

Angular–Momentum–Resolved Aharonov–Bohm Coupling Energy

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The Aharonov–Bohm (AB) effect is conventionally interpreted as a phase shift acquired by charged particles encircling a flux, with no fields acting locally along their paths. Here we show that a confined Dirac electron exhibits a distinct AB *coupling energy* arising from a *local current–potential interaction*, whose form depends on the chosen prescription. In the *wave–particle* (WP) prescription the response is confined to the flux core: only the $l = 0$ mode leaves a finite remnant as the core shrinks, while all higher modes vanish. In the *wave–entity* (WE) prescription the $l = 0$ result coincides with WP, but for $l \geq 1$ the response becomes a quantized, l –linear energy shift. The AB effect thereby emerges as a quantized, mode–resolved energy law that establishes locality through standard field coupling and distinguishes between electron prescriptions.

INTRODUCTION

The Aharonov–Bohm (AB) effect [1–3] is usually presented as a phase shift acquired by charged particles encircling a confined magnetic flux, with no electric or magnetic fields acting locally along their trajectories, only a vector potential. This interpretation, confirmed in electron interferometry and mesoscopic rings [4–7], is often taken to suggest nonlocal quantum influences.

Flux dependence, however, also appears in the discrete eigenenergies of mesoscopic systems, where the AB phase modifies the quantization of confined electrons and produces measurable level shifts [8–10]. This energetic perspective complements the interference view, indicating that the AB effect can manifest not only as phase phenomena but also through explicit energy couplings, where the underlying current density becomes the natural physical carrier of the response.

In earlier work [11–13] we derived analytic Dirac eigenmodes in cylindrical cavities, exposing charge and current densities in closed form. These modes exhibit circulating currents even in the $l = 0$ channel set by spin, supplying the structure needed for energetic couplings. This observation motivates the present study: to examine how the same conserved current interacts locally with the vector potential and to assess whether the resulting energy laws retain or modify the conventional AB picture.

On this basis we adopt a *wave–entity* (WE) prescription, in which the electron is treated as a continuous, current–carrying field governed by the conserved four–current of the Dirac equation. In uniform fields it reproduces the familiar spin–1/2 Zeeman response with $g = 2$, while in structured fields such as the AB geometry it uncovers energetic features absent in the magnetization–field account. By contrasting the current–potential and magnetization–field couplings, we examine the AB response as a manifestation of local field interaction.

CONFINED DIRAC MODES

We consider confined Dirac eigenmodes in a cylindrical cavity, where the current structure is explicit and the Aharonov–Bohm (AB) coupling energy can be evaluated directly. The wavefunction satisfies

$$i\hbar \partial_t \Psi(\mathbf{r}, t) = \left[-i\hbar c \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \gamma^0 m_e c^2 + U(\mathbf{r}) \right] \Psi(\mathbf{r}, t), \quad (1)$$

with confinement potential

$$U(\mathbf{r}) = \begin{cases} 0, & 0 < \rho < R, \quad -d < z < d, \\ U, & \rho > R, \quad -d < z < d, \\ \infty, & |z| > d. \end{cases} \quad (2)$$

Stationary states take the form $\Psi(\mathbf{r}, t) = e^{-i\mathcal{E}t/\hbar} \psi(\rho, \phi, z)$ with angular factor $e^{il\phi}$ ($l = 0, 1, 2, \dots$) and axial wavenumbers $k_m = m\pi/(2d)$ (m odd). Radial parameters ζ_{nlm}, ξ_{nlm} are set by boundary matching at $\rho = R$, giving discrete eigenenergies $\mathcal{E}_{nlm}$. Standard Bessel-function identities and eigenmode expansions are employed [14].

