + \eta_0[\mathtt{W}^{(0)}]_{\gamma_\ell+1-\kappa_1}[\mathtt{V}^{(\ell-2)}]^t_{\gamma_\ell+1-\kappa_1}\right) \equiv \mathbf{0} \pmod{u}. \tag{3.26}$$

Towards this, let $[\mathtt{W}^{(0)}]_{\gamma_\ell+1} = (\mathbf{b}_i)$ and $[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1} = (\mathbf{v}_j)$, where $\mathbf{b}_i$ and $\mathbf{v}_j$ denote the i-th and j-th rows of the matrices $[\mathtt{W}^{(0)}]_{\gamma_\ell+1}$ and $[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1}$, respectively, for each i and j. Further, let $\mathtt{F}^{(\ell-2)}_{i,j}, \mathtt{E}^{(2\ell-4)}_{i,j}, \mathtt{E}^{(\ell+\kappa-2)}_{i,j} \in \mathcal{T}_m$ denote the (i,j)-th entries of the matrices $\mathtt{F}^{(\ell-2)}, \mathtt{E}^{(2\ell-4)}, \mathtt{E}^{(\ell+\kappa-2)}$, respectively. Thus, the system of matrix equations (3.24)-(3.26) is equivalent to the following system of equations in unknowns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{\gamma_\ell+1}}$ over $\mathcal{T}_m$:

$$\mathbf{b}_i \cdot \mathbf{v}_j + \mathbf{b}_j \cdot \mathbf{v}_i \equiv \mathtt{F}^{(\ell-2)}_{i,j} \pmod{u} \quad \text{for } 1 \le i < j \le \Lambda_{\gamma_\ell+1}, \tag{3.27}$$

$$\mathbf{1} \cdot \mathbf{v}_i \equiv (\mathtt{E}^{(2\ell-4)}_{i,i})^{2^{m-1}} \pmod{u} \quad \text{for } 1 \le i \le \Lambda_{s-2\mathrm{f}_\ell+2}, \tag{3.28}$$

$$\mathbf{b}_i \cdot \mathbf{v}_i \equiv \eta_0^{-1}\mathtt{E}^{(\ell+\kappa-2)}_{i,i} \pmod{u} \quad \text{for } 1 \le i \le \Lambda_{\gamma_\ell+1-\kappa_1}. \tag{3.29}$$

For each integer j satisfying $1 \le j \le \Lambda_{\gamma_\ell+1}$, one can easily see that there exists a unique integer r_j satisfying $1 \le r_j \le \gamma_\ell + 1$ and $\Lambda_{r_j-1} + 1 \le j \le \Lambda_{r_j}$. The corresponding unknown vector $\mathbf{v}_j$ can be written as $\mathbf{v}_j = (\mathbf{0}\ \mathbf{v}_j^{n-\Lambda_{r_j+\ell-2}})$, where $\mathbf{0}$ denotes the zero vector of length $\Lambda_{r_j+\ell-2}$ and $\mathbf{v}_j^{n-\Lambda_{r_j+\ell-2}}$ denotes the vector of length $n - \Lambda_{r_j+\ell-2}$ obtained by omitting the first $\Lambda_{r_j+\ell-2}$ coordinates of $\mathbf{v}_j$. This implies, for $\Lambda_{r_j-1} + 1 \le j \le \Lambda_{r_j}$, that the first $\Lambda_{r_j+\ell-2}$ coordinates of $\mathbf{v}_j$ are zero, leaving $n - \Lambda_{r_j+\ell-2}$ variables in $\mathbf{v}_j$. For $1 \le j \le \Lambda_{\gamma_\ell+1}$, let $\widehat{\mathbf{v}}_j = \mathbf{v}_j^{n-\Lambda_{r_j+\ell-2}}$ and $\widehat{\mathbf{b}}_j = \mathbf{b}_j^{n-\Lambda_{r_j+\ell-2}}$ be the vectors of length $n - \Lambda_{r_j+\ell-2}$ obtained from $\mathbf{v}_j$ and $\mathbf{b}_j$ by omitting their first $\Lambda_{r_j+\ell-2}$ coordinates, respectively. Consequently, the system of equations (3.27)-(3.29) is equivalent to the following system of equations in unknowns $\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \ldots, \widehat{\mathbf{v}}_{\Lambda_{\gamma_\ell+1}}$ over $\mathcal{T}_m$:

$$\widehat{\mathbf{b}}_i \cdot \widehat{\mathbf{v}}_j + \widehat{\mathbf{b}}_j \cdot \widehat{\mathbf{v}}_i \equiv \mathtt{F}^{(\ell-2)}_{i,j} \pmod{u} \quad \text{for } 1 \le i < j \le \Lambda_{\gamma_\ell+1}, \tag{3.30}$$

$$\widehat{\mathbf{1}} \cdot \widehat{\mathbf{v}}_i \equiv (\mathtt{E}^{(2\ell-4)}_{i,i})^{2^{m-1}} \pmod{u} \quad \text{for } 1 \le i \le \Lambda_{s-2\mathrm{f}_\ell+2}, \tag{3.31}$$

$$\widehat{\mathbf{b}}_i \cdot \widehat{\mathbf{v}}_i \equiv \eta_0^{-1}\mathtt{E}^{(\ell+\kappa-2)}_{i,i} \pmod{u} \quad \text{for } 1 \le i \le \Lambda_{\gamma_\ell+1-\kappa_1}, \tag{3.32}$$

where $\widehat{\mathbf{1}}$ denotes the all-one vector having the same length as that of the vector $\widehat{\mathbf{v}}_i$ for each i. We further express the above system of equations (3.30)-(3.32) in the matrix form as

$$\mathcal{N}\begin{bmatrix} \widehat{\mathbf{v}}_1^t \\ \widehat{\mathbf{v}}_2^t \\ \widehat{\mathbf{v}}_3^t \\ \vdots \\ \vdots \\ \widehat{\mathbf{v}}^t_{\Lambda_{\gamma_\ell+1-\kappa_1}} \\ \vdots \\ \widehat{\mathbf{v}}^t_{\Lambda_{s-2\mathrm{f}_\ell+2}+1} \\ \vdots \\ \widehat{\mathbf{v}}^t_{\Lambda_{\gamma_\ell+1}-2} \\ \widehat{\mathbf{v}}^t_{\Lambda_{\gamma_\ell+1}-1} \\ \widehat{\mathbf{v}}^t_{\Lambda_{\gamma_\ell+1}} \end{bmatrix} \equiv \begin{bmatrix} (\mathtt{E}^{(2\ell-4)}_{1,1})^{2^{m-1}} \\ \vdots \\ (\mathtt{E}^{(2\ell-4)}_{\Lambda_{s-2\mathrm{f}_\ell+2},\Lambda_{s-2\mathrm{f}_\ell+2}})^{2^{m-1}} \\ \eta_0^{-1}\mathtt{E}^{(\ell+\kappa-2)}_{1,1} \\ \vdots \\ \eta_0^{-1}\mathtt{E}^{(\ell+\kappa-2)}_{\Lambda_{\gamma_\ell+1-\kappa_1},\Lambda_{\gamma_\ell+1-\kappa_1}} \\ \mathtt{F}^{(\ell-2)}_{1,2} \\ \vdots \\ \mathtt{F}^{(\ell-2)}_{1,\Lambda_{\gamma_\ell+1}} \\ \vdots \\ \mathtt{F}^{(\ell-2)}_{\Lambda_{\gamma_\ell+1}-1,\Lambda_{\gamma_\ell+1}} \end{bmatrix} \pmod{u}, \tag{3.33}$$

where the matrix $\mathcal{N}$ of order $\left(\Lambda_{s-2\mathrm{f}_\ell+2} + \Lambda_{\gamma_\ell+1-\kappa_1} + \frac{\Lambda_{\gamma_\ell+1}(\Lambda_{\gamma_\ell+1}-1)}{2}\right) \times \left(\sum_{i=\ell}^{\gamma_\ell+\ell} \lambda_i\Lambda_{i-\ell+1} + \Lambda_{\gamma_\ell+1}(n - \Lambda_{\gamma_\ell+\ell})\right)$

is given by

$$\mathcal{N} = \left[\begin{array}{ccccccccc}
\widehat{\mathbf{1}} & & & & & & & & \\
 & \widehat{\mathbf{1}} & & & & & & & \\
 & & \ddots & & & & & & \\
 & & & \widehat{\mathbf{1}} & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & \widehat{\mathbf{1}} & & & \\
\widehat{\mathbf{b}}_1 & & & & & & & & \\
 & \widehat{\mathbf{b}}_2 & & & & & & & \\
 & & \ddots & & & & & & \\
 & & & \widehat{\mathbf{b}}_{\Lambda_{\gamma_\ell+1-\kappa_1}} & & & & & \\
\widehat{\mathbf{b}}_2 & \widehat{\mathbf{b}}_1 & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & & \\
\widehat{\mathbf{b}}_{\Lambda_{\gamma_\ell+1}} & & & & & & \cdots & & \widehat{\mathbf{b}}_1 \\
 & & & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
 & & & & & & \cdots & \widehat{\mathbf{b}}_{\Lambda_{\gamma_\ell+1}} & \widehat{\mathbf{b}}_{\Lambda_{\gamma_\ell+1}-1}
\end{array}\right].$$

Furthermore, working as in Lemma 4.1 of Yadav and Sharma [31], we observe that the rows of the matrix $\mathcal{N}$ are linearly independent over $\mathcal{T}_m$ if and only if $\mathbf{1} \notin \mathcal{C}^{(\gamma_\ell+1-\kappa_1)}$. Accordingly, we will consider the following two cases separately: (i) $\mathbf{1} \notin \mathcal{C}^{(\gamma_\ell+1-\kappa_1)}$, and (ii) $\mathbf{1} \in \mathcal{C}^{(\gamma_\ell+1-\kappa_1)}$.

(i) Let $\mathbf{1} \notin \mathcal{C}^{(\gamma_\ell+1-\kappa_1)}$. In this case, the rows of the matrix $\mathcal{N}$ are linearly independent over $\mathcal{T}_m$, and hence the row-rank of the matrix $\mathcal{N}$ is $\Lambda_{s-2\mathtt{f}_\ell+2}+\Lambda_{\gamma_\ell+1-\kappa_1}+\frac{\Lambda_{\gamma_\ell+1}(\Lambda_{\gamma_\ell+1}-1)}{2}$. Thus, the matrix equation (3.33) always has a solution. In fact, the number of solutions of the matrix equation (3.33), and hence the number of choices for the matrix $[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1} \in \mathcal{M}_{\Lambda_{\gamma_\ell+1}\times n}(\mathcal{T}_m)$ is given by

$$(2^m)^{\sum\limits_{i=\ell}^{\gamma_\ell+\ell} \lambda_i\Lambda_{i-\ell+1}+\Lambda_{\gamma_\ell+1}(n-\Lambda_{\gamma_\ell+\ell})-\Lambda_{s-2\mathtt{f}_\ell+2}-\Lambda_{\gamma_\ell+1-\kappa_1}-\frac{\Lambda_{\gamma_\ell+1}(\Lambda_{\gamma_\ell+1}-1)}{2}}.$$

(ii) Now, let us assume that $\mathbf{1} \in \mathcal{C}^{(\gamma_\ell+1-\kappa_1)}$. In this case, $\mathbf{1}$ belongs to $\mathcal{T}_m$-span of the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{\Lambda_{\gamma_\ell+1-\kappa_1}}$. Since the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{\Lambda_{\gamma_\ell+1-\kappa_1}}$ are linearly independent over $\mathcal{T}_m$, there exist unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_{\Lambda_{\gamma_\ell+1-\kappa_1}} \in \mathcal{T}_m$ such that $\alpha_1\mathbf{b}_1+\alpha_2\mathbf{b}_2+\cdots+\alpha_{\Lambda_{\gamma_\ell+1-\kappa_1}}\mathbf{b}_{\Lambda_{\gamma_\ell+1-\kappa_1}} \equiv \mathbf{1} \pmod{u}$. Moreover, all the rows of the matrix $\mathcal{N}$ except the last row are linearly independent over $\mathcal{T}_m$. This implies that the row-rank of the matrix $\mathcal{N}$ is $\Lambda_{\gamma_\ell+1-\kappa_1}+\Lambda_{s-2\mathtt{f}_\ell+2}+\frac{\Lambda_{\gamma_\ell+1}(\Lambda_{\gamma_\ell+1}-1)}{2}-1$. Further, one can easily see that the matrix equation (3.33) has a solution if and only if

$$\sum_{h=1}^{\Lambda_{\gamma_\ell+1-\kappa_1}} \alpha_h(\mathtt{E}_{h,h}^{(2\ell-4)})^{2^{m-1}}+\eta_0^{-1}\sum_{g=1}^{\Lambda_{\gamma_\ell+1-\kappa_1}} \alpha_g^2\mathtt{E}_{g,g}^{(\ell+\kappa-2)}+\sum_{1\le i<j\le\Lambda_{\gamma_\ell+1-\kappa_1}} \alpha_i\alpha_j\mathtt{F}_{i,j}^{(\ell-2)} \equiv 0 \pmod{u}. \quad (3.34)$$

As $\mathcal{C}^{(\gamma_\ell+1-\kappa_1)}$ is a doubly even code over $\mathcal{T}_m$ satisfying $\mathbf{1} \in \mathcal{C}^{(\gamma_\ell+1-\kappa_1)}$, we see, by Theorem 3.2 of Yadav and Sharma [31], that $n \equiv 0, 4 \pmod{8}$. We next assert that the equation (3.34) holds (*i.e.,* the matrix equation (3.33) always has a solution). To prove this assertion, let $[\mathtt{V}^{(a)}]_{\gamma_\ell+1} = (\mathbf{v}_j^{(a)})$ for $1 \le a \le \ell-3$, where $\mathbf{v}_j^{(a)}$ denotes the j-th row of the matrix $[\mathtt{V}^{(a)}]_{\gamma_\ell+1}$ for each j and a. Further, let us write

$$\mathbf{1} \equiv \sum_{j=1}^{\Lambda_{\gamma_\ell+1-\kappa_1}} \alpha_j\left(\mathbf{b}_j+u\mathbf{v}_j^{(1)}+\cdots+u^{\ell-3}\mathbf{v}_j^{(\ell-3)}\right)+u^{\ell-2}\mathbf{d}_{\ell-2}+\cdots+u^{\ell+\kappa-2}\mathbf{d}_{\ell+\kappa-2} \pmod{u^{\ell+\kappa-1}}$$

for some $\mathbf{d}_{\ell-2}, \mathbf{d}_{\ell-1}, \ldots, \mathbf{d}_{\ell+\kappa-2} \in \mathcal{T}_m^n$. Note that $\mathbf{1}\cdot\mathbf{1} \equiv n \pmod{u^{\ell+\kappa-1}}$. As $n \equiv 0, 4 \pmod{8}$, we must have $\mathbf{1}\cdot\mathbf{1} \equiv 0 \pmod{u^{\ell+\kappa-1}}$. From this, we get

$$\sum_{h=1}^{\Lambda_{\gamma_\ell+1-\kappa_1}} \alpha_h^2\mathtt{E}_{h,h}^{(2\ell-4)}+\mathbf{d}_{\ell-2}\cdot\mathbf{d}_{\ell-2} \equiv 0 \pmod{u} \quad (3.35)$$

and

$$\eta_0^{-1}\sum_{g=1}^{\Lambda_{\gamma_\ell+1-\kappa_1}}\alpha_g^2\mathtt{E}_{g,g}^{(\ell+\kappa-2)}+\sum_{1\le i<j\le\Lambda_{\gamma_\ell+1-\kappa_1}}\alpha_i\alpha_j\mathtt{F}_{i,j}^{(\ell-2)}+\mathbf{1}\cdot\mathbf{d}_{\ell-2}\equiv 0 \pmod{u}. \tag{3.36}$$

By (3.35) and (3.36), we see that the equation (3.34) holds, and hence the matrix equation (3.33) always has a solution. Further, the number of solutions of the matrix equation (3.33), and hence the number of choices for the matrix $[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1}\in\mathcal{M}_{\Lambda_{\gamma_\ell+1}\times n}(\mathcal{T}_m)$ is given by

$$(2^m)^{\sum\limits_{i=\ell}^{\gamma_\ell+\ell}\lambda_i\Lambda_{i-\ell+1}+\Lambda_{\gamma_\ell+1}(n-\Lambda_{\gamma_\ell+\ell})-\Lambda_{s-2\mathfrak{f}_\ell+2}-\Lambda_{\gamma_\ell+1-\kappa_1}-\frac{\Lambda_{\gamma_\ell+1}(\Lambda_{\gamma_\ell+1}-1)}{2}+1}.$$

Next, for a given choice of the matrix $[V^{(\ell-2)}]_{\gamma_\ell+1}$ satisfying (3.24)-(3.26), we can write

$$\mathtt{F}^{(\ell-2)}+[\mathtt{W}^{(0)}]_{\gamma_\ell+1}[\mathtt{V}^{(\ell-2)}]^t_{\gamma_\ell+1}+[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1}[\mathtt{W}^{(0)}]^t_{\gamma_\ell+1}\ \equiv\ uQ_1 \pmod{u^2} \tag{3.37}$$

for some $Q_1\in\mathtt{Sym}_{\Lambda_{\gamma_\ell+1}}(\mathcal{T}_m)$. This, by (3.17), implies that

$$\begin{aligned}Q_1+\mathtt{F}^{(\ell-1)}&+[\mathtt{W}^{(0)}]_{\gamma_\ell+1}[\mathtt{V}^{(\ell-1)}]^t_{\gamma_\ell+1}+[\mathtt{V}^{(\ell-1)}]_{\gamma_\ell+1}[\mathtt{W}^{(0)}]^t_{\gamma_\ell+1}\\&+[\mathtt{V}^{(1)}]_{\gamma_\ell+1}[\mathtt{V}^{(\ell-2)}]^t_{\gamma_\ell+1}+[\mathtt{V}^{(\ell-2)}]_{\gamma_\ell+1}[\mathtt{V}^{(1)}]^t_{\gamma_\ell+1}\ \equiv\ \mathbf{0} \pmod{u}.\end{aligned} \tag{3.38}$$

Now, working as in Lemma 4.1 of Yadav and Sharma [32], we observe that there exists a matrix $[\mathtt{V}^{(\ell-1)}]_{\gamma_\ell+1}$ satisfying (3.19) and (3.38). Moreover, the matrix $[\mathtt{V}^{(\ell-1)}]_{\gamma_\ell+1}$ satisfying (3.19) and (3.38) has precisely

$$(2^m)^{\sum\limits_{i=\ell+1}^{\gamma_\ell+\ell}\lambda_i\Lambda_{i-\ell}+\Lambda_{\gamma_\ell+1}(n-\Lambda_{\gamma_\ell+\ell})-\Lambda_{s-2\mathfrak{f}_\ell+1}-\frac{\Lambda_{\gamma_\ell+1}(\Lambda_{\gamma_\ell+1}-1)}{2}}$$

distinct choices. Further, working as in Proposition 4.3 of Yadav and Sharma [31], we see that the matrices $\begin{bmatrix}0 & \cdots & 0 & \mathtt{B}^{(\ell-g)}_{\gamma_\ell+g,\gamma_\ell+\ell} & \cdots & \mathtt{B}^{(\ell-g)}_{\gamma_\ell+g,e}\end{bmatrix}$ for $2\le g\le\ell-1$ and $\mathtt{W}^{(\ell)}_{\gamma_\ell+\ell}$ satisfying (3.22) and (3.23) have precisely

$$(2^m)^{(\Lambda_{\gamma_\ell+\ell-1}-\Lambda_{\gamma_\ell+1})(n-\Lambda_{\gamma_\ell+\ell}-\Lambda_{\gamma_\ell+1})}\begin{bmatrix}\lambda_{\gamma_\ell+\ell}+n-\Lambda_{\gamma_\ell+\ell}-\Lambda_{\gamma_\ell+1}\\ \lambda_{\gamma_\ell+\ell}\end{bmatrix}_{2^m}$$

distinct choices. From this, the desired result follows in the case when ℓ is even.

