Automorphisms of valued fields: amalgamation and existential closedness

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Abstract

We study valued fields equipped with an automorphism. We prove that all of them have an extension admitting an equivariant cross-section of the valuation. In residual characteristic zero, and in the presence of such a cross-section, we show that amalgamation problems are solvable precisely when the induced residual problem is, characterise the existentially closed objects of this category, and prove that its positive theory does not have the tree property of the second kind. We prove analogous results with cross-sections replaced by angular components. Along the way, we show that array modelling does not require thickness.

Contents

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Introduction

The study of difference equations may be carried out inside a suitable ring of functions, say in the indeterminate x, equipped with the endomorphism $\sigma(f(x)) = f(x+1)$. For this reason, a field K with a distinguished endomorphism σ is called a *difference field* [Coh65]; when σ is an automorphism, (K, σ) is said to be *inversive*. The typical object of interest in this work will be an inversive *valued difference field*: a valued field (K, v) together with an automorphism σ compatible with the valuation v, that is, fixing the valuation ring $\mathcal{O}$ setwise. If we denote by Γ the value group and by $v\colon K \to \Gamma \cup \{\infty\}$ the (surjective) valuation, we could equivalently specify a pair of automorphisms, one of the field K and one of the ordered abelian group Γ, that we will both denote by σ, such that for all $a \in K$ one has $v(\sigma(a)) = \sigma(v(a))$.

Automorphisms of valued fields appear for instance in [Sen69; Put72; Kel86; KS22; KKS25]. Their model theory has been studied intensively since the beginning of the millennium, for example in [Sca00; Sca07; BMS07; Azg10; AD11; Pal12; DO15; Rid17; BG25], and it has been used to obtain Lang–Weil-type estimates in finite difference fields [Hil+24] and a version of Kapranov's Theorem for tropical difference algebra [Ali25]. A connection with algebraic dynamics was envisaged in the introduction to [CH08]. Although the techniques have recently been extended to the non-inversive case [DH22; DH25; Ram24], all difference fields in this paper will be inversive, hence we will from now on omit this qualifier and tacitly assume the surjectivity of all of our field endomorphisms, unless otherwise specified.

When studying fields, it is often convenient to place oneself in a *universal domain*: a large enough algebraically closed field. It has long been known that this is not a peculiarity of field theory, as one can give a precise definition of "universal domain" via the notion of *existentially closed* object of a category of structures with embeddings (Definition 1.1). If such a category is closed under inductive limits, every object embeds in an existentially closed one. This happens to be the case for the category of valued difference fields; here, a consequence of existential closedness is a form of the Nullstellensatz for difference varieties which takes the valuation into account: for example, whenever a difference variety defined over K has a point in some valued difference field extension $(L, v_L, \sigma_L) \supseteq (K, v, \sigma)$ with all coordinates of valuation zero, then such a point already exists in K. A sufficiently *saturated* (Definition 2.1) existentially closed valued difference field can then be seen as a universal domain for *tropical difference algebra*.

For categories of structures with an automorphism it is customary to refer to the automorphism of an existentially closed object as *generic* [Mac97; CP98]. Note that not every structure carries a generic automorphism; for example, every field that does must be algebraically closed; this follows from the fact that every automorphism of a field can be extended to its algebraic closure. In the valued difference case, some consequences of genericity can be easily deduced using standard results from the model theory of valued fields: for instance (Proposition 2.3) the valued field needs to be algebraically closed, and its residue field and value group, equipped with the induced automorphisms, need to be existentially closed in their respective categories.

After observing this, one is immediately confronted with the fact that, in almost all of the existing literature on fields with a valuation and an auto-

morphism, the behaviour of σ on Γ is severely constrained, and certainly not generic. Namely, most of the currently available results are only valid in the *multiplicative* case [Pal12], two important special cases being the *isometric* one—where $v(\sigma(x)) = v(x)$ for every x, that is, the induced automorphism of Γ is the identity—and the *strongly contractive* one—where $v(\sigma(x)) > n \cdot v(x)$ for every x of positive valuation and every $n \in \mathbb{N}$. Exceptions are [Bey+20], which deals with existentially closed (K, v, σ) where σ is only required to preserve the field structure but not v—hence there is no induced automorphism of Γ at all—and [DO15; Rid17; Ram24], that black-box the behaviour of σ on Γ. Another non-multiplicative example is, on the field of transseries, precomposition by a positive infinite element, studied in [Hoe06, Chapter 5] and [DO15, Section 8].

Existential closedness for ordered abelian groups with an automorphism—from now on, *difference oags*—has recently been studied in [DM24]. Having started to grasp an understanding of existentially closed difference oags, which occur as value groups of existentially closed valued difference fields, it is natural to wonder whether one may leverage knowledge of the former to obtain results on the latter. In this paper, we will see that the answer is affirmative.

The first property one may expect from a universal domain for tropical difference algebra is the existence of a *cross-section* of v—a group morphism $s\colon \Gamma \to K^\times$ such that $v(s(\gamma)) = \gamma$—that is σ-equivariant. In turn, such an object induces a σ-equivariant *angular component*, that is, a multiplicative homomorphism from K to the residue field k extending the residue map. The existence of these maps implies that the inclusion of $\mathbb{Z}[\sigma]$-modules $\mathcal{O}^\times \hookrightarrow K^\times$ is *pure*: every system of multiplicative difference equations with parameters in $\mathcal{O}^\times$ that has a solution in $K^\times$ must have one in $\mathcal{O}^\times$. If said equations are not allowed to mention σ, then this is well known to follow from the fact that Γ is torsion-free. When σ enters the picture, showing this becomes more intricate, and constitutes our first main result.

Theorem A (Theorem 2.14)**.** Suppose (K, v, σ) is an existentially closed valued difference field. Then $\mathcal{O}^\times$ is a pure $\mathbb{Z}[\sigma]$-submodule of $K^\times$. It follows that every valued difference field embeds into one with a cross-section commuting with σ.

While adding a cross-section to the language enlarges the family of definable sets, we prove that this does not drastically change the notion of universal domain. Namely, every valued difference field with a cross-section that is existentially closed is also existentially closed as a valued difference field with no extra structure (Proposition 2.21); we also obtain similar results for other expansions of valued difference fields.

We prove the above results in Section 2 by combining logical and tropical tools, such as the Compactness Theorem, saturation, and initial varieties. We also use a version of the Fundamental Theorem of Tropical Geometry for valuations with arbitrary rank (Theorem 1.12), which we prove in Section 1 by a compactness argument using upper bounds from [JS18] on the minimum degree of a tropical basis. The existence of an equivariant cross-section allows to define equivariant initial forms (Remark 2.16), thereby providing further evidence that this framework is suitable to develop tropical difference algebra. We will also see that, by an argument due to Philip Dittmann, existentially closed valued difference fields are not existentially closed as difference fields (Corollary 2.27).

In Section 3 we work in the case of residual characteristic zero, and study *amalgamation problems*: given a base object A and morphisms $f\colon A \to B$ and

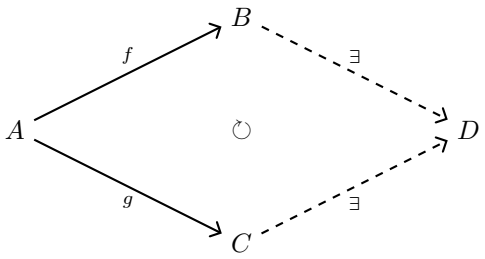

Figure 1: An amalgamation diagram.

$g\colon A \to C$, we seek an object D and morphisms making the diagram in Figure 1 commute.

It is well known that, in the category of valued fields, as well as in that of difference fields, where in both cases we allow as arrows all possible embeddings, not every amalgamation problem can be solved. For difference fields, one may solve amalgamation problems where the arrows are required to preserve certain predicates expressing the possibility to add solutions to difference equations of a specific form [Mac97]. In our setting, even adding the aforementioned predicates to the residue field does not allow to solve all amalgamation problems (by Example 3.2).

As for valued fields, one remedy is to use RV-*structures*, or equivalently [LT25] *stringent valued hyperfields*; to the best of our knowledge, a theory of generic automorphisms of such structures is still lacking. For this reason, we adopt another common approach, and work in the category of valued difference fields endowed with a σ-equivariant angular component—briefly, ac-valued difference fields—of residual characteristic zero, with embeddings required to commute with angular components. We characterise which amalgamation problems have solutions by proving an embedding result (Proposition 3.27) that also allows us to characterise the existentially closed objects of this category in terms of genericity of the automorphisms induced on k and Γ and a generalisation of henselianity commonly used in the context of valued difference fields.

Theorem B (Theorem 3.28)**.** An amalgamation problem of ac-valued difference fields of residual characteristic zero has a solution if and only if the induced amalgamation problem of difference residue fields does.

Theorem C (Theorem 3.29)**.** An ac-valued difference field of residual characteristic zero is existentially closed if and only if it is σ-henselian and the automorphisms induced on its value group and residue field are both generic.

Both theorems also hold if we work with cross-sections instead of angular components, and in both cases we may also add a lift of the residue field to the language. Combining Theorem C with results from [DO15] we show that, if $\mathrm{char}(k) = 0$ and both (k, σ) and (Γ, σ) are existentially closed, then so is their Hahn product $k((\Gamma))$ (Example 3.30).

As observed in [Pil00; Ben03a; BP07], even if a class of existentially closed structures is not *elementary*, which happens to be the case in our setting by [KS02], one can still analyse it through the lenses of model theory by using

positive logic, an approach already followed in e.g. [HK21; Kam23; EKN21; DM24] that constitutes one of the main motivations for the general study of this framework, carried out for instance in [Hay19; DK22; DGK23; Kam24]. Positive logic has also been used to introduce *definability patterns* [Hru20; Seg25] and obtain applications to approximate subgroups [Hru22].

The relevant neostability-theoretic *dividing line* for this paper is the *tree property of the second kind*: the existence of an unbounded definable family of independent partitions. The absence of this property is a tameness condition, known as NTP_2 and studied intensively in recent years, see e.g. [Che14; CH14; MOS20; Kam24]. In our final Section 4 we take the opportunity to advance the study of NTP_2 in the positive setting by generalising to arbitrary positive theories the *Array Modelling Theorem* (Theorem 4.10), a Ramsey-theoretic statement established in [Kam24] for *thick* positive theories. Array modelling implies a number of desirable general properties such as *submultiplicativity of burden* (Corollary 4.18) that, combined with Theorem B, allow us to adapt the arguments of [CH14] to prove that our class enjoys the above tameness property, which for example fails in existentially closed exponential fields [HK21].

Theorem D (Theorem 4.40)**.** Existentially closed valued difference fields of residual characteristic zero are NTP_2 in the sense of positive logic.

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1 Preliminaries

1.1 Model theory

Lowercase letters such as $a, b, x, y, \ldots$ will usually denote finite (except in Section 4) tuples of elements or of variables, e.g. $a = (a_0, \ldots, a_{n-1})$, and their length (in the previous case, n) will be denoted by $|a|$. We abuse the notation and write e.g. $a \in M$ instead of $a \in M^{|a|}$. Similarly, for a function f defined on M, we write $f(a)$ for $(f(a_0), \ldots, f(a_{n-1}))$, etc. We refer the reader to [DM24, Section 1.2] for a quick tour of some model-theoretic terminology and facts, to

[Hod93] for a more extensive treatment of the fundamentals, and to [Dri14] for the basics of valued fields and of their model theory. We nevertheless recall the following standard definition.

Definition 1.1. Let $\mathcal{L}$ be a first-order language, T an $\mathcal{L}$-theory axiomatised by $\forall\exists$-sentences, and $\mathcal{K}$ the category of models of T with embeddings. An object $M \in \mathcal{K}$ is *existentially closed* iff, for every embedding $f\colon M \to N$, every tuple a from M, and every quantifier-free formula $\varphi(x, y)$ without parameters, if $N \vDash \exists x\, \varphi(x, f(a))$ then $M \vDash \exists x\, \varphi(x, a)$.

Remark 1.2. The following is well known.

(1) Every such $\mathcal{K}$ is closed under inductive limits [Hod93, Theorem 6.5.9 and Exercise 5 in Section 6.5], and this in turn ensures that every object embeds in an existentially closed one [Hod93, Theorem 8.2.3].

(2) By enlarging $\mathcal{L}$, hence restricting the class of embeddings between models, every first-order theory can be $\forall\exists$-axiomatised [Hod93, Theorems 2.6.5 and 2.6.6]. However, this process (known as *Morleyisation* or *atomisation*) may change the notion of existentially closed model.

In our context, the following assumption is natural; see Remark 2.4(2).

Assumption 1.3. All difference fields and oags are assumed inversive.

Definition 1.4. We will mostly deal with the following languages.

(1) The *three-sorted language of valued fields* $\mathcal{L}_{\mathrm{vf}}$ has

   (a) a sort VF for the valued field, carrying a copy of the language $\mathcal{L}_{\mathrm{field}} \coloneqq \{+_{\mathrm{VF}}, 0_{\mathrm{VF}}, -_{\mathrm{VF}}, \cdot_{\mathrm{VF}}, 1_{\mathrm{VF}}, (-)^{-1}_{\mathrm{VF}}\}$ of fields, interpreted in the natural way with the convention that $0^{-1} = 0$,

   (b) a sort Γ for the value group, carrying a copy of the language $\mathcal{L}_{\mathrm{oag}} \coloneqq \{+_\Gamma, 0_\Gamma, -_\Gamma, <_\Gamma\}$ of oags and an additional constant symbol ∞, to be interpreted as an infinite absorbing element, with $-\infty = \infty$, equal to the valuation of 0_{VF},

   (c) a sort k for the residue field, carrying a copy of the language of fields $\{+_k, 0_k, -_k, \cdot_k, 1_k, (-)^{-1}_k\}$, interpreted in the natural way with the convention that $0^{-1} = 0$,

   (d) a function symbol $v\colon \mathrm{VF} \to \Gamma$,

   (e) a function symbol $\mathrm{res}\colon \mathrm{VF} \to k$.

   We view a valued field as an $\mathcal{L}_{\mathrm{vf}}$-structure by interpreting v as the (surjective) valuation, res as the map sending x to the residue of x if x lies in the valuation ring, that we denote by $\mathcal{O}$, and to 0 otherwise, and the rest of the symbols in the obvious way suggested by the symbols themselves.

(2) The *language of* ac*-valued fields* $\mathcal{L}_{\mathrm{ac}}$ is the expansion of $\mathcal{L}_{\mathrm{vf}}$ by a function symbol $\mathrm{ac}\colon \mathrm{VF} \to k$, to be interpreted as an angular component map.

(3) The *language of* s*-valued fields* $\mathcal{L}_s$ is the expansion of $\mathcal{L}_{\mathrm{vf}}$ by a function symbol $s\colon \Gamma \to \mathrm{VF}$, to be interpreted as a cross-section of the valuation.

(4) We write $\mathcal{L}_\sigma$ (resp. $\mathcal{L}_{\mathrm{ac},\sigma}, \mathcal{L}_{s,\sigma}$) for the expansion of $\mathcal{L}_{\mathrm{vf}}$ (resp. $\mathcal{L}_{\mathrm{ac}}, \mathcal{L}_s$) by symbols $\sigma_{\mathrm{VF}}, \sigma_{\mathrm{VF}}^{-1}, \sigma_\Gamma, \sigma_\Gamma^{-1}, \sigma_k, \sigma_k^{-1}$, interpreted as automorphisms of the sorts indicated in the subscripts and their inverses, and commuting with the other maps in the language (see below).

(5) We also expand all of the above languages by a function symbol $\iota\colon k \to \mathrm{VF}$, to be interpreted—in the equicharacteristic case—as a lift of the residue field commuting with σ. The resulting languages are denoted e.g. by $\mathcal{L}_{\mathrm{ac},\iota,\sigma}$.

(6) If $\mathcal{L}$ is one of the languages introduced above, we will write *$\mathcal{L}$-valued difference field* to mean a valued difference field expanded by the corresponding structure.

As customary, we treat Γ as an oag even if, strictly speaking, it also contains ∞, conflate K with $\mathrm{VF}(K)$ whenever convenient, and drop subscripts if no confusion may arise. If K is clear from context, we may also write k in place of $k(K)$, etc. We also say e.g. "σ commutes with v" instead of "$\sigma_\Gamma \circ v = v \circ \sigma_{\mathrm{VF}}$", as already done in Definition 1.4(4). When dealing with extensions we may or may not use subscripts to differentiate the interpretation of a particular symbol, typically σ. For instance, we sometimes write $(K, \sigma) \subseteq (L, \sigma)$, and sometimes write $(K, \sigma_K) \subseteq (L, \sigma_L)$ for emphasis. For brevity, we write e.g. (K, v, σ) to refer to a valued difference field K viewed as an $\mathcal{L}_\sigma$-structure, even if $\mathcal{L}_\sigma$ contains other symbols, e.g. σ^{-1}.

Recall that DOAG denotes the $\mathcal{L}_{\mathrm{oag}}$-theory of nontrivial divisible ordered abelian groups. We denote by ACF the $\mathcal{L}_{\mathrm{field}}$-theory of algebraically closed fields, by ACVF the $\mathcal{L}_{\mathrm{vf}}$-theory of nontrivially valued algebraically closed fields, by $\mathsf{ACVF}_{\mathrm{ac}}$ the $\mathcal{L}_{\mathrm{ac}}$-theory of nontrivially valued algebraically closed fields equipped with an angular component map, and similarly for the other languages mentioned above. For p, q prime or zero, we write $\mathsf{ACVF}_{p,q}$ for the completion of ACVF obtained by specifying that $\mathrm{char}(\mathrm{VF}) = p$ and $\mathrm{char}(k) = q$. As customary, in the aforementioned theories of valued fields, the valuation map $v\colon \mathrm{VF} \to \Gamma$ is assumed surjective. All of these theories are easily checked to be $\forall\exists$-axiomatisable in their language, hence by Remark 1.2 each of their models embeds in an existentially closed one.[1] After adding σ, σ^{-1} to the languages above, together with axioms saying that these are respectively an automorphism and its inverse, $\forall\exists$-axiomatisability is clearly preserved.

Remark 1.5. Our choice of language is slightly different from the usual one, in that we have symbols for multiplicative inverses on the sorts VF and k and a unary res map, while most of the literature uses the language of rings and a binary map $\mathrm{Res}(x, y)$ defined as $\mathrm{res}(x/y)$ if $y \neq 0$ and $x/y \in \mathcal{O}$, and 0 otherwise. Once a symbol for the multiplicative inverse is available, $\mathrm{Res}(x, y)$ is piecewise definable by a term, and each piece is quantifier-free definable, hence many results from the literature, e.g. on quantifier elimination, are easily shown to transfer.

Example 1.6. Given (k, σ_k) and (Γ, σ_Γ), the Hahn field $k((\Gamma))$ can be equipped with the structure of a valued difference field, with the natural valuation and

[1] In fact, all of DOAG, ACF, ACVF, $\mathsf{ACVF}_{\mathrm{ac}}$ are *model complete*, that is, all of their models are existentially closed.

$\sigma\colon \mathrm{VF}(L) \to \mathrm{VF}(L)$ given by $(\sigma(f))(\gamma) \coloneqq \sigma_k(f(\sigma_\Gamma^{-1}(\gamma)))$ or, in power series notation,

$$\sigma\left(\sum_{\gamma\in\Gamma} a_\gamma t^\gamma\right) = \sum_{\gamma\in\Gamma} \sigma_k(a_\gamma) t^{\sigma_\Gamma(\gamma)}.$$

As a special case, consider the field of complex Puiseux series with the automorphism sending $\sum_{n\in\mathbb{Q}} a_n t^n$ to $\sum_{n\in\mathbb{Q}} \overline{a_n} t^{2n}$, where $\bar{z}$ denotes the complex conjugate of z.

Observe that such structures have a natural cross-section, namely $\gamma \mapsto t^\gamma$, which is easily checked to commute with σ. In particular, the associated angular component, which maps $f = \sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ to $a_{v(f)}$, is σ-equivariant.

1.2 Tropical geometry

In what follows, we will use results from tropical geometry, for which we refer the reader to [MS15]. In the cited source, all value groups are Archimedean, which is typically not the case in this paper. Nevertheless, all the results we will use from [MS15] still hold in our setting; since this sometimes requires justification, and because it may be of independent interest, we gather in this subsection the results that we will need. We especially want to highlight that many of these are, or can be reduced to, first-order statements; therefore, by the model-theoretic properties of ACVF, they do not depend on the rank of the valuation.

For the rest of this section we use $K[x^{\pm 1}], \mathcal{O}[x^{\pm 1}], k[x^{\pm 1}]$ as shorthands for $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \mathcal{O}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ respectively, for some fixed n. For $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma^n$, we will use $\langle \mathbf{u}, \gamma\rangle$ to denote the scalar product $u_1\gamma_1 + \ldots + u_n\gamma_n$.

Definition 1.7. Let (K, v) be a valued field.

(1) A *tropical polynomial* is an expression of the form $\min\{\gamma_i + \langle \mathbf{u}_i, x\rangle \mid i \leq \ell\}$, where x is a variable ranging over Γ^n, $\mathbf{u}_i \in \mathbb{Z}^n$, and $\gamma_i \in \Gamma$.

(2) We call $\gamma \in \Gamma^n$ a *tropical root* of a tropical polynomial g iff the minimum in g is achieved at least twice at γ.

Let $f \in K[x^{\pm 1}]$ be a Laurent polynomial; write f as $\sum_{\mathbf{u}\in S} c_\mathbf{u} x^\mathbf{u}$, for some finite subset $S \subseteq \mathbb{Z}^n$, with $c_\mathbf{u} \neq 0$ for all $\mathbf{u} \in S$, and let $\gamma \in \Gamma^n$.

(3) The *tropicalisation* $\mathrm{trop}(f)\colon \Gamma^n \to \Gamma$ is defined as

$$(\mathrm{trop}(f))(\gamma) \coloneqq \min\{v(c_\mathbf{u}) + \langle \mathbf{u}, \gamma\rangle \mid \mathbf{u} \in S\}.$$

Fix now a cross-section of the valuation $s\colon \Gamma \to K$.

(4) The *initial form of f with respect to γ* is the Laurent polynomial over the residue field k defined as $\mathrm{in}_\gamma(f) \coloneqq \sum_{\mathbf{u}\in S'} \mathrm{res}(c_\mathbf{u} s(-v(c_\mathbf{u})))x^\mathbf{u}$, where $S' \coloneqq \{\mathbf{u} \in S \mid \forall \mathbf{u}' \in S, v(c_\mathbf{u}) + \langle \mathbf{u}, \gamma\rangle \leq v(c_{\mathbf{u}'}) + \langle \mathbf{u}', \gamma\rangle\}$.

(5) Given an ideal I of $K[x^{\pm 1}]$, the *initial ideal of I with respect to γ* is the ideal of $k[x^{\pm 1}]$ that is generated by the set $\{\mathrm{in}_\gamma(f) \mid f \in I\}$.

(6) If $V \subseteq \mathbb{G}_{\mathrm{m}}^n(K)$ is an algebraic subvariety and I is the ideal of Laurent polynomials over K vanishing on V, then we denote by $\mathrm{in}_\gamma(V)$ the algebraic subvariety of $\mathbb{G}_{\mathrm{m}}^n(k)$ associated to $\mathrm{in}_\gamma(I)$. We call this the *initial variety of V with respect to γ*.

Note that initial forms depend on the choice of a cross-section, even if this is suppressed from the notation. Below, if we mention initial forms without specifying a cross-section, we assume implicitly that a one exists and has been fixed. Recall that cross-sections always exist if the value group is divisible, e.g. in the algebraically closed case, see for example [ADH17, Lemma 3.3.32].

Remark 1.8.

(1) In order to define initial forms one does not really need the cross-section s, rather the corresponding angular component $\mathrm{res}(xs(-v(x)))$. We mostly use the cross-section as that is customary in the literature, but observe that many results in tropical geometry (say from [MS15, Sections 2 and 3]) generalise to initial forms when they are defined using an arbitrary angular component.

(2) Initial varieties are sometimes called *tropical reductions* or *tropical degenerations* in the literature.

It is almost immediate to see that the codimension 1 case of the Fundamental Theorem of Tropical Geometry, also known as Kapranov's Theorem [MS15, Theorem 3.1.3], is a first-order statement. In higher codimension we will need a bound of Joswig and Schröter on the minimum degree of a *tropical basis*.

Definition 1.9. Let K be a valued field.

(1) Let I be a homogeneous ideal of $K[x_0, \ldots, x_n]$. A *universal Gröbner basis* $\mathcal{U}$ of I is a finite subset of I such that, for every $\gamma \in \Gamma(K)^{n+1}$, the set $\mathrm{in}_\gamma(\mathcal{U})$ generates $\mathrm{in}_\gamma(I)$.