The full spinor solutions are given in Ref. [13]. For present purposes, it suffices to note that they yield explicit charge and current densities,

$$\begin{aligned} q_{nlm}(\rho, z) &= -e N_{nlm}^2 \cos^2(k_m z) \\ &\quad \begin{cases} J_l^2(\zeta_{nlm}\rho), & \rho < R, \\ \kappa_{nlm}^2 K_l^2(\xi_{nlm}\rho), & \rho > R, \end{cases} \\ j_{nlm,\phi}(\rho, z) &= -2\mu_B N_{nlm}^2 \cos^2(k_m z) \\ &\quad \begin{cases} \zeta_{nlm} J_l(\zeta_{nlm}\rho) J_{l+1}(\zeta_{nlm}\rho), & \rho < R, \\ \kappa_{nlm}^2 \xi_{nlm} K_l(\xi_{nlm}\rho) K_{l+1}(\xi_{nlm}\rho), & \rho > R, \end{cases} \end{aligned} \quad (3)$$

with $j_{nlm,\rho} = j_{nlm,z} = 0$ and $\mu_B = e\hbar/(2m_e)$. The current vanishes on axis as $J_l J_{l+1} \sim (\zeta\rho)^{2l+1}$, identifying a vortex core even for $l = 0$, reinforcing spin as circulation of current [15].

Normalization follows from charge conservation,

$$N_{nlm}^2 = \frac{1}{\pi R^2 d} \left[\begin{array}{c} -J_{l-1}(\zeta_{nlm} R) J_{l+1}(\zeta_{nlm} R) \\ + \kappa_{nlm}^2 K_{l-1}(\xi_{nlm} R) K_{l+1}(\xi_{nlm} R) \end{array} \right]^{-1}. \quad (4)$$

using $\int q_{nlm}(\rho, z) \rho d\rho d\phi dz = -e$.

These eigenmodes make the conserved four-current fully explicit and provide the quantitative foundation for evaluating the AB coupling in both the WP and WE frameworks.

AB COUPLING IN THE WAVE-PARTICLE PICTURE

In the conventional quantum description, the electron is endowed with a point-like magnetic dipole weighted by the probability density. The interaction then takes the form of a Zeeman-like density $-\mathbf{M} \cdot \mathbf{B}$ [16] confined to the flux core. For a solenoid aligned with the cavity axis, the vector potential takes the form

$$\mathbf{A} = A_\phi(\rho) \hat{\phi}, \quad A_\phi(\rho) = \begin{cases} \frac{\Phi \rho}{2\pi a^2}, & \rho \leq a, \\ \frac{\Phi}{2\pi \rho}, & \rho > a, \end{cases} \quad (5)$$

the magnetic field follows as

$$\mathbf{B} = B_z(\rho) \hat{z}, \quad B_z(\rho) = \begin{cases} \frac{\Phi}{\pi a^2}, & \rho \leq a, \\ 0, & \rho > a, \end{cases} \quad (6)$$

where Φ is the enclosed flux and a regulates the solenoid core. In this configuration, the magnetic field is confined to the solenoid core, whereas the vector potential extends throughout the cavity, providing the field-free region central to the Aharonov–Bohm coupling.

For a spin-up state, the magnetization density is simply the Bohr magneton multiplied by the probability density,

$$M_{nlm,z}(\rho, z) = -\mu_B N_{nlm}^2 \cos^2(k_m z) \begin{cases} J_l^2(\zeta_{nlm} \rho), & \rho \leq R, \\ \kappa_{nlm}^2 K_l^2(\xi_{nlm} \rho), & \rho > R, \end{cases} \quad (7)$$

so that the coupling energy reads

$$\begin{aligned} \Omega_{WP} &= - \int \mathbf{M} \cdot \mathbf{B} d^3r \\ &= - \frac{\Phi}{\pi a^2} \int_0^a \int_0^{2\pi} \int_{-d}^d M_{nlm,z}(\rho, \phi, z) \rho dz d\phi d\rho. \end{aligned} \quad (8)$$

The integrations over z and ϕ yield factors of d and 2π , respectively, leaving a core-localized radial integral,

$$\Omega_{WP} = \Omega(R) \frac{2}{a^2} \int_0^a J_l^2(\zeta \rho) \rho d\rho, \quad (9)$$

where

$$\Omega(R) \equiv \mu_B \Phi d N_{nlm}^2. \quad (10)$$

sets the coupling scale. Using the normalization Eq. 4, this scale takes the explicit form

$$\Omega(R) = \frac{\mu_B \Phi}{\pi R^2} \left[\begin{array}{c} -J_{l-1}(\zeta_{nlm} R) J_{l+1}(\zeta_{nlm} R) \\ + \kappa_{nlm}^2 K_{l-1}(\xi_{nlm} R) K_{l+1}(\xi_{nlm} R) \end{array} \right]^{-1}. \quad (11)$$

