CREATING TRIANGLES IN CONSTRUCTOR-BLOCKER GAMES

CHLOÉ BOISSON, YANNICK MOGGE, ALINE PARREAU, THÉO PIERRON

Abstract. Generalized Turán problems investigate the maximization of the number of certain structures (typically edges) under some constraints in a graph. We study a game version of these problems, the Constructor-Blocker game. We mainly focus on the case where Constructor tries to maximize the number of triangles in her graph, while forbidding her to claim short paths or cycles. We also study a variant of this game, where we impose some planarity constraints on Constructor instead of forbidding certain subgraphs. For all games studied, we obtain (precise) asymptotics or upper and lower bounds.

1. Introduction

1.1. **The Turán problem.** One of the most studied problems in extremal graph theory is the so called Turán problem. Given a family $\mathcal{F}$ of graphs, a graph is called *$\mathcal{F}$-free*, if it does not contain any graph $F \in \mathcal{F}$ as a subgraph. We can now ask the following question: what is the maximum number of edges an $\mathcal{F}$-free graph on n vertices can have?

This number is called the *extremal number* $ex(n, \mathcal{F})$, or simply $ex(n, F)$ if $\mathcal{F}$ consists of a single graph F. In [12], Mantel determined this quantity for a triangle $F = K_3$, and Turán generalized this result in [16] to all complete graphs. Since this first result, the extremal number has been widely studied for variety of graphs and families of graphs; see for instance [15] for a survey.

A next step in understanding $\mathcal{F}$-free graphs can be the study of their subgraphs. Given a graph H, denote by $ex(n, H, \mathcal{F})$ the maximal number of copies of H an n-vertex $\mathcal{F}$-free graph can have. The classical Turán problem corresponds to the case where $H = K_2$. Although some mentions of this problem in particular cases had appeared earlier (for instance in [7]), the study of $ex(n, H, \mathcal{F})$ really intensified in the last decade, starting with Alon and Shikhelman in [2], and since then many beautiful results have been proven (see e.g. [11]).

1.2. **Constructor-Blocker games.** In this paper, we study a game version of the generalized Turán problem, which was first introduced by Patkós, Stojaković, and Vizer [13] under the name *Constructor-Blocker game*. The rules of this game are as follows. Given two fixed graphs H and F, two players, called *Constructor* and *Blocker*, alternately claim an unclaimed edge of the

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complete graph K_n. Constructor can only claim an edge in such a way that her graph remains F-free, while Blocker can claim any unclaimed edge without further restrictions. The game ends when Constructor cannot claim any more edges or when all edges have been claimed. The *score* of the game is the number of copies of H in Constructor's graph at the end of the game. Constructor aims to maximize the score while Blocker aims to minimize it. We denote the score of the game by $g(n, H, F)$, assuming optimal play by both players and Constructor starting the game. For a family $\mathcal{F}$ of graphs, we similarly define $g(n, H, \mathcal{F})$.

To get some intuition of the relation between $ex(h, H, \mathcal{F})$ and, $g(n, H, \mathcal{F})$, observe that the trivial bound $g(n, H, \mathcal{F}) \leqslant ex(n, H, \mathcal{F})$ always holds. Moreover, in the case that H is an edge, then Constructor can secretly pick an $\mathcal{F}$-free graph that maximizes the number of edges, and during the game she will be able to claim at least half of the edges of this graph. Therefore, we have $\frac{1}{2}ex(n, K_2, \mathcal{F}) \leqslant g(n, K_2, \mathcal{F}) \leqslant ex(n, K_2, \mathcal{F})$, which tells us that $ex(n, K_2, \mathcal{F})$ and $g(n, K_2, \mathcal{F})$ are of the same order of magnitude. Actually, the asymptotics of $g(n, K_2, \mathcal{F})$ only depend on the chromatic number of $\mathcal{F}$ and were determined in the context of Avoider-Enforcer games in [4].

So far, the quantity $g(n, H, F)$ has mainly been studied in two papers ([5, 13]). In [13], Patkós, Stojaković, and Vizer deal with the cases where F is a star and H is a smaller star or a tree, and where both H and F are paths. In [5], Balogh, Chen, and English extended these results to the cases where H is a complete graph and F is a graph of high chromatic number, when both graphs are odd cycles, and also obtained bounds for $g(n, K_3, C_4)$.