(2) Let I be an ideal in $K[x^{\pm 1}]$. A *tropical basis* of I is a finite generating subset $\mathcal{T} \subseteq I$ such that for every $\gamma \in \Gamma^n$, if there is $f \in I$ such that γ is not a tropical root of $\mathrm{trop}(f)$, then there is $g \in \mathcal{T}$ such that γ is not a tropical root of $\mathrm{trop}(g)$.

Given the ideal I, denote by I_{proj} its homogenisation: the homogeneous ideal in $K[x_0, \ldots, x_n]$ obtained by homogenising the elements of $K[x^{\pm 1}] \cap I$.

Proposition 1.10. Let K be an algebraically closed valued field with value group $\mathbb{Q}$. For all finite $S \subseteq \mathbb{Z}^n$ there are $d \in \mathbb{N}$ and a finite $H \subseteq \mathbb{N}^n$ such that the following holds. Let $I \subseteq K[x^{\pm 1}]$ be an ideal and $\mathcal{G}$ a finite set generating of I such that every polynomial in $\mathcal{G}$ can be written as $\sum_{\mathbf{u} \in S} c_{\mathbf{u}} x^{\mathbf{u}}$ (with possibly some $c_{\mathbf{u}} = 0$).

(1) There is a universal Gröbner basis $\mathcal{U}$ of I_{proj} such that every polynomial in $\mathcal{U}$ can be written as $\sum_{\mathbf{u} \in \{0,\ldots,d\} \times H} b_{\mathbf{u}} x^{\mathbf{u}}$.

(2) There is a tropical basis $\mathcal{T} \supseteq \mathcal{G}$ of I such that every polynomial in $\mathcal{T}$ can be written as $\sum_{\mathbf{u} \in H} b_{\mathbf{u}} x^{\mathbf{u}}$.

Proof. Because $\Gamma(K) = \mathbb{Q}$, every ideal of $K[x_0, \dots, x_n]$ is defined over a subfield of K with discrete valuation. Hence we may apply [JS18, Theorem 10] to I_{proj} and obtain that there are d, depending only on n and on the minimal degree of a set of generators of I_{proj}, a tropical basis $\mathcal{T}_0$, and a universal Gröbner basis $\mathcal{U}$ of I_{proj}, both consisting of polynomials of degree at most d. We can then take $\mathcal{T}$ to be the union of $\mathcal{G}$ with the set of polynomials from $\mathcal{T}_0$ evaluated in $x_0 = 1$; we will show that $\mathcal{T}$ is a tropical basis, from which it follows that we can take as H the set of exponents $[0, d]^n \subseteq \mathbb{N}^n$.

Equip K with a cross-section, so that we may define initial forms. By [MS15, Theorem 3.2.3], $\mathcal{T}$ is a tropical basis if and only if, for every γ, whenever there is a monomial in $\mathrm{in}_\gamma(I)$, then there is some $g \in \mathcal{T}$ such that $\mathrm{in}_\gamma(g)$ is a monomial.

Let $\gamma \in \Gamma^n$, and assume that $\mathrm{in}_\gamma(I)$ contains a monomial. Then by [MS15, Proposition 2.6.1], $\mathrm{in}_{(0,\gamma)}(I_{\mathrm{proj}})$ also contains a monomial, so there is $f \in \mathcal{T}_0$ such that $\mathrm{in}_{(0,\gamma)}(f)$ is a monomial. Since $\mathrm{in}_{(0,\gamma)}(f(x_0, \dots, x_n)) = \mathrm{in}_\gamma(f(1, x_1, \dots, x_n))$ as shown in the proof of [MS15, Proposition 2.6.1], $\mathcal{T}$ is then a tropical basis. □

Fact 1.11. Let I be an ideal in $K[x^{\pm 1}]$ and $\gamma \in \Gamma^n$. For every $g \in \mathrm{in}_\gamma(I)$ there is $h \in I$ such that $g = \mathrm{in}_\gamma(h)$.

Proof. This is [MS15, Lemma 2.6.2(1)]. One sees by inspection that in its proof (and in the proofs of Lemma 2.4.2 and Proposition 2.6.1, which are used within it) the rank 1 assumption is never used. □

Theorem 1.12. [Fundamental Theorem of Tropical Geometry for valuations of arbitrary rank] Let K be an algebraically closed valued field and $I \subseteq K[x^{\pm 1}]$ an ideal. Then the following properties are equivalent for all $\gamma \in \Gamma^n$.

(1) For all $f \in I$, the point γ is a tropical root of $\mathrm{trop}(f)$.

(2) The ideal $\mathrm{in}_\gamma(I) \subseteq k[x^{\pm 1}]$ is proper.

(3) There is $z \in (K^\times)^n$ such that $f(z) = 0$ for every $f \in I$ and $v(z) = \gamma$.

Proof. If K is trivially valued then the statement is obvious, so we may assume the valuation is nontrivial.

The implications (2) ⇒ (1) and (3) ⇒ (1) are obvious, and (1) ⇒ (2) follows from Fact 1.11.

For (1) ⇒ (3), let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be the algebraic subvariety defined by I. Fix a set $\mathcal{G}$ of generators of I, with support S. Let $H \subseteq \mathbb{N}^n$ be given by Proposition 1.10 applied to S and to an arbitrary algebraically closed valued field L of the same characteristic (p, q) with value group $\mathbb{Q}$. We identify a finite family of Laurent polynomials in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with exponents in S with the tuple t of their coefficients; write $\mathcal{G}_t$ for such a family and I_t for the ideal it generates.

Let $\varphi_H(t, \gamma)$ be the formula saying that for every tuple $(y_{\mathbf{u}})_{\mathbf{u} \in H}$, if the polynomial $f = \sum_{\mathbf{u} \in H} y_{\mathbf{u}} x^{\mathbf{u}}$ vanishes on all z such that all elements of $\mathcal{G}_t$ vanish on z (this is a finite conjunction expressing the fact that f is in the radical of I_t), then γ is a tropical root of $\mathrm{trop}(f)$. In an arbitrary valued field, for all t and γ, the implication (1) ⇒ $\varphi_H(t, \gamma)$ holds trivially.

By Proposition 1.10, in L we have that $\varphi_H(t, \gamma)$ holds if and only if γ satisfies property (1) for I_t; by the Fundamental Theorem of Tropical Geometry [MS15, Theorem 3.2.3], in rank 1 properties (1) and (3) are equivalent, so in L we have that $\varphi_H(t, \gamma)$ holds if and only if γ satisfies property (3) for I_t. This equivalence

is a first-order statement, so it holds in algebraically closed nontrivially valued fields of arbitrary rank by completeness of $\mathsf{ACVF}_{p,q}$. □

The following proposition, which generalises a step of the proof of the Fundamental Theorem of Tropical Geometry [MS15, Proposition 3.2.11], will be needed later on. We consider initial forms as computed with respect to a given angular component map, see Remark 1.8.

Proposition 1.13. Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an algebraic subvariety defined over an algebraically closed nontrivially ac-valued field K, and $\gamma \in \Gamma^n$.

(1) The variety $\mathrm{in}_\gamma(V)(k(K))$ coincides with the set of angular components of points of $V(K)$ of valuation γ.

(2) For each $\alpha \in \mathrm{in}_\gamma(V)(k(K))$, the set of $a \in V(K)$ such that $v(a) = \gamma$ and $\mathrm{ac}(a) = \alpha$ is Zariski dense in $V(K)$.

Proof. Let $I \subseteq K[x^{\pm 1}]$ be the ideal associated to V. Let d and H be given by Proposition 1.10 applied to the set S of exponents of a fixed basis of I. Similarly as in the proof of Theorem 1.12, write $I = I_t$ where t is the tuple of coefficients of a set of generators with exponents in S. Let $J_{t,\gamma}$ be the ideal generated by $\mathrm{in}_\gamma(f)$, as f varies amongst polynomials in I_t with exponents in H. In every nontrivially ac-valued field, for every t and γ, it follows from the definition of initial form that

$$\{\mathrm{ac}(a) \mid a \in V(I_t), v(a) = \gamma\} \subseteq V(\mathrm{in}_\gamma(I_t)) \subseteq V(J_{t,\gamma}).$$

By [MS15, Proposition 3.2.11], in every algebraically closed ac-valued field of rank 1, the first inclusion above is an equality. If furthermore $\Gamma(K) = \mathbb{Q}$, then by Proposition 1.10 there is a universal Gröbner basis of I_{proj} with exponents in $\{0, \dots, d\} \times H$, and it follows that the second inclusion is also an equality.

As the sets $\{\mathrm{ac}(a) \mid a \in V(I_t), v(a) = \gamma\}$ and $V(J_{t,\gamma})$ are first-order definable uniformly in t and γ, completeness of $\mathsf{ACVF}_{p,q,\mathrm{ac}}$ yields that they coincide in every algebraically closed nontrivially ac-valued field. As $V(\mathrm{in}_\gamma(I_t))$ lies between them, this proves (1).

As for (2), fix a polynomial $g \in K[x^{\pm 1}]$. The conclusion follows from the previous point and the first-order expressibility of "if there are points $z_0, z_1 \in V$ with $v(z_0) = \gamma$, $\mathrm{ac}(z_0) = \alpha$ and $g(z_1) \neq 0$, then there is a point $z_2 \in V$ such that $v(z_2) = \gamma$, $\mathrm{ac}(z_2) = \alpha$, and $g(z_2) \neq 0$". □

Lemma 1.14. Let I be an ideal in $K[x^{\pm 1}]$. Then $\mathrm{in}_0(I) = \mathrm{res}(I \cap \mathcal{O}[x^{\pm 1}])$.

Proof. ($\supseteq$) Let $f \in \mathrm{res}(I \cap \mathcal{O}[x^{\pm 1}])$; if $f = 0$ then there is nothing to prove, so assume $f \neq 0$. Then there is $g \in I \cap \mathcal{O}[x^{\pm 1}]$ such that $f = \mathrm{res}(g)$. Since $\mathrm{res}(g) \neq 0$, g has at least one coefficient of valuation 0, and it follows from the definitions that $f = \mathrm{res}(g) = \mathrm{in}_0(g)$.

($\subseteq$) Let $f \in \mathrm{in}_0(I)$. By Fact 1.11 there is $g \in I$ such that $f = \mathrm{in}_0(g)$, say $g = \sum_{\mathbf{u} \in S} c_{\mathbf{u}} x^{\mathbf{u}}$. Let $S' \subseteq S$ be the set of $\mathbf{u}$ for which $v(c_{\mathbf{u}})$ is minimal. If $\mathbf{u}_0 \in S'$, then

$$\mathrm{in}_0(g) = \sum_{\mathbf{u} \in S'} \mathrm{res}(c_{\mathbf{u}} s(-v(c_{\mathbf{u}})) x^{\mathbf{u}} = \mathrm{res}\left(\frac{g}{s(v(c_{\mathbf{u}_0}))}\right).$$

Now $g/(s(v(c_{\mathbf{u}_0}))) \in I \cap \mathcal{O}[x^{\pm 1}]$, and the minimal valuation of its coefficients is 0, so we have

$$\mathrm{in}_0\left(\frac{g}{s(v(c_{\mathbf{u}_0}))}\right) = \mathrm{res}\left(\frac{g}{s(v(c_{\mathbf{u}_0}))}\right) = \mathrm{in}_0(g) = f. \qquad \square$$

Proposition 1.15. Let I be an ideal in $K[x^{\pm 1}]$ and σ an automorphism. Then $\sigma(\mathrm{in}_0(I)) = \mathrm{in}_0(\sigma(I))$.

Proof. By Lemma 1.14, it is enough to show $\sigma(\mathrm{res}(I \cap \mathcal{O}[x^{\pm 1}])) = \mathrm{res}(\sigma(I) \cap \mathcal{O}[x^{\pm 1}])$. This follows easily from $\sigma \circ \mathrm{res} = \mathrm{res} \circ \sigma$. $\square$

2 Purity and reducts

As already recalled, every algebraically closed valued field admits a cross-section of the valuation. In this section, we show that universal domains for tropical difference algebra admit one that is σ-equivariant.

For this purpose, we endow the multiplicative group of a valued difference field (K, v, σ) with a module structure over the polynomial ring $\mathbb{Z}[\sigma]$ by defining $\left(\sum_{i=0}^{n} m_i \sigma^i\right) \cdot z = \prod_{i=0}^{n} \sigma^i(z^{m_i})$. We show that, assuming existential closedness, $\mathcal{O}^\times$ is a *pure* $\mathbb{Z}[\sigma]$*-submodule* of $K^\times$, and we use standard arguments to deduce the existence of a cross-section commuting with σ if enough saturation is present.

Recall that a formula is *existential* iff it is of the form $\exists x\ \varphi(x)$ with $\varphi(x)$ quantifier-free, and *positive* (resp. *positive primitive*) iff it is obtained from atomic formulas by using $\wedge, \vee, \exists$ (resp. $\wedge, \exists$).

Definition 2.1.

(1) Let M be an $\mathcal{L}$-structure. A *partial type* $\pi(x)$ is a set of $\mathcal{L}$-formulas with parameters from M such that, for every n and every $\varphi_0(x), \ldots, \varphi_n(x) \in \pi(x)$, there is $a \in M$ such that, for every $i \leq n$, we have $M \vDash \varphi_i(a)$.

(2) The *quantifier-free type* $\mathrm{qftp}(a/B)$ of a tuple a in an $\mathcal{L}$-structure M over a set $B \subseteq M$ is the set of quantifier-free $\mathcal{L}$-formulas $\varphi(x)$ with parameters from B such that $M \vDash \varphi(a)$.

(3) Let κ be an infinite cardinal. A structure M is *existentially* (resp. *positively*, resp. *quantifier-free*) *κ-saturated* iff, whenever $\pi(x)$ is a partial type consisting of existential (resp. positive, resp. quantifier-free) formulas with parameters from a set $A \subseteq M$ of size $|A| < \kappa$, there is $a \in M$ satisfying all formulas in $\pi(x)$.

So, for example, the $\mathbb{Z}[\sigma]$-module $\mathcal{O}^\times$ is positively $\aleph_1$-saturated if and only if, for every system of countably many equations in the language of $\mathbb{Z}[\sigma]$-modules with parameters from $\mathcal{O}^\times$, if every finite subsystem has a solution in $\mathcal{O}^\times$ then the whole system does.[2] This is in particular the case if (K, v, σ) is existentially $\aleph_1$-saturated.

Remark 2.2. By standard model-theoretic arguments, for every cardinal κ, every valued difference field has an existentially κ-saturated existentially closed extension.

[2] Note that every partial type consisting of positive formulas is implied by a partial type consisting of positive primitive formulas.

Proposition 2.3. In every existentially closed (K, v, σ), the following statements are true.

(1) The field $\mathrm{VF}(K)$ is algebraically closed.

(2) The difference oag $\Gamma(K)$ is existentially closed.

(3) The difference field $k(K)$ is existentially closed.

Proof. As commonly done in model theory, we embed the valued field (K, v) into a $|K|^+$-*strongly homogeneous* valued field $\mathfrak{U}$, that is, one where every *elementary map* with domain of size at most $|K|$ extends to an automorphism. Recall that, whenever a theory has *quantifier elimination*, an elementary map is nothing more than a partial isomorphism, and that quantifier elimination holds for ACVF in $\mathcal{L}_{\mathrm{vf}}$, see [HHM06, Theorem 2.1.1(iii)][3].

(1) As (K, v, σ) is existentially closed, it follows easily from the definitions that, in order to prove its algebraic closedness, it suffices to embed it into an algebraically closed valued difference field. By quantifier elimination, σ may be regarded as an elementary map $\mathfrak{U} \to \mathfrak{U}$, and by $|K|^+$-strong homogeneity we may extend it to some $\sigma_{\mathfrak{U}} \in \mathrm{Aut}(\mathfrak{U})$, thus obtaining the required extension.

(2) View again σ as an elementary map $\mathfrak{U} \to \mathfrak{U}$. Let $(\Gamma', \sigma') \supseteq (\Gamma(K), \sigma_{\Gamma(K)})$. Since $\Gamma(\mathfrak{U}) \models \mathsf{DOAG}$,[4] up to replacing $\mathfrak{U}$ by a larger $\mathfrak{U}'$, we may embed Γ' in it over $\Gamma(K)$. Again by quantifier elimination, $\sigma_K \cup \sigma'$ is an elementary map, hence extends to an automorphism of $\mathfrak{U}$. Every existential formula over $\Gamma(K)$ realised in Γ' is as well realised in $\Gamma(\mathfrak{U})$. By existential closedness of K, it must be also realised in $\Gamma(K)$. This shows that $\Gamma(K)$ is existentially closed.

(3) This can be proved similarly to (2), using ACF in place of DOAG. □

Remark 2.4.

(1) In our framework, the residue and valuation maps are assumed surjective from the start. In fact, as far as existentially closed models are concerned, this is not necessary, as surjectivity of these maps can be proved with arguments similar to those of the proof of Proposition 2.3. We thank Ehud Hrushovski for pointing this out.

(2) Every possibly non-inversive difference field embeds canonically in its inversive closure. In particular, even if we work with non-inversive valued difference fields, the existentially closed ones are going to be automatically inversive. We thank Simone Ramello for pointing this out.

2.1 Purity

Recall that, if R is a ring, a submodule N of an R-module M is *pure* iff, for every matrix $A \in \mathrm{Mat}_{d\times n}(R)$ and vector $b \in N^d$, if the equation $A \cdot x = b$ has a solution in M then it has a solution in N. This passes to localisations, and is in fact a local property, see [Lam99, §4, Exercise 34]. The case of interest for us is that purity as a $\mathbb{Z}[\sigma]$-module implies purity as a $\mathbb{Z}[\sigma, \sigma^{-1}]$-module, hence it suffices to work with $\mathbb{Z}[\sigma]$.

[3] Our language is slightly different, see Remark 1.5.

[4] In characteristic $(0, p)$ we need to add to the language on Γ a constant symbol, interpreted as $v(p)$. We leave to the reader the easy task of modifying this part of the proof accordingly.

Remark 2.5. It follows easily from the definitions that, since $\Gamma = K^\times/\mathcal{O}^\times$ is torsion-free, $\mathcal{O}^\times$ is a pure abelian subgroup of $K^\times$.

The following example shows that, if we do not assume existential closedness, purity may indeed fail.

Example 2.6. Consider the $\mathbb{R}$-*exponentiation* (see [KS22, Definition 3.3.1][5]) $\sigma\colon \mathbb{C}((\mathbb{R})) \to \mathbb{C}((\mathbb{R}))$ given by

$$\sum c_r t^r \mapsto \sum e^r c_r t^r.$$

In this example, $\mathcal{O}^\times$ is not a pure $\mathbb{Z}[\sigma]$-submodule of $K^\times$: the equation $\sigma(x)x^{-1} = et^0$ has parameters in $\mathcal{O}^\times$ and is solved by t^1, but there are no solutions in $\mathcal{O}^\times$ because σ induces the identity on the residue field.

In particular, this automorphism admits no equivariant angular component, hence no equivariant cross-section.

Notation 2.7.

(1) For $r \leq s \in \mathbb{Z}$, we denote by $[r, s]$ the interval $\{h \in \mathbb{Z} \mid r \leq h \leq s\}$.

(2) For a tuple a with all coordinates in the domain of an automorphism σ and $r \leq s \in \mathbb{Z}$ we will write $\mathrm{orb}_{r,s}(a)$ to denote the tuple $(\sigma^r(a), \ldots, \sigma^s(a))$.

Note that, if $\mathbf{u}_1, \ldots, \mathbf{u}_{n_2} \in \mathbb{Z}^{n_1}$ and $\psi\colon \mathbb{G}_{\mathrm{m}}^{n_1} \to \mathbb{G}_{\mathrm{m}}^{n_2}$ is the *monomial map* $z \mapsto (z^{\mathbf{u}_1}, \ldots, z^{\mathbf{u}_{n_2}})$, then $\mathrm{trop}(\psi)\colon \Gamma^{n_1} \to \Gamma^{n_2}$ is the morphism

$$\gamma \mapsto (\langle \mathbf{u}_1, \gamma\rangle, \ldots, \langle \mathbf{u}_{n_2}, \gamma\rangle)\,.$$

Lemma 2.8. Let K be a valued difference field, let $A \in \mathrm{Mat}_{d\times n}(\mathbb{Z}[\sigma])$, and let $b \in (\mathcal{O}^\times)^d$. Assume there is $\beta \in (K^\times)^n$ such that $A \cdot \beta = b$.

There are $\ell \in \mathbb{N}$, an algebraic subgroup G of $\mathbb{G}_{\mathrm{m}}^{n(\ell+1)}$, and $t \in (\mathcal{O}^\times)^{n(\ell+1)}$ such that, for all $z \in (K^\times)^n$, we have $A \cdot z = b$ if and only if $\mathrm{orb}_{0,\ell}(z) \in t \cdot G$.

Proof. For each entry a_{ij} of A, there are $\ell_{ij} \in \mathbb{Z}_{\geq 0}$ and $m_{ij0}, \ldots, m_{ij\ell_{ij}} \in \mathbb{Z}$ such that

$$a_{ij} = m_{ij\ell_{ij}}\sigma^{\ell_{ij}} + m_{ij(\ell_{ij}-1)}\sigma^{\ell_{ij}-1} + \ldots + m_{ij1}\sigma + m_{ij0}.$$

Let $\ell \coloneqq \max_{ij} \ell_{ij}$. For $i \in [1, d]$, let $C_i \in \mathrm{Mat}_{n\times(\ell+1)}(\mathbb{Z})$ be the matrix

$$C_i = (c_{ijh})_{j\in[1,n],h\in[0,\ell]}$$

in which the entry c_{ijh} is the (integer, possibly 0) coefficient of σ^h in the entry a_{ij} of A (see Example 2.9 below). For $i \in [1, d]$, let $p_i((y_{jh})_{j\in[1,n],h\in[0,\ell]})$ be the monomial

$$p_i = \prod_{j=1}^{n}\prod_{h=0}^{\ell} y_{jh}^{c_{ijh}}.$$

Let $G \coloneqq \{y \in \mathbb{G}_{\mathrm{m}}^{n(\ell+1)} \mid \forall i \in [1, d]\ p_i(y) = 1\}$ and $H \coloneqq \{y \in \mathbb{G}_{\mathrm{m}}^{n(\ell+1)} \mid \forall i \in [1, d]\ p_i(y) = b_i\}$. Then G is an algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^{n(\ell+1)}$, and H is a

[5]The reader familiar with this work will see that, if we decompose the automorphism group of $\mathbb{C}((\mathbb{R}))$ as in [KS22, Theorem 3.7.1] with respect to the natural cross-section, then any counterexample to purity must have a nontrivial factor in $\mathrm{Hom}(\mathbb{R}, \mathbb{C}^\times)$.

coset of G. By construction, for all $z \in (K^\times)^n$, we have $A \cdot z = b$ if and only if $p_i(\mathrm{orb}_{0,\ell}(z)) = b_i$ for all $i \in [1,d]$, if and only if $(\sigma^i(z_j))_{i\in[0,\ell],j\in[1,n]} \in H$.

Since $A \cdot \beta = b$, we have that $(\mathrm{orb}_{0,\ell}(\beta)) \in H(K)$. Moreover, by Remark 2.5, there is some $t \in (\mathcal{O}^\times)^{n(\ell+1)}$ such that $H = t \cdot G$. □

Example 2.9. Let $A = \begin{pmatrix} 1-\sigma \\ 1-\sigma^2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then the matrices C_1 and C_2 are $C_1 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$ and $C_2 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$, and for all $a \in K^\times$ we have that $A \cdot a = b$ if and only if $(a, \sigma(a), \sigma^2(a))$ lies in the algebraic subgroup of $(K^\times)^3$ defined by $z_1 z_2^{-1} = 1 \wedge z_1 z_3^{-1} = 1$.

Lemma 2.10. Let $\psi\colon \mathbb{G}_{\mathrm{m}}^{n_1} \to \mathbb{G}_{\mathrm{m}}^{n_2}$ be a monomial map, defined by $\psi(z) = (z^{\mathbf{u}_1}, \dots, z^{\mathbf{u}_{n_2}})$ for appropriate vectors $\mathbf{u}_1, \dots, \mathbf{u}_{n_2} \in \mathbb{Z}^{n_1}$.

There is a monomial map $\xi\colon \mathbb{G}_{\mathrm{m}}^{n_1} \to \mathbb{G}_{\mathrm{m}}^{n_2}$, defined by $\xi(z) = (z^{\mathbf{v}_1}, \dots, z^{\mathbf{v}_{n_2}})$, such that $\ker \xi$ is the connected component of the identity of $\ker \psi$ and that each $\mathbf{v}_i$ is a rational multiple of an element of the group generated by $\mathbf{u}_1, \dots, \mathbf{u}_{n_1}$.

Proof. It suffices to let $\mathbf{v}_1, \dots, \mathbf{v}_{n_2}$ be generators of the unique primitive lattice containing the one generated by $\mathbf{u}_1, \dots, \mathbf{u}_{n_1}$ and of the same rank. See [Sch25, Lemma A.7]. □

Lemma 2.11. Let $G \leq \mathbb{G}_{\mathrm{m}}^n$ be a connected algebraic subgroup. For all $t \in (\mathcal{O}^\times)^n$, the initial variety $\mathrm{in}_0(t \cdot G)$ is irreducible.