(II) When ℓ is odd, the desired result follows by working as in case (I). □

The following proposition lifts a self-orthogonal code of type $\{\Lambda_{s-\kappa_1+1},\lambda_{s-\kappa_1+2},\ldots,\lambda_{s+\kappa_1}\}$ and length n over $\mathscr{R}_{\kappa-1,m}$ satisfying property ($\mathfrak{P}$) to a self-orthogonal code of type $\{\Lambda_{s-\kappa_1},\lambda_{s-\kappa_1+1},\ldots,\lambda_{s+\kappa_1+1}\}$ and the same length n over $\mathscr{R}_{\kappa+1,m}$ satisfying property ($\mathfrak{P}$) when e is even and $2\kappa\le e$. It also enumerates the number of possible ways to perform this lifting.

Proposition 3.4. *Let e be even, $2\kappa\le e$, and let $\ell=\kappa+1$. Let $\mathscr{D}_{\kappa-1}$ be a self-orthogonal code of type $\{\Lambda_{s-\kappa_1+1},\lambda_{s-\kappa_1+2},\ldots,\lambda_{s+\kappa_1}\}$ and length n over $\mathscr{R}_{\kappa-1,m}$ satisfying property ($\mathfrak{P}$). Let $\mathcal{C}^{(s-\kappa)}\subseteq\mathcal{C}^{(s-\kappa+1)}\subseteq\mathcal{C}^{(s-\kappa_1)}$ be a chain of linear subcodes of $Tor_1(\mathscr{D}_{\kappa-1})$, such that $\dim\mathcal{C}^{(s-\kappa)}=\Lambda_{s-\kappa}$, $\dim\mathcal{C}^{(s-\kappa+1)}=\Lambda_{s-\kappa+1}$, $\dim\mathcal{C}^{(s-\kappa_1)}=\Lambda_{s-\kappa_1}$ and the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, (such a chain $\mathcal{C}^{(s-\kappa)}\subseteq\mathcal{C}^{(s-\kappa+1)}\subseteq\mathcal{C}^{(s-\kappa_1)}$ of codes exists for each choice of $\mathscr{D}_{\kappa-1}$, by Lemma 3.2). The following hold.*

(a) There exists a self-orthogonal code $\mathscr{D}_{\kappa+1}$ of type $\{\Lambda_{s-\kappa_1},\lambda_{s-\kappa_1+1},\ldots,\lambda_{s+\kappa_1+1}\}$ and length n over $\mathscr{R}_{\kappa+1,m}$ satisfying property ($\mathfrak{P}$) with $Tor_1(\mathscr{D}_{\kappa+1})=\mathcal{C}^{(s-\kappa_1)}$ and $Tor_{j+1}(\mathscr{D}_{\kappa+1})=Tor_j(\mathscr{D}_{\kappa-1})$ for $1\le j\le\kappa-1$, where such a code exists:

- *unconditionally, if $\mathbf{1}\notin\mathcal{C}^{(s-\kappa)}$;*
- *if and only if either $n\equiv 0 \pmod 8$ or $n\equiv 4 \pmod 8$ with m even when $\mathbf{1}\in\mathcal{C}^{(s-\kappa)}$.*

(b) Moreover, each such choice of the codes $\mathscr{D}_{\kappa-1}$, $\mathcal{C}^{(s-\kappa_1)}$, $\mathcal{C}^{(s-\kappa+1)}$ and $\mathcal{C}^{(s-\kappa)}$ gives rise to precisely

$$2^{\epsilon}(2^m)^{\sum\limits_{i=\kappa+1}^{s+\kappa_1+1}\lambda_i\Lambda_{i-\kappa}+\sum\limits_{j=\kappa+2}^{s+\kappa_1+1}\lambda_j\Lambda_{j-\kappa-1}+Y_{\kappa+1}}\begin{bmatrix}\lambda_{s+\kappa_1+1}+n-\Lambda_{s+\kappa_1+1}-\Lambda_{s-\kappa_1}\\ \lambda_{s+\kappa_1+1}\end{bmatrix}_{2^m}$$

distinct self-orthogonal codes $\mathscr{D}_{\kappa+1}$ of type $\{\Lambda_{s-\kappa_1},\lambda_{s-\kappa_1+1},\ldots,\lambda_{s+\kappa_1+1}\}$ and length n over $\mathscr{R}_{\kappa+1,m}$ satisfying property ($\mathfrak{P}$) with $Tor_1(\mathscr{D}_{\kappa+1})=\mathcal{C}^{(s-\kappa_1)}$ and $Tor_{j+1}(\mathscr{D}_{\kappa+1})=Tor_j(\mathscr{D}_{\kappa-1})$ for $1\le j\le\kappa-1$, where $\epsilon=1$ if $\mathbf{1}\in\mathcal{C}^{(s-\kappa)}$, while $\epsilon=0$ otherwise, and $Y_{\kappa+1}=(\Lambda_{s+\kappa_1}+\Lambda_{s-\kappa_1})(n-\Lambda_{s+\kappa_1+1}-\Lambda_{s-\kappa_1})+\Lambda^2_{s-\kappa_1}+\Lambda_{s-\kappa_1}-\Lambda_{s-\kappa}-2\Lambda_{s-\kappa+1}+\omega_{\kappa+1}$ with $\omega_{\kappa+1}=1$ if $\mathbf{1}\in\mathcal{C}^{(s-\kappa+1)}$, while $\omega_{\kappa+1}=0$ otherwise.

Proof. To prove the result, we assume, without any loss of generality, that the code $\mathscr{D}_{\kappa-1}$ has a generator matrix

$$\mathtt{G}_{\kappa-1}=\begin{bmatrix}\mathtt{W}_1^{(\kappa-1)}\\ \mathtt{W}_2^{(\kappa-1)}\\ \vdots\\ \mathtt{W}_{s-\kappa_1+1}^{(\kappa-1)}\\ u\mathtt{W}_{s-\kappa_1+2}^{(\kappa-1)}\\ \vdots\\ u^{\kappa-2}\mathtt{W}_{s+\kappa_1}^{(\kappa-1)}\end{bmatrix}=\begin{bmatrix}\mathtt{W}_1^{(0)}+u\mathtt{V}_1^{(1)}+u^2\mathtt{V}_1^{(2)}+\cdots+u^{\kappa-2}\mathtt{V}_1^{(\kappa-2)}\\ \mathtt{W}_2^{(0)}+u\mathtt{V}_2^{(1)}+u^2\mathtt{V}_2^{(2)}+\cdots+u^{\kappa-2}\mathtt{V}_2^{(\kappa-2)}\\ \vdots\\ \mathtt{W}_{s-\kappa_1+1}^{(0)}+u\mathtt{V}_{s-\kappa_1+1}^{(1)}+u^2\mathtt{V}_{s-\kappa_1+1}^{(2)}+\cdots+u^{\kappa-2}\mathtt{V}_{s-\kappa_1+1}^{(\kappa-2)}\\ u\mathtt{W}_{s-\kappa_1+2}^{(\kappa-1)}\\ \vdots\\ u^{\kappa-2}\mathtt{W}_{s+\kappa_1}^{(\kappa-1)}\end{bmatrix},$$

where

$$[\mathtt{W}^{(0)}]_{s-\kappa_1+1}=\begin{bmatrix}\mathtt{W}_1^{(0)}\\ \mathtt{W}_2^{(0)}\\ \vdots\\ \mathtt{W}_{s-\kappa_1+1}^{(0)}\end{bmatrix}=\begin{bmatrix}\mathtt{I}_{\lambda_1} & \mathtt{B}_{1,1}^{(0)} & \cdots & \mathtt{B}_{1,s-\kappa_1}^{(0)} & \cdots & \mathtt{B}_{1,e-1}^{(0)} & \mathtt{B}_{1,e}^{(0)}\\ \mathbf{0} & \mathtt{I}_{\lambda_2} & \cdots & \mathtt{B}_{2,s-\kappa_1}^{(0)} & \cdots & \mathtt{B}_{2,e-1}^{(0)} & \mathtt{B}_{2,e}^{(0)}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathtt{I}_{\lambda_{s-\kappa_1+1}} & \cdots & \mathtt{B}_{s-\kappa_1+1,e-1}^{(0)} & \mathtt{B}_{s-\kappa_1+1,e}^{(0)}\end{bmatrix}$$

with $\mathtt{I}_{\lambda_i}$ as the $\lambda_i\times\lambda_i$ identity matrix over $\mathcal{T}_m$, $\mathtt{B}_{i,j}^{(0)}\in\mathcal{M}_{\lambda_i\times\lambda_{j+1}}(\mathcal{T}_m)$ for $1\le i\le s-\kappa_1+1$ and $i\le j\le e$, $[\mathtt{V}^{(a)}]_{s-\kappa_1+1}\in\mathcal{M}_{\Lambda_{s-\kappa_1+1}\times n}(\mathcal{T}_m)$ for $1\le a\le\kappa-2$, and the matrix $\mathtt{W}_{s-\kappa_1-1+g}^{(\kappa-1)}\in\mathcal{M}_{\lambda_{s-\kappa_1-1+g}\times n}(\mathscr{R}_{\kappa-1,m})$ to be considered modulo $u^{\kappa+1-g}$ for $3\le g\le\kappa$. Note that the torsion code $Tor_1(\mathscr{D}_{\kappa-1})$ is a $\Lambda_{s-\kappa_1+1}$-dimensional code over $\mathcal{T}_m$ with a generator matrix $[\mathtt{W}^{(0)}]_{s-\kappa_1+1}$. Here, we also assume, without any loss of generality, that the code $\mathcal{C}^{(s-\kappa_1)}$ has a generator matrix $[\mathtt{W}^{(0)}]_{s-\kappa_1}$, the code $\mathcal{C}^{(s-\kappa+1)}$ has a generator matrix $[\mathtt{W}^{(0)}]_{s-\kappa+1}$ and the code $\mathcal{C}^{(s-\kappa)}$ has a generator matrix $[\mathtt{W}^{(0)}]_{s-\kappa}$. Moreover, by Remark 3.1, we assume that the matrix $(\mathtt{B}^{(0)})_{s-\kappa_1,s+\kappa_1+1}$ has full row-rank.

Since $\mathscr{D}_{\kappa-1}$ is a self-orthogonal code over $\mathscr{R}_{\kappa-1,m}$ satisfying property $(\mathfrak{P})$, we have

$$\begin{aligned}
[\mathtt{W}^{(\kappa-1)}]_{s-\kappa_1}[\mathtt{W}^{(\kappa-1)}]_{s-\kappa_1}^t &\equiv \mathbf{0} \pmod{u^{\kappa-1}},\\
\mathtt{Diag}\left([\mathtt{W}^{(\kappa-1)}]_{s-\kappa_1+1-\mathrm{f}_i}[\mathtt{W}^{(\kappa-1)}]_{s-\kappa_1+1-\mathrm{f}_i}^t\right) &\equiv \mathbf{0} \pmod{u^{\kappa-1+i}} \quad \text{for} \quad i\in\{2,4,6,\ldots,\kappa-1\},\\
\pi_{2\kappa-3}\left(\mathtt{Diag}([\mathtt{W}^{(\kappa-1)}]_{s-\kappa+2}[\mathtt{W}^{(\kappa-1)}]_{s-\kappa+2}^t)\right) &= \mathbf{0},\\
[\mathtt{W}^{(\kappa-1)}]_{s-\kappa_1}\mathtt{W}_{s-\kappa_1-1+g}^{(\kappa-1)t} &\equiv \mathbf{0} \pmod{u^{\kappa+1-g}} \quad \text{for } 2\le g\le\kappa,\\
\mathtt{W}_{s-\kappa_1-1+i}^{(\kappa-1)}\mathtt{W}_{s-\kappa_1-1+j}^{(\kappa-1)t} &\equiv \mathbf{0} \pmod{u^{\kappa+3-i-j}} \text{ for } 2\le i,j\le\kappa \text{ and } i+j\le\kappa+2.
\end{aligned}$$

This implies that

$$\begin{aligned}
[\mathtt{W}^{(\kappa-1)}]_{s-\kappa_1}[\mathtt{W}^{(\kappa-1)}]_{s-\kappa_1}^t &\equiv u^{\kappa-1}\mathtt{F}^{(\kappa-1)}+u^{\kappa}\mathtt{F}^{(\kappa)}+\cdots+u^{2\kappa}\mathtt{F}^{(2\kappa)} \pmod{u^{2\kappa+1}},\\
[\mathtt{W}^{(\kappa-1)}]_{s-\kappa_1}\mathtt{W}_{s-\kappa_1-1+g}^{(\kappa-1)t} &\equiv u^{\kappa+1-g}R_g \pmod{u^{\kappa+2-g}} \quad \text{for } 2\le g\le\kappa,\\
\mathtt{W}_{s-\kappa_1-1+i}^{(\kappa-1)}\mathtt{W}_{s-\kappa_1-1+j}^{(\kappa-1)t} &\equiv \mathbf{0} \pmod{u^{\kappa+3-i-j}} \text{ for } 2\le i,j\le\kappa \text{ and } i+j\le\kappa+2,
\end{aligned}$$

where $\mathtt{F}^{(\kappa-1)},\mathtt{F}^{(\kappa)},\ldots,\mathtt{F}^{(2\kappa)}\in\mathtt{Sym}_{\Lambda_{s-\kappa_1}}(\mathcal{T}_m)$ with $\mathtt{F}_{j,j}^{(\kappa-3+i)}=\mathtt{F}_{j,j}^{(\kappa-2+i)}=0$ for $1\le j\le\Lambda_{s-\kappa_1+1-\mathrm{f}_i}$ and $i\in\{2,4,\ldots,\kappa-1\}$, and the matrix $R_g\in\mathcal{M}_{\Lambda_{s-\kappa_1}\times\lambda_{s-\kappa_1-1+g}}(\mathcal{T}_m)$ for $2\le g\le\kappa$.

Now, to prove the result, we define a matrix $\mathtt{G}_{\kappa+1}$ over $\mathscr{R}_{\kappa+1,m}$ as follows:

$$\mathtt{G}_{\kappa+1}=\begin{bmatrix}\mathtt{W}_1^{(\kappa+1)}\\ \mathtt{W}_2^{(\kappa+1)}\\ \vdots\\ \mathtt{W}_{s-\kappa_1}^{(\kappa+1)}\\ u\mathtt{W}_{s-\kappa_1+1}^{(\kappa+1)}\\ \vdots\\ u^{\kappa}\mathtt{W}_{s+\kappa_1+1}^{(\kappa+1)}\end{bmatrix}=\begin{bmatrix}\mathtt{W}_1^{(\kappa-1)}+u^{\kappa-1}\mathtt{V}_1^{(\kappa-1)}+u^{\kappa}\mathtt{V}_1^{(\kappa)}\\ \mathtt{W}_2^{(\kappa-1)}+u^{\kappa-1}\mathtt{V}_2^{(\kappa-1)}+u^{\kappa}\mathtt{V}_2^{(\kappa)}\\ \vdots\\ \mathtt{W}_{s-\kappa_1}^{(\kappa-1)}+u^{\kappa-1}\mathtt{V}_{s-\kappa_1}^{(\kappa-1)}+u^{\kappa}\mathtt{V}_{s-\kappa_1}^{(\kappa)}\\ u\mathtt{W}_{s-\kappa_1+1}^{(\kappa+1)}\\ \vdots\\ u^{\kappa}\mathtt{W}_{s+\kappa_1+1}^{(\kappa+1)}\end{bmatrix},$$

where the matrices $[\mathtt{V}^{(\mu)}]_{s-\kappa_1} \in \mathcal{M}_{\Lambda_{s-\kappa_1}\times n}(\mathcal{T}_m)$ for $\mu \in \{\kappa-1,\kappa\}$, $\mathtt{W}^{(\kappa+1)}_{s-\kappa_1-1+g} \in \mathcal{M}_{\lambda_{s-\kappa_1-1+g}\times n}(\mathscr{R}_{\kappa+1,m})$ for $2 \leq g \leq \kappa$ and $\mathtt{W}^{(\kappa+1)}_{s+\kappa_1+1} \in \mathcal{M}_{\lambda_{s+\kappa_1+1}\times n}(\mathcal{T}_m)$ are of the forms

$$\begin{bmatrix} \mathtt{V}^{(\mu)}_1 \\ \mathtt{V}^{(\mu)}_2 \\ \vdots \\ \mathtt{V}^{(\mu)}_{s-\kappa_1} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & \mathtt{B}^{(\mu)}_{1,\mu+1} & \mathtt{B}^{(\mu)}_{1,\mu+2} & \cdots & \mathtt{B}^{(\mu)}_{1,s-\kappa_1+\mu} & \cdots & \mathtt{B}^{(\mu)}_{1,e} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathtt{B}^{(\mu)}_{2,\mu+2} & \cdots & \mathtt{B}^{(\mu)}_{2,s-\kappa_1+\mu} & \cdots & \mathtt{B}^{(\mu)}_{2,e} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathtt{B}^{(\mu)}_{s-\kappa_1,s-\kappa_1+\mu} & \cdots & \mathtt{B}^{(\mu)}_{s-\kappa_1,e} \end{bmatrix},$$

$$\mathtt{W}^{(\kappa+1)}_{s-\kappa_1-1+g} = \mathtt{W}^{(\kappa-1)}_{s-\kappa_1-1+g} + u^{\kappa+1-g}\begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathtt{B}^{(\kappa+1-g)}_{s-\kappa_1-1+g,s+\kappa_1+1} & \cdots & \mathtt{B}^{(\kappa+1-g)}_{s-\kappa_1-1+g,e} \end{bmatrix} \text{ and}$$

$$\mathtt{W}^{(\kappa+1)}_{s+\kappa_1+1} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathtt{I}_{\lambda_{s+\kappa_1+1}} & \mathtt{B}^{(0)}_{s+\kappa_1+1,s+\kappa_1+1} & \cdots & \mathtt{B}^{(0)}_{s+\kappa_1+1,e} \end{bmatrix}$$

