MATRIX-WEIGHTED BESOV SPACES ASSOCIATED WITH NON-ISOTROPIC DILATIONS

XIONG LIU AND WENHUA WANG

Abstract. Let $\alpha \in \mathbb{R}$, $p \in [1, \infty)$, $q \in (0, \infty]$, $\mathbf{W}$ be a matrix weight, and A be an expansive dilation on $\mathbb{R}^d$. In this paper, the authors firstly investigate and develop some aspects of homogeneous anisotropic Besov spaces $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ and inhomogeneous anisotropic Besov spaces $B^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ theory in the matrix weight setting. Moreover, we show that these spaces are characterized by the magnitude of the φ-transforms in appropriate sequence spaces. Notably, all these results remain novel even in the diagonal non-isotropic case (when $A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ with $\{\lambda_j\}_{j=1}^d \subset \mathbb{C}$).

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1. Introduction

1.1. **Background and motivation.** As is known to all, the theory of Besov and Triebel-Lizorkin spaces has been developed and plays an essential role in the fields of harmonic analysis and partial differential equations (see e.g. [14, 16, 17, 18, 22, 28, 30]). An important reason is that, such as Lebesgue, Hardy, Sobolev, Lipschitz spaces, etc. are some special cases of either Besov spaces

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or Triebel-Lizorkin spaces (see e.g. [29]). Moreover, these spaces can be characterized by their discrete analogues: the sequence Besov spaces.

In 1997, Volberg presented a different solution to the matrix weighted L^p boundedness of the Hilbert transform via techniques related to classical Littlewood-Paley theory (see e.g. [31]). Soon afterwards, in the matrix weight setting, the theory of function spaces is one of the most fundamental and hot topics in the fields of harmoic analysis and functional analysis. For examples, matrix A_p weighted Besov spaces were introduced by Roudenko [27] and Frazier and Roudenko [20]. For more information about matrix-weighted function spaces, we refer the reader to see [7, 8, 9, 10, 11, 12, 26].

On the other hand, extending classic function spaces arising in harmonic analysis of Euclidean spaces $\mathbb{R}^d$ to other domains and non-isotropic settings is a crucial and hot topic. For instance, in the 1970's, Calderón and Torchinsky [13, 15] studied the parabolic Hardy spaces and established their some real-variable characterizations. In 2003, Bownik [1] investigated the anisotropic Hardy space $H^p_A(\mathbb{R}^d)$ with $p \in (0, \infty)$ and A being a general expensive matrix on $\mathbb{R}^d$, which was a generalization of parabolic Hardy spaces. In 2005, Bownik [2] developed anisotropic version of Besov spaces. In 2007, Bownik [3] also studied anisotropic Triebel-Lizorkin spaces with doubling measures. For more information about anisotropic function spaces, we refer the reader to see [3, 5, 6, 23, 24, 25].

Inspired by the previous papers, the purpose of this paper is to extend some aspects of anisotropic function spaces theory, in particular, the study of Besov spaces, previously obtained with no weights and partially for scalar weights, to the matrix weight setting.

1.2. **Organization.** This article is organized as follows.

In Section 2, we first recall some notations and definitions concerning anosotropic dilations and and matrix A_p weight. Finally, we introduce the anisotropic Besov spaces in the convex of matrix weight.

The aim of Section 3 is to establish the main results and give their proof.

In Section 4, we will introduce the inhomogeneous anisotropic Besov spaces in the convex of matrix weight.

1.3. **Notation.** Finally, we make some conventions on notation. Let $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{Z}_+ := \{1, 2, \dots\}$. For any $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d_+ := (\mathbb{Z}_+)^d$, let $|\alpha| := \alpha_1 + \cdots + \alpha_d$ and

$$\partial^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}.$$

Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. For any $p \in [1, \infty]$, let p' denote the conjugate index of p, which satisfies that $\frac{1}{p} + \frac{1}{p'} = 1$. The *symbol* $D \lesssim F$ means that $D \le CF$. If $D \lesssim F$ and $F \lesssim D$, we then write $D \sim F$. Let $\mathrm{I}_{d\times d}$ denote the $d \times d$ unit matrix, and $\mathbf{0}$ the origin of $\mathbb{R}^d$. If the set $E \subset \mathbb{R}^d$, let $E^\complement := \mathbb{R}^d \setminus E$ and we denote $\mathbf{1}_E$ its *characteristic function*. Denote by $\mathcal{S}(\mathbb{R}^d)$ the *space of all Schwartz functions*.

2. Preliminaries

In this section, let us recall some basic definitions and properties of non-commutative L_p-spaces and anisotropic dilations, and introduce the omatrix-weighted anisotropic Besov spaces.

2.1. **Anisotropic dilations on $\mathbb{R}^d$.** Firstly, we recall the definition of expansive dilations on $\mathbb{R}^d$, which is introduced by Bownik (see [1, p. 5]). A real $d \times d$ matrix A is called an *expansive dilation* (shortly a *dilation*), if $\min_{\lambda \in \sigma(A)} |\lambda| > 1$, where $\sigma(A)$ denotes the set of all *eigenvalues* of A. Let λ_- and λ_+ be two *positive numbers* such that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+.$$

From [1, Lemma 2.2], we know that, for a fixed dilation A, there exist a number $r \in (1, \infty)$ and a set $\Delta := \{x \in \mathbb{R}^d : |Px| < 1\}$, where P is some non-degenerate $d \times d$ matrix, such that $\Delta \subset r\Delta \subset A\Delta$, and one can additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the d-dimensional Lebesgue measure of the set Δ. For $k \in \mathbb{Z}$, let $B_k := A^k \Delta$. Then B_k is open, and

$$B_k \subset rB_k \subset B_{k+1}, \quad |B_k| = |\det A|^k.$$

An ellipsoid $x + B_k$ for some $x \in \mathbb{R}^d$ and $k \in \mathbb{Z}$ is called a *dilated ball*. Define

$$\mathfrak{B} := \left\{x + B_k : x \in \mathbb{R}^d, k \in \mathbb{Z}\right\}. \tag{2.1}$$

Throughout the whole paper, let σ be the *smallest integer* such that $2B_0 \subset A^\sigma B_0$. Then, for all $k, j \in \mathbb{Z}$ with $k \leq j$, it holds true that

$$B_k + B_j \subset B_{j+\sigma}, \tag{2.2}$$

$$B_k + (B_{k+\sigma})^\complement \subset (B_k)^\complement, \tag{2.3}$$

where $E + F$ denotes the *algebraic sum* $\{x + y : x \in E, y \in F\}$ of sets $E, F \subset \mathbb{R}^d$.

Definition 2.1. *A quasi-norm, associated with dilation A, is a Borel measurable mapping ρ_A : $\mathbb{R}^n \to [0, \infty)$, for simplicity, denoted by ρ, satisfying*

(i) $\rho(x) > 0$ *for all* $x \in \mathbb{R}^d \setminus \{\mathbf{0}\}$*, here and hereafter,* $\mathbf{0}$ *denotes the origin of* $\mathbb{R}^d$*;*
(ii) $\rho(Ax) = |\det A|\rho(x)$ *for any* $x \in \mathbb{R}^d$*;*
(iii) $\rho(x + y) \leq C_A [\rho(x) + \rho(y)]$ *for all* $x, y \in \mathbb{R}^d$*, where* $C_A \in [1, \infty)$ *is a constant independent of x and y.*

Remark 2.2. (i) In the standard dyadic case $A := 2\mathrm{I}_{d\times d}$, $\rho(x) := |x|^d$ for all $x \in \mathbb{R}^d$ is an example of homogeneous quasi-norms associated with A, here and hereafter, $\mathrm{I}_{d\times d}$ denotes the $d \times d$ *unit matrix*, $|\cdot|$ always denotes the *Euclidean norm* in $\mathbb{R}^d$.