Proof. By [Sch25, Theorem A.13] and connectedness, G is defined by a set of equations of the form $z^{\mathbf{u}} - 1 = 0$ and the associated lattice $\Lambda \subseteq \mathbb{Z}^n$ is primitive. The subvariety $\mathrm{in}_0(t \cdot G)$ of $\mathbb{G}_{\mathrm{m}}^n(k)$ is contained in the coset by $\mathrm{res}(t)$ of the group H defined by the equations $\mathrm{in}_0(z^{\mathbf{u}} - 1) = 0$, and it is easily checked that $\mathrm{in}_0(z^{\mathbf{u}} - 1) = z^{\mathbf{u}} - 1$. The lattice Λ' associated to H must satisfy $\Lambda' \supseteq \Lambda$. Moreover, $\mathrm{in}_0(t \cdot G) \subseteq \mathrm{res}(t) \cdot H$, and by [MS15, Lemma 3.2.6][6], $\dim(\mathrm{in}_0(G)) = \dim(G)$. It follows that $\dim(H) = \dim(G)$, hence Λ' and Λ have the same rank, and because Λ is primitive $\Lambda = \Lambda'$. □

Definition 2.12.

(1) We define $\sigma_\to$ as the map sending a Laurent polynomial $f = \sum c_{ijh} y_{jh}^i \in K[y_{jh}^{\pm 1} \mid j \in [1,n], h \in \mathbb{Z}]$ to the Laurent polynomial

$$\sigma_\to(f) \coloneqq \sum \sigma(c_{ijh}) y_{j(h+1)}^i.$$

(2) We write $\sigma_\to^p$ for $(\sigma_\to)^p$.

Note that $\sigma_\to$ is invertible and that $f(a) = 0$ if and only if $(\sigma_\to(f))(\sigma(a)) = 0$.

Lemma 2.13. Let (K, σ) be an algebraically closed difference field, and let $I \subseteq K[y_{jh}^{\pm 1} \mid j \in [1,n], h \in [0,\ell]]$ be a proper ideal such that the variety $V(I)$ is a coset of an algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^{n(\ell+1)}$. Assume moreover that there is $\alpha \in (K^\times)^n$ such that, for all $r \leq s \in \mathbb{Z}$, we have that $\mathrm{orb}_{r,s}(\alpha) \in V(\sum_{p\in[r,s]} \sigma_\to^p(I))$.

For each $r \leq s \in \mathbb{Z}$ there is an ideal $J_{r,s}$ of $K[y_{jh}^{\pm 1} \mid j \in [1,n], h \in [r, s+\ell]]$ such that $J_{r,s} \supseteq \sum_{p\in[r,s]} \sigma_\to^p(I)$ and the set $\{J_{r,s}\}_{[r,s]\in\mathbb{Z}}$ satisfies the following properties.

[6] This is for rank 1 algebraically closed valued fields, but note that every algebraic group in this lemma is defined over the prime field.

(1) For all $r \leq s \in \mathbb{Z}$, the variety $H_{r,s} \coloneqq V(J_{r,s})$ is a coset of a connected algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^{n(s-r+\ell+1)}$.

(2) For all $r \leq s \in \mathbb{Z}$, we have $H_{r+1,s+1} = \sigma(H_{r,s})$.

(3) If v is a valuation on K commuting with σ, and $V(I)$ intersects $(\mathcal{O}^\times)^{n(\ell+1)}$, then each $H_{r,s}$ intersects $(\mathcal{O}^\times)^{n(s-r+\ell+1)}$.

Proof. For each interval $[r,s] \subseteq \mathbb{Z}$, define ideals $I_{r,s}$ and $I'_{r,s}$ of $K[y_{jh}^{\pm 1} \mid j \in [1,n], h \in [r, s+\ell]]$ by

$$I_{r,s} \coloneqq \sum_{p \in [r,s]} \sigma_{\to}^{p}(I)$$

and

$$I'_{r,s} \coloneqq \sum_{[r_0,s_0] \supseteq [r,s]} (I_{r_0,s_0} \cap K[y_{jh}^{\pm 1} \mid j \in [1,n], h \in [r, s+\ell]]).$$

By noetherianity of $K[y_{jh}^{\pm 1} \mid j \in [1,n], h \in [r,s+\ell]]$, this sum stabilises, so given $[r,s]$, there is $[r_0,s_0]$ such that for all $[r_1,s_1] \supseteq [r_0,s_0]$ we have

$$I_{r_1,s_1} \cap K[y_{jh}^{\pm 1} \mid j \in [1,n], h \in [r, s+\ell]] = I_{r_0,s_0} \cap K[y_{jh}^{\pm 1} \mid j \in [1,n], h \in [r, s+\ell]] = I'_{r,s},$$

hence $V(I'_{r,s})$ is still a coset of an algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^{n(s-r+\ell+1)}$. Let $J_{r,s}$ be the prime ideal containing $I'_{r,s}$ of polynomials vanishing on the irreducible component $H_{r,s}$ of $V(I'_{r,s})$ which contains the point $\mathrm{orb}_{r,s}(\alpha)$. Then, for all $r \leq s$, the irreducible variety $H_{r,s}$ is again a coset of an algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^{n(s-r+\ell+1)}$, proving (1), and by construction $H_{r+1,s+1} = \sigma(H_{r,s})$, proving (2).

Finally, for (3) note that, under these assumptions, I is generated by polynomials of the form $x^{\mathbf{u}} - b$, with $b \in \mathcal{O}^\times$. It follows that so is $I'_{r,s}$, so $V(I'_{r,s})$ has a point with all coordinates in $\mathcal{O}^\times$ by Remark 2.5. Different irreducible components of a subgroup of $\mathbb{G}_{\mathrm{m}}^N$ are translates of each other by a root of unity. As roots of unity have valuation 0, if one of them intersects $(\mathcal{O}^\times)^N$, then so do all of them, and the conclusion follows. □

Theorem 2.14. Suppose K is an existentially closed valued difference field. Then $\mathcal{O}^\times$ is a pure $\mathbb{Z}[\sigma]$-submodule of $K^\times$.

Proof. Let $A \in \mathrm{Mat}_{d \times n}(\mathbb{Z}[\sigma])$, let $b \in (\mathcal{O}^\times)^d$, let $\beta \in (K^\times)^n$ be such that $A \cdot \beta = b$, and denote by ℓ the highest order of an iterate of σ appearing in the matrix A.

Let $t \cdot G$ be the coset of an algebraic subgroup $G \leq \mathbb{G}_{\mathrm{m}}^{n(\ell+1)}$ obtained from Lemma 2.8, and let I be the ideal of polynomials in the ring $K[y_{jh}^{\pm 1} \mid j \in [1,n], h \in [0,\ell]]$ which vanish on $t \cdot G$. Let $\{H_{r,s}\}_{[r,s] \subseteq \mathbb{Z}}$ be the set of varieties obtained by Lemma 2.13. For each $[r,s]$, as $H_{r,s}$ is irreducible, by Lemma 2.11 so is $\mathrm{in}_0(H_{r,s})$.

Let Θ be the set of formulas in the variables $(y_{jh})_{j \in [1,n], h \in \mathbb{Z}}$, which contains, for each finite subtuple $(y_{jh})_{j \in [1,n], h \in [r,s+\ell]}$, the following.

(1) The formulas saying that $(y_{jh})_{j \in [1,n], h \in [r,s+\ell]}$ is a K-generic point[7] of the irreducible variety $H_{r,s}$, all of whose coordinates have valuation 0.

[7] Recall that a point of an irreducible algebraic variety is K-generic iff it does not lie in any proper subvariety definable over K.

(2) The formulas saying that $(\mathrm{res}(y_{jh}))_{j\in[1,n],h\in[r,s+\ell]}$ is a k-generic point of the irreducible variety $\mathrm{in}_0(H_{r,s})$.

Let Δ be a finite fragment of Θ. Without loss of generality, Δ says that a finite subtuple of variables $(y_{jh})_{j\in[1,n],h\in[r,s+\ell]}$ lies in a certain Zariski open dense subset H' of $H_{r,s}$, that all of its coordinates have valuation 0, and that its residue lies in a certain Zariski open dense subset H'' of $\mathrm{in}_0(H_{r,s})$. Since the residue field k is algebraically closed, there is a point $(a_{jh})_{j\in[1,n],h\in[r,s+\ell]} \in H''(k)$, and by Proposition 1.13 the set of points of valuation 0 in $H_{r,s}(K)$ with residue $(a_{jh})_{j\in[1,n],h\in[r,s+\ell]}$ is Zariski dense in $H_{r,s}$, so in particular there is such a point in $H'(K)$. Therefore Θ is finitely satisfiable, so by the Compactness Theorem it is consistent.

Hence Θ is realised in some valued field extension (K', v') of (K, v) by some infinite tuple $(\alpha_{jh})_{j\in[1,n],h\in\mathbb{Z}}$. Without loss of generality we assume $K' = K((\alpha_{jh})_{j\in[1,n],h\in\mathbb{Z}})$. By using Proposition 1.15, we see that $\sigma(\mathrm{in}_0(H_{r,s})) = \mathrm{in}_0(\sigma(H_{r,s})) = \mathrm{in}_0(H_{r+1,s+1})$. Because $H_{r,s}$ is a coset of an algebraic subgroup isomorphic (as an algebraic group) over the prime field to a power of $\mathbb{G}_\mathrm{m}$, and because in every power of $\mathbb{G}_\mathrm{m}$ there is a unique type of a tuple of valuation 0 with k-generic residue, it follows that Θ is a complete quantifier-free $\mathcal{L}_\mathrm{vf}$-type over K.

Therefore, extending the action of $\sigma_\to$ from polynomials to formulas in the obvious way, as $(\alpha_{jh})_{j\in[1,n],h\in\mathbb{Z}}$ realises Θ, for all $r \le s \in \mathbb{Z}$ we have, in $\mathcal{L}_\mathrm{vf}$,

$$\sigma_\to\left(\mathrm{qftp}((\alpha_{jh})_{j\in[1,n],h\in[r,s+\ell]}/K)\right) = \mathrm{qftp}((\alpha_{jh})_{j\in[1,n],h\in[r+1,s+\ell+1]}/K).$$

Therefore we may extend σ to an automorphism σ' of (K', v') by putting $\sigma'(\alpha_{jh}) \coloneqq \alpha_{j(h+1)}$ for each $j \in [1, n]$ and $h \in \mathbb{Z}$. Then the tuple $(\alpha_{10}, \ldots, \alpha_{n0}) \in (K'^\times)^n$ is a solution of the equation $A\cdot z = b$, whose coordinates have valuation 0. Hence, by existential closedness there is a solution with coordinates of valuation 0 in K. □

Corollary 2.15. Let K be an existentially closed valued difference field. If $\mathcal{O}^\times$ is positively $\aleph_1$-saturated as a $\mathbb{Z}[\sigma]$-module (e.g. if K is existentially $\aleph_1$-saturated), then K admits a cross-section $s\colon \Gamma \to K$ satisfying $s \circ \sigma = \sigma \circ s$, hence an angular component $\mathrm{ac}\colon K \to k$ satisfying $\mathrm{ac} \circ \sigma = \sigma \circ \mathrm{ac}$. In particular, every valued difference field embeds into one admitting an equivariant cross-section.

Proof. Recall that, in an R-module, a *partial positive primitive type* (or *partial pp-type*) is specified by a filter of pp-definable cosets, hence can be written by using at most $|R| + \aleph_0$-many formulas (see e.g. [Pre88] for model theory of modules). In particular, under our assumptions the $\mathbb{Z}[\sigma]$-module $\mathcal{O}^\times$ realises all consistent partial pp-types. By Theorem 2.14 the short exact sequence of $\mathbb{Z}[\sigma]$-modules $1 \to \mathcal{O}^\times \to K^\times \to \Gamma \to 0$ is pure, hence the existence of a cross-section follows from [Pre88, Theorem 2.8]. As standard, a cross-section s induces an angular component $\mathrm{ac}_s(x) \coloneqq \mathrm{res}(x/s(v(x)))$, and clearly if s commutes with σ then so does ac_s, since both res and v do. The "in particular" part follows from Remark 2.2. □

Remark 2.16. If (K, v, σ) is a valued difference field and $s\colon \Gamma \to K$ is a σ-equivariant cross-section, then it is easy to see that, for every Laurent polynomial $f \in K[x^{\pm 1}]$ and $\gamma \in \Gamma^n$, we have $\sigma(\mathrm{in}_\gamma(f)) = \mathrm{in}_{\sigma(\gamma)}(\sigma(f))$; similarly, for initial ideals we get $\sigma(\mathrm{in}_\gamma(I)) = \mathrm{in}_{\sigma(\gamma)}(\sigma(I))$.

2.2 Good reducts

By Corollary 2.15, every existentially closed valued difference field that is sufficiently existentially saturated can be expanded by an equivariant cross-section. In order to use the cross-section as an auxiliary object, and later remove it from the language, it is important to understand whether, given an existentially closed (K, v, σ, s), its reduct (K, v, σ) is still existentially closed. Similar questions can be formulated when passing from (K, v, σ, s) to $(K, v, \sigma, \mathrm{ac})$, etc.

Below, we will prove that these questions have positive answers. As this will be proved separately for each case of interest, and as there will be a nontrivial amount of algebra involved, the reader may wonder whether some abstract argument shows that existential closedness passes to reducts. This is in fact false. We thank Angus Matthews for the following example, and for his permission to include it here. We will encounter another instance of this phenomenon in Corollary 2.27.

Example 2.17 (A. Matthews)**.** Let T_0 be an inductive theory that is not model complete (e.g. the theory of oags with an automorphism, or the theory of fields), and let T_1 be its Morleyisation (see the references given in Remark 1.2). Observe that every model of T_0 expands to a model of T_1, and that every model of T_1 is existentially closed by quantifier elimination. In particular, if $M_0 \models T_0$ is not existentially closed, then its natural expansion to $M_1 \models T_1$ is existentially closed but has a non-existentially closed reduct.

Proposition 2.18. Let $(K, v, \sigma, \mathrm{ac})$ be an ac-valued difference field closed under roots, and let (L, v, σ) be an existentially closed, existentially $\aleph_1$-saturated valued difference field extension. There is an equivariant angular component map on L extending the angular component map of K.

Proof. Let T be the $\mathbb{Z}[\sigma]$-submodule of $L^\times$ generated by $K^\times$ and $\mathcal{O}(L)^\times$, and let $\theta\colon T \to k(L)^\times$ be the unique morphism which coincides with ac on $K^\times$ and with res on $\mathcal{O}(L)^\times$. Enumerate T in a tuple $\bar{b}$. If we prove that the pp-type of $\bar{b}$ in the $\mathbb{Z}[\sigma]$-module $L^\times$ is contained in the pp-type of $\theta(\bar{b})$ in $k(L)^\times$, then we may conclude by using existential saturation of $k(L)^\times$ as a $\mathbb{Z}[\sigma]$-module to apply Point (v) in [Pre88, Theorem 2.8], obtaining a morphism $\mathrm{ac}_L\colon L^\times \to k(L)^\times$ which extends θ.

Let a be some finite subtuple of $\bar{b}$ and $A \in \mathrm{Mat}_{d\times n}(\mathbb{Z}[\sigma])$ a matrix such that $A \cdot z = a$ has a solution in $(L^\times)^n$. Fix such a solution α. We need to show that $A \cdot z = \theta(a)$ has a solution in $(k(L)^\times)^n$. Since $k(L)$ is existentially closed as a difference field by Proposition 2.3, it suffices to find an extension of $k(L)$ where a solution exists.

As in the proof of Lemma 2.8, we associate to A some d matrices $C_1, \ldots, C_d \in \mathrm{Mat}_{n\times(\ell+1)}(\mathbb{Z})$, where ℓ is maximum such that σ^ℓ appears in A, such that the entry c_{ijh} of the matrix C_i is the integer coefficient of σ^h in the entry $a_{ij} \in \mathbb{Z}[\sigma]$ of A.

For all $r \leq s \in \mathbb{Z}$, there is a group homomorphism $\psi_{r,s}\colon \mathbb{G}_{\mathrm{m}}^{n(s+\ell-r+1)} \to \mathbb{G}_{\mathrm{m}}^{d(s-r+1)}$ satisfying $\psi_{r,s}(\mathrm{orb}_{r,s+\ell}(\alpha)) = \mathrm{orb}_{r,s}(a)$, namely we can take the ho-

momorphism which maps $(y_{jh})_{j\in[1,n],h\in[r,s+\ell]}$ to

$$\begin{pmatrix} \prod_{j=1}^{n}\prod_{h=r}^{r+\ell} y_{jh}^{c_{1jh}} \\ \vdots \\ \prod_{j=1}^{n}\prod_{h=r}^{r+\ell} y_{jh}^{c_{djh}} \\ \prod_{j=1}^{n}\prod_{h=r+1}^{r+1+\ell} y_{jh}^{c_{1jh}} \\ \vdots \\ \prod_{j=1}^{n}\prod_{h=r+1}^{r+1+\ell} y_{jh}^{c_{djh}} \\ \vdots \\ \prod_{j=1}^{n}\prod_{h=s}^{s+\ell} y_{jh}^{c_{1jh}} \\ \vdots \\ \prod_{j=1}^{n}\prod_{h=s}^{s+\ell} y_{jh}^{c_{djh}} \end{pmatrix}.$$

By Lemma 2.10 there is a monomial map $\xi_{r,s}$ such that $\ker\xi_{r,s}$ is the connected component of $\ker\psi_{r,s}$. Moreover, each of the exponents of $\xi_{r,s}$ is a rational multiple of an element of the group spanned by the exponents of $\psi_{r,s}$, and therefore every entry of $\xi_{r,s}(\mathrm{orb}_{r,s}(\alpha))$ is a root of a product of the entries of $\mathrm{orb}_{r,s}(a)$. By assumption, T is a multiplicative subgroup that is closed under taking roots hence, since each entry of $\mathrm{orb}_{r,s}(a)$ lies in T, we have $\xi_{r,s}(\mathrm{orb}_{r,s}(\alpha)) \in T^{d(s-r+1)}$.

Let $e = (e_1, \ldots, e_d)$ denote the tuple $\xi_{0,0}(\mathrm{orb}_{0,\ell}(\alpha))$. Consider the set Ω of $\mathcal{L}_{\mathrm{field}}$-formulas over $k(L)$ in variables $(y_{jh})_{j\in[1,n],h\in\mathbb{Z}}$ which says that each finite subtuple $(y_{jh})_{j\in[1,n],h\in[r,s+\ell]}$ is a $k(L)$-generic point of the algebraic variety $V_{r,s}$ defined by the equations $\xi_{r,s}((z_{jh})_{j\in[1,n],h\in[r,s+\ell]}) = \theta(\mathrm{orb}_{r,s}(e))$. Note that the genericity requirement makes sense because, as $V_{r,s}(k(L))$ is a coset of $\mathrm{in}_0(\ker\xi_{r,s})$ (which does not depend on the choice of an angular component by Lemma 1.14), it is irreducible by Lemma 2.11.

To see that Ω is consistent, consider a finite fragment mentioning formulas in the finite subtuple of variables $(y_{jh})_{j\in[1,n],h\in[r,s+\ell]}$. As $k(L)^\times$ is a divisible abelian group, there is a (possibly non-equivariant) angular component $\mathrm{ac}' \colon L \to k(L)$ extending θ. Since, for all $r \le s$, we have $\mathrm{orb}_{r,s}(e) \in \mathrm{im}(\xi_{r,s})$, and ac' is a multiplicative homomorphism, $V_{r,s}(k(L))$ is not empty.

Hence the set Ω is realised in some field extension of $k(L)$; let $(\beta_{jh})_{j\in[1,n],h\in\mathbb{Z}}$ be a realisation, and consider the field $k(L)((\beta_{jh})_{j\in[1,n],h\in\mathbb{Z}})$. We now have that, for all $r \le s$, the finite tuple $(\beta_{j(h+1)})_{j\in[1,n],h\in[r,s+\ell]}$ is generic in the algebraic variety defined by the equations

$$\xi_{r+1,s+1}((z_{j(h+1)})_{j\in[1,n],h\in[r,s+\ell]}) = \theta(\mathrm{orb}_{r+1,s+1}(e)),$$

which since σ and θ commute is the same as

$$\xi_{r+1,s+1}((z_{j(h+1)})_{j\in[1,n],h\in[r,s+\ell]}) = \sigma(\theta(\mathrm{orb}_{r,s}(e))).$$

Hence, applying $\sigma_\to$ to the $\mathcal{L}_{\mathrm{field}}$-quantifier-free type over $k(L)$ of the tuple $(\beta_{jh})_{j\in[1,n],h\in[r,s+\ell]}$ yields the one of $(\beta_{jh})_{j\in[1,n],h\in[r+1,s+1+\ell]}$. So the automorphism σ of $k(L)$ extends to an automorphism of $k(L)((\beta_{jh})_{j\in[1,n],h\in\mathbb{Z}})$, in which the equation $A \cdot z = \theta(a)$ is solved by the tuple $\beta \coloneqq (\beta_{10}, \ldots, \beta_{1n})$. □

Corollary 2.19. If $(K, v, \sigma, \mathrm{ac})$ is existentially closed, then (K, v, σ) is existentially closed.

Proof. As $\mathsf{ACVF}_{\mathrm{ac}}$ eliminates quantifiers[8], arguing as in the proof of Proposition 2.3(1) we see that K is algebraically closed, and in particular closed under taking roots. Suppose that $(L_0, v_{L_0}, \sigma_{L_0}) \supseteq (K, v, \sigma)$ is such that $L_0 \vDash \exists x\, \varphi(x)$, with $\varphi(x)$ a quantifier-free $\mathcal{L}_\sigma(K)$-formula. Let (L, v_L, σ_L) be an existentially $\aleph_1$-saturated existentially closed extension of L_0. After applying Proposition 2.18, L extends K as ac-valued difference fields, and $L \vDash \exists x\, \varphi(x)$. Since $\mathcal{L}_{\mathrm{ac},\sigma} \supseteq \mathcal{L}_\sigma$ and $(K, v, \sigma, \mathrm{ac})$ is existentially closed, we have $K \vDash \exists x\, \varphi(x)$. □

Lemma 2.20. If (K, v, σ, s) is existentially closed, then $(\Gamma(K), \sigma)$ is existentially closed.

Proof. It is well known that the theory ACVF_s of algebraically closed valued fields with a cross-section eliminates quantifiers; see e.g. [Kes24, Theorem 2.2] for a proof for $\mathsf{ACVF}_{s,\iota}$ that can be easily adapted.

Embed (K, v, s) in a monster model $\mathfrak{U} \vDash \mathsf{ACVF}_s$. Let (Δ, σ_Δ) be an extension of $(\Gamma(K), \sigma_{\Gamma(K)})$; we may assume that Δ is embedded in $\Gamma(\mathfrak{U})$ over $\Gamma(K)$. If the map $f \coloneqq \sigma \cup \sigma_\Delta$, viewed as a partial map $\mathfrak{U} \to \mathfrak{U}$, is elementary, then it extends to an automorphism of $\mathfrak{U}$ and we conclude as in Proposition 2.3.

By quantifier elimination it suffices to show that f preserves and reflects atomic formulas. Up to applying the same arguments to f^{-1}, it is enough to show that atomic formulas are preserved. It suffices to check formulas of the form $\varphi(a, \gamma) \coloneqq v(t_0(a, \gamma)) \geq v(t_1(a, \gamma))$, where t_0, t_1 are terms in the language with a cross-section s, and the tuples a,γ are from the sorts VF, Γ respectively. We do this by induction on the maximum depth of nested occurrences of s in t_0, t_1; in the same induction, we simultaneously prove that the valuation of such terms lies in Δ.

If s does not appear at all, then φ is an $\mathcal{L}_{\mathrm{vf}}$-formula, hence the conclusion follows from the fact that σ is an automorphism. If s never appears nested more than once, then φ is, up to equivalence, of the form $v(p(a, s(\gamma))) \geq v(q(a, s(\gamma)))$, where γ is a tuple from $\Delta \setminus \Gamma(K)$ that is $\mathbb{Q}$-linearly independent over $\Gamma(K)$ and p, q are polynomials over $\mathbb{Z}$, say. Then $p(a, x)$ is a polynomial over K; write it as the sum of its monomials, say $p(a, x) = \sum_i m_i(x)$. By linear independence of γ over $\Gamma(K)$, the $v(m_i(s(\gamma)))$ are pairwise distinct, hence there is i_0 such that $v(p(a, s(\gamma))) = v(m_{i_0}(s(\gamma)))$. Similarly we may write $v(q(a, s(\gamma))) = v(n_{j_0}(s(\gamma)))$. As σ_Δ is an automorphism, it is easy to see that $v(p(f(a), f(s(\gamma)))) = v(f(m_{i_0})(f(s(\gamma))))$, and similarly for q, hence f preserves $\varphi(a, \gamma)$; because m_{i_0} is a monomial, we also have $v(f(m_{i_0})(f(s(\gamma)))) \in \Delta$.