1.3. **Connection to other games.** Let us now consider similarities and differences between Constructor-Blocker games and other types of games. Constructor-Blocker games are part of the broader category of *positional games* (for a more comprehensive overview see e.g. [10]), which are perfect information games between two players, who alternately claim elements of a finite set X (called the *board* of the game) equipped with a family $\mathcal{W} \subset \mathcal{P}(X)$ of *winning sets*. The two most common settings are *Maker-Maker games* and *Maker-Breaker games*. In the Maker-Maker convention, the winner is the first player completing a winning set. In the Maker-Breaker convention, Maker wins if she is able to claim all of the elements of a winning set at the end of the game, Breaker wins otherwise. This also means, that in Maker-Maker games there can be a draw, but in Maker-Breaker games there has to be a winner.

As an example for a positional game, we can consider the *Maker-Breaker clique game*. Here, the board is the edge set of a complete graph ($X = E(K_n)$) and the winning sets are the edge sets of all k-cliques. Thanks to Ramsey's theorem, we know that, if n is large enough, there must be a k-clique either in Maker's or in Breaker's graph at the end of the game. We can deduce, that Maker (since she is the first player) has a winning strategy, thanks to the concept of *strategy stealing*. Indeed, one of the two players has to have a winning strategy, and if Breaker would have a winning strategy then Maker could apply this strategy (after playing her first move arbitrarily) and thus win. Therefore, we know that for n large enough this game is a Maker's win. To analyse this game further, we can now consider either the smallest n such that this game is a Breaker's win, or the largest number of k-cliques Maker can actually make.

Going in the second direction, Bagan et al. [3] introduced *scoring positional games*. The two players alternately claim elements of the board until there are none left. In the Maker-Maker convention, the players get a point whenever they complete a winning set and the score of the game is the difference between the number of points of the first and the second player. In the Maker-Breaker convention, the score is the number of winning sets completed by Maker. During the course of the game, Maker wants to maximize the number of winning sets, while Breaker wants to minimize it.

We can relate Constructor-Blocker games very well to scoring positional games. Constructor takes over the role of Maker, Blocker takes over the role of Breaker, the board is the edge set of the complete graph K_n, and the winning sets are the subgraphs of K_n isomorphic to H. But we also have to take into account that Constructor has further restrictions on how she is allowed to claim edges, since she has to avoid claiming a graph isomorphic to F.

Let us also mention two other game settings that can transcribe this aspect of the Constructor-Blocker game – that is, forbidding some subgraphs – very well, but that do not take the maximization of the number of copies of H into account.

The first one we want to mention are *Avoider-Enforcer games*. In an Avoider-Enforcer game, instead of winning sets we have a collection of losing sets. Avoider wins if she does not complete a losing set during the course of the game, Enforcer wins otherwise. Now let us look at a Constructor-Blocker game with $H = K_2$ and F being some graph, then Constructor wants to maximize the number of edges in her graph, while at the same time not creating F as a subgraph. The score of the game actually corresponds to the length of an Avoider-Enforcer game (where the losing sets are all subgraphs of K_n isomorphic to F) before Avoider loses. As a sidenote, let us also mention, that this is actually the method how Balogh and Martin [4] determined $g(n, K_2, F)$ before Constructor-Blocker games were even introduced.

The second game we want to mention are *saturation games*. In a saturation game, both players alternately claim edges of a graph, but these edges do not belong to a specific player. Instead, both players are not allowed to claim some subgraph with the union of their edges. The first player wants to maximize the length of the game – that is, the number of claimed edges when no player can claim any more edges – while the second player wants to minimize it (see [1] for an example).