(ii) From [1, Lemma 3.2], we find that, there exists a positive constant C such that, for all $x \in \mathbb{R}^d$,

$$\begin{aligned} &\frac{1}{C}[\rho(x)]^{\zeta_-} \leq |x| \leq C[\rho(x)]^{\zeta_+}, \quad \text{if } \rho(x) \geq 1, \\ &\frac{1}{C}[\rho(x)]^{\zeta_+} \leq |x| \leq C[\rho(x)]^{\zeta_-}, \quad \text{if } \rho(x) < 1, \end{aligned} \tag{2.4}$$

where $\zeta_+ := \frac{\ln \lambda_+}{\ln |\det A|}$ and $\zeta_- := \frac{\ln \lambda_-}{\ln |\det A|}$.

(iii) Bownik has obtained that, all homogeneous quasi-norms associated with a given dilation A are equivalent, see [1, Lemma 2.4]. Therefore, in what follows, for a given dilation A, for simplicity, we always use the *step homogeneous quasi-norm* ρ defined by setting, for all $x \in \mathbb{R}^d$,

$$\rho(x) := \sum_{k\in\mathbb{Z}} |\det A|^k \mathbf{1}_{B_{k+1}\setminus B_k}(x) \text{ if } x \neq \mathbf{0}, \quad \text{or else} \quad \rho(\mathbf{0}) := 0.$$

By (2.2), we can deduce that, for all $x,\ y \in \mathbb{R}^d$,

$$\rho(x+y) \leq |\det A|^{\sigma} (\max\{\rho(x),\ \rho(y)\}) \leq |\det A|^{\sigma}[\rho(x)+\rho(y)].$$

2.2. **Matrix A_p weight.** In this subsection, we will recall the definitions of anisotropic matrix A_p weight.

Let $\mathcal{M}$ be the cone of nonnegative definite operators on a Hilbert space $\mathcal{H}$ of dimension n (consider $H = \mathbb{C}^n$ or $\mathbb{R}^n$), i.e., for $M \in \mathcal{M}$ we have $\langle Mx, x\rangle_H \geq 0$, for all $x \in \mathcal{H}$. Therefore, a matrix weight $\mathbf{W}$ is an a.e. invertible map $\mathbf{W} : \mathbb{R}^d \to \mathcal{M}$. Define the matrix-weighted Lebesgue space $L^p(\mathbb{R}^d, \mathbf{W})$ as all measurable vector-valued function $\vec{f} := (f_1, f_2, \ldots, f_m)^T : \mathbb{R}^d \to \mathcal{H}$ satisfy

$$\left\|\vec{f}\right\|_{L^p(\mathbb{R}^d,\mathbf{W})} := \left(\int_{\mathbb{R}^d} \left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{f}(x)\right\|_{\mathcal{H}} dx\right)^{\frac{1}{p}} < \infty.$$

Since the A_p condition can be formulated for any family of norms ω_t on a Hilbert space $\mathcal{H}$, we will recall the following norms:

$$\omega_t(y) = \left\|\mathbf{W}^{\frac{1}{p}}(x)y\right\|_{\mathcal{H}}, \text{ where } y \in \mathcal{H}, x \in \mathbb{R}^d.$$

Its dual metric is defined by

$$\omega_t^*(x) = \sup_{z\in\mathcal{H}, z\neq 0} \frac{\langle x, z\rangle_{\mathcal{H}}}{\omega_t(z)} = \left\|\mathbf{W}^{\frac{1}{p}}(x)y\right\|_{\mathcal{H}}, \text{ where } y \in \mathcal{H}, x \in \mathbb{R}^d.$$

Define the norm $\omega_{p,B}$ through the averagings of the metrics ω_t on the ball $B \in \mathfrak{B}$:

$$\omega_{p,B}(x) = \left[\fint_B (\omega_t(x))^p\ dt\right]^{\frac{1}{p}}.$$

Its dual metric is defined by

$$\omega_{p',B}^*(x) = \left[\fint_B (\omega_t^*(x))^{p'}\ dt\right]^{\frac{1}{p'}}.$$

The following is the definition of A_p matrix weight, which is from [**?**].

Definition 2.3. *Let $p \in (1, \infty)$. We say that $\mathbf{W}$ is an A_p matrix weight if $\mathbf{W} : \mathbb{R}^d \to \mathcal{M}$ is such that $\mathbf{W}$ and $\mathbf{W}^{-\frac{p'}{p}}$ are locally integrable and there exists $C < 0$ such that*

$$\omega_{p',B}^* \leq C(\omega_{p,B})^*.$$

Definition 2.4. *Let $p \in [1, \infty)$. We say that matrix weight $\mathbf{W}$ is a doubling matrix of order p, if there exists $C_{p,d}$ such that, for any $y \in \mathcal{H}$, and $B \in \mathfrak{B}$,*

$$\int_{2B} \left\|\mathbf{W}^{\frac{1}{p}}(x)y\right\|_{\mathcal{H}}^p dx \leq C_{p,d} \int_B \left\|\mathbf{W}^{\frac{1}{p}}(x)y\right\|_{\mathcal{H}}^p dx.$$

The following are some equivalent definitions of matrix A_p weight in the anisotropic setting, whose proof is similar to Roudenko's ones [27].

Lemma 2.5. *The following condition are equivalllent:*

(i) $\mathbf{W} \in A_p$;

(ii) $\mathbf{W}^{-\frac{p'}{p}} \in A_p$;

(iii) $\fint_B \left[\fint_B \left\| \mathbf{W}^{\frac{1}{p}}(x) \mathbf{W}^{-\frac{1}{p}}(y) \right\|^{p'} dy \right]^{\frac{p}{p'}} dx \leq C$, *for any* $B \in \mathfrak{B}$;

(iii) $\fint_B \left[\fint_B \left\| \mathbf{W}^{\frac{1}{p}}(x) \mathbf{W}^{-\frac{1}{p}}(y) \right\|^{p} dy \right]^{\frac{p'}{p}} dx \leq C$, *for any* $B \in \mathfrak{B}$,

where $\|\mathbf{W}^{\frac{1}{p}}(x)\mathbf{W}^{-\frac{1}{p}}(y)\|$ *denotes the matrix norm.*

2.3. **Matrix-weighted Besov spaces associated with anisotropic dilations.** In this subsection, we will introduce the definitions of anisotropic matrix-weighted Besov spaces.

A $C^\infty(\mathbb{R}^d)$ function φ is said to belong to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ if, for any $\ell \in \mathbb{Z}_+$ and multi-index α, $\|\varphi\|_{\alpha,\ell} := \sup_{x \in \mathbb{R}^d} [\rho(x)]^\ell |\partial^\alpha \varphi(x)| < \infty$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and satisfy

$$\operatorname{supp} \hat{\varphi} \subset [-\pi, \pi]^d \backslash \{\mathbf{0}\}$$

and

$$\operatorname{supp} \left| \hat{\varphi}((A^*)^{-k}\xi) \right| > 0, \quad \text{for any } \xi \in \mathbb{R}^d \backslash \{\mathbf{0}\}.$$

In what follows, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}^d$, let

$$\varphi_k(x) := |\det A|^{-k} \varphi(A^{-k}x).$$

Definition 2.6. *Let* $\alpha \in \mathbb{R}$, $p \in [1, \infty)$, $q \in (0, \infty]$ *and* $\mathbf{W}$ *be a matrix weight. The Besov space* $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ *is defined by, all vector-valued distributions* $\vec{f} := (f_1, f_2, \dots, f_m)^T$ *with* $f_i \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^d)$, $i = 1, \dots, m$, *such that*

$$\|f\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})} := \left(\sum_{k \in \mathbb{Z}} |\det A|^{k\alpha q} \left\| \vec{f} * \varphi_k \right\|^q_{L^p(\mathbf{W})} \right)^{\frac{1}{q}},$$

where $\vec{f} * \varphi_k := (f_1 * \varphi_k, f_2 * \varphi_k, \dots, f_m * \varphi_k)^T$.