with $\mathtt{B}^{(\mu)}_{i,j} \in \mathcal{M}_{\lambda_i\times\lambda_{j+1}}(\mathcal{T}_m)$ for $1 \leq i \leq s-\kappa_1$ and $i+\mu \leq j \leq e$, $\mathtt{B}^{(\kappa+1-g)}_{s-\kappa_1-1+g,h} \in \mathcal{M}_{\lambda_{s-\kappa_1-1+g}\times\lambda_{h+1}}(\mathcal{T}_m)$ for $s+\kappa_1+1 \leq h \leq e$, and $\mathtt{B}^{(0)}_{s+\kappa_1+1,a} \in \mathcal{M}_{\lambda_{s+\kappa_1+1}\times\lambda_{a+1}}(\mathcal{T}_m)$ for $s+\kappa_1+1 \leq a \leq e$. Let $\mathscr{D}_{\kappa+1}$ be a linear code of length n over $\mathscr{R}_{\kappa+1,m}$ with a generator matrix $\mathtt{G}_{\kappa+1}$. Note that the code $\mathscr{D}_{\kappa+1}$ is of type $\{\Lambda_{s-\kappa_1}, \lambda_{s-\kappa_1+1}, \ldots, \lambda_{s+\kappa_1+1}\}$ over $\mathscr{R}_{\kappa+1,m}$ satisfying $Tor_1(\mathscr{D}_{\kappa+1}) = \mathcal{C}^{(s-\kappa_1)}$ and $Tor_{i+1}(\mathscr{D}_{\kappa+1}) = Tor_i(\mathscr{D}_{\kappa-1})$ for $1 \leq i \leq \kappa-1$. Moreover, by Lemma 2.1, we see that the code $\mathscr{D}_{\kappa+1}$ is a self-orthogonal code over $\mathscr{R}_{\kappa+1,m}$ satisfying property $(\mathfrak{P})$ if and only if there exist matrices $[\mathtt{V}^{(\kappa-1)}]_{s-\kappa_1}$, $[\mathtt{V}^{(\kappa)}]_{s-\kappa_1}$, $\begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} \, \mathtt{B}^{(\kappa+1-g)}_{s-\kappa_1-1+g,s+\kappa_1+1} & \cdots & \mathtt{B}^{(\kappa+1-g)}_{s-\kappa_1-1+g,e} \end{bmatrix}$ for $2 \leq g \leq \kappa$ and $\mathtt{W}^{(\kappa+1)}_{s+\kappa_1+1}$ satisfying the following system of matrix equations:

$$\begin{aligned} &\mathtt{F}^{(\kappa-1)} + [\mathtt{W}^{(0)}]_{s-\kappa_1}[\mathtt{V}^{(\kappa-1)}]^t_{s-\kappa_1} + [\mathtt{V}^{(\kappa-1)}]_{s-\kappa_1}[\mathtt{W}^{(0)}]^t_{s-\kappa_1} + u\Big([\mathtt{W}^{(0)}]_{s-\kappa_1}[\mathtt{V}^{(\kappa)}]^t_{s-\kappa_1} \\ &\quad + [\mathtt{V}^{(\kappa)}]_{s-\kappa_1}[\mathtt{W}^{(0)}]^t_{s-\kappa_1} + [\mathtt{V}^{(1)}]_{s-\kappa_1}[\mathtt{V}^{(\kappa-1)}]^t_{s-\kappa_1} + [\mathtt{V}^{(\kappa-1)}]_{s-\kappa_1}[\mathtt{V}^{(1)}]^t_{s-\kappa_1} + \mathtt{F}^{(\kappa)}\Big) \equiv \mathbf{0} \pmod{u^2}, \quad (3.39) \\ &\mathtt{Diag}\Big(\mathtt{E}^{(2\kappa-2)} + [\mathtt{V}^{(\kappa-1)}]_{s-\kappa+1}[\mathtt{V}^{(\kappa-1)}]^t_{s-\kappa+1}\Big) \equiv \mathbf{0} \pmod{u}, \quad (3.40) \\ &\mathtt{Diag}\Big(\mathtt{E}^{(2\kappa-1)} + \eta_0[\mathtt{W}^{(0)}]_{s-\kappa+1}[\mathtt{V}^{(\kappa-1)}]^t_{s-\kappa+1}\Big) \equiv \mathbf{0} \pmod{u}, \quad (3.41) \\ &\mathtt{Diag}\Big(\mathtt{E}^{(2\kappa)} + [\mathtt{V}^{(\kappa)}]_{s-\kappa}[\mathtt{V}^{(\kappa)}]^t_{s-\kappa} + \eta_0[\mathtt{W}^{(0)}]_{s-\kappa}[\mathtt{V}^{(\kappa)}]^t_{s-\kappa} \\ &\quad + \eta_0[\mathtt{V}^{(1)}]_{s-\kappa}[\mathtt{V}^{(\kappa-1)}]^t_{s-\kappa} + \eta_1[\mathtt{W}^{(0)}]_{s-\kappa}[\mathtt{V}^{(\kappa-1)}]^t_{s-\kappa}\Big) \equiv \mathbf{0} \pmod{u}, \quad (3.42) \\ &R_g + [\mathtt{W}^{(0)}]_{s-\kappa_1}\begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} \, \mathtt{B}^{(\kappa+1-g)}_{s-\kappa_1-1+g,s+\kappa_1+1} & \cdots & \mathtt{B}^{(\kappa+1-g)}_{s-\kappa_1-1+g,e} \end{bmatrix}^t \equiv \mathbf{0} \pmod{u}, \quad (3.43) \\ &[\mathtt{W}^{(0)}]_{s-\kappa_1}\mathtt{W}^{(\kappa+1)t}_{s+\kappa_1+1} \equiv \mathbf{0} \pmod{u}, \quad (3.44) \end{aligned}$$

where $\mathtt{E}^{(2\kappa-2)}$, $\mathtt{E}^{(2\kappa-1)}$ and $\mathtt{E}^{(2\kappa)}$ are $\Lambda_{s-\kappa+1}\times\Lambda_{s-\kappa+1}$, $\Lambda_{s-\kappa+1}\times\Lambda_{s-\kappa+1}$ and $\Lambda_{s-\kappa}\times\Lambda_{s-\kappa}$ matrices over $\mathcal{T}_m$ whose rows are the first $\Lambda_{s-\kappa+1}$, $\Lambda_{s-\kappa+1}$ and $\Lambda_{s-\kappa}$ rows of the matrices $\mathtt{F}^{(2\kappa-2)}$, $\mathtt{F}^{(2\kappa-1)}$ and $\mathtt{F}^{(2\kappa)}$, respectively.

We will now establish the existence of the matrices $[\mathtt{V}^{(\kappa-1)}]_{s-\kappa_1}$, $[\mathtt{V}^{(\kappa)}]_{s-\kappa_1}$, $\begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} \, \mathtt{B}^{(\kappa+1-g)}_{s-\kappa_1-1+g,s+\kappa_1+1} & \cdots & \mathtt{B}^{(\kappa+1-g)}_{s-\kappa_1-1+g,e} \end{bmatrix}$ for $2 \leq g \leq \kappa$ and $\mathtt{W}^{(\kappa+1)}_{s+\kappa_1+1}$ satisfying the system of matrix equations (3.39)-(3.44) and count their choices. To do this, we first recall that the matrix $(\mathtt{B}^{(0)})_{s-\kappa_1,s+\kappa_1+1}$ has full row-rank over $\mathcal{T}_m$. Working as in Proposition 3.3, we see that there exists a matrix $[\mathtt{V}^{(\kappa-1)}]_{s-\kappa_1}$ satisfying the following system of matrix equations:

$$\begin{aligned} \mathtt{F}^{(\kappa-1)} + [\mathtt{W}^{(0)}]_{s-\kappa_1}[\mathtt{V}^{(\kappa-1)}]^t_{s-\kappa_1} + [\mathtt{V}^{(\kappa-1)}]_{s-\kappa_1}[\mathtt{W}^{(0)}]^t_{s-\kappa_1} &\equiv \mathbf{0} \pmod{u}, \quad (3.45) \\ \mathtt{Diag}\Big(\mathtt{E}^{(2\kappa-2)} + [\mathtt{V}^{(\kappa-1)}]_{s-\kappa+1}[\mathtt{V}^{(\kappa-1)}]^t_{s-\kappa+1}\Big) &\equiv \mathbf{0} \pmod{u}, \quad (3.46) \\ \mathtt{Diag}\Big(\mathtt{E}^{(2\kappa-1)} + \eta_0[\mathtt{W}^{(0)}]_{s-\kappa+1}[\mathtt{V}^{(\kappa-1)}]^t_{s-\kappa+1}\Big) &\equiv \mathbf{0} \pmod{u}. \quad (3.47) \end{aligned}$$

Moreover, the matrix $[\mathtt{V}^{(\kappa-1)}]_{s-\kappa_1}$ satisfying (3.45)-(3.47) has precisely

$$(2^m)^{\sum\limits_{i=\kappa+1}^{s+\kappa_1+1}\lambda_i\Lambda_{i-\kappa}+\Lambda_{s-\kappa_1}(n-\Lambda_{s+\kappa_1+1})-2\Lambda_{s-\kappa+1}-\frac{\Lambda_{s-\kappa_1}(\Lambda_{s-\kappa_1}-1)}{2}+\omega_{\kappa+1}}$$

distinct choices, where $\omega_{\kappa+1} = 1$ if $\mathbf{1} \in \mathcal{C}^{(s-\kappa+1)}$, while $\omega_{\kappa+1} = 0$ otherwise. Further, for a given choice of the matrix $[\mathtt{V}^{(\kappa-1)}]_{s-\kappa_1}$ satisfying (3.45)-(3.47), we can write

$$\mathtt{F}^{(\kappa-1)} + [\mathtt{W}^{(0)}]_{s-\kappa_1}[\mathtt{V}^{(\kappa-1)}]^t_{s-\kappa_1} + [\mathtt{V}^{(\kappa-1)}]_{s-\kappa_1}[\mathtt{W}^{(0)}]^t_{s-\kappa_1} \equiv uQ \pmod{u^2} \tag{3.48}$$

for some $Q \in \mathtt{Sym}_{\Lambda_{s-\kappa_1}}(\mathcal{T}_m)$. This, by (3.39), implies that

$$\begin{aligned} Q + \mathtt{F}^{(\kappa)} + [\mathtt{W}^{(0)}]_{s-\kappa_1}[\mathtt{V}^{(\kappa)}]^t_{s-\kappa_1} + [\mathtt{V}^{(\kappa)}]_{s-\kappa_1}[\mathtt{W}^{(0)}]^t_{s-\kappa_1} \\ +[\mathtt{V}^{(1)}]_{s-\kappa_1}[\mathtt{V}^{(\kappa-1)}]^t_{s-\kappa_1} + [\mathtt{V}^{(\kappa-1)}]_{s-\kappa_1}[\mathtt{V}^{(1)}]^t_{s-\kappa_1} \equiv \mathbf{0} \pmod{u}. \end{aligned} \tag{3.49}$$

Furthermore, working as in Lemma 4.1 of Yadav and Sharma [31], we observe that there exists a matrix $[\mathtt{V}^{(\kappa)}]_{s-\kappa_1}$ satisfying (3.42) and (3.49) if and only if exactly one the following two conditions is satisfied: (I) $\mathbf{1} \notin \mathcal{C}^{(s-\kappa)}$, and (II) $\mathbf{1} \in \mathcal{C}^{(s-\kappa)}$ with either $n \equiv 0 \pmod 8$ or $n \equiv 4 \pmod 8$ and m even. Moreover, the matrix $[\mathtt{V}^{(\kappa)}]_{s-\kappa_1}$ satisfying (3.42) and (3.49) has precisely

$$2^{\epsilon}(2^m)^{\sum\limits_{j=\kappa+2}^{s+\kappa_1+1}\lambda_j\Lambda_{j-\kappa-1}+\Lambda_{s-\kappa_1}(n-\Lambda_{s+\kappa_1+1})-\Lambda_{s-\kappa}-\frac{\Lambda_{s-\kappa_1}(\Lambda_{s-\kappa_1}-1)}{2}}$$

distinct choices, where $\epsilon = 1$ if $\mathbf{1} \in \mathcal{C}^{(s-\kappa)}$ with either $n \equiv 0 \pmod 8$ or $n \equiv 4 \pmod 8$ and m is even, while $\epsilon = 0$ otherwise. Further, working as in Proposition 4.3 of Yadav and Sharma [31], we see that the matrices $\begin{bmatrix}\mathbf{0} & \cdots & \mathbf{0} & \mathtt{B}^{(\kappa+1-g)}_{s-\kappa_1-1+g,s+\kappa_1+1} & \cdots & \mathtt{B}^{(\kappa+1-g)}_{s-\kappa_1-1+g,e}\end{bmatrix}$ for $2 \le g \le \kappa$ and $\mathtt{W}^{(\kappa+1)}_{s+\kappa_1+1}$ satisfying (3.43) and (3.44) have precisely

$$(2^m)^{(\Lambda_{s+\kappa_1}-\Lambda_{s-\kappa_1})(n-\Lambda_{s+\kappa_1+1}-\Lambda_{s-\kappa_1})}\begin{bmatrix}\lambda_{s+\kappa_1+1}+n-\Lambda_{s+\kappa_1+1}-\Lambda_{s-\kappa_1}\\ \lambda_{s+\kappa_1+1}\end{bmatrix}_{2^m}$$

distinct choices. From this, the desired result follows immediately. □

The following proposition lifts a self-orthogonal code of type $\{\Lambda_{s-\kappa_1+1}, \lambda_{s-\kappa_1+2}, \ldots, \lambda_{s+\kappa_1+1}\}$ and length n over $\mathscr{R}_{\kappa,m}$ satisfying property $(\mathfrak{P})$ to a self-orthogonal code of type $\{\Lambda_{s-\kappa_1}, \lambda_{s-\kappa_1+1}, \ldots, \lambda_{s+\kappa_1+2}\}$ and the same length n over $\mathscr{R}_{\kappa+2,m}$ satisfying property $(\mathfrak{P})$ when e is odd and $2\kappa \le e$. It also enumerates the number of possible ways to perform this lifting.

Proposition 3.5. *Let e be odd, $2\kappa \le e$, and let $\ell = \kappa+2$. Let $\mathscr{D}_\kappa$ be a self-orthogonal code of type $\{\Lambda_{s-\kappa_1+1}, \lambda_{s-\kappa_1+2}, \ldots, \lambda_{s+\kappa_1+1}\}$ and length n over $\mathscr{R}_{\kappa,m}$ satisfying property $(\mathfrak{P})$. Let $\mathcal{C}^{(s-\kappa_1)}$ and $\mathcal{C}^{(s-\kappa+1)}$ be linear subcodes of $Tor_1(\mathscr{D}_\kappa)$, such that $\mathcal{C}^{(s-\kappa+1)} \subseteq \mathcal{C}^{(s-\kappa_1)}$, $\dim \mathcal{C}^{(s-\kappa+1)} = \Lambda_{s-\kappa+1}$, $\dim \mathcal{C}^{(s-\kappa_1)} = \Lambda_{s-\kappa_1}$, and the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, (such a pair of codes $\mathcal{C}^{(s-\kappa_1)}$ and $\mathcal{C}^{(s-\kappa+1)}$ exists for each choice of $\mathscr{D}_\kappa$, by Lemma 3.2). The following hold.*

(a) There exists a self-orthogonal code $\mathscr{D}_{\kappa+2}$ of type $\{\Lambda_{s-\kappa_1}, \lambda_{s-\kappa_1+1}, \ldots, \lambda_{s+\kappa_1+2}\}$ and length n over $\mathscr{R}_{\kappa+2,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_{\kappa+2}) = \mathcal{C}^{(s-\kappa_1)}$ and $Tor_{j+1}(\mathscr{D}_{\kappa+2}) = Tor_j(\mathscr{D}_\kappa)$ for $1 \le j \le \kappa$, where such a code exists:

- *unconditionally, if $\mathbf{1} \notin \mathcal{C}^{(s-\kappa+1)}$;*
- *if and only if either $n \equiv 0 \pmod 8$ or $n \equiv 4 \pmod 8$ with m even when $\mathbf{1} \in \mathcal{C}^{(s-\kappa+1)}$.*

(b) Moreover, each such choice of the codes $\mathscr{D}_\kappa$, $\mathcal{C}^{(s-\kappa_1)}$, and $\mathcal{C}^{(s-\kappa+1)}$ gives rise to precisely

$$2^{\epsilon}(2^m)^{\sum\limits_{i=\kappa+2}^{s+\kappa_1+2}\lambda_i\Lambda_{i-\kappa-1}+\sum\limits_{j=\kappa+3}^{s+\kappa_1+2}\lambda_j\Lambda_{j-\kappa-2}+Y_{\kappa+2}}\begin{bmatrix}\lambda_{s+\kappa_1+2}+n-\Lambda_{s+\kappa_1+2}-\Lambda_{s-\kappa_1}\\ \lambda_{s+\kappa_1+2}\end{bmatrix}_{2^m}$$

distinct self-orthogonal codes $\mathscr{D}_{\kappa+2}$ of type $\{\Lambda_{s-\kappa_1}, \lambda_{s-\kappa_1+1}, \ldots, \lambda_{s+\kappa_1+2}\}$ and length n over $\mathscr{R}_{\kappa+2,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_{\kappa+2}) = \mathcal{C}^{(s-\kappa_1)}$ and $Tor_{j+1}(\mathscr{D}_{\kappa+2}) = Tor_j(\mathscr{D}_\kappa)$ for $1 \le j \le \kappa$, where $Y_{\kappa+2} = (\Lambda_{s+\kappa_1+1}+\Lambda_{s-\kappa_1})(n-\Lambda_{s+\kappa_1+2}-\Lambda_{s-\kappa_1})+\Lambda^2_{s-\kappa_1}+\Lambda_{s-\kappa_1}-\Lambda_{s-\kappa+1}-\Lambda_{s-\kappa}$ and $\epsilon = 1$ if $\mathbf{1} \in \mathcal{C}^{(s-\kappa+1)}$, while $\epsilon = 0$ otherwise.

Proof. Working as in Proposition 3.4, the desired result follows. □

The following proposition considers the case $2\kappa \leq e$ and provides a method for lifting a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$ to a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and the same length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$, where the integer ℓ satisfies $\kappa+3 \leq \ell \leq e-\kappa+1$ and $\ell \equiv e \pmod 2$. It also enumerates the number of possible ways to perform this lifting.

Proposition 3.6. *Let $2\kappa \leq e$, and let ℓ be a fixed integer satisfying $\kappa+3 \leq \ell \leq e-\kappa+1$ and $\ell \equiv e \pmod 2$. Let us define $\gamma_\ell = s-\mathrm{f}_\ell$. Let $\mathscr{D}_{\ell-2}$ be a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$. Let $\mathcal{C}^{(\gamma_\ell+1)}$ be a $\Lambda_{\gamma_\ell+1}$-dimensional linear subcode of the torsion code $Tor_1(\mathscr{D}_{\ell-2})$. The following hold.*

(a) There exists a self-orthogonal code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{i+1}(\mathscr{D}_\ell) = Tor_i(\mathscr{D}_{\ell-2})$ for $1 \leq i \leq \ell-2$.

(b) Moreover, each such pair of codes $\mathscr{D}_{\ell-2}$ and $\mathcal{C}^{(\gamma_\ell+1)}$) gives rise to precisely

$$(2^m)^{\sum\limits_{i=\ell}^{\gamma_\ell+\ell}\lambda_i\Lambda_{i-\ell+1}+\sum\limits_{j=\ell+1}^{\gamma_\ell+\ell}\lambda_j\Lambda_{j-\ell}+Y_\ell}\begin{bmatrix}\lambda_{\gamma_\ell+\ell}+n-\Lambda_{\gamma_\ell+\ell}-\Lambda_{\gamma_\ell+1}\\ \lambda_{\gamma_\ell+\ell}\end{bmatrix}_{2^m}$$

distinct self-orthogonal codes $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{i+1}(\mathscr{D}_\ell) = Tor_i(\mathscr{D}_{\ell-2})$ for $1 \leq i \leq \ell-2$, where $Y_\ell = (\Lambda_{\gamma_\ell+\ell-1}+\Lambda_{\gamma_\ell+1})(n-\Lambda_{\gamma_\ell+\ell}-\Lambda_{\gamma_\ell+1})+\Lambda_{\gamma_\ell+1}^2+\Lambda_{\gamma_\ell+1}-\Lambda_{\gamma_\ell+1-\kappa_1}-\Lambda_{\gamma_\ell-\kappa_1}$.