If s appears nested more than once in a term, then there must be an occurrence of v in the middle, that is, the term contains a subterm of the form $s(t_2(a, \gamma, v(t_3(a, \gamma))))$, where t_2, t_3 are terms with fewer nested occurrences of s. As by inductive assumption $v(t_3(a, \gamma)) \in \Delta$, we fall back into the previous case. □

Proposition 2.21. If (K, v, σ, s) is existentially closed, and we set $\mathrm{ac}(x) \coloneqq \mathrm{res}(x / s(v(x)))$, then $(K, v, \sigma, \mathrm{ac})$ is existentially closed. Therefore, so is (K, v, σ).

Proof. Given an extension $(L, v_L, \sigma_L, \mathrm{ac}_L)$ of $(K, v, \sigma, \mathrm{ac})$, which we may assume existentially closed and existentially $\aleph_1$-saturated, we will show that there is a cross-section $s_L \colon \Gamma(L) \to L$ which extends s and such that $\mathrm{ac}_L(z) =$

[8] This is folklore, and follows from e.g. the much more general [HH19, Theorem 1]. For a direct proof, use [HKR18, Fact 2.4] in combination with [Tou23, Observation 1.51].

$\mathrm{res}(z/s_L(v_L(z)))$ for every $z \in L$. This will conclude the proof, as it shows that then $(K, v, \sigma, \mathrm{ac})$ is existentially closed as an ac-valued field, and then (K, v, σ) is existentially closed by Corollary 2.19.

It is enough to show that there is a cross-section s_L such that $\mathrm{ac}_L(s_L(\gamma)) = 1$ for every $\gamma \in \Gamma(L)$: if this is the case, then for every $z \in L$ we have $\mathrm{res}(z/s_L(v_L(z))) = \mathrm{ac}_L(z/s_L(v_L(z))) = \mathrm{ac}_L(z)/\mathrm{ac}_L(s_L(v_L(z))) = \mathrm{ac}_L(z)$ as we wanted.

Consider the $\mathbb{Z}[\sigma]$-submodule N of L consisting of elements of L with angular component 1; note that the image of s is contained in N. Let $\bar{b}$ be a tuple enumerating $\Gamma(K)$. If we prove that the pp-type of $\bar{b}$ in $\Gamma(L)$ is contained in the pp-type of $s(\bar{b})$ in N, then we may apply [Pre88, Theorem 2.8(v)] and obtain the desired extension of s to $\Gamma(L)$. In other words, we need to show that if a is a finite subtuple of $\bar{b}$, and $A \in \mathrm{Mat}_{d\times n}(\mathbb{Z}[\sigma])$ is such that $A \cdot z = a$ has a solution in $\Gamma(L)$, then $A \cdot z = s(a)$ has a solution in N. In fact, it suffices to show that $A \cdot z = a$ has a solution in $\Gamma(K)$, as then if γ is such a solution we get that $A \cdot s(\gamma) = s(a)$ and $s(\gamma) \in N$.

This follows from Lemma 2.20, concluding the proof. □

Corollary 2.22. Let K be an existentially closed ac-valued (resp. s-valued) difference field. Then $\mathrm{VF}(K)$ is algebraically closed and (Γ, σ) and (k, σ) are existentially closed in their respective categories. In particular, (K, v, ac) (resp. (K, v, s)) and (K, v) are existentially closed.

Proof. By Proposition 2.21 it suffices to deal with the ac-valued case. By Corollary 2.19 (K, v, σ) is existentially closed, so we may apply Proposition 2.3. Alternatively, we may adapt its proof by using quantifier elimination in $\mathsf{ACVF}_{\mathrm{ac}}$. The "in particular" part follows from the above and Proposition 2.3(1). □

2.3 Bad reducts

Another natural reduct of (K, v, σ) to consider is the difference field (K, σ). In this subsection we will see that, in contrast to what we saw in Subsection 2.2, forgetting the valuation does not preserve existential closedness.

Recall that existentially closed difference fields are axiomatisable by a theory denoted by ACFA, in the language $\{+, 0, -, \cdot, 1, (-)^{-1}, \sigma, \sigma^{-1}\}$. The axioms of this theory say that $(K, \sigma) \models \mathsf{ACFA}$ iff K is algebraically closed and, for all n, all irreducible algebraic subvarieties V, W of K^n and K^{2n} respectively such that $W \subseteq V \times \sigma(V)$ and the projections of W to V and to $\sigma(V)$ are dominant, there is $x \in V(K)$ such that $(x, \sigma(x)) \in W$. See [Mac97, Theorem 7], [CH99, Theorem on p. 3007].

We will see in Example 3.30 that if $k \models \mathsf{ACFA}$ has characteristic zero and Γ is an existentially closed difference oag, then the Hahn field $k((t^\Gamma))$, made into a valued difference field as in Example 1.6, is existentially closed. The following argument, due to Philip Dittmann, shows that it is not existentially closed as a difference field, that is, it is not a model of ACFA.

Below, when we speak of fixed points, fixed fields, etc., we always refer to fixed points of σ, fixed fields of σ, etc.

Proposition 2.23 (P. Dittmann)**.** Let (K, v, σ) be a henselian valued difference field with a fixed point of nonzero valuation. Then $(K, \sigma) \not\models \mathsf{ACFA}$.

Proof. It is well known (see e.g. [CH99, Proposition 1.2]) that the fixed field of every model of ACFA is pseudofinite, and in particular not separably closed. By [FJ23, Corollary 12.5.5][9] every pseudo-algebraically closed (in particular, every pseudofinite) nontrivially valued henselian field is separably closed. Since there is a fixed point of nonzero valuation, the induced valuation on the fixed field K_0 of K is nontrivial. Hence, to obtain a contradiction, it suffices to show that K_0 is henselian.

We check henselianity directly. Let $f \in \mathcal{O}(K_0)[x]$ and $a \in \mathcal{O}(K_0)$ be such that $\mathrm{res}(a)$ is a simple root of $\mathrm{res}(f)$. By henselianity of K there is $b \in \mathcal{O}(K)$ with $f(b) = 0$ and $v(b-a) > 0$. If $\sigma(b) = b$ we are done, so assume this is not the case. As f is over the fixed field, $\sigma(b)$ is a root of f. Therefore $(x-b)(x-\sigma(b))$ divides f, and since $\mathrm{res}(\sigma(b)) = \sigma(\mathrm{res}(b)) = \sigma(\mathrm{res}(a)) = \mathrm{res}(a)$ this contradicts that $\mathrm{res}(a)$ is a simple root of $\mathrm{res}(f)$. □

Corollary 2.24 (P. Dittmann)**.** If $k \vDash$ ACFA and Γ is a difference oag with nonzero fixed points, then the Hahn field $k((t^\Gamma))$ is not a model of ACFA.

Proof. By inspection, if k_0 is the fixed field of k, and Γ_0 is the fixed subgroup of Γ, then the fixed field of $k((t^\Gamma))$ is $k_0((t^{\Gamma_0}))$, hence we are in the assumptions of Proposition 2.23. □

By mining the proof of Proposition 2.23, we can obtain an explicit example of failure of the ACFA axioms.

Lemma 2.25. The fixed field is closed in the valuation topology. Moreover, if $v(a) = 0$ and $\mathrm{res}(\sigma(a)) \neq \mathrm{res}(a)$ then the ball $\{x \mid v(x-a) > 0\}$ does not intersect the fixed field.

Proof. Closedness is clear. For the moreover part, suppose $v(\sigma(x) - \sigma(a)) > 0$. If $\sigma(x) = x$, then $v(a - \sigma(a)) > 0$, against $\mathrm{res}(\sigma(a)) \neq \mathrm{res}(a)$. □

Proposition 2.26. Let (K, v, σ) be a valued difference field containing a fixed point $\varepsilon = \sigma(\varepsilon)$ of positive valuation and an element $a \in \mathcal{O}^\times$ of valuation zero that, for some $n > 1$, is a fixed point of σ^n but such that, for every m with $1 \leq m < n$, its residue $\mathrm{res}(a)$ is not a fixed point of σ_k^m. Then $(K, \sigma) \nvDash$ ACFA.

Proof. For notational simplicity, we will only deal with the case $n = 2$. Define $P(x) \coloneqq (x-a)(x-\sigma(a))$, observe that it is over the fixed field K_0, and set $Q(x,y) \coloneqq P(x)P(y) - \varepsilon$. By the Eisenstein criterion (in the version of [FJ23, Lemma 2.3.10(b)]) applied to $(K[x])[y]$, the vanishing variety V of Q is absolutely irreducible.

Let us show it does not have points in K_0. Assume it does, say $(b_1, b_2) \in V(K_0)$. Then $P(b_1)P(b_2) = \varepsilon$, hence $v(P(b_1)) + v(P(b_2)) > 0$, so for some i we have $v(b_i - a) > 0$ or $v(b_i - \sigma(a)) > 0$. By assumption on $\mathrm{res}(a)$ and Lemma 2.25 we must have $\sigma(b_i) \neq b_i$.

Let W be the diagonal of $V \times V$. Then W, which is absolutely irreducible because it is isomorphic to V, projects surjectively on V and on $\sigma(V) = V$, but a σ-point on W would contradict that $V(K_0) = \emptyset$. □

Corollary 2.27. Every existentially closed valued difference field is not existentially closed as a difference field.

[9] In the third edition of the same book (2008), this is Corollary 11.5.5.

Proof. Let us show that every existentially closed valued difference field satisfies the assumptions of Proposition 2.26. The existence of ε follows from existential closedness of Γ (Proposition 2.3(2)). To find some a with $v(a) = 0$, $\sigma^2(a) = a$, and $\mathrm{res}(\sigma(a)) \neq \mathrm{res}(a)$, it suffices to take two Gauss extensions (see [EP05, Corollary 2.2.2]), let σ swap the added points, and apply existential closedness. □

Remark 2.28. The proof above may be adapted, for instance, to the existentially closed isometric case studied in [AD11], but not to the existentially closed strongly contractive case, where the valued field sort is indeed a model of ACFA [CH14, Fact 2.13].

Corollary 2.27 implies that there must exist a valued difference field (K, v, σ) and a difference field extension $(L, \sigma_L) \supseteq (K, \sigma)$ such that no valuation on L extending v is compatible with σ_L. It is indeed possible to extract a concrete example from the above proofs.

Example 2.29. Let $K = \mathbb{C}((t))$ with the usual valuation and

$$\sigma\left(\sum_m a_m t^m\right) = \sum_m \overline{a_m} t^m$$

where $\bar{z}$ denotes the complex conjugate of z. Take, in the notation of the proof of Proposition 2.26, $a = i$ and $\varepsilon = t$, i.e. $P(x) = (x - i)(x + i) = x^2 + 1$ and $Q(x, y) = (x^2 + 1)(y^2 + 1) - t$.

By quantifier elimination in ACF and the fact that V is defined over the fixed field, hence its generic type is invariant over the fixed field, we can find $L \coloneqq K(b_1, b_2)$ with $\sigma(b_j) = b_j$ and $(b_1, b_2) \in V(Q)$. Assume towards a contradiction that there is an extension of the valuation to L preserved by σ. Because $Q(b_1, b_2) = 0$, there is j such that $v(b_j^2 + 1) > 0$. So $\mathrm{res}(b_j) = \pm i$ is not a fixed point, hence neither is b_j, contradiction.

In other words, what is happening is that the valuation forces points on the variety $(x^2 + 1)(y^2 + 1) - t$ to have a coordinate infinitesimally close to a square root of -1; continuity with respect to the valuation then implies that this coordinate cannot be fixed.

In characteristic (p, p), working with $\mathbb{F}_p^{\mathrm{alg}}((t))$ and replacing complex conjugation by the Frobenius automorphism allows a similar argument to go through.

3 Amalgamation

An amalgamation problem of (expanded) valued fields induces amalgamation problems at the residue field and value group levels. Therefore, an obvious obstruction to the solution of an amalgamation problem is that one of these induced problems cannot be solved.

In the case of valued difference fields, value difference groups can always be amalgamated by [DM24, Theorem 5.10], but residual obstructions may still be present.

Example 3.1. Let K, M and L be the trivially valued difference fields with valued field sorts $\mathrm{VF}(K) = \mathbb{Q}$ and $\mathrm{VF}(M) = \mathrm{VF}(L) = \mathbb{Q}(i)$, and automorphisms $\sigma_M(i) = i$ and $\sigma_L(i) = -i$. Then M and L do not amalgamate over the common substructure K.

Even in the absence of obstacles coming from the residue field (e.g. algebraically closed difference fields are amalgamation bases, see Theorem 3.28 and the following discussion), amalgamation problems may fail to have solutions.

Example 3.2. Let k be algebraically closed, let $\sigma \in \mathrm{Aut}(k)$ be arbitrary, and consider the Hahn product $k((\mathbb{R}))$ of (k, σ) and $(\mathbb{R}, 0, +, \leq, \mathrm{id}_{\mathbb{R}})$. Let $K_0 \coloneqq k((\mathbb{Z})) \subseteq k((\mathbb{R}))$, and let $K \coloneqq K_0(t^\pi)$. By the Zariski–Abhyankar inequality [Dri14, Corollary 3.25], the residue field of K is an algebraic extension of k, so it is equal to k. Let L and M be valued fields isomorphic to $K(\sqrt{t^\pi})$, endowed with automorphisms σ_L and σ_M extending the automorphism on K so that σ_L fixes the square roots of t^π while σ_M swaps them. Then (L, σ_L) and (M, σ_M) cannot be amalgamated over K.

This is not surprising, as similar issues are well known to exist when one tries to amalgamate e.g. henselian valued fields of residual characteristic zero (with no extra structure). Common solutions are to expand the language by an angular component or a cross-section, which in our case will be assumed to be equivariant. In Theorem 3.28 we will show that, in the residual characteristic zero case, this allows to characterise which amalgamation problems have solutions. Our result also works when a lift of the residue field is added to the language.

3.1 Set-up

Definition 3.3. A difference field (k, σ) is *linearly difference closed* iff for all $n \in \mathbb{N}$ and $\alpha_0, \ldots, \alpha_n \in k$ with $\alpha_n \neq 0$ the equation $1 + \alpha_0 x + \ldots + \alpha_n \sigma^n(x) = 0$ has a solution in k.

It is an easy and well-known fact that every model of ACFA is linearly difference closed.

The following definitions appear for example in [AD11, Section 2].

Definition 3.4. Let (K, σ) be a difference field. A *σ-polynomial over K* is an expression of the form $G(x) = g(x, \sigma(x), \ldots, \sigma^n(x))$, where $g \in K[T_0, \ldots, T_n]$.

The *order* of the σ-polynomial $G(x)$ is the smallest n for which there exists $g \in K[T_0, \ldots, T_n]$ such that $G(x) = g(x, \ldots, \sigma^n(x))$. The *complexity* of a σ-polynomial $G(x) = g(x, \sigma(x), \ldots, \sigma^n(x))$ of order n is the ordered triple $(n, \deg_{x_n} g, \deg g)$, where $\deg g$ denotes the total degree of g.

We regard complexities of σ-polynomials as being linearly ordered by the lexicographical ordering on $\mathbb{N}^3$.

Definition 3.5. Let (K, σ) be a difference field, $(K_0, \sigma_{|K_0})$ a difference subfield. An element $a \in K$ is *σ-algebraic* over K_0 iff there is a nonzero σ-polynomial G over K_0 such that $G(a) = 0$, and *σ-transcendental* over K_0 otherwise. If a is σ-algebraic over K_0, a *minimal σ-polynomial* for a over K_0 is a nonzero σ-polynomial G of minimal complexity such that $G(a) = 0$.

For G a σ-polynomial, we refer to [AD11, Section 2] for the definition of the σ-polynomials $G_{(\mathbf{i})}$. Note that if $\mathbf{i} \neq 0$ then $G_{(\mathbf{i})}$ has strictly lower complexity than G.

For G a σ-polynomial and $a \in K$, we refer to [Azg10, Definition 4.2] or [DO15, Definition 4.1] for the definitions of (G, a) being in *σ-Hensel configuration* and

of the value $\gamma(G,a)$. Recall that a valued difference field is *σ-henselian* iff, whenever (G,a) is in σ-Hensel configuration, there is b with $G(b)=0$ and $v(a-b)=\gamma(G,a)$.[10] Our uses of σ-henselianity will be mediated by the following observation.

Fact 3.6 ([DO15, Remark 4.2])**.** Let G be nonconstant, $G(a)\neq 0$, $v(G(a))>0$ and $v(G_{(\mathbf{i})})=0$ for all $\mathbf{i}\neq 0$ with $G_{(\mathbf{i})}\neq 0$. Then (G,a) is in σ-Hensel configuration with $\gamma(G,a)>0$.

Fact 3.7. If (K,v,σ) is σ-henselian, then $(k(K),\sigma)$ is linearly difference closed.

Proof. See [Azg10, Lemma 4.6] or [DO15, Lemma 4.6]. □

Fact 3.8. Assume K is σ-henselian and $\mathrm{char}(k)=0$. For every difference subfield $K_0\subseteq\mathcal{O}$ there is a difference subfield K_1 of K with $K_0\subseteq K_1\subseteq\mathcal{O}$ and $\mathrm{res}(K_1)=k(K)$.

Proof. This is [AD11, Theorem 4.8]. Note that its proof does not assume that the automorphism σ is isometric (i.e. $\sigma(\gamma)=\gamma$ for all $\gamma\in\Gamma$), even if the main results of the cited paper do. Moreover, even though the notion of σ-henselianity used there is different, it can be seen by using Fact 3.6 that the proof still works with the notion of σ-henselianity used here. □

Lemma 3.9. Let $\mathcal{L}$ be either $\mathcal{L}_\sigma$, $\mathcal{L}_{\mathrm{ac},\sigma}$, or the expansion of one of these by a lift. Every existentially closed $\mathcal{L}$-valued difference field of residual characteristic zero is σ-henselian.

Proof. If $\mathcal{L}$ equals $\mathcal{L}_\sigma$ (resp. $\mathcal{L}_{\mathrm{ac},\sigma}$), this follows from Proposition 2.3(3) (resp. Corollary 2.22), the fact that every σ-algebraically maximal (i.e. having no proper σ-algebraic immediate extension) valued difference field with linearly difference closed residue field and residual characteristic zero is σ-henselian [DO15, Corollary 5.6(2)], and the fact that σ-henselianity is expressible by a set of $\forall\exists$-sentences in $\mathcal{L}$. The last fact can be checked by inspection, by observing that, in [DO15, Definition 4.1], for G of fixed complexity, it suffices to check finitely many of the $G_{(\mathbf{j})}$ because the other ones will be null regardless of the coefficients of G.

Assume now $\mathcal{L}_0$ is one of the languages above and $\mathcal{L}=\mathcal{L}_0\cup\{\iota\}$. Let K be an existentially closed $\mathcal{L}$-valued difference field and let K_0 be its reduct to $\mathcal{L}_0$. Extend K_0 to some σ-algebraically maximal M_0, which is σ-henselian as recalled above. Now apply Fact 3.8 and the syntactical considerations above. □

Corollary 3.10. Assume $\mathrm{char}(k)=0$.

(1) If $(K,v,\sigma,\mathrm{ac},\iota)$ is existentially closed, then so are all the reducts obtained by forgetting some subset of $\{\sigma,\mathrm{ac},\iota\}$.

(2) If (K,v,σ,s,ι) is existentially closed, and $\mathrm{ac}(x)=\mathrm{res}(x/s(v(x)))$, then so is $(K,v,\sigma,\mathrm{ac},\iota)$.

[10] There is another notion of σ-henselianity, designed to work for valued difference fields with analytic structure, see [Rid17, Definition 4.10]. In the case where the analytic structure is trivial and $\mathrm{char}(k)=0$, the two notions are equivalent, as they both imply (and are implied by) elementary equivalence to a maximally complete valued difference field with linearly difference closed residue difference field. We thank Silvain Rideau-Kikuchi for pointing this out.

Proof. This is proved similarly to the results in Subsection 2.2.

For $\mathcal{L}_{\mathrm{ac},\sigma}$ we use that, by Fact 3.8, partial lifts of the residue field may always be extended. This also implies the result for $\mathcal{L}_\sigma$ by Corollary 2.19, and for $\mathcal{L}_{\mathrm{ac}}$ and $\mathcal{L}$ by Corollary 2.22. For $\mathcal{L}_{\mathrm{ac},\iota}$ we apply [Kes24, Proposition 1.8], which allows us to repeat the argument with quantifier elimination as in Proposition 2.3.

For $\mathcal{L}_{\iota,\sigma}$, let (L, v, σ, ι) be an extension of (K, v, σ, ι). Then, by taking an existentially closed, existentially $\aleph_1$-saturated extension if necessary, we may assume that L has a σ-equivariant angular component map extending the one on K by Proposition 2.18. Then $(L, v, \sigma, \mathrm{ac}, \iota)$ is an extension of $(K, v, \sigma, \mathrm{ac}, \iota)$, as there are no extra compatibility conditions to be checked between ac and ι. This proves that (K, v, σ, ι) is existentially closed. Finally, for (K, v, ι), we know by the above that $(K, v, \mathrm{ac}, \iota)$ is existentially closed so we can again extend the angular component map to any sufficiently saturated $\mathcal{L}_\iota$-valued field extension of (K, v, ι). This completes the proof of the first part.

The proof of the second part uses the same ideas as the proof of Lemma 3.9, together with Proposition 2.21, and is left to the reader. □

Corollary 3.11. Let $\mathcal{L}$ be $\mathcal{L}_\sigma, \mathcal{L}_{\mathrm{ac},\sigma}, \mathcal{L}_{s,\sigma}$, or an expansion of the previous by a lift. If K is an existentially closed $\mathcal{L}$-valued difference field of residual characteristic zero, then it is σ-henselian.

Proof. By Corollary 3.10 and Lemma 3.9. □

Notation 3.12. If K is an ac-valued difference field, $A \subseteq K$, and $b \in K$, then by $\langle A \rangle$ and $A\langle b \rangle$ we mean the ac-valued difference fields generated by A and by $A \cup \{b\}$, respectively, unless otherwise specified. We use the same notation for structures in other languages, e.g. $k(E)\langle \alpha \rangle$ is the difference field generated by $k(E)$ and α.

The proof of the following fact is straightforward.

Fact 3.13. Let $(K, v_K) \subseteq (L, v)$ be an extension of valued fields.

(1) Let $a \in L$ be such that $0, v(a), \ldots, v(a^n) \in \Gamma(L)$ are in different cosets modulo $\Gamma(K)$. Then for all $c_0, \ldots, c_n \in K$ we have

$$v\left(\sum_i c_i a^i\right) = \min_i \{v_K(c_i) + iv(a)\}.$$

In particular $v(K + Ka + \ldots + Ka^n) \subseteq \Gamma(K) + \mathbb{Z}v(a)$.

(2) If there is $a \in L$ such that $\Gamma(L) = \Gamma(K) + \mathbb{Z}v(a)$, then every element of L is of the form bca^n, for suitable $b \in K$, $c \in \mathcal{O}(L)^\times$, and $n \in \mathbb{Z}$.

The following facts about valued difference fields appear in Sections 2 and 6 of [AD11]. As in Fact 3.8, the isometricity assumption is not used in these three lemmas, hence they hold for arbitrary valued difference fields.

Suppose (K, v, σ) and (K', v, σ) are valued difference fields. Put $\mathcal{O} \coloneqq \mathcal{O}(K)$ and $\mathcal{O}' \coloneqq \mathcal{O}(K')$. Let (E, v, σ) be a valued difference subfield of both K and K'.

Fact 3.14 ([AD11, Lemma 2.5])**.** Let $a \in \mathcal{O}$ and assume $\alpha = \mathrm{res}(a)$ is σ-transcendental over $k(E)$. Then

(1) $v(P(a)) = \min_i\{v(b_i)\}$ for each σ-polynomial $P(x) = \sum b_i\sigma^i(x)$ over E;

(2) $v(E\langle a\rangle^\times) = v(E^\times) = \Gamma(E)$, and $E\langle a\rangle$ has residue field $k(E)\langle\alpha\rangle$;

(3) if $b \in \mathcal{O}'$ is such that $\beta = \mathrm{res}(b)$ is σ-transcendental over $k(E)$, then there is a valued difference field isomorphism $E\langle a\rangle \to E\langle b\rangle$ over E sending a to b.