Let $\varphi \in \mathcal{S}$ and satisfy

$$\operatorname{supp} \hat{\varphi}, \operatorname{supp} \hat{\psi} \subset [-\pi, \pi]^d \backslash \{\mathbf{0}\}$$

and

$$\sum_{k \in \mathbb{Z}} \overline{\hat{\varphi}((A^*)^{-k}\xi)} \hat{\psi}((A^*)^{-k}\xi) = 1, \quad \text{for any } \xi \in \mathbb{R}^d \backslash \{\mathbf{0}\}. \tag{2.5}$$

For any $\mathbf{j} \in \mathbb{Z}^d$, $k \in \mathbb{Z}$, define

$$Q_{\mathbf{j},k} = A^{-k}([0, 1]^d + \mathbf{j})$$

be the dilated cube, $x_{Q_{\mathbf{j},k}} = A^{-k}\mathbf{j}$.

$$\mathcal{Q} = \left\{Q_{\mathbf{j},k} : k \in \mathbb{Z}, \mathbf{j} \in \mathbb{Z}^d\right\}.$$

Define $\varphi_Q = |\det A|^{-\frac{k}{2}}\varphi(A^k x - \mathbf{j}) = |Q|^{\frac{1}{2}}\varphi(x - x_Q)$.

For any $\vec{f}$, $f_i \in \mathcal{S}'$, define φ-transform S_φ:

$$S_\varphi(\vec{f}) = \left\{\langle\vec{f}, \varphi_Q\rangle\right\}_Q = \left\{(\langle f_1, \varphi_Q\rangle, \langle f_2, \varphi_Q\rangle, \dots, \langle f_m, \varphi_Q\rangle)^T\right\}_Q.$$

Moreover, we call $\vec{s}_Q(\vec{f}) = \langle\vec{f}, \varphi_Q\rangle$ the φ-transform coefficients of $\vec{f}$.

Similarly to φ_Q, define

$$\psi_Q = |\det A|^{-\frac{k}{2}}\psi(A^k x - \mathbf{j}) = |Q|^{\frac{1}{2}}\psi(x - x_Q).$$

The inverse φ-transform T_ψ is the map taking a sequence $s = \{s_Q\}$ to $T_\psi s = \sum_{Q\in\mathcal{Q}} s_Q\psi_Q$. In the vector case,

$$T_\psi \vec{s} = \sum_{Q\in\mathcal{Q}} \vec{s}_Q\psi_Q,$$

where $\vec{s}_Q\psi_Q = (s_Q^1\psi_Q, s_Q^2\psi_Q, \dots, s_Q^m\psi_Q)^T$. The φ-transform decomposition show that, for any $f \in \mathcal{S}'\backslash\mathcal{P}(\mathbb{R}^d)$,

$$f = \sum_Q \langle f, \varphi_Q\rangle\psi_Q =: \sum_Q s_Q\psi_Q.$$

Definition 2.7. *Let $\alpha \in \mathbb{R}$, $p \in [1, \infty)$, $q \in (0, \infty]$ and $\mathbf{W}$ be a matrix weight. The Besov space $\dot{b}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ is defined by, all vector-valued sequence $\vec{s} := \{\vec{s}_Q\}_{Q\in\mathcal{Q}}$, where $\vec{s}_Q = (s_Q^1, s_Q^2, \dots, s_Q^m)^T$, $i = 1, \dots, m$, such that*

$$\|\vec{s}\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})} := \left(\sum_{k\in\mathbb{Z}} |\det A|^{k\alpha q} \left\|\sum_{Q\in\mathcal{Q}, |Q|=|\det A|^{-k}} |Q|^{-\frac{1}{2}}|\vec{s}_Q|\mathbf{1}_Q\right\|^q_{L^p(\mathbf{W})}\right)^{\frac{1}{q}}.$$

Remark 2.8. (i) In the isotropic setting, i.e., when $A = 2\mathrm{I}_{d\times d}$, these spaces reduce to the weighted-matrix Besov space $\dot{B}^{\alpha,q}_p(\mathbb{R}^d, \mathbf{W})$.

3. The main results and their proof

3.1. The boundedness of inverse φ-transform.

Definition 3.1. *Let $\alpha \in \mathbb{R}$, $p, q \in (0, \infty]$, $\delta \in (\alpha - \lfloor\alpha\rfloor, 1]$. Set $J = \frac{\beta}{p} + \max\{0, 1 - \frac{1}{p}\}$, $N = \max\{\lfloor\frac{J-\alpha-1}{\zeta_-}\rfloor, -1\}$ We say that $\mathfrak{M}_Q$ is a smooth (M, N, δ)-molecule associated with dyadic cube $Q \in \mathcal{Q}$ with $|Q| = |\det A|^{-k}, k \in \mathbb{Z}$ if there exists $M > J$ such that*

(i) $\left|\mathfrak{M}_Q(x)\right| \leq |\det A|^{\frac{k}{2}} \frac{1}{\left[1+\rho(A^k(x-x_Q))\right]^{\max\left\{M, \frac{(M-\alpha)\zeta_+}{\zeta_-}\right\}}}$;

(ii) $\int_{rr^d} x^\gamma\mathfrak{M}_Q(x)\,dx = 0$, *for* $|\gamma| \leq N$;

(iii) $\left|\partial^\gamma\mathfrak{M}_Q(A^{-k}\cdot)(x)\right| \leq |\det A|^{\frac{k}{2}} \frac{1}{[1+\rho(x-A^k x_Q)]^M}$, *for* $|\gamma| \leq \lfloor\frac{\alpha}{\zeta_-}\rfloor + 1$;

(iv) $\left|\partial^\gamma\mathfrak{M}_Q(A^{-k}\cdot)(x) - \partial^\gamma\mathfrak{M}_Q(A^{-k}\cdot)(x)\right| \leq |\det A|^{\frac{k}{2}-\gamma-\delta}\rho(x-y)^\delta \sup_{\rho(z)\leq\rho(x-y)} \frac{1}{\left[1+\rho(x-z-A^k x_Q)\right]^M}$, *if* $|\gamma| = \lfloor\frac{\alpha}{\zeta_-}\rfloor$.

The proof of the following lemma is similar to [27].

Lemma 3.2. *Let $\alpha \in \mathbb{R}$, $p \in [1,\infty)$, $q \in (0,\infty]$, and $\mathbf{W}$ be a doubling matrix weight of order p. Assume that $\{\mathfrak{M}_Q\}_Q$ is a family of smooth molecules for $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})$, then we have*

$$\left\|\sum_{Q\in\mathcal{Q}} \vec{s}_Q \mathfrak{M}_Q\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})} \le C \left\|\{\vec{s}_Q\}_{Q\in\mathcal{Q}}\right\|_{\dot{b}^{\alpha,q}_{p,A}}.$$

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, $p \in [1,\infty)$, $q \in (0,\infty]$, and $\mathbf{W}$ be a doubling matrix weight of order p. Then, for any sequence $\vec{s} = \{\vec{s}_Q\}_Q \in \dot{b}^{\alpha,q}_{p,A}(\mathbf{W})$, we have*

$$\left\|T_\psi \vec{s}\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})} = \left\|\sum_{Q\in\mathcal{Q}} \vec{s}_Q \psi_Q\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})} \le C \left\|\{\vec{s}_Q\}_{Q\in\mathcal{Q}}\right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})}.$$

Proof. Its proof can be deduced by the fact that $\{\psi_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth molecules for $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})$. □

3.2. **The boundedness of φ-transform.**

Definition 3.3. *For $k \in \mathbb{Z}$, define*

$$E_k := \left\{\vec{f} \in L^p(\mathbf{W}) : \operatorname{supp} \hat{f}_i \subseteq B_{k+1}, i = 1,2,\ldots,m\right\},$$

then we say that E_k consists of vector function of exponential type $|\det A|^{k+1}$.