1.4. **Our results.** We further extend the results in [5] and [13], mainly focusing on the asymptotics of $g(n, K_3, \mathcal{F})$ for several choices of $\mathcal{F}$ and comparing it to $ex(n, K_3, \mathcal{F})$. Our first result considers the case where we relax the constraints on Constructor by not forbidding any subgraph at all (while maximizing the number of triangles). This setting actually corresponds to a regular scoring positional game. We get the following result:

Theorem 1.1. *We have*

$$g(n, K_3, \varnothing) = (1 + o(1))\frac{n^3}{48}\,.$$

Next, we consider the number of triangles while forbidding short paths. In [13], Patkós, Stojaković, and Vizer considered $g(n, K_3, P_5)$ and showed that $g(n, K_3, P_5) = \frac{n}{4} - o(n)$. We extend the length of the forbidden path by one and get the following result:

Theorem 1.2. *We have*

$$g(n, K_3, P_6) = (1 + o(1))\frac{n}{2}\,.$$

We also consider the cases, where instead of a path we forbid Constructor from claiming small stars (basically restricting the maximum degree in her graph). For the case that $F = S_4$, we get the following result:

Theorem 1.3. *We have*

$$g(n, K_3, S_4) = (1 + o(1))\frac{n}{3}\,.$$

We also considered the case $F = S_5$, but for this case the bounds that we obtained are not tight. Here, our result is the following:

Theorem 1.4. *It holds that*

$$(1 + o(1))\frac{2n}{3} \leqslant g(n, K_3, S_5) \leqslant (1 + o(1))n\,.$$

Both of these theorems result from analysing all possible cases, which unfortunately did not help us to generalize our results to larger stars. However, we present an idea for a lower bound on $g(n, K_3, S_k)$ when k goes to infinity (while growing not too fast):

Theorem 1.5. *If $k(n) \underset{n\to\infty}{\longrightarrow} \infty$ and $k(n) = o(\sqrt{n})$, then*

$$g(n, K_3, S_{k(n)+1}) \geqslant (1 + o(1))\frac{n\,(k(n))^2}{48}\,.$$

Lastly, we wonder what would happen in the Constructor-Blocker game if, instead of forbidding some subgraph in Constructor's graph, we would require it to remain planar. As bounded-degreeness (corresponding to forbidding some stars), planarity is among the classical properties studied in a graph. A graph is *planar* if it can be drawn on a plane without any edges crossing.

For a graph to be planar it cannot have certain graphs as a subgraph (for instance a K_5), but planarity cannot be entirely described by the absence of a finitite family of subgraphs. Therefore, this question cannot be stated in terms of Constructor-Blocker games as defined earlier, and we need to extend our framework. More precisely, we will define two new games: the *planar Constructor-Blocker game* (PCB) and the *embedded Constructor-Blocker game* (ECB).

In the planar Constructor-Blocker game, Constructor can only claim edges such that her graph remains planar, while Blocker has no further restrictions. This means, that at any time of the game, there has to exist an embedding of $\mathcal{C}$ into the plane with no crossing edges. The embedding of Constructor's graph into the plane is not fixed at the beginning of the game. Similar to before, we will assume optimal play by both players, and the score of the game remains the number of triangles in $\mathcal{C}$ at the end of the game, which we will denote by $g_{\mathrm{PCB}}(n, K_3)$.

We obtained that Constructor can asymptotically build as many triangles as possible under the planarity constraint.

Theorem 1.6. *We have*

$$g_{\mathrm{PCB}}(n, K_3) = (1 + o(1))3n\,.$$

In the embedded Constructor-Blocker game, we impose even more restrictions on Constructor. When the game starts, we already fix an embedding of K_n in the plane – actually, we place the n vertices as the n^{th} roots of unity in the complex plane. Constructor is only allowed to claim edges inside of the unit disk, and her edges are not allowed to cross. She still aims to maximize the number of triangles in her graph. We will denote the score of this game by $g_{\mathrm{ECB}}(n, K_3)$.

For the embedded Constructor-Blocker game, we were able to provide both a lower and a non-trivial upper bound, but those do not match.

Theorem 1.7. *It holds that*

$$(1 + o(1))\frac{n}{2} \leqslant g_{\mathrm{ECB}}(n, K_3) \leqslant (1 + o(1))\frac{2n}{3}\,.$$

1.5. **Organization of the paper.** In Section 2, notations and preliminary results are given that will be useful later in the paper. We prove the results concerning the Constructor-Blocker game in Section 3, starting with forbidding nothing, then a path and finally a star. Theorems 1.1 to 1.5 are proven in this section. Section 4 is devoted to PCB and ECB games and contain the proofs of Theorem 1.6 and Theorem 1.7.