Fact 3.15 ([AD11, Lemma 2.6])**.** Assume $\mathrm{char}(k(E)) = 0$, and let $G(x)$ be a nonconstant σ-polynomial over the valuation ring of E such that $\mathrm{res}(G(x))$ has the same complexity as $G(x)$. Let $a \in \mathcal{O}$, $b \in \mathcal{O}'$, and assume that $G(a) = 0$, $G(b) = 0$, and that $\mathrm{res}(G(x))$ is a minimal σ-polynomial of $\alpha \coloneqq \mathrm{res}\, a$ and of $\beta \coloneqq \mathrm{res}\, b$ over $k(E)$. Then

(1) $E\langle a\rangle$ has value group $v(E^\times) = \Gamma(E)$ and residue field $k(E)\langle\alpha\rangle$;

(2) if there is a difference field isomorphism $k(E)\langle\alpha\rangle \to k(E)\langle\beta\rangle$ over $k(E)$ sending α to β, then there is a valued difference field isomorphism $E\langle a\rangle \to E\langle b\rangle$ over E sending a to b.

Fact 3.16. Let $E \subseteq F$ and $E' \subseteq F'$ be substructures of ac-valued fields, with $v(F^\times) = v(E^\times)$. Let $f\colon E \to E'$ be an $\mathcal{L}_{\mathrm{ac}}$-isomorphism, and $g\colon F \to F'$ be an $\mathcal{L}_{\mathrm{vf}}$-isomorphism extending f. Suppose that $\mathrm{res}(\mathcal{O}(F)) \subseteq k(E)$ and that $f(\mathrm{res}(u)) = \mathrm{res}'(g(u))$ for all $u \in \mathcal{O}(F)$.

Then $\mathrm{ac}(F) \subseteq k(E)$ and $f(\mathrm{ac}(a)) = \mathrm{ac}'(g(a))$ for all $a \in F$.

Proof. The proof of [AD11, Lemma 6.3] does not use the elementarity requirements in the definition of a good map in [AD11, Section 6], thus we can relax the assumption of being a good map to being an isomorphism of ac-valued fields. □

Below, we extend the tropicalisation operator to σ-polynomials in the obvious way. For example, $\mathrm{trop}(\sigma^2(x^3) + \sigma(x^4)) = \min\{3\sigma^2(x), 4\sigma(x)\}$.

Definition 3.17. Let B be a valued difference field. An element a of a valued difference field extension of B is *regular over* B iff for every σ-polynomial f over B we have $v(f(a)) = \mathrm{trop}(f)(v(a))$.

Remark 3.18. If a is regular over B, and c is such that $(\mathrm{ac}(a), v(a)) = (\mathrm{ac}(c), v(c))$, then c is also regular over B.

Proof. This is essentially [DO15, Lemma 3.2], together with the observation that $v(c-a) > v(a)$ if and only if $(\mathrm{ac}(a), v(a)) = (\mathrm{ac}(c), v(c))$. This equality can be easily shown by using that the pair of maps (ac, v) induces an isomorphism of $k \times \Gamma$ with the RV-*structure* $\mathrm{VF}^\times/(1+\mathfrak{m})$, where $\mathfrak{m}$ denotes the maximal ideal. □

3.2 Amalgamation

Assumption 3.19. From now on, all fields will be assumed to have characteristic zero.

Lemma 3.20. Let U, N be valued difference fields such that no σ^ℓ is the identity on $k(U)$ nor on $k(N)$. Let E be a common valued difference subfield, and let $\gamma \in \Gamma(N) \setminus \Gamma(E)$. Assume U is quantifier-free $|E|^+$-saturated.

(1) There is $a \in U$ that is regular over E with $v(a) = \gamma$.

(2) The value group of $E\langle a\rangle$ is $\Gamma(E)\langle\gamma\rangle$, and its residue field has size at most $|E|$.

(3) Suppose $b \in N$ is regular over E and the identity on $\Gamma(E)$ extends to an isomorphism $\Gamma(E)\langle\gamma\rangle \to \Gamma(E)\langle v(b)\rangle$ sending γ to $v(b)$. Then the identity on E extends to a valued difference field isomorphism $E\langle a\rangle \to E\langle b\rangle$ sending a to b.

Proof. This is a variant of [Pal12, Lemma 8.8]. One modifies its proof by replacing the tacitly used [Pal12, Lemma 8.5] by [DO15, Lemma 3.6] (as in [DO15, Lemma 6.1][11]) and observes that the saturation assumption in [Pal12, Lemma 8.8] is only used to realise a quantifier-free type in U. □

Lemma 3.21. Let M be an ac-valued difference field, and let $a \in M$ be regular over an ac-valued difference subfield B. Let $g_0(x), \ldots, g_n(x)$ be σ-monomials with coefficients from B such that, for all $i \neq j$, the σ-monomial $g_i(x)$ is not of the form $bg_j(x)$ for any $b \in B$. If all $g_i(a)$ have the same valuation, then $\mathrm{ac}(g_0(a) + \ldots + g_n(a)) = \mathrm{ac}(g_0(a)) + \ldots + \mathrm{ac}(g_n(a))$.

Proof. By regularity, $g_0(a) + \ldots + g_n(a)$ has the same valuation as each $g_i(a)$, say γ. In some extension of M, let c be such that $v(c) = \gamma$ and $\mathrm{ac}(c) = 1$. Then both sides of the equality equal $\mathrm{res}((g_0(a) + \ldots + g_n(a))/c)$. □

Notation 3.22. Until the end of the section, $\mathcal{L}$ will denote one of $\mathcal{L}_{\mathrm{ac},\sigma}$, $\mathcal{L}_{s,\sigma}$, $\mathcal{L}_{\mathrm{ac},\iota,\sigma}$, or $\mathcal{L}_{s,\iota,\sigma}$.

Definition 3.23. Let M and N be $\mathcal{L}$-structures. A *k-elementary map* is an isomorphism $f\colon A \to B$ of $\mathcal{L}$-substructures of M and N respectively such that $f_{|k(A)}$ is an elementary map $k(M) \to k(N)$.

We now show how to extend k-elementary maps to make sure that the valuation and residue maps on the domain substructure are surjective (note that, in the absence of a cross-section, the valuation map in a substructure need not be surjective, and similarly for lift and residue).

Lemma 3.24. Let E be a common $\mathcal{L}$-substructure of the $\mathcal{L}$-valued difference fields U and N. Let $f\colon A \to B$ be a k-elementary map extending id_E. Assume that U, N are σ-henselian and that $k(N)$ is $|k(A)|^+$-saturated.

Then, for all $\alpha \in k(U)$, the map f extends to a k-elementary map $f'\colon A' \to B'$, with domain of cardinality $|A'| = |A|$, such that $\alpha \in \mathrm{res}(\mathcal{O}(A'))$ and $\Gamma(A') = \Gamma(A)$.

Proof. Let $\alpha \in k(U) \setminus \mathrm{res}(\mathcal{O}(A))$. As $f|_{k(A)}\colon k(U) \to k(N)$ is an elementary map, by saturation there is some $\beta \in k(N)$ such that the map $f|_{k(A)} \cup \{(\alpha, \beta)\}$ is elementary. We now give a separate argument for each language.

If $\mathcal{L} = \mathcal{L}_{\mathrm{ac},\iota,\sigma}$, then we may take as A' and B' the structures generated by $A\alpha$ and $B\beta$ respectively. It is then clear that $f \cup \{(\iota(\alpha), \iota(\beta))\}$ extends to the desired k-elementary map.

[11] Point (ii) of the cited lemma incorrectly states that, in its notation, the residue field of $E\langle a\rangle$ is the same as that of E. In fact, the residue field will in general grow. We thank Simone Ramello for bringing this fact to our attention.

Suppose now $\mathcal{L} = \mathcal{L}_{s,\iota,\sigma}$. For all $i \in \mathbb{Z}$, let $a_i \coloneqq \iota(\sigma^i(\alpha))$ and $b_i \coloneqq \iota(\sigma^i(\beta))$. We prove by induction on $i \in \mathbb{N}$ that the k sort of the $\mathcal{L}_{s,\iota}$-structure generated by $k(A) \cup \mathrm{VF}(A)(a_{-i}, \dots, a_i)$ equals $k(A)(\sigma^{-i}(\alpha), \dots, \sigma^i(\alpha))$ and that there is an $\mathcal{L}_{s,\iota}$-isomorphism f_i extending f and sending the tuple $(a_{-i}, \dots, a_i)$ to $(b_{-i}, \dots, b_i)$. Suppose this holds for some $i \in \mathbb{N}$. Applying [Kes24, Proposition 1.7] we obtain an $\mathcal{L}_{s,\iota}$-isomorphism extending f_i and sending a_{i+1} to b_{i+1}. Applying [Kes24, Proposition 1.7] again, this time to a_{-i-1}, we obtain the desired k-elementary map f_{i+1}. We now conclude by taking $f' \coloneqq \bigcup_{i \in \mathbb{N}} f_i$.

Suppose now that $\mathcal{L} = \mathcal{L}_{\mathrm{ac},\sigma}$. We may assume $\alpha \in k(A)$ and $\beta \in k(B)$. If α is σ-transcendental over $\mathrm{res}(\mathcal{O}(A))$, then β is σ-transcendental over $\mathrm{res}(\mathcal{O}(B))$. Pick $a \in \mathcal{O}(U)$ and $b \in \mathcal{O}(N)$ with $\mathrm{res}(a) = \alpha$ and $\mathrm{res}(b) = \beta$. By Fact 3.14(3) f extends to an $\mathcal{L}_\sigma$-isomorphism $f' \colon A\langle a \rangle \to B\langle b \rangle$ sending a to b, and by Fact 3.14(2) $\Gamma(A) = \Gamma(A\langle a \rangle)$. By the choice of β, $f'|_{k(A\langle a \rangle)}$ is an elementary map $k(U) \to k(N)$. We then use Fact 3.16 to see that f' commutes with ac. Thus, f' is a k-elementary map.

Now assume that α is σ-algebraic over $\mathrm{res}(\mathcal{O}(A))$, and fix a minimal σ-polynomial $g(x)$ witnessing this. Let $G(x)$ be a σ-polynomial over $\mathcal{O}(A)$ with residue $g(x)$ and of the same complexity as g. Let $a_0 \in \mathcal{O}(U)$ be such that $\mathrm{res}(a_0) = \alpha$. We will find $a \in \mathcal{O}(U)$ with $G(a) = 0$ and $\mathrm{res}(a) = \alpha$. If $G(a_0) = 0$ we simply set $a \coloneqq a_0$. Otherwise, as U is σ-henselian, it suffices to show that (G, a_0) is in σ-Hensel configuration with $\gamma(G, a_0) > 0$. To this end, we check that the assumptions of Fact 3.6 are satisfied.[12] Clearly G is nonconstant, and we have assumed $G(a_0) \neq 0$. Furthermore, by construction $v(G(a_0)) > 0$. If there was $\mathbf{i}$ such that $G_{(\mathbf{i})} \neq 0$ but $v(G_{(\mathbf{i})}(a_0)) > 0$, then we would obtain a σ-polynomial of lower complexity over $\mathrm{res}(\mathcal{O}(A))$ vanishing on α, contradicting minimality of $g = \mathrm{res}(G)$. Similarly, σ-henselianity of N allows us to obtain $b \in \mathcal{O}(N)$ with $(f(G))(b) = 0$ and $\mathrm{res}(b) = \beta$. By Fact 3.15 f extends to an $\mathcal{L}_\sigma$-isomorphism $f' \colon A\langle a \rangle \to B\langle b \rangle$ sending a to b and $\Gamma(A) = \Gamma(A\langle a \rangle)$. By Fact 3.16, f' commutes with ac, and by the choice of β, the map $f'|_{k(A\langle a \rangle)} \colon k(U) \to k(N)$ is partial elementary. Thus, f' is a k-elementary map.

If $\mathcal{L} = \mathcal{L}_{s,\sigma}$, we use the same proof as in the previous case, except that we do not need to invoke Fact 3.16. Commutation with the cross-section is given by the fact that Γ does not grow. □

Lemma 3.25. Let E be a common substructure of the $\mathcal{L}$-valued difference fields M and N such that there is an embedding e of $\Gamma(M)$ into $\Gamma(N)$ over $\Gamma(E)$. Let $f \colon A \to B$ be a k-elementary map extending id_E with $f|_{\Gamma(A)} = e|_{\Gamma(A)}$. Let U be an extension of M, and assume that U, N are σ-henselian, that $k(N)$ is $|k(A)|^+$-saturated, and that U is quantifier-free $|M|^+$-saturated.

Then, for all $\gamma \in \Gamma(M)$, the map f extends to a k-elementary map $f' \colon A' \to B'$, with domain $A' \subseteq U$ of cardinality $|A'| = |A|$, such that $\gamma \in v(A')$ and $\Gamma(A') \subseteq \Gamma(M)$.

Proof. Let $\gamma \in \Gamma(M) \setminus v(A)$ and $\delta \coloneqq e(\gamma)$. We give a separate argument for each language.

Suppose $\mathcal{L} = \mathcal{L}_{\mathrm{ac},\sigma}$. By Fact 3.7 we are in the assumptions of Lemma 3.20, hence there is $a \in U$ regular over $\mathrm{VF}(A)$ with $v(a) = \gamma$. By Lemma 3.24, we may extend f to some element of $\mathcal{O}(U)$ with residue $\mathrm{ac}(a)$. Pick $b \in N$ with $v(b) = \delta$

[12] While we will not needs this, let us remark that one can check the definition σ-henselianity directly, which yields that $\gamma(G, a_0) \in \{v(G(a_0)), \sigma^n(v(G(a_0)))\}$.

and $\mathrm{ac}(b) = f(\mathrm{ac}(a))$. By Remark 3.18, b is regular over B. By Lemma 3.20, we may extend f to a valued difference field isomorphism sending a to b. From the fact that $\mathrm{ac}(b) = f(\mathrm{ac}(a))$ and Lemma 3.21, we conclude that f commutes with ac.

If $\mathcal{L} = \mathcal{L}_{\mathrm{ac},\iota,\sigma}$, argue as in the previous case and then, to check that f also commutes with the lift, use the fact $\mathrm{ac}(a)$ is transcendental by regularity of a.

Suppose now that $\mathcal{L} = \mathcal{L}_{s,\sigma}$. For all $i \in \mathbb{Z}$, let $a_i \coloneqq s(\sigma^i(\gamma))$ and $b_i \coloneqq s(\sigma^i(\delta))$. Denote by $\langle X\rangle$ the structure generated by X in the language $\mathcal{L}_s$. We will prove by induction on $i \in \mathbb{N}$ that $\Gamma(A\langle a_{-i},\dots,a_i\rangle) = \Gamma(A) + \mathbb{Z}\sigma^{-i}(\gamma) + \ldots + \mathbb{Z}\sigma^i(\gamma)$ and that there is an s-valued field isomorphism $f_i\colon A\langle a_{-i},\dots,a_i\rangle \to B\langle b_{-i},\dots,b_i\rangle$ extending f and sending the tuple $(a_{-i},\dots,a_i)$ to $(b_{-i},\dots,b_i)$, which is enough to prove the lemma as then $f' \coloneqq \bigcup_{i\in\mathbb{N}} f_i$ is the desired isomorphism. Suppose this holds for some $i \in \mathbb{N}$. We will extend f_i to an $\mathcal{L}_s$-isomorphism

$$f_i^+\colon A\langle a_{-i},\dots,a_i,a_{i+1}\rangle \to B\langle b_{-i},\dots,b_i,b_{i+1}\rangle.$$

We then repeat the argument to deal with a_{-i-1} and b_{-i-1}.

Suppose first that for all $\ell \in \mathbb{N}\setminus\{0\}$ we have $\ell\sigma^{i+1}(\gamma) \notin \Gamma(A) + \mathbb{Z}\sigma^{-i}(\gamma) + \ldots + \mathbb{Z}\sigma^i(\gamma) = \Gamma(A\langle a_{-i},\dots,a_i\rangle)$. Then by Fact 3.13(1) we have that a_{i+1} and b_{i+1} are transcendental over $\mathrm{VF}(A\langle a_{-i},\dots,a_i\rangle)$ and $\mathrm{VF}(B\langle b_{-i},\dots,b_i\rangle)$ respectively, and that the field isomorphism f_i^+ extending f_i and sending a_{i+1} to b_{i+1} commutes with the valuation, and by construction it commutes with the cross-section. Fact 3.13(1) also gives us that

$$v(A\langle a_{-i},\dots,a_{i+1}\rangle) = \Gamma(A) + \mathbb{Z}a_{-i} + \ldots + \mathbb{Z}a_{i+1}.$$

If instead there is some positive $\ell \in \mathbb{N}$ such that $\ell\sigma^{i+1}(\gamma) \in \Gamma(A) + \mathbb{Z}\sigma^{-i}(\gamma) + \ldots + \mathbb{Z}\sigma^i(\gamma)$, take the minimal such ℓ, together with integer coefficients h_i, and $\gamma_0 \in \Gamma(A)$ such that $\ell\sigma^{i+1}(\gamma) = \gamma_0 + \sum_{-i\le j\le i} h_i\sigma^i(\gamma)$. Applying the cross-section, we get that $a_{i+1}^\ell = s(\gamma_0)\prod_{-i\le j\le i} a_j^{h_j}$ and $b_{i+1}^\ell = s(f(\gamma_0))\prod_{-i\le j\le i} b_j^{h_j}$. It follows then by Fact 3.13(1) that $x^\ell - s(\gamma_0)\prod_{-i\le j\le i} a_j^{h_j}$ is the minimal polynomial of a_{i+1} over $\mathrm{VF}(A\langle a_{-i},\dots,a_i\rangle)$, and $x^\ell - f(s(\gamma_0))\prod_{-i\le j\le i} b_j^{h_j}$ is the minimal polynomial of b_{i+1} over $\mathrm{VF}(B\langle b_{-i},\dots,b_i\rangle)$. Because we have $f_i(\prod_{-i\le j\le i} a_j^{h_j}) = \prod_{-i\le j\le i} b_j^{h_j}$, we get that f_i extends to a map

$$f_i^+\colon A\langle a_{-i},\dots,a_{i+1}\rangle \to B\langle b_{-i},\dots,b_{i+1}\rangle$$

sending a_{i+1} to b_{i+1} and whose VF part is a field isomorphism. It also follows from Fact 3.13(1) that f_i^+ commutes with the valuation map, and it commutes with the cross-section by construction. Fact 3.13(1) also gives us that $v(A\langle a_{-i},\dots,a_{i+1}\rangle) = \Gamma(A) + \mathbb{Z}a_{-i} + \ldots + \mathbb{Z}a_{i+1}$, as required.

If $\mathcal{L} = \mathcal{L}_{s,\iota,\sigma}$, the proof is analogous to the proof of the case $\mathcal{L}_{s,\iota,\sigma}$ in Lemma 3.24 with the roles of the sorts Γ and k interchanged, the role of ι now played by s, and the applications of [Kes24, Proposition 1.7] replaced by applications of [Kes24, Proposition 2.1]. □

Remark 3.26. Above, if $\mathcal{L}$ contains a cross-section symbol s, we may furthermore ensure that $A' \subseteq M$ and that $\mathrm{res}(\mathcal{O}(A')) = \mathrm{res}(\mathcal{O}(A))$. As we will never use this, we leave it to the reader to check that this follows from the constructions given.

We are now ready to prove an embedding result, using which we will be able to characterise the solvable amalgamation problems, as well as the existentially closed $\mathcal{L}$-valued difference fields.

Proposition 3.27. Let E, M, N be substructures of $\mathcal{L}$-valued difference fields of residual characteristic zero, with M and N extending E and such that:

(1) N is quantifier-free $|M|^+$-saturated, σ-henselian, with $k(N)$ an $|M|^+$-saturated model of ACFA,

(2) $\Gamma(M)$ embeds in $\Gamma(N)$ over $\Gamma(E)$, and

(3) $k(M)$ embeds in $k(N)$ over $k(E)$.

Then there is an embedding of M into N over E.

Proof. By extending M and N, we may assume that they are $\mathcal{L}$-valued difference fields.

Since $k(M)$ embeds into $k(N)$ over $k(E)$, there is an existentially closed (in particular, σ-henselian by Corollary 3.11) and quantifier-free $|M|^+$-saturated extension U of M such that the restrictions to the algebraic closure of $k(E)$ of the automorphisms of $k(U)$ and $k(N)$ are isomorphic. By [CH99, Theorem 1.3] id_E is then a k-elementary map between substructures of U and N.

By a standard inductive argument, in order to embed M into N over E, it is enough to show that if $f\colon A \to B$ is a k-elementary map with $|A| \leq |M|$ and $a \in U$, then there is a k-elementary map $f'\colon A' \to B'$ with $a \in A'$ and $|A'| \leq |M|$.[13] Additionally, we inductively maintain the condition that $\Gamma(A) \subseteq \Gamma(M)$.

By repeated applications of Lemma 3.25, we extend f to a k-elementary map $f_0\colon A_0 \to B_0$ with $v(A_0) = \Gamma(A_0) = \Gamma(M)$ and $|A_0| \leq |M|$. By repeated applications of Lemma 3.24, we then find a k-elementary map $f_1\colon A_1 \to B_1$ extending f_0 with $\mathrm{res}(\mathcal{O}(A_1)) = k(A_1) \supseteq k(M)$, $v(A_1) = \Gamma(A_1) = \Gamma(M)$ and $|A_1| \leq |M|$. By further iterating Lemma 3.24 if necessary, we may assume that $k(A_1)$ is linearly difference closed. Since A_1 has the same value group as M and larger residue field, the maximal immediate extension A_2 of A_1 inside U contains M; observe that $|A_2| \leq |k(A_1)|^{|\Gamma(M)|} \leq 2^{|M|}$.

Claim 3.27.1. The k-elementary map $f_1\colon A_1 \to B_1$ extends to a k-elementary map $f_2\colon A_2 \to B_2$.

Proof of Claim. Observe that the proof of [DO15, Corollary 5.10][14] only uses saturation to realise sets of quantifier-free formulas. Namely, the only step where saturation is used is in the proof of [AD11, Lemma 5.9], in saying that, in our notation, certain *pseudo-Cauchy sequences* from B_1 have pseudolimits (see for example [DO15, Section 2] for the definitions of pseudo-Cauchy sequence, or *pc-sequence*, and of pseudolimit). To conclude, observe that if $\pi(x)$ is a set of quantifier-free formulas over B_1 stating that x is a pseudolimit of a certain pc-sequence then, by definition of pc-sequence, $\pi(x)$ is finitely satisfiable in B_1, and a fortiori in N. □ CLAIM

[13] Note that A' may contain points of $U \setminus M$.

[14] For the reader's convenience, we point out that most of the proofs in [DO15, Section 5], which culminates in the corollary of interest, are omitted as they heavily rely on similar material presented in [AD11, Section 5].

As the extension $A_1 \subseteq A_2$ is immediate, in order to conclude that f_2 is a k-elementary map it remains to be shown that, if $\mathcal{L}$ contains an angular component symbol, then f_2 commutes with ac. This follows from the fact, because $a \in A_2$ is a pseudolimit of a pseudo-Cauchy sequence, there is $b \in A_1$ such that $v(b-a) > v(a)$.

We conclude the proof by simply restricting f_2 to M. □

Recall that K is an *amalgamation base* iff every amalgamation problem $M \leftarrow K \rightarrow L$ has a solution.

Theorem 3.28. An amalgamation problem of valued difference fields of residual characteristic zero with either an angular component or a cross-section, and possibly with a lift, has a solution if and only if the induced problem on the residue difference fields has a solution.

In particular, K is an amalgamation base if and only if $k(K)$ is an amalgamation base, if and only if σ has a unique extension to the algebraic closure of $k(K)$ up to isomorphism.

Proof. If the amalgamation problem $M \leftarrow K \rightarrow L$ has a solution, then this clearly descends to the residue difference field level, so we prove the converse. Let N_0 be a valued difference field with one of the additional structures mentioned above, whose residue difference field $k(N_0)$ solves the amalgamation problem $k(M) \leftarrow k(K) \rightarrow k(L)$. Let N be an existentially closed, existentially $|M|^+$-saturated extension of N_0, so that $k(M)$ and $k(L)$ embed into $k(N)$ over $k(K)$. By Corollary 2.22 and Corollary 3.11, assumption (1) of Proposition 3.27 is satisfied. Moreover, as $\Gamma(N)$ is existentially closed and existentially $|\Gamma(M)|^+$-saturated, it follows from [DM24, Theorem 5.10] that assumption (2) is satisfied as well, while assumption (3) is satisfied by choice of N_0. Hence M embeds into N over K, solving the amalgamation problem.

For the "in particular" clause, the first part is clear. The fact that if the residue automorphism extends uniquely to the algebraic closure then $k(K)$ is an amalgamation base for difference fields follows from [Mac97, Lemma 8]. Conversely, if $(k(K), \sigma)$ is an amalgamation base and σ_1, σ_2 are two extensions to the algebraic closure, then any solution of the amalgamation problem $(k(K)^{\mathrm{alg}}, \sigma_1) \leftarrow (k(K), \sigma) \rightarrow (k(K)^{\mathrm{alg}}, \sigma_2)$ induces an isomorphism between $(k(K)^{\mathrm{alg}}, \sigma_1)$ and $(k(K)^{\mathrm{alg}}, \sigma_2)$. □

Solvable amalgamation problems appear in the difference algebra literature under the name of *compatible extensions*. See e.g. [Coh65, Chapter 7]. See [CH99, Lemmas 2.8 and 2.9(4)] for a condition ensuring that a field automorphism extends uniquely to the algebraic closure.