Lemma 3.4. *Let $f \in \mathcal{S}'$, $g \in \mathcal{S}(\mathbb{R}^d)$, and $\operatorname{supp}\hat{f}$, $\operatorname{supp}\hat{g} \subseteq (A^*)^k[-\pi,\pi]^d$ with $k \in \mathbb{Z}$. Then we have*

$$f * g(x) = \sum_{\mathbf{j}\in\mathbb{Z}^d} |\det A|^{-k} f(A^{-k}\mathbf{j}) g(x - A^{-k}\mathbf{j}).$$

The following is maximal operator type inequality.

Lemma 3.5. *Let $p \in (1,\infty)$, $\mathbf{W} \in A_p$ and $\vec{f} \in E_0$. Then we have*

$$\sum_{\mathbf{j}\in\mathbb{Z}^d} \int_{Q_{\mathbf{j},0}} \left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{f}(\mathbf{j})\right\|^p dx \le C_{p,d} \left\|\vec{f}\right\|_{L^p(\mathbf{W})}.$$

Proof. Choose a smooth function $\phi \in \mathcal{S}$ satisfying: if $\xi \in B_1$, $\hat{\phi} = 1$, and $\operatorname{supp}\hat{\phi} \subseteq [-\pi,\pi]^d$. Then for any $\vec{f} \in E_0$, we have $\vec{f} = \vec{f} * \phi$. Therefore,

$$\begin{aligned}\sum_{\mathbf{j}\in\mathbb{Z}^d} \int_{Q_{\mathbf{j},0}} \left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{f}(\mathbf{j})\right\|^p dx &= \sum_{\mathbf{j}\in\mathbb{Z}^d} \int_{Q_{\mathbf{j},0}} \left\|\mathbf{W}^{\frac{1}{p}}(x) \int_{\mathbb{R}^d} \vec{f}(y)\phi(\mathbf{j}-y)\,dy\right\|^p dx \\ &\lesssim \sum_{\mathbf{j}\in\mathbb{Z}^d} \int_{Q_{\mathbf{j},0}} \left[\int_{\mathbb{R}^d} \left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{f}(y)\frac{1}{[\rho(1+(\mathbf{j}-y))]^N}\right\| dy\right]^p dx,\end{aligned}$$

where $N > d + \frac{\beta p}{p'}$, β is the doubling exponent of matrix weight $\mathbf{W}$.

By the discrete Hölder inequality, we obtain

$$\sum_{\mathbf{j}\in\mathbb{Z}^d} \int_{Q_{\mathbf{j},0}} \left[\int_{\mathbb{R}^d} \left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{f}(y)\frac{1}{[\rho(1+(\mathbf{j}-y))]^N}\right\| dy\right]^p dx$$

$$\lesssim \sum_{\mathbf{j}\in\mathbb{Z}^d}\int_{Q_{\mathbf{j},0}}\left[\sum_{\mathbf{j}'\in\mathbb{Z}^d}\int_{Q_{\mathbf{j}',0}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{f}(y)\frac{1}{[\rho(1+(\mathbf{j}-y))]^N}\right\|dy\right]^p dx$$

$$\lesssim \sum_{\mathbf{j}\in\mathbb{Z}^d}\int_{Q_{\mathbf{j},0}}\sum_{\mathbf{j}'\in\mathbb{Z}^d}\left[\int_{Q_{\mathbf{j}',0}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{f}(y)\right\|dy\right]^p\frac{1}{[\rho(1+(\mathbf{j}-y))]^N}\cdot\left(\sum_{\mathbf{j}'\in\mathbb{Z}^d}\frac{1}{[\rho(1+(\mathbf{j}-y))]^N}\right)^{\frac{p}{p'}}dx$$

$$\lesssim \sum_{\mathbf{j}\in\mathbb{Z}^d}\int_{Q_{\mathbf{j},0}}\sum_{\mathbf{j}'\in\mathbb{Z}^d}\left[\int_{Q_{\mathbf{j}',0}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{f}(y)\right\|dy\right]^p\frac{1}{[\rho(1+(\mathbf{j}-y))]^N}dx.$$

It follows from the Hölder inequality that

$$\begin{aligned}\left[\int_{Q_{\mathbf{j}',0}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{f}(y)\right\|dy\right]^p &\le \left[\int_{Q_{\mathbf{j}',0}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\mathbf{W}^{-\frac{1}{p}}(y)\right\|\left\|\mathbf{W}^{\frac{1}{p}}(y)\vec{f}(y)\right\|dy\right]^p\\ &\le \left[\int_{Q_{\mathbf{j}',0}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\mathbf{W}^{-\frac{1}{p}}(y)\right\|^{p'}dy\right]^{\frac{p}{p'}}\cdot\int_{Q_{\mathbf{j}',0}}\left\|\mathbf{W}^{\frac{1}{p}}(y)\vec{f}(y)\right\|^p dy.\end{aligned}$$

Now we can prove that

$$\sum_{\mathbf{j}\in\mathbb{Z}^d}\int_{Q_{\mathbf{j},0}}\sum_{\mathbf{j}'\in\mathbb{Z}^d}\left[\int_{Q_{\mathbf{j}',0}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\mathbf{W}^{-\frac{1}{p}}(y)\right\|^{p'}dy\right]^{\frac{p}{p'}}\frac{1}{[\rho(1+(\mathbf{j}-y))]^N}dx\lesssim 1.$$

Therefore, we have

$$\begin{aligned}\sum_{\mathbf{j}\in\mathbb{Z}^d}\int_{Q_{\mathbf{j},0}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{f}(\mathbf{j})\right\|^p dx &\lesssim \sum_{\mathbf{j}'\in\mathbb{Z}^d}\int_{Q_{\mathbf{j}',0}}\left\|\mathbf{W}^{\frac{1}{p}}(y)\vec{f}(y)\right\|^p dy\\ &\sim \int_{\mathbb{R}^d}\left\|\mathbf{W}^{\frac{1}{p}}(y)\vec{f}(y)\right\|^p dy\\ &\sim \left\|\vec{f}\right\|^p_{L^p(\mathbf{W})}.\end{aligned}$$

□

Lemma 3.6. *Let* $\mathbf{W}$ *be a doubling matrix of order* p *with doubling exponent* $\beta < p$*, and* $\vec{f}\in E_0$*. Then we have*

$$\sum_{\mathbf{j}\in\mathbb{Z}^d}\int_{Q_{\mathbf{j},0}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{f}(\mathbf{j})\right\|^p dx\le C_{p,d}\left\|\vec{f}\right\|_{L^p(\mathbf{W})}.$$

Theorem 3.2. *Let* $\alpha\in\mathbb{R}$*,* $q\in(0,\infty]$*,* $p\in[1,\infty)$*,* $\mathbf{W}\in A_p$*. Then for any* $f\in\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})$*, we have*

$$\left\|\{\vec{s}_Q\}_Q\right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})}\lesssim\left\|\vec{f}\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})},$$

where $\vec{s}_Q=S_\varphi(\vec{f})=\langle\vec{f},\varphi_Q\rangle=(\langle f_1,\varphi_Q\rangle,\langle f_2,\varphi_Q\rangle,\dots,\langle f_m,\varphi_Q\rangle)^T$.