Proof. Working as in Proposition 4.3 of Yadav and Sharma [31], the desired result follows immediately. □

The following proposition considers the case $2\kappa > e$ and provides a method for lifting a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$ to a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$, where the integer ℓ satisfies $e-\kappa+2-2\theta_e \leq \ell \leq \kappa-\mathrm{f}_{2\kappa-e}+1$ and $\ell \equiv e \pmod 2$. It also enumerates the number of possible ways to perform this lifting.

Proposition 3.7. *Let $2\kappa > e$, and let ℓ be a fixed integer satisfying $e-\kappa+2-2\theta_e \leq \ell \leq \kappa-\mathrm{f}_{2\kappa-e}+1$ and $\ell \equiv e \pmod 2$. Let us define $\gamma_\ell = s-\mathrm{f}_\ell$. Let $\mathscr{D}_{\ell-2}$ be a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$. Let $\mathcal{C}^{(\gamma_\ell+1)}$ be a $\Lambda_{\gamma_\ell+1}$-dimensional linear subcode of the torsion code $Tor_1(\mathscr{D}_{\ell-2})$. The following hold.*

(a) There exists a self-orthogonal code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{i+1}(\mathscr{D}_\ell) = Tor_i(\mathscr{D}_{\ell-2})$ for $1 \leq i \leq \ell-2$.

(b) Moreover, each such pair of codes $\mathscr{D}_{\ell-2}$ and $\mathcal{C}^{(\gamma_\ell+1)}$ gives rise to precisely

$$(2^m)^{\sum\limits_{i=\ell}^{\gamma_\ell+\ell}\lambda_i\Lambda_{i-\ell+1}+\sum\limits_{j=\ell+1}^{\gamma_\ell+\ell}\lambda_j\Lambda_{j-\ell}+Y_\ell}\begin{bmatrix}\lambda_{\gamma_\ell+\ell}+n-\Lambda_{\gamma_\ell+\ell}-\Lambda_{\gamma_\ell+1}\\ \lambda_{\gamma_\ell+\ell}\end{bmatrix}_{2^m}$$

distinct self-orthogonal codes $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{i+1}(\mathscr{D}_\ell) = Tor_i(\mathscr{D}_{\ell-2})$ for $1 \leq i \leq \ell-2$, where $Y_\ell = (\Lambda_{\gamma_\ell+\ell-1}+\Lambda_{\gamma_\ell+1})(n-\Lambda_{\gamma_\ell+\ell}-\Lambda_{\gamma_\ell+1})+\Lambda_{\gamma_\ell+1}^2+\Lambda_{\gamma_\ell+1}-\Lambda_{s-2\mathrm{f}_\ell+2}-\Lambda_{s-2\mathrm{f}_\ell+1}$.

Proof. By applying Lemma 4.1 of Yadav and Sharma [32] and working as in Proposition 3.3, the desired result follows immediately. □

The following proposition lifts a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$ to a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and the same length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$, where the integer ℓ satisfies $\ell \equiv e \pmod 2$ and $e-\kappa+2 \leq \ell \leq e$ if $2\kappa \leq e$, while $\kappa-\mathrm{f}_{2\kappa-e}+1 < \ell \leq e$ if $2\kappa > e$. It also enumerates the number of possible ways to perform this lifting.

Proposition 3.8. *Let ℓ be a fixed integer satisfying $\ell \equiv e \pmod 2$ and $e-\kappa+2 \leq \ell \leq e$ if $2\kappa \leq e$, while $\kappa-\mathrm{f}_{2\kappa-e}+1 < \ell \leq e$ if $2\kappa > e$. Let us define $\gamma_\ell = s-\mathrm{f}_\ell$. Let $\mathscr{D}_{\ell-2}$ be a self-orthogonal code of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$. Let $\mathcal{C}^{(\gamma_\ell+1)}$ be a $\Lambda_{\gamma_\ell+1}$-dimensional linear subcode of the torsion code $Tor_1(\mathscr{D}_{\ell-2})$. The following hold.*

(a) *There exists a self-orthogonal code* $\mathscr{D}_\ell$ *of type* $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ *and length* n *over* $\mathscr{R}_{\ell,m}$ *satisfying property* $(\mathfrak{P})$ *with* $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ *and* $Tor_{i+1}(\mathscr{D}_\ell) = Tor_i(\mathscr{D}_{\ell-2})$ *for* $1 \le i \le \ell-2$.

(b) *Moreover, each such pair of codes* $\mathscr{D}_{\ell-2}$ *and* $\mathcal{C}^{(\gamma_\ell+1)}$ *gives rise to precisely*

$$(2^m)^{\sum\limits_{i=\ell}^{\gamma_\ell+\ell} \lambda_i \Lambda_{i-\ell+1} + \sum\limits_{j=\ell+1}^{\gamma_\ell+\ell} \lambda_j \Lambda_{j-\ell} + Y_\ell} \begin{bmatrix} \lambda_{\gamma_\ell+\ell} + n - \Lambda_{\gamma_\ell+\ell} - \Lambda_{\gamma_\ell+1} \\ \lambda_{\gamma_\ell+\ell} \end{bmatrix}_{2^m}$$

distinct self-orthogonal codes $\mathscr{D}_\ell$ *of type* $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ *and length* n *over* $\mathscr{R}_{\ell,m}$ *satisfying property* $(\mathfrak{P})$ *with* $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ *and* $Tor_{i+1}(\mathscr{D}_\ell) = Tor_i(\mathscr{D}_{\ell-2})$ *for* $1 \le i \le \ell-2$, *where* $Y_\ell = (\Lambda_{\gamma_\ell+\ell-1} + \Lambda_{\gamma_\ell+1})(n - \Lambda_{\gamma_\ell+\ell} - \Lambda_{\gamma_\ell+1}) + \Lambda_{\gamma_\ell+1}^2 + \Lambda_{\gamma_\ell+1}$.

Proof. Working as in Proposition 4.3 of Yadav and Sharma [31], the desired result follows immediately. □

In the following lemma, we show that for every self-orthogonal code of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$, there exists a chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, such that $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \le i \le s+\theta_e$, the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, and the code $\mathcal{C}^{(s-\kappa+\theta_e)}$ satisfies the additional condition $\mathbf{1} \notin \mathcal{C}^{(s-\kappa+\theta_e)}$ provided that $2\kappa \le e$, $n \equiv 4 \pmod 8$ and m is odd.

Lemma 3.5. *For every self-orthogonal code of type* $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ *and length* n *over* $\mathscr{R}_{e,m}$, *there exists a chain* $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ *of self-orthogonal codes of length* n *over* $\mathcal{T}_m$, *where* $\dim \mathcal{C}^{(i)} = \Lambda_i$ *for* $1 \le i \le s+\theta_e$, *the code* $\mathcal{C}^{(s-\kappa_1)}$ *is doubly even, and the code* $\mathcal{C}^{(s-\kappa+\theta_e)}$ *satisfies the additional condition* $\mathbf{1} \notin \mathcal{C}^{(s-\kappa+\theta_e)}$ *provided that* $2\kappa \le e$, $n \equiv 4 \pmod 8$ *and* m *is odd.*

Proof. To prove the result, let $\mathscr{D}_e$ be a self-orthogonal code of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$. By Lemma 2.2, we see that the i-th torsion code $Tor_i(\mathscr{D}_e)$ is a self-orthogonal code of length n and dimension Λ_i over $\mathcal{T}_m$ for $1 \le i \le s+\theta_e$. It is easy to see that $Tor_1(\mathscr{D}_e) \subseteq Tor_2(\mathscr{D}_e) \subseteq \cdots \subseteq Tor_{s+\theta_e}(\mathscr{D}_e)$. Further, by Lemma 3.3, we note that the torsion code $Tor_{s-\kappa_1}(\mathscr{D}_e)$ is a doubly even code over $\mathcal{T}_m$.

We next assert that $\mathbf{1} \notin Tor_{s-\kappa+\theta_e}(\mathscr{D}_e)$ when $2\kappa \le e$, $n \equiv 4 \pmod 8$ and m is odd. To prove this assertion, let us suppose that the code $\mathscr{D}_e$ has a generator matrix $\mathcal{G}_e$ as defined in (3.1). Further, corresponding to the code $\mathscr{D}_e$, let us consider a linear code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ with a generator matrix $\mathcal{G}_\ell$ (as defined in (3.2)) for each integer ℓ satisfying $2 \le \ell \le e-2$ and $\ell \equiv e \pmod 2$. By Lemma 3.1, we see that $\mathscr{D}_\ell$ is a self-orthogonal code over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$. In particular, $\mathscr{D}_{\kappa-1+\theta_e}$ is a self-orthogonal code of type $\{\Lambda_{s-\kappa_1+1}, \lambda_{s-\kappa_1+2}, \ldots, \lambda_{s+\kappa_1+\theta_e}\}$ over $\mathscr{R}_{\kappa-1+\theta_e,m}$ satisfying property $(\mathfrak{P})$ with $\mathscr{D}_{\kappa+1+\theta_e}$ is a self-orthogonal code of type $\{\Lambda_{s-\kappa_1}, \lambda_{s-\kappa_1+1}, \ldots, \lambda_{s+\kappa_1+1+\theta_e}\}$ over $\mathscr{R}_{\kappa+1+\theta_e,m}$ satisfying property $(\mathfrak{P})$. Now, working as in Propositions 3.4 and 3.5, it is easy to see that $\mathbf{1} \notin Tor_{s-\kappa+\theta_e}(\mathscr{D}_e)$ when $2\kappa \le e$, $n \equiv 4 \pmod 8$ and m is odd. On taking $\mathcal{C}^{(i)} = Tor_i(\mathscr{D}_e)$ for $1 \le i \le s+\theta_e$, the desired result follows. □

Now, let $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ be a chain of self-orthogonal codes of length n over $\mathcal{T}_m$, where $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \le i \le s+\theta_e$, the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, and the code $\mathcal{C}^{(s-\kappa+\theta_e)}$ satisfies the additional condition $\mathbf{1} \notin \mathcal{C}^{(s-\kappa+\theta_e)}$, provided that $2\kappa \le e$, $n \equiv 4 \pmod 8$ and m is odd. Note that the proofs of Propositions 3.1-3.8 provide a method to construct a self-orthogonal code $\mathscr{D}_e$ of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ satisfying $Tor_i(\mathscr{D}_e) = \mathcal{C}^{(i)}$ for $1 \le i \le s+\theta_e$. Below, we outline this construction method by distinguishing the following two cases: (A) $2\kappa \le e$ and (B) $2\kappa > e$.

(A) Construction of a self-orthogonal code of length n over $\mathscr{R}_{e,m}$ when $2\kappa \le e$

Input: A chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, such that

- $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \le i \le s+\theta_e$,
- the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, and
- $\mathbf{1} \notin \mathcal{C}^{(s-\kappa+\theta_e)}$ if $n \equiv 4 \pmod 8$ and m is odd.

Step 1: If e is even, use Proposition 3.1 to construct a self-orthogonal code $\mathscr{D}_2$ of type $\{\Lambda_s, \lambda_{s+1}\}$ and length n over $\mathscr{R}_{2,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_2) = \mathcal{C}^{(s)}$.
On the other hand, if e is odd, employ Proposition 3.2 to construct a self-orthogonal code $\mathscr{D}_3$ of type $\{\Lambda_s, \lambda_{s+1}, \lambda_{s+2}\}$ and length n over $\mathscr{R}_{3,m}$ satisfying property $(\mathfrak{P})$, $Tor_1(\mathscr{D}_3) = \mathcal{C}^{(s)}$ and $Tor_2(\mathscr{D}_3) = \mathcal{C}^{(s+1)}$.

Step 2: For $4 \le \ell \le \kappa$ and $\ell \equiv e \pmod 2$, use Proposition 3.3 to construct a self-orthogonal code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{j+1}(\mathscr{D}_\ell) = Tor_j(\mathscr{D}_{\ell-2})$ for $1 \le j \le \ell-2$ from a self-orthogonal code $\mathscr{D}_{\ell-2}$ of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$.

Step 3: When e is even, employ Proposition 3.4 to construct a self-orthogonal code $\mathscr{D}_{\kappa+1}$ of type $\{\Lambda_{s-\kappa_1}, \lambda_{s-\kappa_1+1}, \ldots, \lambda_{s+\kappa_1+1}\}$ and length n over $\mathscr{R}_{\kappa+1,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_{\kappa+1}) = \mathcal{C}^{(s-\kappa_1)}$ and $Tor_{j+1}(\mathscr{D}_{\kappa+1}) = Tor_j(\mathscr{D}_{\kappa-1})$ for $1 \leq j \leq \kappa - 1$ from a self-orthogonal code $\mathscr{D}_{\kappa-1}$ of type $\{\Lambda_{s-\kappa_1+1}, \lambda_{s-\kappa_1+2}, \ldots, \lambda_{s+\kappa_1}\}$ and length n over $\mathscr{R}_{\kappa-1,m}$ satisfying property $(\mathfrak{P})$.

On the other hand, when e is odd, use Proposition 3.5 to construct a self-orthogonal code $\mathscr{D}_{\kappa+2}$ of type $\{\Lambda_{s-\kappa_1}, \lambda_{s-\kappa_1+1}, \ldots, \lambda_{s+\kappa_1+2}\}$ and length n over $\mathscr{R}_{\kappa+2,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_{\kappa+2}) = \mathcal{C}^{(s-\kappa_1)}$ and $Tor_{j+1}(\mathscr{D}_{\kappa+2}) = Tor_j(\mathscr{D}_{\kappa})$ for $1 \leq j \leq \kappa$ from a self-orthogonal code $\mathscr{D}_{\kappa}$ of type $\{\Lambda_{s-\kappa_1+1}, \lambda_{s-\kappa_1+2}, \ldots, \lambda_{s+\kappa_1+1}\}$ and length n over $\mathscr{R}_{\kappa,m}$ satisfying property $(\mathfrak{P})$.

Step 4: For $\kappa + 3 \leq \ell \leq e - \kappa + 1$ and $\ell \equiv e \pmod 2$, use Proposition 3.6 to construct a self-orthogonal code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{i+1}(\mathscr{D}_\ell) = Tor_i(\mathscr{D}_{\ell-2})$ for $1 \leq i \leq \ell - 2$ from a self-orthogonal code $\mathscr{D}_{\ell-2}$ of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$.

Step 5: For $e - \kappa + 2 \leq \ell \leq e$ and $\ell \equiv e \pmod 2$, employ Proposition 3.8 to construct a self-orthogonal code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{i+1}(\mathscr{D}_\ell) = Tor_i(\mathscr{D}_{\ell-2})$ for $1 \leq i \leq \ell - 2$ from a self-orthogonal code $\mathscr{D}_{\ell-2}$ of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$.

Output: A self-orthogonal code $\mathscr{D}_e$ of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ satisfying $Tor_i(\mathscr{D}_e) = \mathcal{C}^{(i)}$ for $1 \leq i \leq s + \theta_e$.

(B) Construction of a self-orthogonal code of length n over $\mathscr{R}_{e,m}$ when $2\kappa > e$.

Input: A chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, where $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \leq i \leq s + \theta_e$, and the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even.

Step 1: If e is even, apply Proposition 3.1 to construct a self-orthogonal code $\mathscr{D}_2$ of type $\{\Lambda_s, \lambda_{s+1}\}$ and length n over $\mathscr{R}_{2,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_2) = \mathcal{C}^{(s)}$.

On the other hand, if e is odd, apply Proposition 3.2 to construct a self-orthogonal code $\mathscr{D}_3$ of type $\{\Lambda_s, \lambda_{s+1}, \lambda_{s+2}\}$ and length n over $\mathscr{R}_{3,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_3) = \mathcal{C}^{(s)}$ and $Tor_2(\mathscr{D}_3) = \mathcal{C}^{(s+1)}$.

Step 2: For $4 \leq \ell \leq e - \kappa + 1 - 2\theta_e$ and $\ell \equiv e \pmod 2$, apply Proposition 3.3 to construct a self-orthogonal code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{j+1}(\mathscr{D}_\ell) = Tor_j(\mathscr{D}_{\ell-2})$ for $1 \leq j \leq \ell - 2$ from a self-orthogonal code $\mathscr{D}_{\ell-2}$ of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$.

Step 3: For $e - \kappa + 2 - 2\theta_e \leq \ell \leq \kappa - \mathrm{f}_{2\kappa-e} + 1$ and $\ell \equiv e \pmod 2$, apply Proposition 3.7 to construct a self-orthogonal code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{i+1}(\mathscr{D}_\ell) = Tor_i(\mathscr{D}_{\ell-2})$ for $1 \leq i \leq \ell - 2$ from a self-orthogonal code $\mathscr{D}_{\ell-2}$ of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$.

Step 4: For $\kappa - \mathrm{f}_{2\kappa-e} + 1 < \ell \leq e$ and $\ell \equiv e \pmod 2$, apply Proposition 3.8 to construct a self-orthogonal code $\mathscr{D}_\ell$ of type $\{\Lambda_{\gamma_\ell+1}, \lambda_{\gamma_\ell+2}, \ldots, \lambda_{\gamma_\ell+\ell}\}$ and length n over $\mathscr{R}_{\ell,m}$ satisfying property $(\mathfrak{P})$ with $Tor_1(\mathscr{D}_\ell) = \mathcal{C}^{(\gamma_\ell+1)}$ and $Tor_{i+1}(\mathscr{D}_\ell) = Tor_i(\mathscr{D}_{\ell-2})$ for $1 \leq i \leq \ell - 2$ from a self-orthogonal code $\mathscr{D}_{\ell-2}$ of type $\{\Lambda_{\gamma_\ell+2}, \lambda_{\gamma_\ell+3}, \ldots, \lambda_{\gamma_\ell+\ell-1}\}$ and length n over $\mathscr{R}_{\ell-2,m}$ satisfying property $(\mathfrak{P})$.

Output: A self-orthogonal code $\mathscr{D}_e$ of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ satisfying $Tor_i(\mathscr{D}_e) = \mathcal{C}^{(i)}$ for $1 \leq i \leq s + \theta_e$.

When $\lambda_j = \lambda_{e-j+2}$ for $1 \leq j \leq e$, we see, by Lemma 2.1, that the above methods give rise to construction of a self-dual code $\mathscr{D}_e$ of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ satisfying $Tor_i(\mathscr{D}_e) = \mathcal{C}^{(i)}$ for $1 \leq i \leq s + \theta_e$. Additionally, the recursive construction method described above coincides with the one used in [32] when $\mathfrak{s} = 1$. In the following example, we illustrate that the self-orthogonal code $\mathcal{C}_0$, as defined in Example 2.1, can be lifted to a self-orthogonal code over $\mathscr{R}_{8,2}$ by applying the above recursive construction method (A), as $2\kappa = 6 < 8 = e$ in this case.