Theorem 3.29. A valued difference field K of residual characteristic zero with either an angular component or a cross-section, and possibly with a lift, is existentially closed if and only if it satisfies the following conditions.

(1) K is σ-henselian.

(2) The automorphisms induced on $\Gamma(K)$ and $k(K)$ are generic.

Proof. The conditions are necessary by Corollaries 2.22, 3.10 and 3.11.

For the converse, suppose K satisfies the given conditions. Let N be a $|K|^+$-saturated elementary extension of K. By elementarity, N is σ-henselian and

$k(N) \models \mathsf{ACFA}$. Suppose that $\varphi(x, a)$ is a quantifier-free formula with $a \in K$ which is satisfied by some b in some extension of K.

Let $\bar{\gamma}$ be a tuple enumerating $\Gamma(K\langle b\rangle)$. Note that $\mathrm{qftp}(\bar{\gamma}/\Gamma(K))$ is finitely satisfiable in $\Gamma(K)$, by existential closedness of the latter. In particular, it is finitely satisfiable in $\Gamma(N)$, and so by saturation of N it is realised in N by some tuple $\bar{\gamma}'$. Hence, there is an embedding $\Gamma(K\langle b\rangle) \to \Gamma(N)$ over $\Gamma(K)$. As ACFA is model complete, the embedding of $k(K)$ in any model of ACFA containing b is elementary, hence there is an embedding $k(K\langle b\rangle) \to k(N)$ by saturation of $k(N)$. We may therefore apply Proposition 3.27, hence $N \models \exists x\, \varphi(x, a)$, and we conclude by elementarity. □

Note that, in this characterisation of existentially closed structures, the only non-first-order part is the assumption that σ_Γ is generic.

Example 3.30. If (k, σ) and (Γ, σ) are both existentially closed and $\mathrm{char}(k) = 0$, then the Hahn field $k((\Gamma))$, with σ and ac as in Example 1.6, is σ-henselian by [DO15, Corollary 5.6(2)]. By Theorem 3.29, it is existentially closed.

The results above are in languages with at least a definable angular component. Without auxiliary maps, the following problems remain open.

Problem 3.31. In the language $\mathcal{L}_\sigma$, characterise

(1) the solvable amalgamation problems, and

(2) the existentially closed models.

4 The tree property of the second kind

In this section we prove that, for $\mathcal{L}$ any of the languages considered above (see Notation 3.22), the positive theory of $\mathcal{L}$-valued difference fields of residual characteristic zero is NTP_2. We do this by adapting the proof of [CH14, Theorem 4.1]. This will require generalising certain statements about indiscernible arrays to the positive setting, which we do in the first two subsections. Along the way, we prove the Array Modelling Theorem for arbitrary positive theories, removing the thickness assumption used in [Kam24], and deduce submultiplicativity of burden from it. We also provide an example showing that Lemma 2.3 in [Che14] is false (this, however, does not affect the correctness of any other results of [Che14]).

Henceforth we assume familiarity with positive logic. We refer the reader to [BP07] for the fundamentals; condensed accounts are available for instance in [DK22; DGK23; DM24]. Terminology-wise, let us just mention that we say *positive theory* for *h-inductive theory*, and that we talk of *positively existentially closed* (*pec*) models, while "existentially closed" is reserved for the classical notion. We refer to [DGK23, Definitions 4.1 and 4.5] for the definitions of IP and TP_2 in the positive setting. From now on, unless otherwise stated, we work in a monster model $\mathfrak{U}$ of a positive theory with the *Joint Continuation Property* (*JCP*). By a *formula* we will mean a positive formula, and by a *type* we will mean a (partial) positive type, we write $\equiv_A$ for having the same positive type over A, if we call a sequence *indiscernible* we mean with respect to positive formulas, etc. Similarly, every mention of NTP_2 and similar properties refers to their versions for positive logic.

We now recall some definitions and facts regarding generalised indiscernibles.

Definition 4.1. Let $\mathfrak{L}$ be a language and I, J two $\mathfrak{L}$-structures indexing some tuples $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$.

(1) The $\mathrm{EM}_{\mathfrak{L}}$-*type* $\mathrm{EM}_{\mathfrak{L}}((a_i)_{i\in I}/A)$ of $(a_i)_{i\in I}$ over A is the set of formulas $\varphi(x_{\bar\eta})$ over A, as $\bar\eta$ varies among finite tuples in I, such that, for every tuple $\bar\xi$ from I with the same quantifier-free $\mathfrak{L}$-type as $\bar\eta$, we have $\vDash \varphi(a_{\bar\xi})$.

(2) We say that $(a_i)_{i\in I}$ is $\mathfrak{L}$-*indiscernible over* A iff

$$\mathrm{EM}_{\mathfrak{L}}((a_i)_{i\in I}/A) = \mathrm{tp}((a_i)_{i\in I}/A).$$

(3) We say that $(b_j)_{j\in J}$ is $\mathrm{EM}_{\mathfrak{L}}$-*based on* $(a_i)_{i\in I}$ *over* A iff, for all $\varphi(x_{\bar\eta}) \in \mathrm{EM}_{\mathfrak{L}}((a_i)_{i\in I}/A)$ and all tuples $\bar\xi$ from J with the same quantifier-free $\mathfrak{L}$-type as $\bar\eta$, we have $\varphi(b_{\bar\xi}) \in \mathrm{EM}_{\mathfrak{L}}((b_i)_{i\in I}/A)$.

(4) We say that $(b_j)_{j\in J}$ is *based on* $(a_i)_{i\in I}$ *over* A iff, for all tuples $\bar\xi \subseteq J$ there is $\bar\eta \subseteq I$, with the same quantifier-free $\mathfrak{L}$-type as $\bar\xi$, such that $b_{\bar\xi} \equiv_A a_{\bar\eta}$.

As usual, when using the above definitions, if we omit the "over A" we mean "over $\emptyset$".

Remark 4.2.

(1) $\mathrm{EM}_{\mathfrak{L}}$-basedness is most significant (and will only be used below) when I and J realise the same finitary quantifier-free $\mathfrak{L}$-types, in which case it is a transitive notion [Kam24, Remark 3.8]. Note that, if $I = J$, then being "$\mathrm{EM}_{\mathfrak{L}}$-based on" is simply inclusion of $\mathrm{EM}_{\mathfrak{L}}$-types.

(2) If $(b_j)_{j\in J}$ is based on $(a_i)_{i\in I}$ over A, then it is also $\mathrm{EM}_{\mathfrak{L}}$-based on it over A, but the converse is false in general [Kam24, Proposition 3.10 and Example 3.11].

Recall that tuples are called *compatible* when they belong to the same cartesian product of sorts.

When index sets are linear orders, we will write $\mathrm{EM}_<$ instead of $\mathrm{EM}_{\{<\}}$. In the same setting, the following fact is standard, see e.g. [Ben03b, Lemma 1.2].

Fact 4.3. Let A be a parameter set, κ a cardinal, and let[15] $\lambda := \beth((2^{|T|+|A|+\kappa})^+)$. Then for every sequence $(a_i)_{i<\lambda}$ of compatible κ-tuples there is a sequence $(b_i)_{i<\omega}$ that is A-indiscernible and based on $(a_i)_{i<\lambda}$ over A.

For some purposes, $\mathrm{EM}_<$-basing, as opposed to basing, suffices. In that case, one may start from shorter sequences.

Fact 4.4 ([DK22, Corollary 5.4])**.** Let A be a parameter set. Then for every sequence $(a_i)_{i<\omega}$ of compatible tuples there is a sequence $(b_i)_{i<\omega}$ that is A-indiscernible and $\mathrm{EM}_<$-based on $(a_i)_{i<\omega}$ over A.

[15] For the sake of readability, we will use the notation $\beth(\alpha)$ instead of the more commonly used $\beth_\alpha$.

4.1 Array Modelling and submultiplicativity of burden

The first objective of this subsection is to generalise [Kam24, Theorems 1.3 and 1.4] to arbitrary positive theories, dropping the thickness assumption. We follow notation from there.

Recall that $\mathfrak{L}_{\mathrm{ar}}$ is the language $\{<_1, <_2\}$, interpreted in a subset Z of a product of linear orders $I \times J$ as $(i,j) <_1 (i',j')$ iff $i < i'$ and $(i,j) <_2 (i',j')$ iff $i = i'$ and $j < j'$. That is, $<_1$ is the order *on* rows and $<_2$ is the order *internal to* rows. For λ an ordinal, $\mathfrak{L}_\lambda$ denotes the expansion of $\mathfrak{L}_{\mathrm{ar}}$ by predicates P_α, for $\alpha < \lambda$, which we will interpret in a subset Z of $\lambda \times \mu$ as $Z \cap (\{\alpha\} \times \mu)$.

If Z is as above, we call $(a_{(ij)})_{(i,j)\in Z}$ an *array*. In order to keep the notation concise, we will also simply write a_Z, and extend this usage to restrictions, e.g. if $Z' \subseteq Z$ we will write $a_{Z'}$ with the obvious meaning. Similarly for e.g. $a_{<\alpha,\geq\beta}$. We will use $\bar{}$ to indicate rows, e.g. $\bar{a}_\alpha$ typically will mean the α-th row of a_Z, and we similarly write e.g. $\bar{a}_{<\alpha}$ for the sequence whose elements are the rows of a_Z up to the α-th, etc.

Remark 4.5.

(1) An array is $\mathfrak{L}_\lambda$-indiscernible exactly when it is *mutually indiscernible*, i.e. each of its rows is indiscernible over the union of all other rows.

(2) An array indexed on $I \times J$ is $\mathfrak{L}_{\mathrm{ar}}$-indiscernible exactly when each of its rows is indiscernible over the union of all other rows and, furthermore, its rows form an indiscernible sequence.

We use *strongly indiscernible* as a synonym for "$\mathfrak{L}_{\mathrm{ar}}$-indiscernible"; note that [Kam24] uses the terminology *array-indiscernible* instead.

Lemma 4.6. Let I be an $\mathfrak{L}$-structure, J, L be $\mathfrak{L}'$-structures, and E a set of parameters. If c_L is based on b_J over $a_I E$ and a_I is $\mathfrak{L}$-indiscernible over $b_J E$, then a_I is also $\mathfrak{L}$-indiscernible over $c_L E$.

Proof. By naming parameters, we may assume $E = \emptyset$. Consider any finite tuples $\bar{\eta}, \bar{\nu} \subseteq I$ with $\mathrm{qftp}_{\mathfrak{L}}(\bar{\eta}) = \mathrm{qftp}_{\mathfrak{L}}(\bar{\nu})$, as well as a finite tuple $\bar{\ell} \subseteq L$. As c_L is based on b_J over a_I, there is a tuple $\bar{\jmath} \subseteq J$ with $b_{\bar{\jmath}} \equiv_{a_I} c_{\bar{\ell}}$. As a_I is $\mathfrak{L}$-indiscernible over b_J, we have that $a_{\bar{\nu}} \equiv_{b_{\bar{\jmath}}} a_{\bar{\eta}}$. It follows that $a_{\bar{\nu}} \equiv_{c_{\bar{\ell}}} a_{\bar{\eta}}$, hence that $a_{\bar{\nu}} \equiv_{c_L} a_{\bar{\eta}}$. □

For an ordinal λ and $I \subseteq \lambda$, by $\mathrm{ot}(I)$ we will mean the order type of I with the order induced from λ. For $Z \subseteq \lambda \times \mu$ we write $Z_I \coloneqq Z \cap (I \times \mu)$ and consider it as an $\mathfrak{L}_{\mathrm{ot}(I)}$-structure in the obvious way.

By inspecting atomic $\mathfrak{L}_\lambda$-formulas, we easily see that if $\lambda = I \dot{\cup} J$, $\bar{\nu}, \bar{\nu}' \subseteq Z_I$ and $\bar{\xi}, \bar{\xi}' \subseteq Z_J$, then $\mathrm{qftp}_{\mathfrak{L}_\lambda}(\nu, \xi) = \mathrm{qftp}_{\mathfrak{L}_\lambda}(\nu', \xi')$ if and only if $\mathrm{qftp}_{\mathfrak{L}_{\mathrm{ot}(I)}}(\nu) = \mathrm{qftp}_{\mathfrak{L}_{\mathrm{ot}(I)}}(\nu')$ and $\mathrm{qftp}_{\mathfrak{L}_{\mathrm{ot}(J)}}(\xi) = \mathrm{qftp}_{\mathfrak{L}_{\mathrm{ot}(J)}}(\xi')$.

Lemma 4.7. Let $\lambda = I \dot{\cup} J$ and $Z, Z' \subseteq \lambda \times \mu$ with $Z_J = Z'_J$. Suppose $b_{Z'_I}$ is $\mathrm{EM}_{\mathfrak{L}_{\mathrm{ot}(I)}}$-based on a_{Z_I} over $a_{Z_J} E$. Let

$$c_{ij} \coloneqq \begin{cases} a_{ij} & \text{when } (i,j) \in Z'_J = Z_J \\ b_{ij} & \text{when } (i,j) \in Z'_I \end{cases}.$$

Then $c_{Z'}$ is $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on a_Z over E.

Proof. We may assume that $E = \emptyset$. Let $\bar{\nu} \subseteq Z_I$, $\bar{\xi} \subseteq Z_J$, and φ be such that $\varphi(x_{\bar{\nu}}, x_{\bar{\xi}}) \in \mathrm{EM}_{\mathfrak{L}_\lambda}(a_Z)$. We have to prove that $\vDash \varphi(c_{\bar{\nu}}, c_{\bar{\xi}})$.

For every $\bar{\nu}'$ with $\mathrm{qftp}_{\mathfrak{L}_{\mathrm{ot}(I)}}(\bar{\nu}') = \mathrm{qftp}_{\mathfrak{L}_{\mathrm{ot}(I)}}(\bar{\nu})$ we have that $\mathrm{qftp}_{\mathfrak{L}_\lambda}(\bar{\nu}', \bar{\xi}) = \mathrm{qftp}_{\mathfrak{L}_\lambda}(\bar{\nu}, \bar{\xi})$, so $\vDash \varphi(a_{\bar{\nu}'}, a_{\bar{\xi}})$. Hence $\varphi(x_{\bar{\nu}'}, a_{\bar{\xi}}) \in \mathrm{EM}_{\mathfrak{L}_{\mathrm{ot}(I)}}(a_{Z_I}/a_{Z_J})$ and so $\vDash \varphi(b_{\bar{\nu}}, a_{\bar{\xi}})$ by the assumption on $b_{Z_I'}$. This means that $\vDash \varphi(c_{\bar{\nu}}, c_{\bar{\xi}})$, as required. □

Lemma 4.8. Let $\lambda = I \dot{\cup} J$ and $Z \subseteq \lambda \times \mu$. Suppose a_Z is such that a_{Z_J} is $\mathfrak{L}_{\mathrm{ot}(J)}$-indiscernible over $a_{Z_I}E$ and a_{Z_I} is $\mathfrak{L}_{\mathrm{ot}(I)}$-indiscernible over $a_{Z_J}E$. Then a_Z is $\mathfrak{L}_\lambda$-indiscernible over E.

Proof. We may assume $E = \emptyset$. Let $\bar{\nu}, \bar{\nu}' \subseteq Z_I$ and $\bar{\xi}, \bar{\xi}' \subseteq Z_J$ be such that $\mathrm{qftp}_{\mathfrak{L}_\lambda}(\bar{\nu}, \bar{\xi}) = \mathrm{qftp}_{\mathfrak{L}_\lambda}(\bar{\nu}', \bar{\xi}')$. It follows easily from the assumptions that $a_{\bar{\nu},\bar{\xi}} \equiv a_{\bar{\nu},\bar{\xi}'} \equiv a_{\bar{\nu}',\bar{\xi}'}$. □

Proposition 4.9. Let $A = (a_{ij})_{i<\lambda, j<\omega}$ be an array with each row consisting of compatible tuples, and E a set of parameters. Then there is an array $(b_{ij})_{i<\lambda,j<\omega}$ which is $\mathfrak{L}_\lambda$-indiscernible over E and $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on A over E.

Proof. We proceed by induction on λ. For $\lambda = 0$ the statement is trivial.

Limit step. Let λ be a nonzero limit ordinal and let A be as in the statement of the proposition. We may assume that $E = \emptyset$. We will inductively replace the rows of A, making the new rows mutually indiscernible over the remaining ones and keeping the array $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on A. More formally, we construct recursively on $\alpha < \lambda$ arrays $A^\alpha = (a^\alpha_{ij})_{i<\lambda,j<\omega}$ such that

(1) $\bar{a}^\alpha_i = \bar{a}^{i+1}_i$ for every $i < \alpha$,

(2) $\bar{a}^\alpha_i = \bar{a}_i$ for every $i \geq \alpha$,

(3) A^α is $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on A, and

(4) $a^\alpha_{<\alpha,<\omega}$ is $\mathfrak{L}_\alpha$-indiscernible over $a^\alpha_{\geq\alpha,<\omega}$.

First, let us argue that this is enough to complete the limit step of the induction. Suppose we have A^α as above, and put $b_{ij} \coloneqq a^{i+1}_{ij}$ for all $i < \lambda$ and $j < \omega$. To see that $b_{<\lambda,<\omega}$ is $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on A, consider any $\varphi(x_{\bar{\nu}})$ in the $\mathrm{EM}_{\mathfrak{L}_\lambda}$-type of A. Then there is $\alpha < \lambda$ such that $\bar{\nu} \subseteq \alpha \times \omega$. As A^α is $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on A, we get that $\vDash \varphi(a^\alpha_{\bar{\nu}})$, but $a^\alpha_{\bar{\nu}} = b_{\bar{\nu}}$, so $\vDash \varphi(b_{\bar{\nu}})$. Similarly, to see that $(b_{ij})_{i<\lambda,j<\omega}$ is $\mathfrak{L}_\lambda$-indiscernible, let $\bar{\nu}, \bar{\nu}' \subseteq \lambda \times \omega$ be finite tuples with $\mathrm{qftp}_{\mathfrak{L}_\lambda}(\bar{\nu}) = \mathrm{qftp}_{\mathfrak{L}_\lambda}(\bar{\nu}')$ and choose $\alpha < \lambda$ such that $\bar{\nu}, \bar{\nu}' \subseteq \alpha \times \omega$. As $\bar{a}^\alpha_{<\alpha}$ is in particular $\mathfrak{L}_\alpha$-indiscernible over $\emptyset$, we get that $a^\alpha_{\bar{\nu}} \equiv a^\alpha_{\bar{\nu}'}$, i.e. $b_{\bar{\nu}} \equiv b_{\bar{\nu}'}$.

We now carry out the construction of the arrays A^α. If $\alpha < \lambda$ is a limit ordinal, we put $a^\alpha_{ij} \coloneqq a^{i+1}_{ij}$ for $i < \alpha$ and $a^\alpha_{ij} = a_{ij}$ for $i \geq \alpha$. Then (1) and (2) are clearly satisfied by this choice and the inductive assumption, and (3) and (4) are satisfied as well, due to the inductive assumption, the fact that every finite tuple of elements of $\alpha \times \omega$ is contained in $\beta \times \omega$ for some $\beta < \alpha$, and arguments similar to those in the previous paragraph.

For the successor step of the construction, suppose we have constructed A^α satisfying conditions (1)–(4). Let $\kappa \coloneqq \beth((2^{\lambda+|A|+|T|})^+)$ and let $\bar{c} = (c_i)_{i<\kappa}$ be a sequence $\mathrm{EM}_<$-based on $\bar{a}_\alpha$ over $\bar{a}^\alpha_{\neq\alpha}$, which exists by compactness.[16] By

[16] Note that we do not need to require this sequence to be indiscernible; this could nonetheless be ensured by invoking Fact 4.4.

Lemma 4.7, the array $\bar{a}^\alpha_{<\alpha} \frown \bar{c} \frown \bar{a}^\alpha_{>\alpha}$ (obtained by replacing the α-th row of A^α with $\bar{c}$) is $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on A^α, and hence on A by inductive assumption on α. By the main inductive assumption, there is an array $(\hat{a}^\alpha_{ij})_{i<\alpha}$ which is $\mathrm{EM}_{\mathfrak{L}_\alpha}$-based on $\bar{a}^\alpha_{<\alpha}$ over $\bar{a}^\alpha_{>\alpha}, \bar{c}$ and $\mathfrak{L}_\alpha$-indiscernible over $\bar{a}^\alpha_{>\alpha}, \bar{c}$. By Lemma 4.7 again, $\hat{a}^\alpha_{<\alpha} \frown \bar{c} \frown \bar{a}^\alpha_{>\alpha}$ is $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on $\bar{a}^\alpha_{<\alpha} \frown \bar{c} \frown \bar{a}^\alpha_{>\alpha}$, and hence on A. As $\bar{a}^\alpha_{<\alpha}$ is $\mathfrak{L}_\alpha$-indiscernible over $\bar{a}^\alpha_{>\alpha}$, we have $\mathrm{EM}_{\mathfrak{L}_\lambda}(\bar{a}^\alpha_{<\alpha}/\bar{a}^\alpha_{>\alpha}, \bar{c}) \supseteq \mathrm{tp}(\bar{a}^\alpha_{<\alpha}/\bar{a}^\alpha_{>\alpha})$, so $\hat{a}^\alpha_{<\alpha}\bar{a}^\alpha_{>\alpha} \equiv \bar{a}^\alpha_{<\alpha}\bar{a}^\alpha_{>\alpha}$. Let f be an automorphism over $\bar{a}^\alpha_{>\alpha}$ sending $\hat{a}^\alpha_{<\alpha}$ to $\bar{a}^\alpha_{<\alpha}$. Then $\bar{a}^\alpha_{<\alpha}$ is $\mathfrak{L}_\alpha$-indiscernible over $f(\bar{c}), \bar{a}^\alpha_{>\alpha}$, and $\bar{a}^\alpha_{<\alpha} \frown f(\bar{c}) \frown \bar{a}^\alpha_{>\alpha}$ is $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on A.

Now by Fact 4.3 we can choose $\bar{d} = (d_j)_{j<\omega}$ to be $\{<\}$-based on $f(\bar{c})$ over $\bar{a}^\alpha_{<\alpha}, \bar{a}^\alpha_{>\alpha}$ and indiscernible over $\bar{a}^\alpha_{<\alpha}, \bar{a}^\alpha_{>\alpha}$. By Lemma 4.7, $\bar{a}^\alpha_{<\alpha} \frown \bar{d} \frown \bar{a}^\alpha_{>\alpha}$ is $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on $\bar{a}^\alpha_{<\alpha} \frown f(\bar{c}) \frown \bar{a}^\alpha_{>\alpha}$, and hence on A. Note that, since $\bar{a}^\alpha_{<\alpha}$ is $\mathfrak{L}_\alpha$-indiscernible over $f(\bar{c}), \bar{a}^\alpha_{>\alpha}$, by Lemma 4.6 it is also $\mathfrak{L}_\alpha$-indiscernible over $\bar{d}, \bar{a}^\alpha_{>\alpha}$. As $\bar{d}$ is indiscernible over $\bar{a}^\alpha_{<\alpha}, \bar{a}^\alpha_{>\alpha}$, it follows from Lemma 4.8 that $\bar{a}^\alpha_{<\alpha} \frown \bar{d}$ is $\mathfrak{L}_{\alpha+1}$-indiscernible over $\bar{a}^\alpha_{>\alpha}$. Thus putting $a^{\alpha+1}_{ij} \coloneqq a^\alpha_{ij}$ for $i \neq \alpha$, and $a^{\alpha+1}_{\alpha,j} \coloneqq d_j$ for all $j < \omega$, we get that $A^{\alpha+1}$ satisfies conditions (3) and (4). It also clearly satisfies conditions (1) and (2) by the inductive assumption about A^α and the choice of $a^{\alpha+1}_{ij}$. This finishes the recursive construction of $(A^\alpha)_{\alpha<\lambda}$, and hence the limit step of the inductive proof of the proposition.

Successor step. We proceed similarly as in the successor step of the above construction. Namely, suppose the statement holds for some λ and consider an array $(a_{ij})_{i<\lambda+1, j<\omega}$. We may assume that $E = \emptyset$. By compactness there is a sufficiently long sequence $\bar{c}$ that is $\mathrm{EM}_<$-based on $\bar{a}_\lambda$ over $\bar{a}_{<\lambda}$. By the inductive hypothesis, there is an $\mathfrak{L}_\lambda$-indiscernible over $\bar{c}$ array $(b_{ij})_{i<\lambda, j<\omega}$ that is $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on $\bar{a}_{<\lambda}$ over $\bar{c}$.