Proof. By the definition, we have

$$\left\|\{\vec{s}_Q\}_Q\right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})}=\left(\sum_{k\in\mathbb{Z}}|\det A|^{k\alpha q}\left\|\sum_{Q\in\mathcal{Q},|Q|=|\det A|^{-k}}|Q|^{-\frac{1}{2}}|\vec{s}_Q|\mathbf{1}_Q\right\|^q_{L^p(\mathbf{W})}\right)^{\frac{1}{q}}$$

$$=\left(\sum_{k\in\mathbb{Z}}|\det A|^{k\alpha q}\left\|\sum_{Q\in\mathcal{Q},|Q|=|\det A|^{-k}}|Q|^{-\frac{1}{2}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{s}_Q\right\|\mathbf{1}_Q(x)\right\|_{L^p(dx)}^q\right)^{\frac{1}{q}}.$$

Fix $k\in\mathbb{Z}$, let $Q_{\mathbf{j},k}=A^{-k}([0,1]^d+\mathbf{j})$, $\mathbf{j}\in\mathbb{Z}^d$. Then we have

$$\vec{s}_Q=\left\langle\vec{f},\varphi_Q\right\rangle=|Q|^{\frac{1}{2}}\left(\vec{f}*\widetilde{\varphi}_k\right)(A^{-k}\mathbf{j})$$

and

$$\left\|\sum_{Q\in\mathcal{Q},|Q|\|\det A|^{-k}}|Q|^{-\frac{p}{2}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{s}_Q\right\|\mathbf{1}_Q(x)\right\|_{L^p(dx)}^p=\sum_{Q\in\mathcal{Q},|Q|=|\det A|^{-k}}|Q|^{-\frac{p}{2}}\int_Q\left\|\mathbf{W}^{\frac{1}{p}}(x)|Q|^{\frac{1}{2}}(\vec{f}*\widetilde{\varphi}_k)(A^{-k}\mathbf{j})\right\|^p dx$$

$$=\sum_{\mathbf{j}\in\mathbb{Z}^d}\int_{Q_{\mathbf{j},k}}\left\|\mathbf{W}^{\frac{1}{p}}(x)(\vec{f}*\widetilde{\varphi}_k)(A^{-k}\mathbf{j})\right\|^p dx.$$

Set $\vec{f}_k(x):=\vec{f}(A^{-k}x)$ and $\mathbf{W}_k(x):=\mathbf{W}(A^kx)$, then $\vec{f}*\widetilde{\varphi}_k(A^{-k}\mathbf{j})=\vec{f}_k*\widetilde{\varphi}(\mathbf{j})$. Therefore,

$$\begin{aligned}\left\|\sum_{Q\in\mathcal{Q},|Q|=|\det A|^{-k}}|Q|^{-\frac{p}{2}}\left\|\mathbf{W}^{\frac{1}{p}}(x)\vec{s}_Q\right\|\mathbf{1}_Q(x)\right\|_{L^p(dx)}^p&=|\det A|^{-k}\sum_{\mathbf{j}\in\mathbb{Z}^d}\int_{Q_{\mathbf{j},0}}\left\|\mathbf{W}_k^{\frac{1}{p}}(x)(\vec{f}_k*\widetilde{\varphi})(\mathbf{j})\right\|^p dx\\&\lesssim|\det A|^{-k}\int_{\mathbb{R}^d}\left\|\mathbf{W}^{\frac{1}{p}}(x)(\vec{f}*\widetilde{\varphi}_k)(x)\right\|^p dx\\&\sim\int_{\mathbb{R}^d}\left\|\mathbf{W}_k^{\frac{1}{p}}(x)(\vec{f}_k*\widetilde{\varphi}_k)(x)\right\|^p dx\\&\sim\left\|\vec{f}_k*\widetilde{\varphi}_k\right\|_{L^p(\mathbf{W})}^p.\end{aligned}$$

From the above estimates, we conclude that

$$\begin{aligned}\left\|\{\vec{s}_Q\}_Q\right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})}&\lesssim\left(\sum_{k\in\mathbb{Z}}|\det A|^{k\alpha q}\left\|\vec{f}_k*\widetilde{\varphi}_k\right\|_{L^p(\mathbf{W})}^q\right)^{\frac{1}{q}}\\&\sim\|f\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})}.\end{aligned}$$

Finally, we only need to show the equivalence of $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W},\varphi)=\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W},\widetilde{\varphi})$. It is easy to see that $\hat{\widetilde{\varphi}}=\overline{\hat{\varphi}}$ and $\hat{\widetilde{\psi}}=\overline{\hat{\psi}}$. Therefore, we know that $(\widetilde{\varphi},\widetilde{\psi})$ satisfies (4.6). From the decomposition of

$$\vec{f}=\sum_{Q\in\mathcal{Q}}\left\langle\vec{f},\widetilde{\varphi}_Q\right\rangle\widetilde{\psi}_Q,$$

the fact that $\{\widetilde{\psi}_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth molecules for $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})$, and Lemma 3.2, we conclude that

$$\left\|\vec{f}\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W},\widetilde{\varphi})}\lesssim\left\|\left\{\left\langle\vec{f},\widetilde{\varphi}_Q\right\rangle\right\}_{Q\in\mathcal{Q}}\right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})}.$$

By this and the above estimate, we obtain

$$\left\|\vec{f}\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W},\widetilde{\varphi})}\lesssim\left\|\vec{f}\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W},\widetilde{\widetilde{\varphi}})}\sim\left\|\vec{f}\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W},\varphi)}.$$

Therefore, we have

$$\left\|\left\{\left\langle\vec{f},\widetilde{\varphi}_Q\right\rangle\right\}_{Q\in\mathcal{Q}}\right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})}=\left\|\{\vec{s}_Q\}_{Q\in\mathcal{Q}}\right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})}\lesssim\left\|\vec{f}\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W},\varphi)}.$$

This completes our proof.

□

Remark 3.7. Notice that, if (φ', ψ'), (φ'', ψ'') satisfy the condition (4.6), $\alpha \in \mathbb{R}$, $q \in (0, \infty]$, $p \in [1, \infty)$ and $\mathbf{W} \in A_p$, then

$$\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W}, \varphi') = \dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W}, \varphi'').$$

Indeed, Let (φ', ψ'), (φ'', ψ'') satisfy (4.6). We can decompose $\vec{f}$ as the following:

$$\vec{f} = \sum_{Q \in \mathcal{Q}} \left\langle \vec{f}, \varphi''_Q \right\rangle \psi''_Q = \sum_{Q \in \mathcal{Q}} \vec{s}''_Q \psi''_Q.$$

From the fact that, for any $Q \in \mathcal{Q}$, ψ''_Q is a molecule, Lemma 3.2 and Theorem 3.2, we conclude that

$$\left\| \vec{f} \right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W}, \varphi')} \lesssim \left\| \left\{ \vec{s}''_Q \right\}_{Q \in \mathcal{Q}} \right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})} \lesssim \left\| \vec{f} \right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W}, \varphi'')}.$$

Reversing the role of φ' and φ'', we obtain the equivalence of $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W}, \varphi')$ and $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W}, \varphi'')$. Therefore, we know that the definition of $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ is independent of the choice of the test function φ.

3.3. **The relationship with reducing operators.**

In this subsection, we will establish the relation of Besov spaces and reducing operators.

Recall that, for any matrix weight $\mathbf{W}$, we know that there exist a sequence of reducing operators $\{\mathcal{A}_Q\}_{Q \in \mathcal{Q}}$ such that, for any $\vec{u} \in \mathcal{H}$,

$$\omega_{p,Q} = \left[\fint_Q \left\| \mathbf{W}^{\frac{1}{p}}(x) \vec{u} \right\|_{\mathcal{H}} \mathbf{1}_Q(x)\, dx \right]^{\frac{1}{p}}.$$

Lemma 3.8. *Let $\{\mathcal{A}_Q\}_{Q \in \mathcal{Q}}$ be the reducing operators for $\mathbf{W}$, $\alpha \in \mathbb{R}$, $q \in (0, \infty]$, $p \in [1, \infty)$. Then we have*

$$\left\| \{\vec{s}_Q\}_Q \right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})} = \left\| \{\vec{s}_Q\}_Q \right\|_{\dot{b}^{\alpha,q}_{p,A}(\{\mathcal{A}_Q\})}.$$