Example 3.1. *Let $\mathscr{R}_{8,2}$ be the finite commutative chain ring as defined in Example 2.1. Let $n = 4$, $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 0$. Let $\mathcal{C}_0$ be a self-orthogonal code of length 4 and dimension 1 over $\mathcal{T}_2$ with a generator matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ (as defined in Example 2.1). Note that $\pi_1(\boldsymbol{d} \cdot \boldsymbol{d}) = \pi_3(\boldsymbol{d} \cdot \boldsymbol{d}) = 0$ for all $\boldsymbol{d} \in \mathcal{C}_0$, where each $\boldsymbol{d} \cdot \boldsymbol{d}$ is viewed as an element of $\mathscr{R}_{8,2}$.*

Here, we will show that the code $\mathcal{C}_0$ can be lifted to a self-orthogonal code of type $\{1,0,0,0,0,0,0,0\}$ and length 4 over $\mathscr{R}_{8,2}$ by applying the above recursive construction method (A). Towards this, let us consider a linear code $\mathscr{C}_2$ of type $\{1,0\}$ and length 4 over $\mathscr{R}_{2,2}$ with a generator matrix

$$\mathcal{G}_2 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + u\begin{bmatrix}0 & a_1 & b_1 & c_1\end{bmatrix},$$

where $a_1, b_1, c_1 \in \mathcal{T}_2$. Now, by the recursive construction method (A), we see that the code $\mathcal{C}_0$ can be lifted to a self-orthogonal code of type $\{1,0,0,0,0,0,0,0\}$ over $\mathscr{R}_{8,2}$ if and only if $\mathscr{C}_2$ is a self-orthogonal code over $\mathscr{R}_{2,2}$ satisfying property $(\mathfrak{P})$, which holds if and only if $a_1^2+b_1^2+c_1^2 \equiv 0 \pmod{u^2}$, or equivalently, $a_1+b_1+c_1 \equiv 0 \pmod{u}$ holds. It is easy to see that $a_1, b_1, c_1 \in \mathcal{T}_2$ satisfying $a_1+b_1+c_1 \equiv 0 \pmod{u}$ have precisely 16 distinct choices. Thus, for a fixed choice of $a_1, b_1, c_1 \in \mathcal{T}_2$ satisfying $a_1+b_1+c_1 \equiv 0 \pmod{u}$, the code $\mathscr{C}_2$ of length 4 over $\mathscr{R}_{2,2}$ with a generator matrix $\mathcal{G}_2$ is a self-orthogonal code satisfying property $(\mathfrak{P})$. Further, note that $\mathcal{G}_2\mathcal{G}_2^t \equiv u^2(a_1^2+b_1^2+c_1^2)+u^4(a_1+b_1+c_1)+u^6 \pmod{u^7}$. Since $a_1+b_1+c_1 \equiv 0 \pmod{u}$, let us suppose that $a_1+b_1+c_1 \equiv u\gamma_1+u^2\gamma_2+u^3\gamma_3+u^4\gamma_4+u^5\gamma_5+u^6\gamma_6 \pmod{u^7}$, where $\gamma_i \in \mathcal{T}_2$ for $1 \leq i \leq 6$. This implies that $\mathcal{G}_2\mathcal{G}_2^t \equiv u^4\gamma_1^2+u^5\gamma_1+u^6(1+\gamma_2+\gamma_2^2) \pmod{u^7}$.

Now, corresponding to such a choice of the code $\mathscr{C}_2$, consider a linear code $\mathscr{C}_4$ of type $\{1,0,0,0\}$ and length 4 over $\mathscr{R}_{4,2}$ with a generator matrix

$$\mathcal{G}_4 = \mathcal{G}_2 + u^2\begin{bmatrix}0 & a_2 & b_2 & c_2\end{bmatrix} + u^3\begin{bmatrix}0 & a_3 & b_3 & c_3\end{bmatrix},$$

where $a_2, b_2, c_2, a_3, b_3, c_3 \in \mathcal{T}_2$. Using again the recursive construction method (A), we see that the code $\mathscr{C}_4$ can be lifted to a self-orthogonal code of type $\{1,0,0,0,0,0,0,0\}$ over $\mathscr{R}_{8,2}$ if and only if $\mathscr{C}_4$ is a self-orthogonal code over $\mathscr{R}_{4,2}$ satisfying property $(\mathfrak{P})$, which holds if and only if $\mathcal{G}_4\mathcal{G}_4^t \equiv 0 \pmod{u^7}$. This holds if and only if $\gamma_1^2+a_2^2+b_2^2+c_2^2+u(\gamma_1+a_2+b_2+c_2)+u^2(1+\gamma_2+\gamma_2^2+a_3^2+b_3^2+c_3^2+a_3+b_3+c_3) \equiv 0 \pmod{u^3}$. Let us first choose $a_2, b_2, c_2 \in \mathcal{T}_2$ satisfying $a_2+b_2+c_2+\gamma_1 \equiv 0 \pmod{u}$, (note that a_2, b_2, c_2 have precisely 16 distinct choices). Next, for a fixed choice of $a_2, b_2, c_2 \in \mathcal{T}_2$ satisfying $a_2+b_2+c_2+\gamma_1 \equiv 0 \pmod{u}$, let us suppose that $a_2+b_2+c_2+\gamma_1 \equiv uz_1+u^2z_2 \pmod{u^3}$. Thus, we need to choose a_3, b_3, c_3 satisfying

$$1+\gamma_2+\gamma_2^2+a_3^2+b_3^2+c_3^2+a_3+b_3+c_3+z_1+z_1^2 \equiv 0 \pmod{u}. \tag{3.50}$$

It is easy to see that there are precisely 32 distinct choices for $a_3, b_3, c_3 \in \mathcal{T}_2$ satisfying (3.50). Now, for a fixed choice of a_3, b_3, c_3 satisfying (3.50), we see that the code $\mathscr{C}_4$ of length 4 over $\mathscr{R}_{4,2}$ with a generator matrix $\mathcal{G}_4$ is a self-orthogonal code satisfying property $(\mathfrak{P})$, and we have that $\mathcal{G}_4\mathcal{G}_4^t \equiv u^7w \pmod{u^8}$, where $w \in \mathcal{T}_2$.

Further, corresponding to such a choice of the code $\mathscr{C}_4$, consider a linear code $\mathscr{C}_6$ of type $\{1,0,0,0,0,0\}$ and length 4 over $\mathscr{R}_{6,2}$ with a generator matrix

$$\mathcal{G}_6 = \mathcal{G}_4 + u^4\begin{bmatrix}0 & a_4 & b_4 & c_4\end{bmatrix} + u^5\begin{bmatrix}0 & a_5 & b_5 & c_5\end{bmatrix},$$

where $a_4, b_4, c_4, a_5, b_5, c_5 \in \mathcal{T}_2$. By the recursive construction method (A), we see that the code $\mathscr{C}_6$ can be lifted to a self-orthogonal code of type $\{1,0,0,0,0,0,0,0\}$ over $\mathscr{R}_{8,2}$ if and only if $\mathscr{C}_6$ is a self-orthogonal code over $\mathscr{R}_{6,2}$ satisfying property $(\mathfrak{P})$, which holds if and only if $w+a_4+b_4+c_4 \equiv 0 \pmod{u}$. It is easy to see that $a_4, b_4, c_4, a_5, b_5, c_5 \in \mathcal{T}_2$ satisfying $w+a_4+b_4+c_4 \equiv 0 \pmod{u}$ have precisely 1024 distinct choices. Thus, for a fixed choice of $a_4, b_4, c_4, a_5, b_5, c_5 \in \mathcal{T}_2$ satisfying $w+a_4+b_4+c_4 \equiv 0 \pmod{u}$, we see that the code $\mathscr{C}_6$ of length 4 over $\mathscr{R}_{6,2}$ with a generator matrix $\mathcal{G}_6$ is a self-orthogonal code over $\mathscr{R}_{6,2}$ satisfying property $(\mathfrak{P})$.

Finally, consider a linear code $\mathscr{C}_8$ of type $\{1,0,0,0,0,0,0,0\}$ and length 4 over $\mathscr{R}_{8,2}$ with a generator matrix

$$\mathcal{G}_8 = \mathcal{G}_6 + u^6\begin{bmatrix}0 & a_6 & b_6 & c_6\end{bmatrix} + u^7\begin{bmatrix}0 & a_7 & b_7 & c_7\end{bmatrix},$$

where $a_6, b_6, c_6, a_7, b_7, c_7 \in \mathcal{T}_2$. Note that the code $\mathscr{C}_8$ is self-orthogonal for all choices of $a_6, b_6, c_6, a_7, b_7, c_7 \in \mathcal{T}_2$. This shows that the code $\mathcal{C}_0$ can be lifted to a self-orthogonal code of type $\{1,0,0,0,0,0,0,0\}$ and length 4 over $\mathscr{R}_{8,2}$ with the help of the recursive construction method (A).

4 Enumeration formulae for self-orthogonal and self-dual codes of length n over $\mathscr{R}_{e,m}$

In this section, we will obtain explicit enumeration formulae for self-orthogonal and self-dual codes of an arbitrary length over $\mathscr{R}_{e,m}$. In particular, when $e=2$, one can easily see that either $\mathscr{R}_{e,m} \simeq \mathbb{F}_{2^m}[u]/\langle u^2\rangle$ or $\mathscr{R}_{e,m} \simeq GR(4,m)$. The enumeration formula for self-dual codes over $\mathbb{F}_{2^m}[u]/\langle u^2\rangle$ was derived by Gaborit [14, Th. 3], while the enumeration formula for self-orthogonal codes over $\mathbb{F}_{2^m}[u]/\langle u^2\rangle$ was obtained by Galvez *et al.* [15, Th. 2]. Yadav and Sharma [31, Th. 5.1] counted all self-orthogonal and self-dual codes of an arbitrary length over $GR(4,m)$. In view of this, we assume, throughout this section, that $e \geq 3$. We also assume that n is a positive integer and $\lambda_1, \lambda_2, \ldots, \lambda_{e+1}$ are non-negative integers satisfying $n = \lambda_1 + \lambda_2 + \cdots + \lambda_{e+1}$. Now, let $\mathfrak{S}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e)$ and $\mathfrak{U}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e)$ denote the number of distinct self-orthogonal and self-dual codes of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$, respectively. We note, by Lemma 2.1 and Remark 2.1, that $\mathfrak{S}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e) = 0$ if $2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{e-j+1} + \lambda_{e-j+2} + \cdots + \lambda_j > n$ holds for some integer j satisfying $s+1 \leq j \leq e$, while $\mathfrak{U}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e) = 0$ if $\lambda_j \neq \lambda_{e-j+2}$ for some integer j satisfying $1 \leq j \leq e$. Further, if $\mathfrak{S}_e(n)$ and $\mathfrak{U}_e(n)$ denote the total number of self-orthogonal and self-dual codes of length n over $\mathscr{R}_{e,m}$, respectively, then by Lemma 2.1 and Remark 2.1, we get

$$\mathfrak{S}_e(n) = \textstyle\sum_1 \mathfrak{S}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e) \text{ and } \mathfrak{U}_e(n) = \textstyle\sum_2 \mathfrak{U}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e), \tag{4.1}$$

where the summation Σ_1 runs over all e-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_e)$ of non-negative integers satisfying $\lambda_1 + \lambda_2 + \cdots + \lambda_e \leq n$ and $2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{e-j+1} + \lambda_{e-j+2} + \cdots + \lambda_j \leq n$ for $s+1 \leq j \leq e$, while the summation Σ_2 runs over all e-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_e)$ of non-negative integers satisfying $\lambda_j = \lambda_{e-j+2}$ for $2 \leq j \leq e$ and $2(\lambda_1 + \lambda_2 + \cdots + \lambda_s) + (1+\theta_e)\lambda_{s+1} = n$. Therefore, to determine the numbers $\mathfrak{S}_e(n)$ and $\mathfrak{U}_e(n)$, it suffices to obtain enumeration formulae for the numbers $\mathfrak{S}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e)$ and $\mathfrak{U}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e)$. Further, in view of Remark 2.1, from now on, throughout this section, we assume that n is a positive integer and $\lambda_1, \lambda_2, \ldots, \lambda_{e+1}$ are non-negative integers satisfying $n = \lambda_1 + \lambda_2 + \cdots + \lambda_{e+1}$ and $2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{e-j+1} + \lambda_{e-j+2} + \cdots + \lambda_j \leq n$ for $s+1 \leq j \leq e$. We also assume that $\Lambda_0 = 0$ and $\Lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i$ for $1 \leq i \leq e+1$. Now, to obtain enumeration formulae for the numbers $\mathfrak{S}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e)$ and $\mathfrak{U}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e)$, we need to prove the following lemmas.

In the following lemma, we assume that $2\kappa \leq e$, and we consider a chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, such that (i) $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \leq i \leq s+\theta_e$, (ii) the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, and (iii) $\mathbf{1} \notin \mathcal{C}^{(s-\kappa+\theta_e)}$ if $n \equiv 4 \pmod 8$ and m is odd. Here, we will count the choices for a self-orthogonal code $\mathscr{D}_e$ of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ satisfying $Tor_i(\mathscr{D}_e) = \mathcal{C}^{(i)}$ for $1 \leq i \leq s+\theta_e$.

Lemma 4.1. *Let $e \geq 3$ be an integer satisfying $2\kappa \leq e$. Let $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ be a chain of self-orthogonal codes of length n over $\mathcal{T}_m$, such that (i) $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \leq i \leq s+\theta_e$, (ii) the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, and (iii) $\mathbf{1} \notin \mathcal{C}^{(s-\kappa+\theta_e)}$ if $n \equiv 4 \pmod 8$ and m is odd. The chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes over $\mathcal{T}_m$ gives rise to precisely*

$$2^{\epsilon}(2^m)^{\sum\limits_{i=1}^{s}\Lambda_i(n-\Lambda_{i+1})+\sum\limits_{j=1}^{s-1+\theta_e}\Lambda_{s+j}(n-\Lambda_{s+j+1}-\Lambda_{s+\theta_e-j})-\sum\limits_{a=1}^{s-\kappa_1-1}\Lambda_a-(1-\theta_e)\frac{\Lambda_s(\Lambda_s-1)}{2}+\mu}\prod_{\ell=s+1+\theta_e}^{e}\begin{bmatrix}\lambda_\ell+n-\Lambda_\ell-\Lambda_{e+1-\ell}\\ \lambda_\ell\end{bmatrix}_{2^m}$$

distinct self-orthogonal codes $\mathscr{D}_e$ of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ satisfying $Tor_i(\mathscr{D}_e) = \mathcal{C}^{(i)}$ for $1 \leq i \leq s+\theta_e$, where

$$(\epsilon,\mu) = \begin{cases} (0,\omega) & \text{if there exists an integer } \omega \text{ satisfying } 1 \leq \omega \leq \kappa_1-\theta_e,\ \mathbf{1} \in \mathcal{C}^{(s-\kappa_1-\omega)} \text{ and } \mathbf{1} \notin \mathcal{C}^{(s-\kappa_1-\omega-1)};\\ (1,\kappa_1) & \text{if } \mathbf{1} \in \mathcal{C}^{(s-\kappa+\theta_e)} \text{ with either } n \equiv 0 \pmod 8 \text{ or } n \equiv 4 \pmod 8 \text{ and } m \text{ being even};\\ (0,0) & \text{otherwise.}\end{cases}$$

Proof. The desired result follows by applying Propositions 3.1, 3.3, 3.4, 3.6 and 3.8 when e is even, whereas the desired result follows by applying Propositions 3.2, 3.3, 3.5, 3.6 and 3.8 when e is odd. □

In the following lemma, we assume that $2\kappa > e$, and we consider a chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, where $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \leq i \leq s+\theta_e$ and the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even. Here, we count the choices for a self-orthogonal code $\mathscr{D}_e$ of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ satisfying $Tor_i(\mathscr{D}_e) = \mathcal{C}^{(i)}$ for $1 \leq i \leq s+\theta_e$.

Lemma 4.2. *Let $e \geq 3$ be an integer satisfying $2\kappa > e$. Let $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ be a chain of self-orthogonal codes of length n over $\mathcal{T}_m$, such that (i) $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \leq i \leq s+\theta_e$, and (ii) the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even. The chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes over $\mathcal{T}_m$ gives rise to precisely*

$$(2^m)^{\sum\limits_{i=1}^{s}\Lambda_i(n-\Lambda_{i+1})+\sum\limits_{j=1}^{s-1+\theta_e}\Lambda_{s+j}(n-\Lambda_{s+j+1}-\Lambda_{s+\theta_e-j})-\sum\limits_{a=1}^{s-\kappa_1-1}\Lambda_a-(1-\theta_e)\frac{\Lambda_s(\Lambda_s-1)}{2}+\mu}\prod_{\ell=s+1+\theta_e}^{e}\begin{bmatrix}\lambda_\ell+n-\Lambda_\ell-\Lambda_{e+1-\ell}\\ \lambda_\ell\end{bmatrix}_{2^m}$$

distinct self-orthogonal codes $\mathscr{D}_e$ of type $\{\lambda_1, \lambda_2, \ldots, \lambda_e\}$ and length n over $\mathscr{R}_{e,m}$ satisfying $Tor_i(\mathscr{D}_e) = \mathcal{C}^{(i)}$ for $1 \leq i \leq s+\theta_e$, where

$$\mu = \begin{cases} \omega & \text{if there exists an integer } \omega \text{ satisfying } 1 \leq \omega \leq s-\kappa_1-2,\ \mathbf{1} \in \mathcal{C}^{(s-\kappa_1-\omega)} \text{ and } \mathbf{1} \notin \mathcal{C}^{(s-\kappa_1-\omega-1)};\\ s-\kappa_1-1 & \text{if } \mathbf{1} \in \mathcal{C}^{(1)};\\ 0 & \text{otherwise.}\end{cases}$$

Proof. When e is even, the desired result follows by applying Propositions 3.1, 3.3, 3.7 and 3.8, while in the case when e is odd, the desired result follows by applying Propositions 3.2, 3.3, 3.7 and 3.8. $\square$

Remark 4.1. *By Lemma 3.4, we see that there are precisely $\widehat{\sigma}_m(n;\mathfrak{d})$ distinct doubly even codes of length n and dimension $\mathfrak{d}$ over $\mathcal{T}_m$ containing the all-one vector and that there are precisely $\sigma_m(n;\mathfrak{d})$ distinct doubly even codes of length n and dimension $\mathfrak{d}$ over $\mathcal{T}_m$ that do not contain the all-one vector, where the numbers $\widehat{\sigma}_m(n;\mathfrak{d})$ and $\sigma_m(n;\mathfrak{d})$ are as obtained in Theorems 3.2 and 3.3 of Yadav and Sharma [31], respectively.*

We next proceed to count the choices for the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, where (i) $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \le i \le s+\theta_e$, (ii) the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, and (iii) $\mathbf{1} \notin \mathcal{C}^{(s-\kappa+\theta_e)}$ if $2\kappa \le e$, $n \equiv 4 \pmod 8$ and m is odd. Towards this, we note that every self-orthogonal code of length n over $\mathcal{T}_m$ is contained in the set $\mathcal{I}(\mathcal{T}_m^n) = \{\mathbf{a} \in \mathcal{T}_m^n : B_m(\mathbf{a},\mathbf{a}) = 0\}$, where $B_m(\mathbf{v},\mathbf{w}) = \pi_0(\mathbf{v}\cdot\mathbf{w}) = \pi_0(\sum\limits_{i=1}^{n} v_i w_i)$ for all $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathcal{T}_m^n$. Furthermore, one can easily see that the set $\mathcal{I}(\mathcal{T}_m^n)$ is an $(n-1)$-dimensional $\mathcal{T}_m$-linear subspace of $\mathcal{T}_m^n$ and that $\mathbf{1} \in \mathcal{I}(\mathcal{T}_m^n)$ if and only if n is even. We will now distinguish the following two cases: **(I)** $\mathbf{1} \notin \mathcal{C}^{(s-\kappa_1)}$ and **(II)** $\mathbf{1} \in \mathcal{C}^{(s-\kappa_1)}$.