Use Fact 4.3 to find a $\bar{b}_{<\lambda}$-indiscernible sequence $(b_{\lambda,i})_{i<\omega}$ that is based on $\bar{c}$ over $\bar{b}_{<\lambda}$. By Lemma 4.6, $\bar{b}_{<\lambda}$ is $\mathfrak{L}_\lambda$-indiscernible over $\bar{b}_\lambda$, hence we may apply Lemma 4.8 and conclude that $\bar{b}_{<\lambda+1}$ is $\mathrm{EM}_{\mathfrak{L}_{\lambda+1}}$-indiscernible.

We conclude by observing that, by repeated applications of Lemma 4.7, $\bar{b}_{<\lambda+1}$ is $\mathrm{EM}_{\mathfrak{L}_{\lambda+1}}$-based on $\bar{b}_{<\lambda} \frown \bar{c}$, in turn $\mathrm{EM}_{\mathfrak{L}_{\lambda+1}}$-based on $\bar{a}_{<\lambda} \frown \bar{c}$, itself $\mathrm{EM}_{\mathfrak{L}_{\lambda+1}}$-based on $\bar{a}_{<\lambda+1}$. □

Theorem 4.10 (Array Modelling)**.** Let $(a_{ij})_{i,j<\omega}$ be an array of compatible tuples and E a set of parameters. Then there is an array $(b_{ij})_{i,j<\omega}$ which is strongly indiscernible over E and $\mathrm{EM}_{\mathrm{ar}}$-based on $(a_{ij})_{i,j<\omega}$ over E.

Proof. Without loss of generality $E = \emptyset$. Let $\lambda \coloneqq \beth((2^{|T|+|a_{0,0}|})^+)$. By compactness we can extend $(a_{ij})_{i,j<\omega}$ to an array $(a_{ij})_{i<\lambda, j<\omega}$ which is $\mathrm{EM}_{\mathfrak{L}_{\mathrm{ar}}}$-based on $(a_{ij})_{i,j<\omega}$. By Proposition 4.9, there is an $\mathfrak{L}_\lambda$-indiscernible array $(c_{ij})_{i<\lambda, j<\omega}$ which is $\mathrm{EM}_{\mathfrak{L}_\lambda}$-based on $(a_{ij})_{i<\lambda, j<\omega}$. In particular, by Remark 4.2(1), $(c_{ij})_{i<\lambda, j<\omega}$ is $\mathrm{EM}_{\mathfrak{L}_{\mathrm{ar}}}$-based on $(a_{ij})_{i<\lambda, j<\omega}$. By Fact 4.3 there is an indiscernible sequence $(\bar{b}_i)_{i<\omega}$ based on the sequence $\bar{c}_{<\lambda}$. Then, by Remark 4.5, $(b_{ij})_{i,j<\omega}$ is an $\mathfrak{L}_{\mathrm{ar}}$-indiscernible array $\mathrm{EM}_{\mathfrak{L}_{\mathrm{ar}}}$-based on $(a_{ij})_{i,j<\omega}$. □

In [Kam24], the thickness assumption was used in proving Array Modelling, but not in deriving its consequences. In other words, having proved Array Modelling without assuming thickness, we may derive, by the same proofs as in [Kam24, Lemma 6.2 and Theorem 1.4], the following corollaries.

Corollary 4.11. A formula $\varphi(x,y)$ has $k{-}\mathsf{TP}_2$ for $k \geq 2$ if and only if there is a strongly indiscernible array $(a_{ij})_{i,j<\omega}$ such that

(1) $\{\varphi(x,a_{i,0}) \mid i<\omega\}$ is consistent, and

(2) $\{\varphi(x,a_{0,i}) \mid i<\omega\}$ is k-inconsistent.

Corollary 4.12. If $\varphi(x,y)$ has $k{-}\mathsf{TP}_2$ for some $k \geq 2$, then $\bigwedge_{i=1}^{n} \varphi(x,y_i)$ has $2{-}\mathsf{TP}_2$ for some n.

Remark 4.13. Suppose $\varphi(x,y)$, k, $\psi(x,y_1,\ldots,y_k)$ and $(a_{ij})_{i,j<\omega}$ witness TP_2. Assume that in the monster model $\varphi \equiv \bigwedge_{i\in I}\varphi_i$ and $\psi \equiv \bigwedge_{j\in J}\psi_j$. Then by compactness there are finite $I_0 \subseteq I$ and $J_0 \subseteq J$ such that, putting $\varphi' \coloneqq \bigwedge_{i\in I_0}\varphi_i$ and $\psi' \coloneqq \bigwedge_{j\in J_0}\psi_j$ the formula $\psi'(y_1,\ldots,y_k) \wedge \varphi'(x,y_1) \wedge \ldots \wedge \varphi'(x,y_n)$ is inconsistent. Then $\varphi'(x,y)$, k, $\psi'(x,y_1,\ldots,y_k)$ and $(a_{ij})_{i,j<\omega}$ also witness TP_2.

We now move towards submultiplicativity of burden.

Definition 4.14. Let $\pi(x)$ be a partial type and κ be a (possibly finite) cardinal. An inp-*pattern* in $\pi(x)$ of depth κ consists of $(a_{\alpha,i})_{\alpha<\kappa,i<\omega}$ and, for each $\alpha<\kappa$, a formula $\varphi_\alpha(x,y_\alpha)$, such that

(1) for every $\alpha<\kappa$ there are $k_\alpha<\omega$ and $\psi_\alpha(y_{\alpha,0},\ldots,y_{\alpha,k_\alpha-1})$ such that

$$\vDash \neg\left(\exists x, y_{\alpha,0},\ldots,y_{\alpha,k_\alpha-1}\ \left(\psi_\alpha(y_{\alpha,0},\ldots,y_{\alpha,k_\alpha-1}) \wedge \bigwedge_{i<k_\alpha}\varphi_\alpha(x,y_{\alpha,i})\right)\right)$$

and $\vDash \psi_\alpha(a_{\alpha,i_0},\ldots,a_{\alpha,i_{k_\alpha-1}})$ for all $i_0<\ldots<i_{k_\alpha-1}<\omega$; and

(2) for every $f\colon \kappa\to\omega$, the set $\pi(x)\cup\{\varphi_\alpha(x,a_{\alpha,f(\alpha)}) \mid \alpha<\kappa\}$ is consistent.

The *burden* of $\pi(x)$, denoted by $\mathrm{bdn}(\pi)$, is the supremum of the depths of all inp-patterns in $\pi(x)$, if one exists, and ∞ otherwise. By $\mathrm{bdn}(a/E)$ we mean $\mathrm{bdn}(\mathrm{tp}(a/E))$.

Clearly, $\pi(x)\subseteq\rho(x)$ implies $\mathrm{bdn}(\pi)\geq\mathrm{bdn}(\rho)$. Note that, by pigeonhole and compactness, we have $\mathrm{bdn}(\pi)<\infty \iff \mathrm{bdn}(\pi)<|T|^+$. In particular, T has TP_2 if and only if there is partial type of infinite burden.

Following [Che14], we will say that an array $(a_{ij})_{i<\alpha,j<\beta}$ is *almost mutually indiscernible* over E iff for every $i_0<\alpha$ we have that $(a_{i_0,j})_{j<\beta}$ is an indiscernible sequence over $a_{<i_0,<\beta}a_{>i_0,0}E$.

In the next lemma we correct [Che14, Lemma 2.3], which we will see in Example 4.20 not to hold as stated in [Che14].

Lemma 4.15. Let $\bar{a}=(a_i)_{i<\omega}$ be an E-indiscernible sequence and let b be arbitrary. Let $p(x,a_0)=\mathrm{tp}(b/a_0E)$, and assume that $p_\infty(x)=\bigcup_{i<\omega}p(x,a_i)$ is consistent. Then there is $\bar{a}'$ such that

(1) $\bar{a}'\equiv_{a_0E}\bar{a}$ and

(2) $\bar{a}'$ is indiscernible over Eb.

Proof. We may assume $E = \emptyset$. Let $b' \vDash p_\infty(x)$ and use Fact 4.4 to obtain some $\bar{a}''$ indiscernible over b' and $\mathrm{EM}_<$-based on $\bar{a}$ over b'.

Note that $b'a''_0 \equiv ba_0$, because for every i we have $b'a_i \equiv ba_0$ by choice of b'. Therefore, there is an automorphism f sending $b'a''_0$ to ba_0. Put $\bar{a}' \coloneqq f(\bar{a}'')$. Clearly $\bar{a}' \equiv_{a_0} \bar{a}$ and, as $\bar{a}''$ is indiscernible over b', we get that $\bar{a}' = f(\bar{a}'')$ is indiscernible over $b = f(b')$. □

Lemma 4.16. The following are equivalent for a partial type $\pi(x)$ over E.

(1) There is no inp-pattern of depth κ in π.

(2) For every $b \vDash \pi(x)$ and every almost mutually indiscernible over E array $(\bar{a}_\alpha)_{\alpha<\kappa}$, there are $\beta < \kappa$ and $\bar{a}'$ indiscernible over bE and such that $\bar{a}' \equiv_{a_{\beta,0}E} \bar{a}_\beta$.

(3) For every $b \vDash \pi(x)$ and every mutually indiscernible over E array $(\bar{a}_\alpha)_{\alpha<\kappa}$, there are $\beta < \kappa$ and $\bar{a}'$ indiscernible over bE and such that $\bar{a}' \equiv_{a_{\beta,0}E} \bar{a}_\beta$.

Proof. We may assume $E = \emptyset$.

(1) ⇒ (2) Let $b \vDash \pi(x)$ and an almost mutually indiscernible $(\bar{a}_\alpha)_{\alpha<\kappa}$ be given. To find $\bar{a}'$ as in the conclusion it suffices to check that we are in the assumptions of Lemma 4.15. Let $p_\alpha(x, a_{\alpha,0}) \coloneqq \mathrm{tp}(b/a_{\alpha,0})$ and $p_{\alpha,\infty}(x) \coloneqq \bigcup_{i<\omega} p_\alpha(x, a_{\alpha,i})$. If $p_{\alpha,\infty}$ is inconsistent for every α then by compactness and indiscernibility of $\bar{a}_\alpha$ we can find for every α integers k_α and formulas $\varphi_\alpha(x, y) \in \mathrm{tp}(b, a_{\alpha,0})$ and ψ_α satisfying condition (1) in Definition 4.14. As we have that $b \vDash \{\varphi_\alpha(x, a_{\alpha,0})\}_{\alpha<\kappa}$, by almost indiscernibility of $(\bar{a}_\alpha)_{\alpha<\kappa}$ we get that $(\bar{a}_\alpha)_{\alpha<\kappa}$ is an inp-pattern of depth κ in π, a contradiction. Thus $p_{\beta,\infty}(x)$ is consistent for some $\beta < \kappa$.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) Assume that there is an inp-pattern $(\bar{a}_\alpha, \varphi_\alpha)_{\alpha<\kappa}$ of depth κ in $\pi(x)$. We may assume $(\bar{a}_\alpha)_{\alpha<\kappa}$ is mutually indiscernible by Proposition 4.9. Let $b \vDash \pi(x) \cup \{\varphi_\alpha(x, a_{\alpha,0}) \mid \alpha < \kappa\}$. Then for every $\beta < \kappa$ we have that $\vDash \varphi_\beta(b, a_{\beta,0})$ and $\{\varphi_\beta(x, a_{\beta,i})\}_{i<\omega}$ is inconsistent, so it is impossible to find an $\bar{a}'$ as required. □

Theorem 4.17. If there is an inp-pattern of depth $\kappa_1 \times \kappa_2$ in $\mathrm{tp}(b_1b_2/E)$, then either there is an inp-pattern of depth κ_1 in $\mathrm{tp}(b_1/b_2E)$ or there is an inp-pattern of depth κ_2 in $\mathrm{tp}(b_2/E)$.

Proof. Assume not. We may assume $E = \emptyset$. We show that condition (3) in Lemma 4.16 is satisfied.

Let $(\bar{a}_\gamma)_{\gamma\in\kappa_1\times\kappa_2}$ be a mutually indiscernible array, where we consider the product $\kappa_1 \times \kappa_2$ lexicographically ordered.

By induction on $\alpha < \kappa_1$ we choose $\bar{a}'_\alpha$ and $\beta_\alpha < \kappa_2$ such that:

(1) $\bar{a}'_\alpha$ is indiscernible over $b_2\bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)}$,

(2) $\bar{a}'_\alpha$ has the same type as $\bar{a}_{(\alpha,\beta_\alpha)}$ over $a_{(\alpha,\beta_\alpha),0}\bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)}$, and

(3) $\bar{a}'_{\leq\alpha} \cup \bar{a}_{\geq(\alpha+1,0)}$ is a mutually indiscernible array.

Assume we have managed up to α, and we need to choose $\bar{a}'_\alpha$ and β_α. Let $D = \bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)}$. As $\left(\bar{a}_{(\alpha,\delta)}\right)_{\delta\in\kappa_2}$ is a mutually indiscernible array over D (by assumption in the case $\alpha = 0$ and by (3) of the inductive hypothesis in the other cases) and there is no inp-pattern of depth κ_2 in $\mathrm{tp}(b_2/D)$, by Lemma 4.16 there are $\bar{a}'_\alpha$ indiscernible over b_2D (which gives us (1)) and $\beta_\alpha < \kappa_2$ such that $\bar{a}'_\alpha \equiv_{a_{(\alpha,\beta_\alpha),0}D} \bar{a}_{(\alpha,\beta_\alpha)}$ (which gives us (2)), and as by the inductive assumption (3) we have in particular that $\bar{a}'_{<\alpha} \frown \bar{a}_{(\alpha,\beta_\alpha)} \frown \bar{a}_{\geq(\alpha+1,0)}$ is mutually indiscernible, the condition $\bar{a}'_\alpha \equiv_D \bar{a}_{(\alpha,\beta_\alpha)}$ gives us that (3) is preserved as well. This completes the induction.

As, by (2), we have $a'_{\alpha,0} = a_{(\alpha,\beta_\alpha),0}$ for every $\alpha < \kappa_1$, we get by (1) that $(\bar{a}'_\alpha)_{\alpha<\kappa_1}$ is an almost mutually indiscernible array over b_2.

As there is no inp-pattern of depth κ_1 in $\mathrm{tp}(b_1/b_2)$, by Lemma 4.16 there are some $\delta < \kappa_1$ and $\bar{a}''$ indiscernible over b_1b_2 and such that $\bar{a}'' \equiv_{a_{(\delta,\beta_\delta),0}} \bar{a}'_\delta \equiv_{a_{(\delta,\beta_\delta),0}} \bar{a}_{(\delta,\beta_\delta)}$ (for the second equivalence, note that $a'_{\delta,0} = a_{(\delta,\beta_\delta),0}$). Therefore, $\bar{a}''$ witnesses that condition (3) in Lemma 4.16 is satisfied. □

Corollary 4.18 (Submultiplicativity of burden)**.** If $\mathrm{bdn}(a_i/E) < k_i$ for $i < n$ with $k_i < \omega$, then $\mathrm{bdn}(a_0, \ldots, a_{n-1}/E) < \prod_{i<n} k_i$.

Corollary 4.19. If T has TP_2, then some formula $\varphi(x, y)$ with $|x| = 1$ has TP_2.

We now provide a counterexample to [Che14, Lemma 2.3], and also to its relaxation to "almost mutually indiscernible". The lemma is used in that paper only with $\kappa = 1$ in which case it is true (it is Lemma 4.15 above, specialised to first-order logic), hence this does not affect correctness of any other results in there.

Example 4.20. Let $\mathcal{L}$ be the language with three sorts A, B, C and a single function symbol $f\colon \mathrm{A} \times \mathrm{B} \to \mathrm{C}$. One checks amalgamation of finite $\mathcal{L}$-structures easily, obtaining a Fraïssé class. Let T be the theory of its Fraïssé limit (see [HM23, Definition 1.2(viii)] for an axiomatisation).

Let $\bar{c} = (c_i)_{i<\omega}$ be a sequence of pairwise distinct elements of sort C, and let a_0 be an arbitrary element of sort A. Consider the 2-row array with first row $\bar{c}$ and second row $\bar{a}$ constantly equal to a_0. Let $C = \emptyset$. This is a counterexample to [Che14, Lemma 2.3].

Proof. By quantifier elimination, this array is mutually indiscernible over $\emptyset$.

Choose any $b \in \mathrm{B}$ such that $f(a_0, b) = c_0$. By quantifier elimination and genericity of f, the partial type p^∞ in [Che14, Lemma 2.3] is consistent (in fact, it defines the whole sort B).

Now take the promised b-mutually indiscernible array, say $\bar{c}' \frown \bar{a}'$. By clause (1) of the lemma, the first column is the same as that of $\bar{c} \frown \bar{a}$, from which it follows that $\bar{a}' = \bar{a}$ and that $f(a'_0, b) = c'_0$. But the c_i are pairwise distinct, hence by clause (1) so are the c'_i, hence we cannot also have $f(a'_0, b) = c'_1$. This contradicts mutual indiscernibility over b, i.e. clause (2). □

4.2 Weak quantifier elimination and weak stable embeddedness

In this subsection we introduce the notions of weak quantifier elimination and (strict) weak stable embeddedness, and generalise to positive logic certain

lemmas from [CH14] concerning array extension.

Recall that usual inductive theories may be identified with *Pillay* positive theories; see [DM24, Subsection 6.2] for details.

Lemma 4.21. Let T be a Pillay theory, $E \subseteq \mathfrak{U}$ be small, x a variable of sort[17] X, and let $\mathcal{F}$ be a family of quantifier-free formulas $\varphi(x)$ over E, closed under Boolean combinations. Assume that, for all $a, b \in \mathfrak{U}$ of sort X, if a and b satisfy the same formulas in $\mathcal{F}$ then $\mathrm{tp}(a/E) = \mathrm{tp}(b/E)$. Then, in $\mathfrak{U}$, every quantifier-free formula over E of sort X is equivalent to a formula in $\mathcal{F}$.

Proof. This is a standard application of a general topological fact, see e.g. [TZ12, Exercise 3.1.1]. In the notation of the cited source, given $\varphi(x)$ a quantifier-free formula, take as X the space of maximal types in variables x, as Y_1 the set given by $\varphi(x)$, which is clopen because T is Pillay, and as Y_2 its complement. □

Definition 4.22. The theory T admits *weak quantifier elimination* iff, whenever a, b are compatible tuples from $\mathfrak{U}$, if $\mathrm{qftp}(a) = \mathrm{qftp}(b)$ then $\mathrm{tp}(a) = \mathrm{tp}(b)$.

Definition 4.23. We say that a $\emptyset$-definable set D is

(1) *weakly stably embedded* iff every formula with parameters $\varphi(x)$ implying $x \in D$ is equivalent to a partial type $\pi(x)$ with parameters in D, in the sense that they have the same set of solutions in every $|T|^+$-saturated pec model; and

(2) *strictly weakly stably embedded* iff furthermore, for every c over which $\varphi(x)$ is definable, $\pi(x)$ can be chosen over $D \cap \mathrm{dcl}(c)$.

Remark 4.24. If T is Boolean (that is, it is the translation in positive logic of a full first-order theory), then weak quantifier elimination is the same as quantifier elimination and weakly stably embedded is the same as stably embedded.

Definition 4.25. Let D be a $\emptyset$-definable set. By $D_{\mathrm{ind}}(\mathfrak{U})$ we denote the structure with universe $D(\mathfrak{U})$ and, for all $n < \omega$ and every formula $\varphi(x_1, \ldots, x_n)$ over $\emptyset$, a predicate for $\varphi(\mathfrak{U}) \cap D(\mathfrak{U})^n$.

We now generalise [CH14, Lemmas 3.6, 3.7 and 3.8].

Lemma 4.26. Let $(c_{ij})_{i,j<\omega}$ be a strongly indiscernible array such that $(\bar{c}_i)_{i<\omega}$ is indiscernible over a, and let b be arbitrary. Then there are $(b_{ij})_{i,j<\omega}$ such that $(b_{ij}c_{ij})_{i,j<\omega}$ is a strongly indiscernible array, $b_{ij}c_{ij} \equiv bc_{00}$ for all i, j, and $(\bar{b}_i\bar{c}_i)_{i<\omega}$ is indiscernible over a.

Proof. Let $\lambda = \beth((2^{|T|+|abc_{00}|})^+)$. By compactness, we can extend $(c_{ij})_{i,j<\omega}$ to a strongly indiscernible array $(c_{ij})_{i<\lambda,j<\omega}$ such that $\bar{c}_{<\lambda}$ is a-indiscernible. For all $i < \lambda$ and $j < \omega$, let b'_{ij} be such that $b'_{ij}c_{ij} \equiv b_{00}c_{00}$. By Theorem 4.10 we can find a strongly indiscernible array $(b''_{ij}c''_{ij})_{i<\lambda,j<\omega}$ which is $\mathrm{EM}_{\mathrm{ar}}$-based on $(\bar{b}'_i\bar{c}_i)_{i<\lambda}$. As $\bar{c}_{<\lambda}$ was strongly indiscernible, $\mathrm{EM}_{\mathrm{ar}}((\bar{b}'_i\bar{c}_i)_{i<\lambda}) \supseteq \mathrm{tp}(\bar{c}_{<\lambda})$ so $\bar{c}''_{<\lambda} \equiv \bar{c}_{<\lambda}$. Hence, by applying an automorphism, we may assume $\bar{c}''_{<\lambda} = \bar{c}_{<\lambda}$. By Fact 4.3, we can find an a-indiscernible sequence $(\bar{b}_i\bar{c}'''_i)_{i<\omega}$ based on $(\bar{b}''_i\bar{c}_i)_{i<\lambda}$ over a. As $\bar{c}_{<\lambda}$ was already indiscernible over a, we get that $\bar{c}'''_{<\omega} \equiv_a \bar{c}_{<\omega}$, so, by applying an automorphism over a, we may assume $\bar{c}'''_{<\omega} = \bar{c}_{<\omega}$. Then $(b_{ij})_{i,j<\omega}$ satisfies the requirements. □

[17] Here we regard the family of sorts as being closed under cartesian products.

Definition 4.27. We say that a structure M (as opposed to a positive theory) is (positively) NTP_2 iff there is no infinite inp-pattern with constant φ_α, k_α, ψ_α and parameters in M.

We will exclusively use the notion just introduced with $M = D_{\mathrm{ind}}(\mathfrak{U})$ for D some $\emptyset$-definable strictly weakly stably embedded set.

Lemma 4.28. Let D be a $\emptyset$-definable strictly weakly stably embedded set such that $D_{\mathrm{ind}}(\mathfrak{U})$ is NTP_2. Let $b \subseteq D(\mathfrak{U})$ and assume $(\bar{c}_i)_{i<\kappa}$ is a mutually indiscernible array over C. If $\kappa \geq (|T| + |b|)^+$, then there are $i < \kappa$ and $\bar{c}' \equiv_{Cc_{i0}} \bar{c}_i$ which is indiscernible over Cb.

Proof. We may assume $C = \emptyset$. By making each c_{ij} longer, we may also assume $\mathrm{dcl}(c_{ij}) = c_{ij}$ for all i, j. Let $p_i(x, c_{i0}) = \mathrm{tp}(b/c_{i0})$ for every $i < \kappa$. By Lemma 4.15, it is enough to show that $\bigcup_{j\in\omega} p_i(x, c_{ij})$ is consistent for some i. Suppose not. By strict weak stable embeddedness, for every $i < \kappa$ we have $p_i(x, c_{ij}) \equiv p_i(x, c_{ij})|_{D(\mathfrak{U})\cap c_{ij}}$ and hence there is a formula $\psi_i(x, y') \in p_i(x, y')$ such that $\{\psi_i(x, D(\mathfrak{U}) \cap c_{ij}) \mid j < \omega\}$ is k_i-inconsistent for some $k_i < \omega$. By pigeonhole we can assume ψ_i and k_i are constantly equal to some fixed ψ and k. Now $(D(\mathfrak{U}) \cap c_{ij})_{i,j<\omega}$, $(x \in D) \wedge \psi(x, y)$, and k witness that $D_{\mathrm{ind}}(\mathfrak{U})$ has TP_2, because the consistency condition with the 0^κ-path is witnessed by b, hence holds for all paths by mutual indiscernibility. □

Lemma 4.29. Suppose C is a set of parameters and $(a_i)_{i<\lambda,j<\omega}$ is almost mutually indiscernible over C. Then there is a mutually indiscernible over C array $(a'_{ij})_{i<\lambda,j<\omega}$ such that $\bar{a}'_i \equiv_{a_{i0}C} \bar{a}_i$ for every $i < \lambda$.