2. Preliminaries

2.1. **Notation.** Most of the notation used in this paper is standard and is chosen according to [17]. We set $[n] := \{k \in \mathbb{N} : 1 \leqslant k \leqslant n\}$ for every positive integer n. Let G be a graph. Then we write $V(G)$ and $E(G)$ for the *vertex set* and the *edge set* of G, respectively. Let v, w be vertices in G. If $\{v, w\}$ is an edge in G, we write vw for short. The *neighbourhood* of v is $N_G(v) := \{w \in V(G) : vw \in E(G)\}$, and we call its size the *degree* of v. Moreover, the *minimum degree* in G is denoted $\delta(G)$, and the *maximum degree* in G is denoted $\Delta(G)$. Often, when the graph G is clear from the context, we omit the subscript G in all definitions above.

Let G and H be graphs. We say that H is a *subgraph* of G, denoted $H \subset G$, if both $V(H) \subset V(G)$ and $E(H) \subset E(G)$ hold. The number of copies of H in G is the number of subgraphs of G isomorphic to H. For a subset $A \subset V(G)$ we let $G - A$ be the subgraph of G with vertex set $V(G) \backslash A$ and edge set $\{vw \in E(G) : v, w \in V(G) \backslash A\}$.

We write P_n for the *path* on n vertices, i.e. with vertex set $V(P_n) = \{v_1, \ldots, v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1} : i \in [n-1]\}$. Additionally, we call a P_n with one additional vertex that is connected to all n vertices of P_n a *fan* of size n, denoted by F_n. Similarly, we write C_n for the *cycle* on n vertices, i.e. with vertex set $V(C_n) = \{v_1, \ldots, v_n\}$ and edge set $E(C_n) = \{v_i v_{i+1} : i \in [n-1]\} \cup \{v_n v_1\}$, and call a C_n with one additional vertex that is connected to all n vertices of C_n a *wheel* of size n, denoted by W_n. We write K_n for the *complete graph* on n vertices. In particular, K_3 is a triangle. A *triangle with a pendant leg* is a triangle with one additional edge attached to one of its vertices. We denote a *star* with n leaves as S_n. Additionally, we call an S_2 a *cherry* and call the central vertex its root.

For functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f(n) = o(g(n))$ if $\lim_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| = 0$ and $f(n) = O(g(n))$ if $\limsup_{n\to\infty} \frac{|f(n)|}{g(n)} < \infty$.

Assume a Constructor-Blocker game is in progress, then we will denote by $\mathcal{C}$ (resp. $\mathcal{B}$) *Constructor's* (resp. *Blocker's*) *graph*, that is the graph consisting of the edges claimed by Constructor (resp. Blocker). If an edge has not been claimed by either player, we call such an edge *free*.

2.2. **Relevant game tools.** The Erdős-Selfridge Criterion [8] is a classical result for Maker-Breaker games, which (if applicable) gives an easy argument for a win by Breaker. It is also often used in cases where Maker takes over the role of Breaker and proceeds to claim at least one element of every winning set. It was stated in its original form as follows:

Lemma 2.1 (Erdős-Selfridge-Criterion [8])**.** *Let X be a finite set and let $\mathcal{W} \subseteq \mathcal{P}(X)$ satisfying*

$$\sum_{W \in \mathcal{W}} 2^{-|W|+1} < 1 \,.$$

Then in the Maker-Breaker game on board X with winning sets $\mathcal{W}$, Breaker has a strategy to claim at least one element in each of the winning sets in $\mathcal{W}$.

In [3], this criterion was adapted for scoring positional games, and we state here a version corresponding to the players claiming edges of a graph.