In the following lemma, we count the choices for the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, such that (i) $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \le i \le s+\theta_e$, and (ii) $\mathcal{C}^{(s-\kappa_1)}$ is a doubly even code satisfying $\mathbf{1} \notin \mathcal{C}^{(s-\kappa_1)}$.

Lemma 4.3. *Let $e \ge 3$ be an integer. Let $N(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e})$ denote the number of distinct choices for the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, such that (i) $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \le i \le s+\theta_e$, and (ii) $\mathcal{C}^{(s-\kappa_1)}$ is a doubly even code satisfying $\mathbf{1} \notin \mathcal{C}^{(s-\kappa_1)}$. We have*

$$N(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) = \begin{cases} \sigma_m(n;\Lambda_{s-\kappa_1}) \prod\limits_{i=1}^{s-\kappa_1} \begin{bmatrix} \Lambda_i \\ \lambda_i \end{bmatrix}_{2^m} \prod\limits_{j=s-\kappa_1+1}^{s+\theta_e} \begin{bmatrix} \Lambda_j - \Lambda_{s-\kappa_1} \\ \lambda_j \end{bmatrix}_{2^m} \prod\limits_{\ell=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-1} \left(\dfrac{2^{m(n-2\ell-1)}-1}{2^{m(\ell+1-\Lambda_{s-\kappa_1})}-1} \right) \\ \text{if } n \text{ is odd;} \\ \left(\mathfrak{D}_0 \left(\dfrac{2^{m(n-\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-1}{2^{m(\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-1} \right) + \mathfrak{B}_0 \left(\dfrac{2^{m(n-2\Lambda_{s+\theta_e})}+2^{m(\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-2}{2^{m(\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-1} \right) \right) \\ \times \prod\limits_{i=1}^{s-\kappa_1} \begin{bmatrix} \Lambda_i \\ \lambda_i \end{bmatrix}_{2^m} \prod\limits_{j=s-\kappa_1+1}^{s+\theta_e} \begin{bmatrix} \Lambda_j - \Lambda_{s-\kappa_1} \\ \lambda_j \end{bmatrix}_{2^m} \prod\limits_{\ell=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-2} \left(\dfrac{2^{m(n-2\ell-2)}-1}{2^{m(\ell+1-\Lambda_{s-\kappa_1})}-1} \right) \\ \text{if } n \text{ is even with } \Lambda_{s+\theta_e} \neq \Lambda_{s-\kappa_1}; \\ (\mathfrak{D}_0 + \mathfrak{B}_0) \prod\limits_{i=1}^{s-\kappa_1} \begin{bmatrix} \Lambda_i \\ \lambda_i \end{bmatrix}_{2^m} \quad \text{if } n \text{ is even and } \Lambda_{s+\theta_e} = \Lambda_{s-\kappa_1}, \end{cases}$$

where

$$\mathfrak{D}_0 = \begin{cases} 1 & \text{if } \Lambda_{s-\kappa_1} = 0; \\ \dfrac{(2^{m(\frac{n}{2}-1)}-1)(2^{m(\frac{n}{2}-\Lambda_{s-\kappa_1}-1)}+1)}{2^m-1} \prod\limits_{i=1}^{\Lambda_{s-\kappa_1}-1} \left(\dfrac{2^{m(n-2-2i)}-1}{2^{m(i+1)}-1} \right) & \text{if } 1 \le \Lambda_{s-\kappa_1} \le \frac{n}{2}-1 \text{ and } n \equiv 2, 6 \pmod 8; \\ \prod\limits_{i=0}^{\Lambda_{s-\kappa_1}-1} \left(\dfrac{2^{m(n-2i-3)}+2^{m(\frac{n}{2}-1-i)}-2^{m(\frac{n}{2}-i-2)}-1}{2^{m(i+1)}-1} \right) & \text{if } 1 \le \Lambda_{s-\kappa_1} \le \frac{n}{2}-1 \text{ with either } n \equiv 0 \pmod 8 \\ & \text{or } n \equiv 4 \pmod 8 \text{ and } m \text{ is even;} \\ \prod\limits_{i=0}^{\Lambda_{s-\kappa_1}-1} \left(\dfrac{2^{m(n-2i-3)}-2^{m(\frac{n}{2}-1-i)}+2^{m(\frac{n}{2}-i-2)}-1}{2^{m(i+1)}-1} \right) & \text{if } 1 \le \Lambda_{s-\kappa_1} \le \frac{n}{2}-1, n \equiv 4 \pmod 8 \text{ and} \\ & m \text{ is odd;} \\ 0 & \text{if } \Lambda_{s-\kappa_1} > \frac{n}{2}-1 \end{cases} \tag{4.2}$$

and

$$\mathfrak{B}_0=\begin{cases}\left(2^{m(n-2\Lambda_{s-\kappa_1}-1)}-2^{m(\frac{n}{2}-\Lambda_{s-\kappa_1}-1)}\right)\prod\limits_{i=0}^{\Lambda_{s-\kappa_1}-2}\left(\dfrac{2^{m(n-2i-3)}+2^{m(\frac{n}{2}-1-i)}-2^{(m\frac{n}{2}-2-i)}-1}{2^{m(i+1)}-1}\right)\\ \textit{if } 1\le\Lambda_{s-\kappa_1}\le\frac{n}{2}-1 \textit{ and } n\equiv 2,6 \pmod 8;\\ \left(2^{m(n-2\Lambda_{s-\kappa_1}-1)}+2^{m(\frac{n}{2}-\Lambda_{s-\kappa_1})}-2^{m(\frac{n}{2}-\Lambda_{s-\kappa_1}-1)}-1\right)\prod\limits_{i=0}^{\Lambda_{s-\kappa_1}-2}\left(\dfrac{2^{m(n-2i-3)}+2^{m(\frac{n}{2}-1-i)}-2^{m(\frac{n}{2}-i-2)}-1}{2^{m(i+1)}-1}\right)\\ \textit{if } 1\le\Lambda_{s-\kappa_1}\le\frac{n}{2}-1 \textit{ with either } n\equiv 0 \pmod 8 \textit{ or } n\equiv 4 \pmod 8 \textit{ and } m \textit{ is even;}\\ \left(2^{m(n-2\Lambda_{s-\kappa_1}-1)}-2^{m(\frac{n}{2}-\Lambda_{s-\kappa_1})}+2^{m(\frac{n}{2}-\Lambda_{s-\kappa_1}-1)}-1\right)\prod\limits_{i=0}^{\Lambda_{s-\kappa_1}-2}\left(\dfrac{2^{m(n-2i-3)}+2^{m(\frac{n}{2}-i-1)}-2^{m(\frac{n}{2}-i-2)}-1}{2^{m(i+1)}-1}\right)\\ \textit{if } 1\le\Lambda_{s-\kappa_1}\le\frac{n}{2}-1, n\equiv 4 \pmod 8 \textit{ and } m \textit{ is odd;}\\ 0 \qquad \textit{if either } \Lambda_{s-\kappa_1}=0 \textit{ or } \Lambda_{s-\kappa_1}>\frac{n}{2}-1.\end{cases} \tag{4.3}$$

Proof. To prove the result, we recall that $\mathbf{1}\in\mathcal{I}(\mathcal{T}_m^n)$ if and only if n is even. Accordingly, we will consider the following two cases separately: (I) n is odd, and (II) n is even.

(I) Let n be odd. Here, we note that $(\mathcal{I}(\mathcal{T}_m^n),B_m(\cdot,\cdot)\restriction_{\mathcal{I}(\mathcal{T}_m^n)\times(\mathcal{I}(\mathcal{T}_m^n))})$ is a symplectic space of dimension $n-1$ over $\mathcal{T}_m$. Further, by Remark 4.1, one can easily see that there are precisely $\sigma_m(n;\Lambda_{s-\kappa_1})\prod\limits_{i=1}^{s-\kappa_1}\begin{bmatrix}\Lambda_i\\ \lambda_i\end{bmatrix}_{2^m}$ distinct choices for the chain $\mathcal{C}^{(1)}\subseteq\mathcal{C}^{(2)}\subseteq\cdots\subseteq\mathcal{C}^{(s-\kappa_1)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, where $\mathcal{C}^{(s-\kappa_1)}$ is a doubly even code satisfying $\mathbf{1}\notin\mathcal{C}^{(s-\kappa_1)}$.

Now, for a given choice of $\mathcal{C}^{(s-\kappa_1)}$, we will count the choices for a self-orthogonal code $\mathcal{C}^{(s+\theta_e)}$ satisfying $\mathcal{C}^{(s-\kappa_1)}\subseteq\mathcal{C}^{(s+\theta_e)}\subseteq(\mathcal{C}^{(s-\kappa_1)})^{\perp_{B_m}}$. Towards this, we see, working as in case (I) in the proof of Theorem 3.1 of Yadav and Sharma [31], that $\mathcal{C}^{(s-\kappa_1)}=\langle\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{\Lambda_{s-\kappa_1}}\rangle$, where $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{\Lambda_{s-\kappa_1}}$ are mutually orthogonal singular vectors in $\mathcal{I}(\mathcal{T}_m^n)$ that are linearly independent over $\mathcal{T}_m$. We further see, by [29, pp. 69-70], that the symplectic space $(\mathcal{I}(\mathcal{T}_m^n),B_m(\cdot,\cdot)\restriction_{\mathcal{I}(\mathcal{T}_m^n)\times(\mathcal{I}(\mathcal{T}_m^n))})$ admits an orthogonal direct sum decomposition of the form: $\mathcal{I}(\mathcal{T}_m^n)=\mathcal{X}\perp\mathcal{Y}$ with $\mathcal{X}=\langle\mathbf{v}_1,\mathbf{v}_1'\rangle\perp\langle\mathbf{v}_2,\mathbf{v}_2'\rangle\perp\cdots\perp\langle\mathbf{v}_{\Lambda_{s-\kappa_1}},\mathbf{v}'_{\Lambda_{s-\kappa_1}}\rangle$ and $\mathcal{Y}=\langle\mathbf{v}_{\Lambda_{s-\kappa_1}+1},\mathbf{v}'_{\Lambda_{s-\kappa_1}+1}\rangle\perp\langle\mathbf{v}_{\Lambda_{s-\kappa_1}+2},\mathbf{v}'_{\Lambda_{s-\kappa_1}+2}\rangle\perp\cdots\perp\langle\mathbf{v}_{\frac{n-1}{2}},\mathbf{v}'_{\frac{n-1}{2}}\rangle$, where $(\mathbf{v}_i,\mathbf{v}_i')$ is a hyperbolic pair in $\mathcal{I}(\mathcal{T}_m^n)$ for $1\le i\le\frac{n-1}{2}$. We further note that each $\mathbf{w}\in(\mathcal{C}^{(s-\kappa_1)})^{\perp_{B_m}}$ can be uniquely written as $\mathbf{w}=\sum\limits_{i=1}^{\frac{n-1}{2}}(\alpha_i\mathbf{v}_i+\beta_i\mathbf{v}_i')$, where $\alpha_i,\beta_i\in\mathcal{T}_m$ for $1\le i\le\frac{n-1}{2}$. As $\mathbf{w}\in(\mathcal{C}^{(s-\kappa_1)})^{\perp_{B_m}}$, we have $B_m(\mathbf{w},\mathbf{v}_j)=0$ for $1\le j\le\Lambda_{s-\kappa_1}$, which implies that $\beta_j=0$ for $1\le j\le\Lambda_{s-\kappa_1}$. Thus, each $\mathbf{w}\in(\mathcal{C}^{(s-\kappa_1)})^{\perp_{B_m}}$ is of the form $\mathbf{w}=\sum\limits_{i=1}^{\Lambda_{s-\kappa_1}}\alpha_i\mathbf{v}_i+\sum\limits_{j=\Lambda_{s-\kappa_1}+1}^{\frac{n-1}{2}}(\alpha_j\mathbf{v}_j+\beta_j\mathbf{v}_j')$, which implies that $\langle\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{\Lambda_{s-\kappa_1}},\mathbf{w}\rangle=\langle\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{\Lambda_{s-\kappa_1}},\sum\limits_{j=\Lambda_{s-\kappa_1}+1}^{\frac{n-1}{2}}(\alpha_j\mathbf{v}_j+\beta_j\mathbf{v}_j')\rangle$. In view of this, we assume, without any loss of generality, that $\mathcal{C}^{(s+\theta_e)}=\langle\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{\Lambda_{s-\kappa_1}}\rangle\perp\langle\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}}\rangle$, where $\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}}$ are mutually orthogonal isotropic vectors in $\mathcal{Y}$ that are linearly independent over $\mathcal{T}_m$. We further note that $\mathcal{Y}$ is a symplectic space over $\mathcal{T}_m$ of dimension $n-1-2\Lambda_{s-\kappa_1}$ and Witt index $\left(\frac{n-1}{2}\right)-\Lambda_{s-\kappa_1}$. One can easily observe that the number of choices for a self-orthogonal code $\mathcal{C}^{(s+\theta_e)}$ satisfying $\mathcal{C}^{(s-\kappa_1)}\subseteq\mathcal{C}^{(s+\theta_e)}\subseteq(\mathcal{C}^{(s-\kappa_1)})^{\perp_{B_m}}$ is equal to the number of $(\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})$-dimensional totally isotropic subspaces of the symplectic space $\mathcal{Y}$, which, by Exercise 8.1 (ii) of [29], has precisely $\prod\limits_{i=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-1}\left(\frac{2^{m(n-2i-1)}-1}{2^{m(i+1-\Lambda_{s-\kappa_1})}-1}\right)$ distinct choices. Finally, for a given choice of $\mathcal{C}^{(s-\kappa_1)}$, we note that the chain $\mathcal{C}^{(s-\kappa_1+1)}\subseteq\mathcal{C}^{(s-\kappa_1+2)}\subseteq\cdots\subseteq\mathcal{C}^{(s+\theta_e)}$ of codes has precisely

$$\prod_{i=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-1}\left(\frac{2^{m(n-2i-1)}-1}{2^{m(i+1-\Lambda_{s-\kappa_1})}-1}\right)\prod_{j=s-\kappa_1+1}^{s+\theta_e}\begin{bmatrix}\Lambda_j-\Lambda_{s-\kappa_1}\\ \lambda_j\end{bmatrix}_{2^m}$$

distinct choices. From this, the desired result follows.

(II) Next, let n be even. In this case, we have $\mathbf{1}\in\mathcal{I}(\mathcal{T}_m^n)$. Let us choose an $(n-2)$-dimensional $\mathcal{T}_m$-linear subspace $\mathcal{U}_m'$ of $\mathcal{I}(\mathcal{T}_m^n)$ satisfying $\mathbf{1}\notin\mathcal{U}_m'$. It is easy to see that $\mathcal{I}(\mathcal{T}_m^n)=\mathcal{U}_m'\perp\langle\mathbf{1}\rangle$. Note that $(\mathcal{U}_m',B_m(\cdot,\cdot)\restriction_{\mathcal{U}_m'\times\mathcal{U}_m'})$ is an $(n-2)$-dimensional symplectic space over $\mathcal{T}_m$.

Now, we will count the choices for the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, where $\mathcal{C}^{(s-\kappa_1)}$ is a doubly even code satisfying $\mathbf{1} \notin \mathcal{C}^{(s-\kappa_1)}$.

Towards this, we first note that $N(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) = 1$ if $\Lambda_{s+\theta_e} = 0$. So we assume, throughout the proof, that $\Lambda_{s+\theta_e} \geq 1$.

When $\Lambda_{s-\kappa_1} = 0$, working as in case (II) in the proof of Proposition 3.1 of Sharma and Kaur [28], we get

$$N(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) = \prod_{i=0}^{\Lambda_{s+\theta_e}-2} \left(\frac{2^{m(n-2i-2)} - 1}{2^{m(i+1)} - 1} \right) \left(\frac{2^{m(n-\Lambda_{s+\theta_e})} - 1}{2^{m\Lambda_{s+\theta_e}} - 1} \right) \prod_{j=s-\kappa_1+1}^{s+\theta_e} \begin{bmatrix} \Lambda_j \\ \lambda_j \end{bmatrix}_{2^m}.$$

On the other hand, when $\Lambda_{s-\kappa_1} \geq 1$, working as in case (II) in the proof of Theorem 3.1 of Yadav and Sharma [31], we see that the code $\mathcal{C}^{(s-\kappa_1)}$ is of the following two forms:

- $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1}} \rangle$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1}} \in \mathcal{U}'_m$ are mutually orthogonal singular vectors that are linearly independent over $\mathcal{T}_m$.
- $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1}-1}, \mathbf{1}+\mathbf{v}_{\Lambda_{s-\kappa_1}} \rangle$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1}} \in \mathcal{U}'_m \setminus \{\mathbf{0}\}$ are such that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1}-1}$, $\mathbf{1} + \mathbf{v}_{\Lambda_{s-\kappa_1}}$ are mutually orthogonal singular vectors that are linearly independent over $\mathcal{T}_m$.

Accordingly, we will consider the following two cases:

(i) First of all, suppose that the code $\mathcal{C}^{(s-\kappa_1)}$ is of the form $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1}} \rangle$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1}} \in \mathcal{U}'_m$ are mutually orthogonal singular vectors that are linearly independent over $\mathcal{T}_m$. Using Lemma 3.4 and working as in case (II) in the proof of Theorem 3.1 of Yadav and Sharma [31], we see that such a code $\mathcal{C}^{(s-\kappa_1)}$ has precisely $\mathfrak{D}_0$ distinct choices, where the number $\mathfrak{D}_0$ is given by (4.2). Given such a choice of $\mathcal{C}^{(s-\kappa_1)}$, we will count the choices for the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$.

Towards this, one can easily see that there are precisely $\mathfrak{D}_0 \prod_{j=1}^{s-\kappa_1} \begin{bmatrix} \Lambda_j \\ \lambda_j \end{bmatrix}_{2^m}$ distinct choices for the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s-\kappa_1)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$.

Now, if $\Lambda_{s+\theta_e} = \Lambda_{s-\kappa_1}$, it is easy to see that the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ has precisely $\mathfrak{D}_0 \prod_{j=1}^{s-\kappa_1} \begin{bmatrix} \Lambda_j \\ \lambda_j \end{bmatrix}_{2^m}$ distinct choices.