Proof. By the definition, we have

$$\begin{aligned}
\left\| \{\vec{s}_Q\}_Q \right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})} &= \left[\sum_{k \in \mathbb{Z}} |\det A|^{kq\alpha} \left\| \sum_{Q \in \mathcal{Q}, |Q| = |\det A|^{-k}} |Q|^{-\frac{1}{2}} \left\| \mathbf{W}^{\frac{1}{p}}(x) \vec{s}_Q \right\| \mathbf{1}_Q(x) \right\|^q_{L^p(dx)} \right]^{\frac{1}{q}} \\
&= \left[\sum_{k \in \mathbb{Z}} |\det A|^{kq\alpha} \left[\sum_{Q \in \mathcal{Q}, |Q| = |\det A|^{-k}} |Q|^{1-\frac{p}{2}} \left(\omega_{p,Q}(\vec{s}_Q) \right)^p \right]^{\frac{q}{p}} \right]^{1q} \\
&\sim \left[\sum_{k \in \mathbb{Z}} |\det A|^{kq\alpha} \left(\sum_{Q \in \mathcal{Q}, |Q| = |\det A|^{-k}} |Q|^{-\frac{p}{2}} \left\| \mathcal{A}_Q \vec{s}_Q \right\|^p_{\mathcal{H}} \int_Q \mathbf{1}(x)\, dx \right)^{\frac{q}{p}} \right]^{1q} \\
&\sim \left[\sum_{k \in \mathbb{Z}} |\det A|^{kq\alpha} \left\| \sum_{Q \in \mathcal{Q}, |Q| = |\det A|^{-k}} |Q|^{-\frac{1}{2}} \left\| \mathcal{A}_Q \vec{s}_Q \right\| \mathbf{1}_Q(x) \right\|^q_{L^p(dx)} \right]^{\frac{1}{q}}
\end{aligned}$$

$$\sim \left\| \{\vec{s}_Q\}_Q \right\|_{\dot{b}^{\alpha,q}_{p,A}(\{\mathcal{A}_Q\})}.$$

□

Theorem 3.3. *Let $\alpha \in \mathbb{R}$, $q \in (0,\infty]$, $p \in [1,\infty)$ and $\mathbf{W} \in A_p$. Then the Besov space $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ is complete.*

Proof. Let $\{\vec{f}_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$. From the above lemma, we know that $\{\{\vec{s}_Q(\vec{f}_i)\}_Q\}_{i\in\mathbb{N}}$ be a Cauchy sequence in $\dot{b}^{\alpha,q}_{p,A}(\{\mathcal{A}_Q\})$. Therefore, we have

$$\begin{aligned}
&\left\| \sum_{Q\in\mathcal{Q}, |Q|=|\det A|^{-k}} |Q|^{-\frac{1}{2}} \left\| \mathcal{A}_Q\left(\vec{s}_Q(\vec{f}_i) - \vec{s}_Q(\vec{f}_l)\right)\right\|_{\mathcal{H}} \mathbf{1}_Q(x) \right\|^p_{L^p(dx)} \\
&= |\det A|^{(\frac{p}{2}-1)k} \sum_{Q\in\mathcal{Q}, |Q|=|\det A|^{-k}} \left\| \mathcal{A}_Q\left(\vec{s}_Q(\vec{f}_i) - \vec{s}_Q(\vec{f}_l)\right)\right\|^p_{\mathcal{H}} \\
&\to 0, \quad \text{as } i, l \to \infty,
\end{aligned}$$

which implies that, for any $Q \in \mathcal{Q}$,

$$\left\| \mathcal{A}_Q\left(\vec{s}_Q(\vec{f}_i) - \vec{s}_Q(\vec{f}_l)\right)\right\|^p_{\mathcal{H}} \to 0, \quad \text{as } i, l \to \infty.$$

By the fact that $\mathcal{A}_Q$ is invertible, we obtain that, for any $Q \in \mathcal{Q}$, $\{\vec{s}_Q(\vec{f}_i)\}_{i\in\mathbb{N}}$ is a vector-valued Cauchy sequence in $\mathcal{H}$. Furthermore, we define $\vec{s}_Q = \lim_{i\to\infty} \vec{s}_Q(\vec{f}_i)$. Let

$$\vec{f} = \sum_{q\in\mathcal{Q}} \vec{s}_Q \psi_Q.$$

Then we have

$$\begin{aligned}
\left\| \vec{f}_i - \vec{f} \right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})} &= \left\| \sum_{Q\in\mathcal{Q}} \left(\vec{s}_Q(\vec{f}_i) - \vec{s}_Q\right)\psi_Q \right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})} \\
&\lesssim \left\| \left\{ \vec{s}_Q(\vec{f}_i) - \vec{s}_Q \right\}_{Q\in\mathcal{Q}} \right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})} \\
&\lesssim \liminf_{l\to\infty} \left\| \left\{ \vec{s}_Q(\vec{f}_i) - \vec{s}_Q(\vec{f}_l) \right\}_{Q\in\mathcal{Q}} \right\|_{\dot{b}^{\alpha,q}_{p,A}(\{\mathcal{A}_Q\})} \to 0, \quad \text{as } l \to \infty.
\end{aligned}$$

From this, the Fatou Lemma and the fact that $\{\{\vec{s}_Q(\vec{f}_i)\}_{Q\in\mathcal{Q}}\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\dot{b}^{\alpha,q}_{p,A}(\{\mathcal{A}_Q\})$, we deduce that

$$\vec{f} = (\vec{f} + \vec{f}_i) - \vec{f}_i \in \dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W}).$$

This implies that the Besov space $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ is complete.

□

Corollary 3.9. *Let $\alpha \in \mathbb{R}$, $q \in (0,\infty]$, $p \in (1,\infty)$, and $\mathbf{W}$ be a matrix weight with corresponding reducing operators $\mathcal{A}_Q$ and $\widetilde{\mathcal{A}}_Q$. Then we have*

(i) $\dot{b}^{\alpha,q}_{p,A}(\{\mathcal{A}_Q\}) \subseteq \dot{b}^{\alpha,q}_{p,A}(\{(\widetilde{\mathcal{A}}_Q)^{-1}\})$;

(ii) *If $\omega \in A_p$, $\dot{b}^{\alpha,q}_{p,A}(\{(\widetilde{\mathcal{A}}_Q)^{-1}\}) \subseteq \dot{b}^{\alpha,q}_{p,A}(\{\mathcal{A}_Q\})$.*

4. Inhomogeneous matrix-weighted Besov spaces

In this section, we will investgate the inhomogeneous Besov spaces. Before we define the vector-valued inhomogeneous Besov space $B^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ with matrix weight $\mathbf{W}$, and briefly give some basic properties of $B^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$. Most of them can be deduced by a straightforward modification of the homogeneous results, therefore, we only outline require changes.

Let $\Phi, \varphi \in \mathcal{S}(\mathbb{R}^d)$ and satisfy

$$\operatorname{supp} \hat{\Phi} \subset [-\pi, \pi]^d$$

and

$$\sup_{k\geq 1} \left|\hat{\varphi}((A^*)^{-k}\xi)\right| > 0, \ |\hat{\Phi}(\xi)| > 0 \ \text{ for any } \xi \in \mathbb{R}^d.$$

In what follows, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $k \in \mathbb{Z}$ with $k > 0$, and $x \in \mathbb{R}^d$, let

$$\varphi_k(x) := |\det A|^{-k} \varphi(A^{-k}x).$$

Definition 4.1. *Let $\alpha \in \mathbb{R}$, $p, q \in (0, \infty]$. We define the matrix-weighted anisotropic inhomogeneous Besov space $B^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ as the set of all $\vec{f} \in \mathcal{S}'$ such that*

$$\left\|\vec{f}\right\|_{B^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})} := \left\|\vec{f} * \Phi\right\|_{L^p(\mathbf{W})} + \left(\sum_{k=1}^{\infty} |\det A|^{\alpha q k} \left\|\vec{f} * \varphi_k\right\|^q_{L^p(\mathbf{W})}\right)^{\frac{1}{q}} < \infty.$$

The corresponding inhomogeneous weighted sequence Besov space $b^{\alpha,q}_{p,A}(\mathbf{W})$ is defined for the vector sequences enumerated by the dyadic cubes Q with $|Q| \leq 1$.