The integral in Eq. 9 can be carried out in closed form using the Bessel identity

$$\frac{2}{x^2} \int_0^x J_l^2(t) t dt = J_l^2(x) - J_{l-1}(x) J_{l+1}(x),$$

so that

$$\Omega_{WP} = \Omega(R) C_l(a), \quad (12)$$

with

$$C_l(a) = J_l^2(\zeta_{nlm} a) - J_{l-1}(\zeta_{nlm} a) J_{l+1}(\zeta_{nlm} a). \quad (13)$$

Here $C_l(a)$ is a dimensionless function carrying the explicit dependence on the regulator radius a .

In the small-core limit $a \rightarrow 0$, the Bessel expansions $J_l(x) \sim (x/2)^l/l!$ for $l \geq 1$ and $J_0(x) \sim 1 - x^2/4$ show that $C_l(a) \rightarrow \delta_{l0}$. Thus,

$$\Omega_{WP} = \Omega(R) \delta_{l0}, \quad (14)$$

at fixed Φ and cavity radius R , so that the flux remains well defined even as the solenoid core shrinks.

Equation 14 makes clear that within WP only the $l = 0$ channel survives, while all $l \geq 1$ vanish algebraically as $(\zeta a)^{2l}$. The finite remnant at $l = 0$, though at first sight surprising, is physically lucid: the concentrated dipole density couples directly to the magnetic field confined in the infinitesimal solenoid core, yielding a nonzero interaction, whereas higher modes are radially suppressed near the axis. This lone energetic trace of the AB effect in WP may therefore be identified with the *spin part* of the coupling. In what follows we show that the WE framework not only reproduces this spin term, but does so entirely through the continuous current density—without invoking a separate spin–orbital decomposition—and further reveals quantized, l -linear increments that extend the AB response across higher angular channels.

AB COUPLING IN THE WAVE-ENTITY FRAMEWORK

The *wave-entity* (WE) framework treats the conserved current as the defining quantity of a physical entity. Accordingly, the Aharonov–Bohm (AB) interaction arises

from the local coupling $-\mathbf{j} \cdot \mathbf{A}$, making the physical response naturally distributed wherever the vector potential is finite, rather than confined to the solenoid core:

$$\begin{aligned}\Omega_{\text{WE}} &= -\int \mathbf{j} \cdot \mathbf{A} d^3r \\ &= -\int_0^\infty \int_0^{2\pi} \int_{-d}^d j_{nlm,\phi}(\rho, z) A_\phi(\rho) \rho d\rho d\phi dz.\end{aligned}\quad (15)$$

This prescription is gauge invariant. Under $\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi$ the variation is

$$\delta\Omega = -\int \mathbf{j} \cdot \nabla\chi d^3r = -\oint \chi \mathbf{j} \cdot d\mathbf{S} + \int \chi (\nabla \cdot \mathbf{j}) d^3r.$$

For stationary modes $\nabla \cdot \mathbf{j} = 0$. With hard walls and evanescent exterior, the surface term vanishes for any admissible gauge χ , leaving Ω_{WE} unchanged.

Substituting Eqs. 3 and 5 gives

$$\begin{aligned}\Omega_{\text{WE}} &= \Omega(R) \left\{ 2 \zeta_{nlm} \int_0^a J_l(\zeta_{nlm}\rho) J_{l+1}(\zeta_{nlm}\rho) \frac{\rho^2}{a^2} d\rho \right. \\ &\quad + 2 \zeta_{nlm} \int_a^R J_l(\zeta_{nlm}\rho) J_{l+1}(\zeta_{nlm}\rho) d\rho \\ &\quad \left. + 2 \kappa_{nlm}^2 \xi_{nlm} \int_R^\infty K_l(\xi_{nlm}\rho) K_{l+1}(\xi_{nlm}\rho) d\rho \right\}.\end{aligned}\quad (16)$$

Each of the three integrals in Eq. 16 can be reduced using Bessel recurrence relations. Evaluating at the boundaries $\rho = R$ yields

$$\Omega_{\text{WE}} = \Omega_{\text{WP}} + l\Omega(R) [C_l(a) + F_l(R)], \quad (17)$$

where

$$F_l(R) = 2 \left[\int_a^R \frac{J_l^2(\zeta_{nlm}\rho)}{\rho} d\rho + \kappa_{nlm}^2 \int_R^\infty \frac{K_l^2(\xi_{nlm}\rho)}{\rho} d\rho \right] \quad (18)$$

is dimensionless and explicitly dependent on R .