Next, if $\Lambda_{s+\theta_e} \neq \Lambda_{s-\kappa_1}$, for a given choice of $\mathcal{C}^{(s-\kappa_1)}$, we need to count the choices for a self-orthogonal code $\mathcal{C}^{(s+\theta_e)}$ satisfying $\mathcal{C}^{(s-\kappa_1)} \subseteq \mathcal{C}^{(s+\theta_e)} \subseteq (\mathcal{C}^{(s-\kappa_1)})^{\perp_{B_m}}$. Here, we see, by [29, pp. 69-70], that the space $(\mathcal{I}(\mathcal{T}_m^n), B_m(\cdot,\cdot)\restriction_{\mathcal{I}(\mathcal{T}_m^n)\times(\mathcal{I}(\mathcal{T}_m^n))})$ admits an orthogonal direct sum decomposition of the form: $\mathcal{I}(\mathcal{T}_m^n) = \mathcal{X}_0 \perp \mathcal{Y}_0 \perp \langle \mathbf{1} \rangle$ with $\mathcal{X}_0 = \langle \mathbf{v}_1, \mathbf{v}'_1 \rangle \perp \langle \mathbf{v}_2, \mathbf{v}'_2 \rangle \perp \cdots \perp \langle \mathbf{v}_{\Lambda_{s-\kappa_1}}, \mathbf{v}'_{\Lambda_{s-\kappa_1}} \rangle$ and $\mathcal{Y}_0 = \langle \mathbf{v}_{\Lambda_{s-\kappa_1}+1}, \mathbf{v}'_{\Lambda_{s-\kappa_1}+1} \rangle \perp \langle \mathbf{v}_{\Lambda_{s-\kappa_1}+2}, \mathbf{v}'_{\Lambda_{s-\kappa_1}+2} \rangle \perp \cdots \perp \langle \mathbf{v}_{\frac{n-2}{2}}, \mathbf{v}'_{\frac{n-2}{2}} \rangle$, where $(\mathbf{v}_i, \mathbf{v}'_i)$ is a hyperbolic pair in $\mathcal{I}(\mathcal{T}_m^n)$ for $1 \leq i \leq \frac{n-2}{2}$. We further note that each $\mathbf{z} \in (\mathcal{C}^{(s-\kappa_1)})^{\perp_{B_m}}$ can be uniquely written as $\mathbf{z} = \sum_{i=1}^{\frac{n-2}{2}} (\alpha_i \mathbf{v}_i + \beta_i \mathbf{v}'_i) + \gamma \mathbf{1}$, where $\alpha_i, \beta_i, \gamma \in \mathcal{T}_m$ for $1 \leq i \leq \frac{n-2}{2}$. Since $\mathbf{z} \in (\mathcal{C}^{(s-\kappa_1)})^{\perp_{B_m}}$, we have $B_m(\mathbf{z}, \mathbf{v}_j) = 0$ for $1 \leq j \leq \Lambda_{s-\kappa_1}$, which implies that $\beta_j = 0$ for $1 \leq j \leq \Lambda_{s-\kappa_1}$. Thus, each $\mathbf{z} \in (\mathcal{C}^{(s-\kappa_1)})^{\perp_{B_m}}$ is of the form $\mathbf{z} = \sum_{i=1}^{\Lambda_{s-\kappa_1}} \alpha_i \mathbf{v}_i + \sum_{j=\Lambda_{s-\kappa_1}+1}^{\frac{n-2}{2}} (\alpha_j \mathbf{v}_j + \beta_j \mathbf{v}'_j) + \gamma \mathbf{1}$, which implies that $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1}}, \mathbf{z} \rangle = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1}}, \sum_{j=\Lambda_{s-\kappa_1}+1}^{\frac{n-2}{2}} (\alpha_j \mathbf{v}_j + \beta_j \mathbf{v}'_j) + \gamma \mathbf{1} \rangle$. In view of this, we assume, without any loss of generality, that the code $\mathcal{C}^{(s+\theta_e)}$ is of the following two forms:

($\star$) $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1}} \rangle \perp \langle \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}} \rangle$, where $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}}$ are mutually orthogonal isotropic vectors in $\mathcal{Y}_0$ that are linearly independent over $\mathcal{T}_m$.

($\dagger$) $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1}} \rangle \perp \langle \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}-1}, \mathbf{1} + \mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}} \rangle$, where $\mathbf{w}_1, \mathbf{w}_2, \ldots,$ $\mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}-1}$ are mutually orthogonal isotropic vectors in $\mathcal{Y}_0$, $\mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}} \in \mathcal{Y}_0$, and the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}-1}, \mathbf{1} + \mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}}$ are linearly independent over $\mathcal{T}_m$.

Here, working as in case (II) in the proof of Proposition 3.1 of Sharma and Kaur [28] and by Exercise 8.1 (ii) of [29], we observe that the subspaces of the forms ($\star$) and ($\dagger$), and hence the code $\mathcal{C}^{(s+\theta_e)}$ has precisely

$$\prod_{i=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-2}\left(\frac{2^{m(n-2i-2)}-1}{2^{m(i+1-\Lambda_{s-\kappa_1})}-1}\right)\left(\frac{2^{m(n-\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-1}{2^{m(\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-1}\right)$$

distinct choices. Furthermore, for a given choice of $\mathcal{C}^{(s-\kappa_1)}$, it is easy to see that the desired chain $\mathcal{C}^{(s-\kappa_1+1)} \subseteq \mathcal{C}^{(s-\kappa_1+2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of codes has precisely

$$\prod_{i=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-2}\left(\frac{2^{m(n-2i-2)}-1}{2^{m(i+1-\Lambda_{s-\kappa_1})}-1}\right)\left(\frac{2^{m(n-\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-1}{2^{m(\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-1}\right)\prod_{j=s-\kappa_1+1}^{s+\theta_e}\begin{bmatrix}\Lambda_j-\Lambda_{s-\kappa_1}\\ \lambda_j\end{bmatrix}_{2^m}$$

distinct choices.

(ii) Let us suppose that the code $\mathcal{C}^{(s-\kappa_1)}$ is of the form $\langle \mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{\Lambda_{s-\kappa_1}-1},\mathbf{1}+\mathbf{v}_{\Lambda_{s-\kappa_1}}\rangle$ with $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{\Lambda_{s-\kappa_1}}\in\mathcal{U}'_m\setminus\{\mathbf{0}\}$, where $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{\Lambda_{s-\kappa_1}-1},\mathbf{1}+\mathbf{v}_{\Lambda_{s-\kappa_1}}$ are mutually orthogonal singular vectors that are linearly independent over $\mathcal{T}_m$. Using Lemma 3.4 and working as in case (II) in the proof of Theorem 3.1 of Yadav and Sharma [31], we see that such a code $\mathcal{C}^{(s-\kappa_1)}$ has precisely $\mathfrak{B}_0$ distinct choices, where the number $\mathfrak{B}_0$ is given by (4.3). Given such a choice of $\mathcal{C}^{(s-\kappa_1)}$, we will count the choices for the chain $\mathcal{C}^{(1)}\subseteq\mathcal{C}^{(2)}\subseteq\cdots\subseteq\mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$. Here, one can easily see that there are precisely $\mathfrak{B}_0\prod\limits_{j=1}^{s-\kappa_1}\left[\begin{smallmatrix}\Lambda_j\\ \lambda_j\end{smallmatrix}\right]_{2^m}$ distinct choices for the chain $\mathcal{C}^{(1)}\subseteq\mathcal{C}^{(2)}\subseteq\cdots\subseteq\mathcal{C}^{(s-\kappa_1)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$.

Now, if $\Lambda_{s+\theta_e}=\Lambda_{s-\kappa_1}$, it is easy to see that the chain $\mathcal{C}^{(1)}\subseteq\mathcal{C}^{(2)}\subseteq\cdots\subseteq\mathcal{C}^{(s+\theta_e)}$ has precisely $\mathfrak{B}_0\prod\limits_{j=1}^{s-\kappa_1}\left[\begin{smallmatrix}\Lambda_j\\ \lambda_j\end{smallmatrix}\right]_{2^m}$ distinct choices.

Next, if $\Lambda_{s+\theta_e}\neq\Lambda_{s-\kappa_1}$, for a given choice of $\mathcal{C}^{(s-\kappa_1)}$, we need to count the choices for a self-orthogonal code $\mathcal{C}^{(s+\theta_e)}$ satisfying $\mathcal{C}^{(s-\kappa_1)}\subseteq\mathcal{C}^{(s+\theta_e)}\subseteq(\mathcal{C}^{(s-\kappa_1)})^{\perp_{B_m}}$. Here, working as in case (i), we observe that the code $\mathcal{C}^{(s+\theta_e)}$ is either of the form $\langle \mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{\Lambda_{s-\kappa_1}-1},\mathbf{1}+\mathbf{v}_{\Lambda_{s-\kappa_1}}\rangle\perp\langle\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}}\rangle$, or of the form $\langle \mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{\Lambda_{s-\kappa_1}-1},\mathbf{1}+\mathbf{v}_{\Lambda_{s-\kappa_1}}\rangle\perp\langle\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}-1},\mathbf{1}\rangle$, where $\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_{\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1}}\in\mathcal{Y}_0$ are mutually orthogonal, isotropic and linearly independent vectors over $\mathcal{T}_m$. Again, working as in case (II) in the proof of Proposition 3.1 of Sharma and Kaur [28] and by Exercise 8.1 (ii) of [29], we see that such a subspace, and hence the code $\mathcal{C}^{(s+\theta_e)}$ has precisely

$$\left(\frac{2^{m(n-2\Lambda_{s+\theta_e})}+2^{m(\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-2}{2^{m(\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-1}\right)\prod_{i=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-2}\left(\frac{2^{m(n-2i-2)}-1}{2^{m(i+1-\Lambda_{s-\kappa_1})}-1}\right)$$

distinct choices. Furthermore, for a given choice of $\mathcal{C}^{(s-\kappa_1)}$, it is easy to see that the desired chain $\mathcal{C}^{(s-\kappa_1+1)} \subseteq \mathcal{C}^{(s-\kappa_1+2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of codes has precisely

$$\left(\frac{2^{m(n-2\Lambda_{s+\theta_e})}+2^{m(\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-2}{2^{m(\Lambda_{s+\theta_e}-\Lambda_{s-\kappa_1})}-1}\right)\prod_{i=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-2}\left(\frac{2^{m(n-2i-2)}-1}{2^{m(i+1-\Lambda_{s-\kappa_1})}-1}\right)\prod_{j=s-\kappa_1+1}^{s+\theta_e}\begin{bmatrix}\Lambda_j-\Lambda_{s-\kappa_1}\\ \lambda_j\end{bmatrix}_{2^m}$$

distinct choices.

On combining the cases (i) and (ii) above, the desired result follows in the case when n is even. $\square$

We next see, by Theorem 3.2 of Yadav and Sharma [31], that if a doubly even code of length n over $\mathcal{T}_m$ contains $\mathbf{1}$, then $n\equiv 0,4 \pmod 8$. We will now consider the case $n\equiv 0,4\pmod 8$ and count all distinct choices for the chain $\mathcal{C}^{(1)}\subseteq\mathcal{C}^{(2)}\subseteq\cdots\subseteq\mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, where (i) $\dim\mathcal{C}^{(i)}=\Lambda_i$ for $1\leq i\leq s+\theta_e$, and (ii) $\mathcal{C}^{(s-\kappa_1)}$ is a doubly even code satisfying $\mathbf{1}\in\mathcal{C}^{(s-\kappa_1)}$. Here, the following three cases arise: (**I**) there exists an integer ω satisfying $0\leq\omega\leq\kappa_1-\theta_e$ if $2\kappa\leq e$, while $0\leq\omega\leq s-\kappa_1-2$ if $2\kappa>e$, $\mathbf{1}\in\mathcal{C}^{(s-\kappa_1-\omega)}$ and $\mathbf{1}\notin\mathcal{C}^{(s-\kappa_1-\omega-1)}$, (**II**) $\mathbf{1}\in\mathcal{C}^{(s-\kappa+\theta_e)}$ and $2\kappa\leq e$ with either $n\equiv 0\pmod 8$ or $n\equiv 4\pmod 8$ and m being even, and (**III**) $\mathbf{1}\in\mathcal{C}^{(1)}$ and $2\kappa>e$. In the following lemma, we count the choices for the chain $\mathcal{C}^{(1)}\subseteq\mathcal{C}^{(2)}\subseteq\cdots\subseteq\mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$ corresponding to case (**I**).

Lemma 4.4. *Let $e\geq 3$ be an integer. Let ω be a fixed integer satisfying $0\leq\omega\leq\kappa_1-\theta_e$ if $2\kappa\leq e$, while $0\leq\omega\leq s-\kappa_1-2$ if $2\kappa>e$. Let $Y_\omega(\lambda_1,\lambda_2,\ldots,\lambda_{s+\theta_e})$ denote the number of distinct choices for the chain $\mathcal{C}^{(1)}\subseteq\mathcal{C}^{(2)}\subseteq\cdots\subseteq\mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, where (i) $\dim\mathcal{C}^{(i)}=\Lambda_i$ for $1\leq i\leq s+\theta_e$,* *(ii) the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, and (iii) $\mathbf{1} \in \mathcal{C}^{(s-\kappa_1-\omega)}$ and $\mathbf{1} \notin \mathcal{C}^{(s-\kappa_1-\omega-1)}$. We have $n \equiv 0, 4 \pmod 8$, and $Y_\omega(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) = 0$ if $\Lambda_{s-\kappa_1-\omega} = 0$. Further, when $\Lambda_{s-\kappa_1-\omega} \geq 1$, we have*

$$\begin{aligned} Y_\omega(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) &= \widehat{\sigma}_m(n; \Lambda_{s-\kappa_1})(2^m)^{\Lambda_{s-\kappa_1-\omega-1}} \begin{bmatrix} \Lambda_{s-\kappa_1} - 1 \\ \Lambda_{s-\kappa_1} - \Lambda_{s-\kappa_1-\omega} \end{bmatrix}_{2^m} \begin{bmatrix} \Lambda_{s-\kappa_1-\omega} - 1 \\ \Lambda_{s-\kappa_1-\omega-1} \end{bmatrix}_{2^m} \prod_{i=1}^{s-\kappa_1-\omega-1} \begin{bmatrix} \Lambda_i \\ \lambda_i \end{bmatrix}_{2^m} \\ &\times \prod_{a=s-\kappa_1-\omega+1}^{s-\kappa_1} \begin{bmatrix} \Lambda_a - \Lambda_{s-\kappa_1-\omega} \\ \lambda_a \end{bmatrix}_{2^m} \prod_{b=s-\kappa_1+1}^{s+\theta_e} \begin{bmatrix} \Lambda_b - \Lambda_{s-\kappa_1} \\ \lambda_b \end{bmatrix}_{2^m} \prod_{g=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-1} \Big(\frac{2^{m(n-2g)} - 1}{2^{m(g+1-\Lambda_{s-\kappa_1})} - 1} \Big) \end{aligned}$$

Proof. To prove the result, we first note that $\mathbf{1} \in \mathcal{C}^{(s-\kappa_1-\omega)}$, which, by Theorem 3.2 of Yadav and Sharma [31], implies that $n \equiv 0, 4 \pmod 8$. We further observe, by Remark 4.1, that the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s-\kappa_1-\omega)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$ has precisely

$$\widehat{\sigma}_m(n; \Lambda_{s-\kappa_1-\omega}) \Big(\begin{bmatrix} \Lambda_{s-\kappa_1-\omega} \\ \Lambda_{s-\kappa_1-\omega-1} \end{bmatrix}_{2^m} - \begin{bmatrix} \Lambda_{s-\kappa_1-\omega} - 1 \\ \Lambda_{s-\kappa_1-\omega-1} - 1 \end{bmatrix}_{2^m} \Big) \prod_{i=1}^{s-\kappa_1-\omega-1} \begin{bmatrix} \Lambda_i \\ \lambda_i \end{bmatrix}_{2^m}$$

distinct choices. Next, by the Pascal's Identity for Gaussian binomial coefficients, we note that

$$\begin{bmatrix} \Lambda_{s-\kappa_1-\omega} \\ \Lambda_{s-\kappa_1-\omega-1} \end{bmatrix}_{2^m} - \begin{bmatrix} \Lambda_{s-\kappa_1-\omega} - 1 \\ \Lambda_{s-\kappa_1-\omega-1} - 1 \end{bmatrix}_{2^m} = (2^m)^{\Lambda_{s-\kappa_1-\omega-1}} \begin{bmatrix} \Lambda_{s-\kappa_1-\omega} - 1 \\ \Lambda_{s-\kappa_1-\omega-1} \end{bmatrix}_{2^m}.$$

This implies that the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s-\kappa_1-\omega)}$ of codes has precisely

$$\widehat{\sigma}_m(n; \Lambda_{s-\kappa_1-\omega})(2^m)^{\Lambda_{s-\kappa_1-\omega-1}} \begin{bmatrix} \Lambda_{s-\kappa_1-\omega} - 1 \\ \Lambda_{s-\kappa_1-\omega-1} \end{bmatrix}_{2^m} \prod_{i=1}^{s-\kappa_1-\omega-1} \begin{bmatrix} \Lambda_i \\ \lambda_i \end{bmatrix}_{2^m}$$

distinct choices. Now, for a given choice of a doubly even code $\mathcal{C}^{(s-\kappa_1-\omega)}$ containing $\mathbf{1}$, we need to count the choices for a doubly even code $\mathcal{C}^{(s-\kappa_1)}$ satisfying $\mathcal{C}^{(s-\kappa_1-\omega)} \subseteq \mathcal{C}^{(s-\kappa_1)} \subseteq (\mathcal{C}^{(s-\kappa_1-\omega)})^{\perp_{B_m}}$. Towards this, we see, by the proof of Theorem 3.2 of Yadav and Sharma [31], that $\mathcal{C}^{(s-\kappa_1-\omega)} = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1-\omega}-1}, \mathbf{1} \rangle$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\Lambda_{s-\kappa_1-\omega}-1}$ are mutually orthogonal singular vectors in $\mathcal{I}(\mathcal{T}_m^n)$ that are linearly independent over $\mathcal{T}_m$. Now, working as in case (II) in the proof of Lemma 4.3 and by applying Theorem 3.1 of Yadav and Sharma [31], we see that for a given choice of $\mathcal{C}^{(s-\kappa_1-\omega)}$, the chain $\mathcal{C}^{(s-\kappa_1-\omega+1)} \subseteq \mathcal{C}^{(s-\kappa_1-\omega+2)} \subseteq \cdots \subseteq \mathcal{C}^{(s-\kappa_1)}$ of codes has precisely

$$\prod_{i=\Lambda_{s-\kappa_1-\omega}}^{\Lambda_{s-\kappa_1}-1} \Big(\frac{2^{m(n-2i-1)} + \epsilon_1 2^{m(\frac{n}{2}-i)} - \epsilon_1 2^{m(\frac{n}{2}-i-1)} - 1}{2^{m(i-\Lambda_{s-\kappa_1-\omega}+1)} - 1} \Big) \prod_{t=s-\kappa_1-\omega+1}^{s-\kappa_1} \begin{bmatrix} \Lambda_t - \Lambda_{s-\kappa_1-\omega} \\ \lambda_t \end{bmatrix}_{2^m}$$

distinct choices, where $\epsilon_1 = 1$ if either $n \equiv 0 \pmod 8$ or $n \equiv 4 \pmod 8$ and m is even, while $\epsilon_1 = -1$ if $n \equiv 4 \pmod 8$ and m is odd. Furthermore, one can easily observe that

$$\widehat{\sigma}_m(n; \Lambda_{s-\kappa_1}) \begin{bmatrix} \Lambda_{s-\kappa_1} - 1 \\ \Lambda_{s-\kappa_1} - \Lambda_{s-\kappa_1-\omega} \end{bmatrix}_{2^m} = \widehat{\sigma}_m(n; \Lambda_{s-\kappa_1-\omega}) \prod_{i=\Lambda_{s-\kappa_1-\omega}}^{\Lambda_{s-\kappa_1}-1} \Big(\frac{2^{m(n-2i-1)} + \epsilon_1 2^{m(\frac{n}{2}-i)} - \epsilon_1 2^{m(\frac{n}{2}-i-1)} - 1}{2^{m(i-\Lambda_{s-\kappa_1-\omega}+1)} - 1} \Big).$$

Finally, working as in case (II) in the proof of Lemma 4.3, we see that for a given choice of $\mathcal{C}^{(s-\kappa_1)}$, the chain $\mathcal{C}^{(s-\kappa_1+1)} \subseteq \mathcal{C}^{(s-\kappa_1+2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of codes has precisely

$$\prod_{j=s-\kappa_1+1}^{s+\theta_e} \begin{bmatrix} \Lambda_j - \Lambda_{s-\kappa_1} \\ \lambda_j \end{bmatrix}_{2^m} \prod_{i=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-1} \Big(\frac{2^{m(n-2i)} - 1}{2^{m(i+1-\Lambda_{s-\kappa_1})} - 1} \Big)$$

distinct choices. From this, the desired result follows immediately. □

In the following lemma, we consider the case $2\kappa \leq e$ and count the choices for the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, such that (i) $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \leq i \leq s+\theta_e$, (ii) the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, and (iii) $\mathbf{1} \in \mathcal{C}^{(s-\kappa+\theta_e)}$ with either $n \equiv 0 \pmod 8$ or $n \equiv 4 \pmod 8$ and m being even.