Definition 4.2. *Let $\alpha \in \mathbb{R}$, $p, q \in (0, \infty]$ and $\mathbf{W}$ be a matrix weight. We define the matrix-weighted sequence anisotropic inhomogeneous Besov space $b^{\alpha,q}_{p,A}(\mathbf{W})$ as the set of all all vector-valued sequences $\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{Q}, l(Q)\leq 1}$ such that*

$$\left\|\vec{f}\right\|_{b^{\alpha,q}_{p,A}(\mathbf{W})} := \left(\sum_{k=1}^{\infty} |\det A|^{\alpha q k} \left\|\sum_{|Q|=|\det A|^{-k}} |Q|^{-\frac{1}{2}} \vec{s}_Q \mathbf{1}_Q\right\|^q_{L^p(\mathbf{W})}\right)^{\frac{1}{q}} < \infty.$$

When $q = \infty$, the ℓ^q-norm can be replaced by the supremum on $k \geq 0$.

Next we also introduce the definitions of φ-transform S_φ and the inverse φ-transform T_φ in the inhomogeneous setting.

Let $\Phi, \Psi \in \mathcal{S}$, $\varphi, \psi \in \mathcal{S}$ and satisfy

$$\operatorname{supp} \hat{\varphi}, \operatorname{supp} \hat{\psi} \subset [-\pi, \pi]^d \backslash \{0\}$$

and

$$\overline{\hat{\Phi}}(\xi)\hat{\Psi}(\xi) + \sum_{k=1}^{\infty} \overline{\hat{\varphi}((A^*)^{-k}\xi)}\hat{\psi}((A^*)^{-k}\xi) = 1, \quad \text{for any } \xi \in \mathbb{R}^d. \tag{4.6}$$

For any $\mathbf{j} \in \mathbb{Z}^d$, $k \in \mathbb{Z}$ with $k > 0$, define

$$Q_{\mathbf{j},k} = A^{-k}([0, 1]^d + \mathbf{j})$$

be the dilated cube, $x_{Q_{\mathbf{j},k}} = A^{-k}\mathbf{j}$.

$$\mathcal{Q} = \left\{Q_{\mathbf{j},k} : k \in \mathbb{Z}, \mathbf{j} \in \mathbb{Z}^d\right\}.$$

Define

$$\Phi_Q = |\det A|^{-\frac{k}{2}}\Phi(A^k x - \mathbf{j}) = |Q|^{\frac{1}{2}}\Phi(x - x_Q)$$

and

$$\varphi_Q = |\det A|^{-\frac{k}{2}}\varphi(A^k x - \mathbf{j}) = |Q|^{\frac{1}{2}}\varphi(x - x_Q).$$

Definition 4.3. *Let $\Phi, \Psi \in \mathcal{S}$, and $\varphi, \psi \in \mathcal{S}$ satisfy (4.6). For any vector-valued distribute $\vec{f}$, $f_i \in \mathcal{S}'$, define the inhomogeneous φ-transform $S_\varphi = S_{\Phi,\varphi}$: $S_\varphi(\vec{f}) := \{S_\varphi(\vec{f})\}_Q$, if $k > 0$,*

$$\{S_\varphi(\vec{f})\}_Q = \left\{\langle \vec{f}, \varphi_Q\rangle\right\}_Q = \left\{(\langle f_1, \varphi_Q\rangle, \langle f_2, \varphi_Q\rangle, \dots, \langle f_m, \varphi_Q\rangle)^T\right\}_Q;$$

if $k = 0$,

$$\{S_\varphi(\vec{f})\}_Q = \left\{\langle \vec{f}, \Phi_Q\rangle\right\}_Q = \left\{(\langle f_1, \Phi_Q\rangle, \langle f_2, \Phi_Q\rangle, \dots, \langle f_m, \Phi_Q\rangle)^T\right\}_Q;$$

For any vector-valued sequence $\vec{s} = \{s_Q\}_{Q,|Q|\le 0}$, define the inhomogeneous inverse φ-transform $T_\psi = T_{\Psi,\psi}$:

$$T_\psi(\vec{s}) := \sum_{Q,|Q|=1} \vec{s}_Q \Psi_Q + \sum_{Q\in\mathcal{Q},|Q|<1} \vec{s}_Q \Psi_Q,$$

Given $\Phi, \varphi \in \mathcal{S}$ satisfying (4.6), we can show that there exist $\Psi, \psi \in \mathcal{S}$ such that

$$\overline{\hat{\Phi}(\xi)}\hat{\Psi}(\xi) + \sum_{k=1}^{\infty} \overline{\hat{\varphi}((A^*)^{-k}\xi)}\hat{\psi}((A^*)^{-k}\xi) = 1, \quad \text{for any } \xi \in \mathbb{R}^d. \tag{4.7}$$

Moreover, we have the following identity for vector-valued distribution $\vec{f}$, $f_i \in \mathcal{S}'$,

$$\vec{f} = \sum_{Q\in\mathcal{Q},|Q|=1} \left\langle \vec{f}, \Phi_Q\right\rangle \Psi_Q + \sum_{k=1}^{\infty} \sum_{Q\in\mathcal{Q},|Q|=|\det A|^{-k}} \left\langle \vec{f}, \varphi_Q\right\rangle \Psi_Q,$$

with convergence in $\mathcal{S}'$.

We can then show that main results of φ-transform adapt directly to inhomogeneous setting. That is, S_φ is a bounded operator from $B^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ to $b^{\alpha,q}_{p,A}(\mathbf{W})$ and T_ψ is a bounded operator from $b^{\alpha,q}_{p,A}(\mathbf{W})$ to $B^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$.

5. Almost diagonal operators

In this section, we study a class of almost diagonal operators on Besov spaces, which was introduced in the dyadic case by Frazier et al. [19]. The interest of these operators comes from their close connection to operators on Besov spaces.

Definition 5.1. *Let $\alpha \in \mathbb{R}$, $p, q \in (0, \infty]$. Suppose that $J = \beta/p + \max\{0, 1 - 1/p\}$. We say that a matrix $\{a_{Q,P}\}_{Q,P\in\mathcal{Q}}$ is almost diagonal, $A \in \mathbf{ad}^{\alpha,q}_{p,A}(\beta)$, if there exists an $c > 0$ such that,*

$$\sup_{Q,P\in\mathcal{Q}} \left|a_{Q,P}\right| < c\left(\frac{|Q|}{|P|}\right)^{\alpha}\left[1 + \frac{\rho_A(x_Q - x_P)}{\max\{|P|, |Q|\}}\right]^{-J-c} \min\left\{\left(\frac{|Q|}{|P|}\right)^{\frac{1+c}{2}}, \left(\frac{|P|}{|Q|}\right)^{\frac{1+c}{2}+J-1}\right\}.$$

The following conclusion demonstrate that we study some property of operators on Besov space $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ by considering corresponding operators on $\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})$.

Theorem 5.1. *Let $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $1 \le p < \infty$, and let $\mathbf{W}$ be a doubling matrix of order p with doubling exponent β. If $A \in \mathbf{ad}^{\alpha,q}_{p,A}(\beta)$, then $A : \dot{b}^{\alpha,q}_{p,A}(\mathbf{W}) \to \dot{b}^{\alpha,q}_{p,A}(\mathbf{W})$ is bounded.*

Definition 5.2. *Let T be a continuous linear operator from $\mathcal{S}'$ to $\mathcal{S}'$. We say that T is an almost diagonal operator for $B^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$, and write $T \in \mathbf{AD}^{\alpha,q}_{p,A}(\beta)$, if for some pair of mutually admissible kernels (φ, ψ), the matrix $\{a_{Q,P}\} \in \mathbf{ad}^{\alpha,q}_{p,A}(\beta)$, where $a_{Q,P} = \langle T\psi_P, \varphi_Q\rangle$.*

Remark 5.3. It is easy to see that, the definition of $T \in \mathbf{AD}^{\alpha,q}_{p,A}(\beta)$ is independent of the choice of the pair (φ, ψ).