Equation 17 encapsulates the central result. The $l = 0$ channel reproduces the conventional WP contribution, where the spin interaction emerges directly from the current–potential coupling. The same current density, however, also gives rise to a quantized *Wave–Entropy* contribution for $l \geq 1$:

$$\Delta\Omega_{\text{WE}} = l\Omega(R) [C_l(a) + F_l(R)]. \quad (19)$$

This shift separates naturally into two parts with distinct physical meaning. The term $l\Omega(R) C_l(a) = l\Omega_{\text{WP}}$, using Eq. 12, is regulator dependent, reflecting how higher modes probe the solenoid core as multiples of the spin coupling. The term $l\Omega(R) F_l(R)$ is regulator independent and global, accounting for the centrifugal weight $\langle 1/\rho^2 \rangle$ of the extended wave. Together these contributions establish the Aharonov–Bohm energy shift as

a quantized, l –linear energetic law absent in the WP picture.

In the small-core limit $a \rightarrow 0$ one obtains

$$\Omega_{\text{WE}} = \Omega(R) \delta_{l0} + l\Omega(R) F_l(R), \quad (20)$$

to be directly compared with Eq. 14, exposing a regulator-independent sequence across higher modes.

For order-of-magnitude estimates of experimental scales, we evaluate the energy factor $\mu_B \Phi/(\pi R^2)$ in $\Omega(R)$. A cavity of radius $R = 100$ nm threaded by one flux quantum $\Phi_0 = h/e$ gives $\mu_B \Phi/(\pi R^2) \approx 7.6$ μeV , corresponding to 1.8 GHz. These shifts are within reach of modern flux-tuned cavity spectroscopy, comparable to persistent-current splittings in mesoscopic loops [9].

Thus the WE framework unifies spin and orbital responses within a single local interaction, yielding a quantized, mode-resolved energetic law that is both experimentally testable and free from nonlocal interpretation.

CONCLUSION

We have shown that the Aharonov–Bohm (AB) response of a confined Dirac electron admits two sharply distinct energetic outcomes, depending on how the coupling is prescribed. In the dipole–field (WP) picture, the interaction is confined to the solenoid core: as the core radius shrinks, only the $l=0$ channel retains a finite remnant corresponding to the spin coupling, while all higher modes vanish. In the current–potential (WE) framework, evaluated using the same eigenmodes, the local coupling $-\mathbf{j} \cdot \mathbf{A}$ yields an additional *quantized, l –linear* contribution composed of two parts. The regulator–dependent term manifests multiples of the spin–baseline unit, while the regulator–independent term, through the factor $F_l(R)$, expresses the centrifugal weight $\langle 1/\rho^2 \rangle$ of the extended Dirac wave. Together these establish a quantized, mode-resolved energetic law that distinguishes the WE framework from the WP prescription and renders the AB response a local field interaction rather than a confined dipole effect.

For comparison, the pioneering AB ring analyses [1, 17] first established that a confined magnetic flux quantizes the electronic spectrum, yielding the canonical form $E_l(\Phi) \propto (l - \Phi/\Phi_0)^2$ with both quadratic and linear l –dependence and an l –independent flux offset. That formulation, geometrically elegant yet simplified, describes an effectively one–dimensional ring that excludes the solenoid core. The flux then acts through the boundary condition, shifting the quantized momentum by Φ/Φ_0 and producing a global energy displacement without local field coupling. In contrast, the present analysis extends this framework by treating the confined Dirac electron as a spatially distributed current that couples directly and locally to the vector potential, with both spin and centrifugal contributions entering explicitly.

The predicted l -linear, regulator-independent shift can be detected through high-resolution cavity spectroscopy or flux-tuned level splitting. Its scale, $\mu_B \Phi / (\pi R^2)$, lies in the microwave regime for $R \sim 100$ nm, comparable to persistent-current oscillations observed in mesoscopic loops. Observation of this mode-resolved shift across higher l channels would provide direct evidence of the real, spatially extended current-potential coupling central to the wave-entity framework.

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