Theorem 2.1 (Theorem 10 in [3])**.** *Let $G = (V, E)$ be a graph on n vertices with m edges. Let $\mathcal{W} \subset \mathcal{P}(E)$ and consider the scoring Maker-Breaker positional game on the board $X = E$ in which the winning sets are the elements of $\mathcal{W}$. Set $\ell := \max_{e \neq e' \in E} |\{W \in \mathcal{W} \mid e, e' \in W\}|$. If Maker starts, then the score of the game is at least $\sum_{W\in\mathcal{W}} 2^{-|W|} - \frac{m\ell}{8}$. If Breaker starts, then the score is at most $\sum_{W\in\mathcal{W}} 2^{-|W|}$.*

Let us also prove a helpful lemma, which provides Constructor with a strategy to claim a triangle with a pendant leg while keeping the maximum degree in her graph at 3. We prove this lemma here, since we will use it on two different occasions in Section 3.3 and Section 4.1.

Lemma 2.2. *If $v_1, \ldots, v_5$ are isolated in $\mathcal{C}$ and if there is no Blocker edge between these vertices, then Constructor can build a triangle with a pendant leg within four moves (if this subgraph is not forbidden).*

Proof. We give a strategy for Constructor to achieve this goal. In her first move she plays v_1v_2. Blocker might play an edge between two vertices among $v_1, \ldots, v_5$, without loss of generality assume this edge to be adjacent to v_5. Then Constructor plays v_1v_3.

- If Blocker plays v_2v_3, then Constructor plays v_1v_4. Blocker cannot prevent her from creating a triangle. Indeed, he might play v_2v_4 or v_3v_4, she will play the other one.
- Otherwise Constructor plays v_2v_3. Then even after Blocker's move she will be able to connect v_4 or v_5 to one of v_1, v_2, v_3.

An example is given in Figure 2.1. □

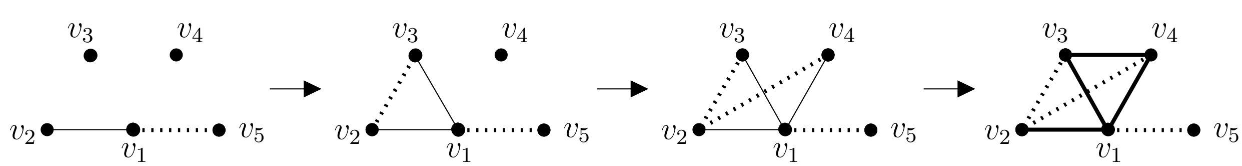


Figure 2.1. How to build a triangle with a pendant leg? An example of Constructor's (solid lines) and Blocker's (dotted lines) successive moves, Constructor being the first player.

The idea behind this lemma is that Constructor cannot build a triangle on her first trial, but she can create many threats so that Blocker will not be able to handle all of them, thus Constructor will still be able to create triangles.

Lastly, we will also prove a helpful lemma for Blocker, which helps to bound the number of Constructor's edges inside each of her components. Blocker can achieve this goal by answering in the component where Constructor played the move just before. It implies that, sometimes, following this strategy is enough for Blocker to prevent Constructor from achieving her goal, and we will use it later on.

Lemma 2.3. *Assume that Blocker always answers a move by Constructor in the same connected component of $\mathcal{C}$ if possible. If at the end of the game some connected component C of $\mathcal{C}$ contains r vertices, then at least $\frac{\binom{r}{2}-\lfloor\frac{r}{2}\rfloor}{2}$ edges between vertices of C are not claimed by Constructor.*

Proof. We proceed by induction on r.

For $r \in \{1, 2\}$, the bound is 0, so this is trivially true.

Let $r \geqslant 3$ and assume this to be true for any $s < r$. Let C be a connected component of $\mathcal{C}$ of size r. Look at the time when Constructor connects two disjoint connected components C_1 and C_2 of sizes $1 \leqslant r_1, r_2 < r$ with $C_1 \cup C_2 = C$ (and $r_1 + r_2 = r$). Up to this point, Blocker has played in C_1 whenever Constructor has done so, respectively in C_2. If Constructor plays to maximize the number of edges in C, without loss of generality, we can assume that from now on she plays in C_1 (resp. in C_2, resp. an edge between C_1 and C_2) whenever Blocker does so.

This ensures that, in total, at least $\frac{\binom{r_1}{2}-\lfloor\frac{r_1}{2}\rfloor}{2} + \frac{\binom{r_2}{2}-\lfloor\frac{r_2}{2}\rfloor}{2} + \lfloor\frac{r_1 r_2}{2}\rfloor$ edges are not claimed by Constructor.