As a straightforward consequence of Theorem 5.1, we can deduce the following conclusion:

Corollary 5.4. *Let $T \in \mathbf{AD}^{\alpha,q}_{p,A}(\beta)$, $\alpha \in \mathbb{R}$, $p \in [1, \infty)$ and $q \in (0, \infty)$. Then T extends to a bounded operator on $\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d, \mathbf{W})$ if $\mathbf{W} \in A_p$.*

Proof. By the density, we need to consider $\vec{f}$ with $f_i \in \mathcal{S}$, $i = 1, 2, \ldots, m$. Let (φ, ψ) be a pair of mutually admissible kernels. Let

$$\vec{t}_Q = \sum_{P\in\mathcal{Q}} \langle T\psi_P,\, \varphi_Q\rangle \vec{s}_P(\vec{f}).$$

Notice that $\{\langle T\psi_P, \varphi_Q\rangle\}_{Q,P} \in \mathbf{ad}^{\alpha,q}_{p,A}(\beta)$. From the φ-transform decomposition

$$\vec{f} = \sum_{P\in\mathcal{Q}} \vec{s}_P(\vec{f})\psi_P,$$

and Theorem 5.1, we can deduce that

$$\begin{aligned}
\left\|T(\vec{f})\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})} &= \left\|\sum_{P\in\mathcal{Q}} \vec{s}_P(\vec{f})T\psi_P\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})} \\
&= \left\|\sum_{Q\in\mathcal{Q}} \left(\sum_{P\in\mathcal{Q}} \langle T\psi_P, \varphi_Q\rangle\, \vec{s}_P(\vec{f})\right)\psi_Q\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})} \\
&= \left\|\sum_{Q\in\mathcal{Q}} \vec{t}_Q\psi_Q\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbb{R}^d,\mathbf{W})} \lesssim \left\|\left\{\vec{t}_Q\right\}_{Q\in\mathcal{Q}}\right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})} \\
&\lesssim \left\|\{\vec{s}_Q\}_{Q\in\mathcal{Q}}\right\|_{\dot{b}^{\alpha,q}_{p,A}(\mathbf{W})} \lesssim \left\|\vec{f}\right\|_{\dot{B}^{\alpha,q}_{p,A}(\mathbf{W})}.
\end{aligned}$$

□

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References

[1] M. Bownik, Anisotropic Hardy spaces and wavelets, *Mem. Amer. Math. Soc.*, **164** (2003), 1-122. 2, 3, 4

[2] M Bownik, Atomic and molecular decompositions of anisotropic Besov spaces, *Math. Z.*, **250** (2005), 539-571. 2

[3] M. Bownik, Anisotropic Triebel-Lizorkin spaces with doubling measures, *J. Geom. Anal.*, **17** (2007), 387-424. 2

[4] J. Bergh and J. Löfström, Interpolation Spaces, An Introduction, Springer, New York, 1976.

[5] M. Bownik and K-P. Ho, Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces, *Trans. Amer. Math. Soc.*, **358** (2006), 1469-1510. 2

[6] M. Bownik, B. Li, D. Yang and Y. Zhou, Weighted anisotropic product Hardy spaces and boundedness of sublinear operators, *Math. Nachr.*, **283** (2010), 392-442. 2

[7] F. Bu, Y. Chen, D. Yang and W. Yuan, Maximal function and atomic characterizations of matrix-weighted Hardy spaces with their applications to boundedness of Calderón-Zygmund operators, arXiv:2501.18800. 2

[8] F. Bu, T. Hytonen, D. Yang, and W. Yuan, Matrix-weighted Besov-type and Triebel- Lizorkin-type spaces I: A_p-dimensions of matrix weights and ψ-transform characterizations, *Math. Ann.*, **391** (2025), 6105-6185. 2

[9] F. Bu, T. Hytonen, D. Yang and W. Yuan, Matrix-weighted Besov-type and Triebel-Lizorkin-type spaces II: Sharp boundedness of almost diagonal operators, *J. Lond. Math. Soc. (2)*, **111** (2025), Paper No. e70094. 2

[10] F. Bu, T. Hytonen, D. Yang and W. Yuan, Matrix-weighted Besov-type and Triebel-Lizorkin-type spaces III: characterizations of molecules and wavelets, trace theorems, and boundedness of pseudo-differential operators and Calderón-Zygmund operators, *Math. Z.* **308** (2024), Paper No. 32, 67 pp. 2

[11] F. Bu, T. Hytonen, D. Yang and W. Yuan, New characterizations and properties of matrix A_∞ weights, arXiv:2311.05974. 2

[12] F. Bu, T. Hytonen, D. Yang and W. Yuan, Besov-Triebel-Lizorkin-type spaces with matrix A_∞ weights, *Sci. China Math.*, (2025), https://doi.org/10.1007/s11425-024-2385-x. 2

[13] A.-P. Calderón, An atomic decomposition of distributions in parabolic spaces, *Adv. Math.*, **25** (1977), 216-225. 2

[14] R.R. Coifman, A real variable characterization of H^p, *Studia Math.*, **51** (1974), 269-274. 1

[15] A.-P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution, *Adv. Math.*, **16** (1975), 1-64. 2

[16] R.R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.*, **83** (1977), 569-645. 1

[17] X. Fan, Global $C^{1,\alpha}$, regularity for variable exponent elliptic equations in divergence form, *J. Differential Equations*, **235** (2007), 397-417. 1

[18] C. Fefferman and E.M. Stein, H^p spaces of several variables, *Acta Math*, **129** (1972), 137-193. 1

[19] M. Frazier and B. Jawerth, A discrete tTransform and decomposition of distribution spaces, *J. Funct. Anal.*, **93** (1989), 34-170. 13

[20] M. Frazier and S. Roudenko, Matrix-weighted Besov spaces and conditions of A_p type for $0 < p \leq 1$, *Indiana Univ. Math. J.*, **53** (2004), 1225-1254. 2

[21] M. Goldberg, Matrix A_p weights via maximal functions, *Pacific J. Math.*, **211** (2003), 201-220.

[22] L. Grafakos, Classical Fourier Analysis, Third edition, Graduate Texts in Mathematics, 249. Springer, New York, 2014. 1

[23] B. Li, M. Bownik and D. Yang, Littlewood-Paley characterization and duality of weighted anisotropic product Hardy spaces, *J. Funct. Anal.*, **266** (2014), 2611-2661. 2

[24] J. Liu, F. Weisz, D. Yang and W. Yuan, Littlewood-Paley and finite atomic characterizations of anisotropic variable Hardy-Lorentz spaces and their applications, *J. Fourier Anal. Appl.*, **25** (2019), 874-9220. 2

[25] J. Liu, D. Yang and W. Yuan, Anisotropic Hardy-Lorentz spaces and their applications, *Sci. China Math.*, **59** (2016), 1669-1720. 2

[26] X. Liu and W. Wang, Matrix-weighted anisotropic Triebel-Lizorkin spaces on $\mathbb{R}^d$, (2025), in progress. 2

[27] S. Roudenko, Matrix-weighted Besov spaces, *Trans. Amer. Math. Soc.*, **355** (2003), 273-314. 2, 5, 6

[28] E.M. Stein and G. Weiss, On the theory of harmonic functions of several variables. I. The theory of H^p-spaces, *Acta Math.*, **103** (1960), 25-62. 1

[29] H. Triebel, Theory of Function spaces, Monographs in Math., Birkhäuser-Verlag, Basel, (1983). 2

[30] L. Tang, $L^{\vec{p},\lambda(\cdot)}$ regularity for fully nonlinear elliptic equations, *Nonlinear Anal.*, **149** (2017), 117-129. 1

[31] A. Volberg, Matrix A_p weights via S-functions, *J. Amer. Math. Soc.*, **10** (1997), 445-466. 2

Xiong Liu, School of Mathematics and Physics, Gansu Center for Fundamental Research in Complex Systems Analysis and Control, Lanzhou Jiaotong University, Lanzhou 730070, P. R. China
Email address: liuxmath@126.com

Wenhua Wang (Corresponding author), Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Harbin 150001, China
Email address: whwangmath@whu.edu.cn