Lemma 4.5. *Let $e \geq 3$ be an integer satisfying $2\kappa \leq e$. Let $M(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta})$ denote the number of distinct choices for the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, such that (i) $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \leq i \leq s+\theta_e$, (ii) the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, and (iii) $\mathbf{1} \in \mathcal{C}^{(s-\kappa+\theta_e)}$ with either $n \equiv 0$*

(mod 8) *or* $n \equiv 4 \pmod 8$ *and* m *being even. We have* $M(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) = 0$ *if* $\Lambda_{s-\kappa+\theta_e} = 0$. *Further, when* $\Lambda_{s-\kappa+\theta_e} \geq 1$, *we have*

$$\begin{aligned} M(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) &= \widehat{\sigma}_m(n; \Lambda_{s-\kappa_1}) \prod_{g=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-1} \left(\frac{2^{m(n-2g)}-1}{2^{m(g+1-\Lambda_{s-\kappa_1})}-1} \right) \begin{bmatrix} \Lambda_{s-\kappa_1}-1 \\ \Lambda_{s-\kappa_1}-\Lambda_{s-\kappa+\theta_e} \end{bmatrix}_{2^m} \\ &\times \prod_{\ell=1}^{s-\kappa+\theta_e} \begin{bmatrix} \Lambda_\ell \\ \lambda_\ell \end{bmatrix}_{2^m} \prod_{b=s-\kappa+1+\theta_e}^{s-\kappa_1} \begin{bmatrix} \Lambda_b - \Lambda_{s-\kappa+\theta_e} \\ \lambda_b \end{bmatrix}_{2^m} \prod_{d=s-\kappa_1+1}^{s+\theta_e} \begin{bmatrix} \Lambda_d - \Lambda_{s-\kappa_1} \\ \lambda_d \end{bmatrix}_{2^m}. \end{aligned}$$

Proof. Working as in Lemma 4.4, the desired result follows immediately. $\square$

In the following lemma, we consider the case $2\kappa > e$ and count the choices for the chain $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ of self-orthogonal codes of length n over $\mathcal{T}_m$, such that (i) $\dim \mathcal{C}^{(i)} = \Lambda_i$ for $1 \leq i \leq s+\theta_e$, (ii) the code $\mathcal{C}^{(s-\kappa_1)}$ is doubly even, and (iii) $\mathbf{1} \in \mathcal{C}^{(1)}$.

Lemma 4.6. *Let* $e \geq 3$ *be an integer satisfying* $2\kappa > e$. *Let* $Z(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta})$ *denote the number of distinct choices for the chain* $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \cdots \subseteq \mathcal{C}^{(s+\theta_e)}$ *of self-orthogonal codes of length* n *over* $\mathcal{T}_m$, *such that (i)* $\dim \mathcal{C}^{(i)} = \Lambda_i$ *for* $1 \leq i \leq s+\theta_e$ *(ii) the code* $\mathcal{C}^{(s-\kappa_1)}$ *is doubly even, and (iii)* $\mathbf{1} \in \mathcal{C}^{(1)}$. *We have* $n \equiv 0, 4 \pmod 8$, *and* $Z(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) = 0$ *if* $\Lambda_1 = 0$. *Further, when* $\Lambda_1 \geq 1$, *we have*

$$\begin{aligned} Z(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) &= \widehat{\sigma}_m(n; \Lambda_{s-\kappa_1}) \begin{bmatrix} \Lambda_{s-\kappa_1}-1 \\ \Lambda_{s-\kappa_1}-\Lambda_1 \end{bmatrix}_{2^m} \prod_{g=\Lambda_{s-\kappa_1}}^{\Lambda_{s+\theta_e}-1} \left(\frac{2^{m(n-2g)}-1}{2^{m(g+1-\Lambda_{s-\kappa_1})}-1} \right) \\ &\times \prod_{d=2}^{s-\kappa_1} \begin{bmatrix} \Lambda_d - \Lambda_1 \\ \lambda_d \end{bmatrix}_{2^m} \prod_{b=s-\kappa_1+1}^{s+\theta_e} \begin{bmatrix} \Lambda_b - \Lambda_{s-\kappa_1} \\ \lambda_b \end{bmatrix}_{2^m}. \end{aligned}$$

Proof. Working as in Lemma 4.4, the desired result follows immediately. $\square$

In the following theorem, we provide an explicit enumeration formula for the number $\mathfrak{S}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e)$.

Theorem 4.1. *Let* $e \geq 3$ *be an integer. Let* n *be a positive integer, and let* $\lambda_1, \lambda_2, \ldots, \lambda_{e+1}$ *be non-negative integers satisfying* $n = \lambda_1 + \lambda_2 + \cdots + \lambda_{e+1}$ *and* $2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{e-j+1} + \lambda_{e-j+2} + \cdots + \lambda_j \leq n$ *for* $s+1 \leq j \leq e$. *Let* $\Lambda_0 = 0$ *and* $\Lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i$ *for* $1 \leq i \leq e+1$. *We have*

$$\begin{aligned} \mathfrak{S}_e(n; \lambda_1, \lambda_2, \ldots, \lambda_e) = (2^m)&^{\sum\limits_{i=1}^{s} \Lambda_i(n-\Lambda_{i+1}) + \sum\limits_{j=1}^{s-1+\theta_e} \Lambda_{s+j}(n-\Lambda_{s+j+1}-\Lambda_{s-j+\theta_e}) - \sum\limits_{a=1}^{s-\kappa_1-1} \Lambda_a - (1-\theta_e)\frac{\Lambda_s(\Lambda_s-1)}{2}} \\ &\times \mathfrak{B}_{\theta_e}(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) \prod_{\ell=s+1+\theta_e}^{e} \begin{bmatrix} \lambda_\ell + n - \Lambda_\ell - \Lambda_{e+1-\ell} \\ \lambda_\ell \end{bmatrix}_{2^m}, \end{aligned}$$

where

$$\mathfrak{B}_{\theta_e}(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) = \begin{cases} N(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) & \text{if } n \equiv 1,2,3,5,6,7 \pmod 8; \\ N(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) + 2(2^m)^{\kappa_1} M(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) & \\ \quad + \sum\limits_{\omega=0}^{\kappa_1-\theta_e} (2^m)^\omega Y_\omega(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) & \text{if } 2\kappa \leq e \text{ with either } n \equiv 0 \pmod 8 \\ & \text{or } n \equiv 4 \pmod 8 \text{ and } m \text{ is even;} \\ N(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) + \sum\limits_{\omega=0}^{\kappa_1-\theta_e} (2^m)^\omega Y_\omega(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) & \text{if } 2\kappa \leq e, n \equiv 4 \pmod 8 \text{ and} \\ & m \text{ is odd;} \\ N(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) + (2^m)^{s-\kappa_1-1} Z(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) & \\ \quad + \sum\limits_{\omega=0}^{s-\kappa_1-2} (2^m)^\omega Y_\omega(\lambda_1, \lambda_2, \ldots, \lambda_{s+\theta_e}) & \text{if } 2\kappa > e \text{ and } n \equiv 0,4 \pmod 8. \end{cases}$$

Proof. When $2\kappa \leq e$, the desired results follows by Lemmas 4.1, 4.3, 4.4 and 4.5. On the other hand, when $2\kappa > e$, we get the desired result by Lemmas 4.2, 4.3, 4.4 and 4.6. $\square$

$\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$	$\mathfrak{S}_4(3;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$	$\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$	$\mathfrak{S}_4(3;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$	$\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$	$\mathfrak{S}_4(3;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$
$\{0,0,0,1\}$	7	$\{0,0,0,2\}$	7	$\{0,0,0,3\}$	1
$\{0,0,1,0\}$	28	$\{0,0,1,1\}$	42	$\{0,0,1,2\}$	7
$\{0,0,2,0\}$	28	$\{0,0,2,1\}$	7	$\{0,0,3,0\}$	1
$\{0,1,0,0\}$	48	$\{0,1,0,1\}$	72	$\{0,1,0,2\}$	12
$\{0,1,1,0\}$	24	$\{0,1,1,1\}$	6	$\{1,0,0,0\}$	0
$\{1,0,0,1\}$	0	$\{1,0,1,0\}$	0		

Table 1: The number $\mathfrak{S}_4(3;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ of self-orthogonal codes of length 3 and type $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ over $\mathscr{R}_{4,1}=GR(4,1)[y]/\langle y^3+2,2y\rangle$

$\{\lambda_1,\lambda_2,\ldots,\lambda_5\}$	$\mathfrak{S}_5(3;\lambda_1,\lambda_2,\ldots,\lambda_5)$	$\{\lambda_1,\lambda_2,\ldots,\lambda_5\}$	$\mathfrak{S}_5(3;\lambda_1,\lambda_2,\ldots,\lambda_5)$	$\{\lambda_1,\lambda_2,\ldots,\lambda_5\}$	$\mathfrak{S}_5(3;\lambda_1,\lambda_2,\ldots,\lambda_5)$
$\{0,0,0,0,1\}$	7	$\{0,0,0,0,2\}$	7	$\{0,0,0,0,3\}$	1
$\{0,0,0,1,0\}$	28	$\{0,0,0,1,1\}$	42	$\{0,0,0,1,2\}$	7
$\{0,0,0,2,0\}$	28	$\{0,0,0,2,1\}$	7	$\{0,0,0,3,0\}$	1
$\{0,0,1,0,0\}$	48	$\{0,0,1,0,1\}$	72	$\{0,0,1,0,2\}$	12
$\{0,0,1,1,0\}$	72	$\{0,0,1,1,1\}$	18	$\{0,0,1,2,0\}$	3
$\{0,1,0,0,0\}$	96	$\{0,1,0,0,1\}$	144	$\{0,1,0,0,2\}$	24
$\{0,1,0,1,0\}$	48	$\{0,1,0,1,1\}$	12	$\{1,0,0,0,0\}$	0
$\{1,0,0,0,1\}$	0	$\{1,0,0,1,0\}$	0		

Table 2: The number $\mathfrak{S}_5(3;\lambda_1,\lambda_2,\ldots,\lambda_5)$ of self-orthogonal codes of length 3 and type $\{\lambda_1,\lambda_2,\ldots,\lambda_5\}$ over $\mathscr{R}_{5,1}=GR(4,1)[y]/\langle y^3+2,2y^2\rangle$

$\{\lambda_1,\lambda_2,\ldots,\lambda_6\}$	$\mathfrak{S}_6(2;\lambda_1,\lambda_2,\ldots,\lambda_6)$	$\{\lambda_1,\lambda_2,\ldots,\lambda_6\}$	$\mathfrak{S}_6(2;\lambda_1,\lambda_2,\ldots,\lambda_6)$	$\{\lambda_1,\lambda_2,\ldots,\lambda_6\}$	$\mathfrak{S}_6(2;\lambda_1,\lambda_2,\ldots,\lambda_6)$
$\{0,0,0,0,0,1\}$	5	$\{0,0,0,0,0,2\}$	1	$\{0,0,0,0,1,0\}$	20
$\{0,0,0,0,1,1\}$	5	$\{0,0,0,0,2,0\}$	1	$\{0,0,0,1,0,0\}$	80
$\{0,0,0,1,0,1\}$	20	$\{0,0,0,1,1,0\}$	5	$\{0,0,0,2,0,0\}$	1
$\{0,0,1,0,0,0\}$	64	$\{0,0,1,0,0,1\}$	16	$\{0,0,1,0,1,0\}$	4
$\{0,1,0,0,0,0\}$	0	$\{0,1,0,0,0,1\}$	0	$\{1,0,0,0,0,0\}$	0

Table 3: The number $\mathfrak{S}_6(2;\lambda_1,\lambda_2,\ldots,\lambda_6)$ of self-orthogonal codes of length 2 and type $\{\lambda_1,\lambda_2,\ldots,\lambda_6\}$ over $\mathscr{R}_{6,2}=GR(4,2)[y]/\langle y^3+2,2y^3\rangle$

$\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$	$\mathfrak{S}_4(4;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$	$\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$	$\mathfrak{S}_4(4;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$	$\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$	$\mathfrak{S}_4(4;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$
$\{0,0,0,1\}$	15	$\{0,0,0,2\}$	35	$\{0,0,0,3\}$	15
$\{0,0,0,4\}$	1	$\{0,0,1,0\}$	120	$\{0,0,1,1\}$	420
$\{0,0,1,2\}$	210	$\{0,0,1,3\}$	15	$\{0,0,2,0\}$	560
$\{0,0,2,1\}$	420	$\{0,0,2,2\}$	35	$\{0,0,3,0\}$	120
$\{0,0,3,1\}$	15	$\{0,0,4,0\}$	1	$\{0,1,0,0\}$	448
$\{0,1,0,1\}$	1568	$\{0,1,0,2\}$	784	$\{0,1,0,3\}$	56
$\{0,1,1,0\}$	1344	$\{0,1,1,1\}$	1008	$\{0,1,1,2\}$	84
$\{0,1,2,0\}$	112	$\{0,1,2,1\}$	14	$\{0,2,0,0\}$	384
$\{0,2,0,1\}$	288	$\{0,2,0,2\}$	24	$\{1,0,0,0\}$	256
$\{1,0,0,1\}$	384	$\{1,0,0,2\}$	64	$\{1,0,1,0\}$	384
$\{1,0,1,1\}$	96	$\{1,0,2,0\}$	16	$\{1,1,0,0\}$	384
$\{1,1,0,1\}$	96	$\{2,0,0,0\}$	0		

Table 4: The number $\mathfrak{S}_4(4;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ of self-orthogonal codes of length 4 and type $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ over $\mathscr{R}_{4,1}=GR(4,1)[y]/\langle y^3+2,2y\rangle$

Example 4.1. *Using Magma, we provide the number $\mathfrak{S}_4(3;\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ of self-orthogonal codes of length 3 and type $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ over $\mathscr{R}_{4,1}=GR(4,1)[y]/\langle y^3+2,2y\rangle$ in Table 1, the number $\mathfrak{S}_5(3;\lambda_1,\lambda_2,\ldots,\lambda_5)$ of self-orthogonal codes of length 3 and type $\{\lambda_1,\lambda_2,\ldots,\lambda_5\}$ over $\mathscr{R}_{5,1}=GR(4,1)[y]/\langle y^3+2,2y^2\rangle$ in Table 2, the number $\mathfrak{S}_6(2;\lambda_1,\lambda_2,\ldots,\lambda_6)$ of self-orthogonal codes of length 2 and type $\{\lambda_1,\lambda_2,\ldots,\lambda_6\}$ over $\mathscr{R}_{6,2}=GR(4,2)[y]/\langle y^3+2,2y^3\rangle$ in Table 3, and the number $\mathfrak{S}_4(4;\lambda_1,\lambda_2,\lambda,\lambda_4)$ of self-orthogonal codes of length 4 and type $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ over $\mathscr{R}_{4,1}=GR(4,1)[y]/\langle y^3+2,2y\rangle$ in Table 4. These computed values agree exactly with Theorem 4.1.*

From the above theorem, one can deduce Theorem 5.1 of Yadav and Sharma [32] on taking $\mathfrak{s}=1$ in the case when $e=\kappa$ is odd. Now, in the following theorem, we provide an explicit enumeration formula for the number

$\mathfrak{U}_e(n;\lambda_1,\lambda_2,\ldots,\lambda_e)$, where $e \geq 3$.

Theorem 4.2. *Let $e \geq 3$ be an integer. Let n be a positive integer, and let $\lambda_1,\lambda_2,\ldots,\lambda_{e+1}$ be non-negative integers satisfying $n=\lambda_1+\lambda_2+\cdots+\lambda_{e+1}$ and $\lambda_j=\lambda_{e-j+2}$ for $1 \leq j \leq e+1$. Let $\Lambda_0=0$ and $\Lambda_i=\lambda_1+\lambda_2+\cdots+\lambda_i$ for $1\leq i\leq e+1$. We have*

$$\mathfrak{U}_e(n;\lambda_1,\lambda_2,\ldots,\lambda_e)=\mathfrak{B}_{\theta_e}(\lambda_1,\lambda_2,\ldots,\lambda_{s+\theta_e})(2^m)^{\sum\limits_{i=1}^{s}\Lambda_i(n-\Lambda_{i+1})-\sum\limits_{a=1}^{s-\kappa_1-1}\Lambda_a-(1-\theta_e)\frac{\Lambda_s(\Lambda_s-1)}{2}},$$

where the number $\mathfrak{B}_{\theta_e}(\lambda_1,\lambda_2,\ldots,\lambda_{s+\theta_e})$ is as obtained in Theorem 4.1.

Proof. By Lemma 2.1 and on substituting $\lambda_j=\lambda_{e-j+2}$ for $1\leq j\leq e+1$ in Theorem 4.1, we get the desired result. □

One can deduce Theorem 5.3 of Yadav and Sharma [32] from the above theorem on taking $\mathfrak{s}=1$ in the case when $e=\kappa$ is odd.

5 Conclusion and future work

In this paper, explicit enumeration formulae for all self-orthogonal and self-dual codes of an arbitrary length over a finite commutative chain ring $\mathscr{R}_{e,m}$ of even characteristic are derived, under the condition that $2 \in \langle u^\kappa\rangle \setminus \langle u^{\kappa+1}\rangle$ for some odd positive integer κ satisfying $3 \leq \kappa \leq e$, where $\mathscr{R}_{e,m}$ is a finite commutative chain ring with maximal ideal $\langle u\rangle$ of nilpotency index $e \geq 3$. A subsequent study [30] will address the complementary case where κ is even. Enumeration formulae for all self-orthogonal and self-dual codes of an arbitrary length over Galois rings of even characteristic (*i.e.,* when $\kappa=1$) and finite commutative chain rings of odd characteristic are explicitly derived in [31] and [33], respectively. Together, the results derived in this paper, along with those in [30, 31] and [33], provide a complete solution to the problem of enumeration of self-orthogonal and self-dual codes of an arbitrary length over any finite commutative chain ring. Future research directions include extending enumeration to codes over more general finite commutative Frobenius rings and exploring structural classifications of self-orthogonal and self-dual codes over various classes of finite commutative Frobenius rings.

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