Proof. We may assume that $C = \emptyset$. By Proposition 4.9 there is a mutually indiscernible array $(\bar{a}''_i)_{i<\lambda}$ which is $\mathrm{EM}_{\mathfrak{L}_\lambda}$–based on $\bar{a}_{<\lambda}$. By the assumption on $\bar{a}_{<\lambda}$ we have that $\bar{a}_i$ is indiscernible for every $i < \lambda$, hence $\bar{a}''_i \equiv \bar{a}_i$, and that for all $k < \omega$, $i_0 < \ldots < i_{k-1} < \lambda$ and $j_0, \ldots, j_{k-1} < \omega$ we have $a_{i_0j_0} \ldots a_{i_{k-1}j_{k-1}} \equiv_C a_{i_00} \ldots a_{i_{k-1}0}$, hence $(a''_{i0})_{i<\lambda} \equiv (a_{i0})_{i<\lambda}$. Thus, we can find an automorphism f sending $(a''_{i0})_{i<\lambda}$ to $(a_{i0})_{i<\lambda}$ and define $\bar{a}'_i = f(\bar{a}''_i)$ for each $i < \lambda$. □

In Lemma 4.31, we assume that T is thick; recall that this means that "$(x_0, x_1, \ldots)$ is an indiscernible sequence" is equivalent, in sufficiently saturated pec models, to a partial type. Direct usage of thickness will be confined to the proof of Lemma 4.31, and mediated by its following consequence.

Remark 4.30. Assume T is thick.

(1) If $\bar{c}$ is a sequence indiscernible over $\bar{d}E$ and $\bar{d}'$ is $\mathrm{EM}_<$-based on $\bar{d}$ over $\bar{c}E$ then $\bar{c}$ is indiscernible over $\bar{d}'E$.

(2) Let $(\bar{a}_i)_{i<\lambda}$ be EM_λ-based on $(\bar{b}_j)_{j<\lambda}$ over E. If $\bar{b}_{<\lambda}$ is mutually indiscernible over E, then so is $\bar{a}_{<\lambda}$.

Lemma 4.31. Assume T is thick. Let D be a $\emptyset$-definable strictly weakly stably embedded set and assume that $D_{\mathrm{ind}}(\mathfrak{U})$ is NTP_2. Let $b \subseteq D(\mathfrak{U})$ be arbitrary and $(c_{ij})_{i,j<\omega}$ be a strongly indiscernible array such that $(\bar{c}_i)_{i<\omega}$ is indiscernible over a. Then there are $(c^*_{ij})_{i,j<\omega}$ and $(b^*_{ij})_{i,j<\omega}$ such that

(1) $(\bar{b}^*_i \bar{c}^*_i)_{i<\omega}$ is indiscernible over a

(2) $(\bar{b}^*_i \bar{c}^*_i)_{i<\omega}$ is mutually indiscernible

(3) $\bar{c}^*_i \equiv_{c_{i0}} \bar{c}_i$ for all $i < \omega$ (so in particular $c^*_{i0} = c_{i0}$)

(4) $b^*_{00} c^*_{00} \equiv_a bc_{00}$

Proof.

Claim 4.31.1. It is enough to find, for every $n < \omega$, sequences $\bar{c}^{**}_{<n}$ and $\bar{b}^{**}_{<n}$ satisfying (2), (3) and such that:

(4′) $b^{**}_{k0} c^{**}_{k0} \equiv_a bc_{00}$ for all $k < n$.

Proof of Claim. Indeed, assume we can find such $\bar{c}^{**}_{<n}$ and $\bar{b}^{**}_{<n}$. Then, for $\kappa := \beth((2^{|T|+|abc_{00}|})^+)$, by compactness and Remark 4.30(2) we may find $\bar{c}^{**}_{<\kappa}$ and $\bar{b}^{**}_{<\kappa}$ satisfying (2), together with (3) (for all $i < \omega$), and with the analogue of (4′) but for $i < \kappa$ instead of $k < n$, with $(c^{**}_{i0})_{i<\kappa}$ indiscernible over a. Then we can apply Fact 4.3 to $(\bar{b}^{**}_i \bar{c}^{**}_i)_{i<\kappa}$ over a, obtaining an array $(\bar{b}'_i \bar{c}'_i)_{i<\omega}$ satisfying (1), (2), and (4). As $(c^{**}_{i0})_{i<\kappa}$ was indiscernible over a, we get that $(c'_{i0})_{i<\omega} \equiv_a (c^{**}_{i0})_{i<\omega} = (c_{i0})_{i<\omega}$. Let f be an automorphism over a sending $(c'_{i0})_{i<\omega}$ to $(c_{i0})_{i<\omega}$. Let $\bar{b}^*_i = f(\bar{b}'_i)$ and $\bar{c}^*_i = f(\bar{c}'_i)$. Then for every i we have that $\bar{c}^*_i \equiv \bar{c}_i$ and $c^*_{i0} = c_{i0}$, yielding (3), while (1), (2) and (4) are clearly preserved by f, hence they are satisfied by $(\bar{b}^*_i \bar{c}^*_i)_{i<\omega}$ as well. $\square_{\text{CLAIM}}$

Fix any $n < \omega$. By compactness, stretch $(\bar{c}_i)_{i<\omega}$ to a strongly indiscernible over a array $(\bar{c}_{(i,\ell)})_{(i,\ell)\in(n\times\omega,<_{\text{lex}})}$ with $\bar{c}_{(0,i)} = \bar{c}_i$ for all $i < \omega$. For each $(i,\ell) \in n \times \omega$ find $b_{(i,\ell)0}$ with $b_{(i,\ell)0} c_{(i,\ell)0} \equiv_a bc_{00}$. By induction on $k < n$ we will find $(b^*_{k,j} c^*_{k,j})_{j\in\omega}$ such that the following hold.

(i) $(\bar{b}^*_i \bar{c}^*_i)_{i\le k}$ is almost mutually indiscernible over $(b_{(i,\ell)0} c_{(i,\ell)0})_{i\in[k+1,n-1],\ell<\omega}$.

(ii) $\bar{c}^*_{\le k} \frown (\bar{c}_{(i,\ell)})_{(i,\ell)\in([k+1,n-1]\times\omega,<_{\text{lex}})}$ is based on $(\bar{c}_i)_{i<\omega}$.

(iii) $(c^*_{i0})_{i\le k} \frown (c_{(i,\ell)0})_{(i,\ell)\in([k+1,n-1]\times\omega,<_{\text{lex}})}$ is based on $(c_{i0})_{i<\omega}$ over a.

(iv) $b^*_{k0} c^*_{k0} \equiv_a bc_{00}$

Suppose we have constructed $(\bar{b}^*_i \bar{c}^*_i)_{i<k}$. Put $\lambda := (|T| + |bc_{00}|)^+$ and find by compactness a sequence $(b^+_i \bar{c}^+_i)_{i<\lambda}$ that is $\text{EM}_<$-based on $(b_{(k,\ell)0} \bar{c}_{(k,\ell)})_{\ell<\omega}$ over aBC, where

$$C := \bar{c}^*_{<k} (\bar{c}_{(i,\ell)})_{i\in[k+1,n-1],\ell<\omega},$$
$$B := \bar{b}^*_{<k} (b_{(i,\ell)0})_{i\in[k+1,n-1],\ell<\omega}.$$

Then $b^+_i c^+_{i0} \equiv_a bc_{00}$ for every $i < \lambda$, and, by the inductive assumption and Remark 4.30(1) applied k times, one for each row of $(\bar{b}^*_i \bar{c}^*_i)_{i<k}$,

$$\begin{aligned}&(\bar{b}^*_i \bar{c}^*_i)_{i<k} \text{ is almost mutually indiscernible}\\ &\text{over } (b^+_i \bar{c}^+_i)_{i<\lambda} (b_{(i,\ell)0} \bar{c}_{(i,\ell)})_{i\in[k+1,n-1],\ell<\omega}.\end{aligned} \tag{†}$$

Observe that, since (ii) inductively holds, by choice of $b^+_i \bar{c}^+_i$ the sequence

$$\bar{c}^*_{<k} \frown \bar{c}^+_{<\lambda} \frown (\bar{c}_{(i,\ell)})_{(i,\ell)\in([k+1,n-1]\times\omega,<_{\text{lex}})}$$

is based on $\bar{c}_{<\omega}$.

It also follows by Remark 4.30(2) that $\bar{c}^+_{<\lambda}$ is a mutually indiscernible over C array, hence, by Lemma 4.28, there are $m < \lambda$ and $\bar{d} = (d_i)_{i<\omega} \equiv_{Cc^+_{m0}} \bar{c}^+_m$ with $(d_i)_{i<\omega}$ indiscernible over BC. For each $j < \omega$, let e_j be such that $e_j d_j \equiv_{BC} b^+_m c^+_{m0}$. By Fact 4.4 there is an indiscernible over BC sequence $(s_j t_j)_{j<\omega}$ which is $\mathrm{EM}_<$-based on $(e_j d_j)_{j<\omega}$ over BC. Let f be an automorphism over BC sending $s_0 t_0$ to $b^+_m c^+_{m0}$. Then $\bar{b}^*_k \coloneqq f(\bar{s})$ and $\bar{c}^*_k \coloneqq f(\bar{t})$ (together with $\bar{b}^*_{<k}\bar{c}^*_{<k}$) satisfy (i) because $(s_j t_j)_{j<\omega}$ is indiscernible over BC, hence so is $(b^*_{kj} c^*_{kj})_{j<\omega}$, and because by (†) the array $(b^*_i \bar{c}^*_i)_{i<k}$ is almost mutually indiscernible over $b^*_{k0} c^*_{k0} (b_{(i,\ell)0} c_{(i,\ell)0})_{i\in[k+1,n-1],\ell<\omega}$.

By the same choice, because f fixes C pointwise, and because $(\bar{c}_{(k,\ell)})_{\ell<\omega}$ is indiscernible over C, the sequence $\bar{c}_{\leq k}$ also satisfies (ii). Also, (iii) follows by the fact that $c^*_{m0} = c^+_{m0}$ and the choice of $(b^+_i \bar{c}^+_i)_{i<\lambda}$, which implies, as $(c_{(k,\ell)0})_{\ell<\omega}$ is indiscernible over aC, that $(c^+_{i0})_{i<\lambda}$ is based on $(c_{(k,\ell)0)})_{\ell<\omega}$ over aC. Point (iv) holds because $b^*_{k0} c^*_{k0} = b^+_m c^+_{m0}$. This completes the construction of $(\bar{b}^*_i \bar{c}^*_i)_{i<n}$.

Applying Lemma 4.29 to $(\bar{b}^*_i \bar{c}^*_i)_{i<n}$ we obtain an array $(\bar{b}^{***}_i \bar{c}^{***}_i)_{i<n}$ satisfying (2), (4′), and $\bar{c}^{***}_i \equiv \bar{c}_i$ with $c^{***}_{i0} = c^*_{i0}$ for every $i < n$. By (iii) we have in particular that $(c^*_{i0})_{i<n} \equiv_a (c_{i0})_{i<n}$, hence we can find an automorphism g over a with $g((c^*_{i0})_{i<n}) = (c_{i0})_{i<n}$. Then $\bar{b}^{**}_i \coloneqq g(\bar{b}^{***}_i)$ and $\bar{c}^{**}_i \coloneqq g(\bar{c}^{***}_i)$ satisfy (2), (3) and (4′), as required. □

4.3 NTP_2 for existentially closed valued difference fields of residual characteristic zero

In our proof of NTP_2 for valued difference fields of residual characteristic zero we will need to Morleyise the residue field sort. For details on (partial) Morleyisation in positive logic see [DK22, Remark 2.2] and [DM24, proof of Remark 6.14].

Notation 4.32. In this subsection we denote by $\mathcal{L}$ one of the four languages in Notation 3.22. We let Mor_k be the language of the Morleyisation of ACFA. We write $\widehat{\mathcal{L}}$ for the expansion of $\mathcal{L}$ by the symbols in Mor_k, added on the k sort. We write $\mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}}$ for the h-inductive theory of $\widehat{\mathcal{L}}$-valued difference fields of residual characteristic zero. By $\mathsf{ACVFA}_{0,0}$ we mean the h-inductive theory of valued difference fields of residual characteristic zero in the language $\mathcal{L}_\sigma$ of Definition 1.4.

Note that, contrary to ACFA, the axioms of $\mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}}$ and of $\mathsf{ACVFA}_{0,0}$ do not require any form of genericity of σ. This is common in the literature (e.g. [HK21; DM24]), and due to the fact that, at any rate, we focus on the existentially closed models.

Remark 4.33. $\mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}}$ is a Pillay theory, as the negations of the formulas $x <_\Gamma y$, $x =_\Gamma y$, and $x =_{\mathrm{VF}} y$ are equivalent to the positive formulas $x >_\Gamma y \vee x =_\Gamma y$, $x <_\Gamma y \vee x >_\Gamma y$, and $(x-y)^{-1}(x-y) =_{\mathrm{VF}} 1$, respectively, and the negation of every predicate coming from Morleyising the sort k is also added to the language in the Morleyisation process.

Proposition 4.34. The JCP-refinements of $\mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}}$ and of $\mathsf{ACVFA}_{0,0}$ are determined by fixing a characteristic zero completion of ACFA.

Proof. Indeed, suppose T is such a completion and $A, B \models \mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}} \cup T$. Then the prime subfields of $k(A)$ and of $k(B)$ are Mor_k-isomorphic. By identifying both of them with the same Mor_k-expansion of $\mathbb{Q}$, we get that $k(A)$ and $k(B)$ amalgamate over $\mathbb{Q}$, so, by Theorem 3.28, A and B amalgamate over $\mathbb{Q}$. This proves JCP.

Conversely, if $T \supseteq \mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}}$ is a JCP-refinement, then for every $A, B \models T$ we can embed A and B in a model $C \models T$ and so the copies of $\mathbb{Q}^{\mathrm{alg}}$ in $k(A)$ and $k(B)$ are isomorphic. By [CH99, Proposition 1.4], this means that T implies a completion of ACFA on k.

For $\mathsf{ACVFA}_{0,0}$, the only difference is that in the proof of the first implication we pass from A and B to some positively $\aleph_1$-saturated extensions, and equip them with angular components by using Corollary 2.15. On $\mathbb{Q}$ the angular component coincides with the residue map, hence $\mathbb{Q}$ is still a common substructure of A and B in the expanded language, hence we can apply Theorem 3.28. □

Proposition 4.35. $\mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}}$ admits weak quantifier elimination in $\widehat{\mathcal{L}}$.

Proof. By [Hod93, Theorems 8.5.2 and 8.5.4], it suffices to show that the class of substructures of models of $\mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}}$ has the Amalgamation Property.

Let $L \leftarrow M \rightarrow N$ be substructures of $\widehat{\mathcal{L}}$-valued difference fields. By Theorem 3.28 it suffices to show that the amalgamation problem $k(L) \leftarrow k(M) \rightarrow k(N)$ has a solution. If k_1 and k_2 are characteristic zero models of ACFA in the language Mor_k extending $k(L)$ and $k(N)$, respectively, such that k_2 is $|k_1|^+$-saturated, then $k(M)$ is a Mor_k-substructure of k_1 and of k_2, hence $\mathrm{id}_{k(M)}$ is a partial Mor_k-elementary map between k_1 and k_2. Hence $\mathrm{id}_{k(M)}$ extends to an embedding of k_1 into k_2. □

Corollary 4.36. In every existentially closed model of $\mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}}$ the sorts Γ and k are *orthogonal*, i.e. for every pair of maximal types $p(x)$ and $q(y)$ with x a tuple of variables of the sort Γ and y a tuple of variables of the sort k, the partial type $p(x) \cup q(y)$ implies a maximal type in variables xy.

Proof. By Proposition 4.35 it suffices to check that if $\alpha \models p$ and $\gamma, \gamma' \models q$ then the structures generated by $\alpha\gamma$ and $\alpha\gamma'$ are isomorphic. This is easy and left to the reader. □

Proposition 4.37. In every existentially closed $\mathfrak{U} \models \mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}}$ the following hold.

(1) The sorts k and Γ are strictly weakly stably embedded.

(2) The formulas in $k_{\mathrm{ind}}(\mathfrak{U})$ are equivalent in $\mathfrak{U}$ to infinitary conjunctions of formulas in $\mathrm{Mor}_k(k(\mathfrak{U}))$, and the formulas in $\Gamma_{\mathrm{ind}}(\mathfrak{U})$ are equivalent to infinitary conjunctions of quantifier-free formulas over $\Gamma(\mathfrak{U})$ in the language $\mathcal{L}_{\mathrm{oag}} \cup \{\sigma, \sigma^{-1}\}$.

Proof. We prove both statements in the case of Γ; the case of k is completely analogous. In both cases, we apply Lemma 4.21. We first deal with the language with lift and section.

For (1), it is enough to show that if M is a substructure of $\mathfrak{U}$ and $\gamma, \delta \in \Gamma(\mathfrak{U})^n$ are such that $\mathrm{qftp}(\gamma/\Gamma(M)) = \mathrm{qftp}(\delta/\Gamma(M))$, then $\mathrm{tp}(\gamma/M) = \mathrm{tp}(\delta/M)$. Indeed, if we have this, let $\varphi(x, c)$ be a quantifier-free formula, with x a tuple of

variables of sort Γ, and set $M \coloneqq \mathrm{dcl}(c)$. Apply Lemma 4.21 with $\mathcal{F}$ the family of quantifier-free formulas over $\Gamma(M)$, obtaining that $\varphi(x,c)$ is equivalent to such a formula. By Proposition 4.35, it suffices to check the definition of strict weak stable embeddedness on quantifier-free formulas, hence this suffices.

Observe that, in the proofs of [Kes24, Lemmas 4.9 and 4.10], the assumption that σ_Γ is multiplicative is never used. We apply [Kes24, Lemma 4.10] for the Γ case, the other lemma being used in the k case. By the cited result, we have an isomorphism $f\colon \langle M,\gamma\rangle \to \langle M,\delta\rangle$ extending id_M and sending γ to δ. As $k(\langle M,\gamma\rangle) = k(M) = k(\langle M,\delta\rangle)$, in particular, f restricted to $k(\langle M,\gamma\rangle)$ is Mor_k-elementary. Hence, by weak quantifier elimination we get that f is a partial immersion, so in particular $\mathrm{tp}(\gamma/M) = \mathrm{tp}(\delta/M)$.

For (2) we show that, if $\gamma,\delta \in \Gamma(\mathfrak{U})^n$ and $\mathrm{qftp}_\Gamma(\gamma) = \mathrm{qftp}_\Gamma(\delta)$, then $\mathrm{tp}(\gamma) = \mathrm{tp}(\delta)$, and then apply Lemma 4.21 and Proposition 4.35. Again by [Kes24, Lemma 4.10], we have an isomorphism $\langle \mathbb{Q},\gamma\rangle \to \langle \mathbb{Q},\delta\rangle$ sending γ to δ. As $k(\langle \mathbb{Q},\gamma\rangle) = \mathbb{Q} = k(\langle \mathbb{Q},\delta\rangle)$, we have in particular that $f|_{k(\langle \mathbb{Q},\gamma\rangle)} = \mathrm{id}_{\mathbb{Q}}$ is elementary. Hence, by weak quantifier elimination we get that f is a partial immersion.

For the other three languages, the proof is analogous, and uses analogues of [Kes24, Lemmas 4.9 and 4.10] that are easier to prove and left to the reader. □

Lemma 4.38. If $K \subseteq L$ is an immediate extension of $\widehat{\mathcal{L}}$-valued difference subfields of $\mathfrak{U}$, and $a \in L$, then every realisation in $\mathfrak{U}$ of the quantifier-free type of a over K in the language $\mathcal{L}$ (see Notation 4.32) realises $\mathrm{tp}(a/K)$.

Proof. This follows by Proposition 4.35 and inspection of $\widehat{\mathcal{L}}$. □

Remark 4.39. By Theorem 3.28, as observed in [DM24, Remark 6.55], the positive theory $\mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}}$ is *Hausdorff*, see [DK22, Definition 2.13]. In particular, it is thick.

Theorem 4.40. The positive theories $\mathsf{ACVFA}_{0,0,\widehat{\mathcal{L}}}$ and $\mathsf{ACVFA}_{0,0}$ are NTP_2.

Proof. If M is an $\widehat{\mathcal{L}}$-valued difference field, or a valued difference field in the language $\mathcal{L}_\sigma$, as in the proof of Corollary 3.10 we may expand a suitable pec extension of it to an $\widehat{\mathcal{L}}_{s,\iota,\sigma}$-valued difference field. It follows that it suffices to show that $\mathsf{ACVFA}_{0,0,s,\iota}$ is NTP_2.

Suppose for a contradiction that a formula $\varphi(x,y)$ has TP_2. We may assume that x is a single variable by Corollary 4.19. As the theory of Γ is NIP (hence NTP_2, see [DGK23, Theorem 1.1]) by [DM24, Corollary 6.39], by Proposition 4.37(2) and Remark 4.13 the structure $\Gamma_{\mathrm{ind}}(\mathfrak{U})$ is NTP_2. Similarly for $k_{\mathrm{ind}}(\mathfrak{U})$, using the fact that $k(\mathfrak{U}) \vDash \mathsf{ACFA}$, hence is simple [CH99]. Therefore, by Lemmas 4.16 and 4.28 we may further assume that x is of sort VF. By Proposition 4.35 and Remark 4.13 we may furthermore assume that $\varphi(x,y)$ is quantifier-free.

Let $(a_{ij})_{i,j<\omega}$ be a strongly indiscernible array witnessing that $\varphi(x,y)$ has TP_2. Let $a \vDash \{\varphi(x,a_{i0}) \mid i<\omega\}$. By Fact 4.4 we may assume that the sequence $(\bar{a}_i)_{i<\omega}$ is a-indiscernible. We run the analogues for the language $\widehat{\mathcal{L}}_{s,\iota,\sigma}$ of steps (I) to (V) in the proof of [CH14, Theorem 4.1], but we replace [CH14, Lemmas 3.6 and 3.8] by Lemmas 4.26 and 4.31. We obtain, for $\beta<\omega$, strongly indiscernible arrays $(a^\beta_{ij})_{i,j<\omega}$ such that $K \coloneqq \bigcup_{\beta<\omega} a^\beta_{00}$ is a substructure of $\mathfrak{U}$ (in fact, a valued difference subfield because, since ι and s are in the language, the maps

res and v are automatically surjective), the union is increasing, $L \coloneqq K\langle a\rangle$ is an immediate extension of K,

(1) for every β, the sequence $(\bar{a}_i^\beta)_{i<\omega}$ is indiscernible over a, and

(2) for every $\alpha < \beta$ and $i, j < \omega$ we have a decomposition $a_{ij}^\beta = (a_{ij}^*, b_{ij}^*)$ such that $\bar{a}_i^* \equiv_{a_{i0}^\alpha} \bar{a}_i^\alpha$.

Note that it follows from (2) that $a_{i,0}^\beta \supseteq a_{i,0}$ for all $i, \beta < \omega$, so $a \vDash \{\varphi(x, a_{i0}^\beta) \mid i < \omega\}$ and the rows $\{\varphi(x, a_{ij}^\beta) \mid j < \omega\}$ are inconsistent for all $i, \beta < \omega$.

We now deviate from the proof of [CH14, Theorem 4.1]: instead of constructing $(a_{ij}^\omega)_{i,j<\omega}$ and running their step (VI), we argue as follows. By Lemma 4.38 and compactness there is a quantifier-free $\mathcal{L}$-formula $\psi(x, c) \in \mathrm{qftp}(a/K)$ such that $\psi(\mathfrak{U}, c) \subseteq \varphi(\mathfrak{U}, a_{00})$. As $K = \bigcup_{\beta<\omega} a_{00}^\beta$ is an increasing union, we may assume $c = a_{00}^\beta$ for some $\beta < \omega$. Then $\psi(x, a_{00}^\beta)$ is a quantifier-free formula, and by indiscernibility of the rows and strong homogeneity of $\mathfrak{U}$ we have $\psi(\mathfrak{U}, a_{ij}^\beta) \subseteq \varphi(\mathfrak{U}, a_{ij}^\beta)$ for all $i, j < \omega$, preserving inconsistency, while $a \vDash \psi(x, a_{i0}^\beta)_{i<\omega}$ by indiscernibility of $(a_{i0}^\beta)_{i<\omega}$ over a, so $(a_{ij}^\beta)_{i,j<\omega}$ witnesses that ψ has TP_2. In particular, ψ has IP witnessed by some $(a_i)_{i<\omega}$ and $(b_w)_{w\subseteq\omega}$. But then, writing $\psi(x, y) = \rho(x, \sigma(x), \ldots, \sigma^n(x), y, \sigma(y), \ldots, \sigma^n(y))$ for some n and a formula ρ in the language of $\mathcal{L}_{s,\iota}$-valued fields, we get that ρ, $(a_i^\beta, \sigma(a_i^\beta), \ldots, \sigma^n(a_i^\beta))_{i<\omega}$ and $(b_w, \sigma(b_w), \ldots, \sigma^n(b_w))_{w\subseteq\omega}$ witness that the $\mathcal{L}_{s,\iota}$–valued algebraically closed field $\mathfrak{U}$ has IP. But algebraically closed $\mathcal{L}_{s,\iota}$-valued fields of residual characteristic zero are NIP, as can be shown, e.g. by using [Kes24, Theorem 2.2]. This is a contradiction. □

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