If $r_1 = 2k_1$ and $r_2 = 2k_2$ with k_1, k_2 positive integers, then this gives $k_1(k_1 - 1) + k_2(k_2 - 1) + 2k_1k_2 = (k_1 + k_2)(k_1 + k_2 - 1) = \frac{\binom{r}{2}-\lfloor\frac{r}{2}\rfloor}{2}$.
The other possibilities for the parities of r_1 and r_2 give the same result.

Hence by induction this lemma is true for any $r \geqslant 1$. □

3. Results on the Constructor-Blocker game

In this section we will give proofs for the cases when Constructor has no restrictions (Theorem 1.1), when Constructor is not allowed to claim short paths (Theorem 1.2), and when Constructor is not allowed to claim small stars (Theorem 1.3 and Theorem 1.4) or large stars (Theorem 1.5).

3.1. **Not forbidding anything.** In this subsection we will prove Theorem 1.1. In this setting, both players are allowed to claim edges without restrictions, and at the end of the game we count the number of triangles in Constructor's graph. This setting actually corresponds to a scoring positional game. The board is the edge set of a complete graph, and the winning sets are all triangles. Before we prove our result, we will first compute the maximum number of triangles a graph on n vertices can have.

Proposition 3.1. *The maximum number of triangles in an n-vertex graph is*

$$ex(n, K_3, \varnothing) = \binom{n}{3} = (1 + o(1))\frac{n^3}{6}\,.$$

This result comes from the fact that computing the number of triangles in a graph is exactly computing the number of sets of three different vertices u, v, w such that all the edges uv, uw, and vw are present in the graph. Thus there are at most $\binom{n}{3}$ triangles in a graph on n vertices, and there is equality if and only if the graph is complete.

In the game version, Constructor will not be able to claim all the edges of the complete graph – she will basically claim half of them. Therefore, maybe a more relevant question would be the maximum number of triangles that a graph on n vertices with at most half of the edges can have. This was answered by Rivin in [14] who found that this number is (asymptotically) $\frac{n^3}{12\sqrt{2}}$ and is achieved by a clique of size $\frac{n}{\sqrt{2}}$. In the Constructor-Blocker game, we obtain the same order of magnitude, but our constant factor is smaller.

Proof of Theorem 1.1. In order to prove our result, we make use of Theorem 2.1. We apply this theorem to the complete graph K_n, with $\mathcal{W}$ be the set of all triangles, Constructor playing as Maker and Blocker playing as Breaker. Since a fixed pair of edges can only be in at most one triangle, $\ell = 1$. Constructor starts, thus

$$g(n, K_3, \varnothing) \geqslant \binom{n}{3} \cdot 2^{-3} - \frac{n(n-1)}{2 \cdot 8} = (1 + o(1))\frac{n^3}{48}\,.$$

In order to get an upper bound, assume temporarily that Blocker starts. Theorem 2.1 states that the score of the game is at most $\binom{n}{3} \cdot 2^{-3} = (1 + o(1))\frac{n^3}{48}$. But the edge Blocker claimed at the beginning blocks at most $n - 2$ triangles, which is negligible with respect to n^3. Thus this minor change on the first player does not affect our asymptotic equivalent and therefore we have

$$g(n, K_3, \varnothing) = (1 + o(1))\frac{n^3}{48}\,.$$

□

Note that $g(n, K_3, \varnothing)$ is the expected number of triangles if both players play randomly during the whole game – saying it differently, the number of triangles in Constructor's graph at the end of the game is the expected number of triangles in a uniform random graph.

3.2. **Forbidding a path.** Next, we turn to the case where we forbid Constructor from claiming a (short) path. Before considering the game, we give the extremal numbers when forbidding a path:

Proposition 3.2 (Corollary 1.7 in [11]). *For $n \geqslant k \geqslant 4$ and $s \geqslant 2$ the following holds:*

$$ex(n, K_s, P_k) = \frac{n}{k-1} \cdot \binom{k-1}{s}.$$

In particular, this tells us that $ex(n, K_3, P_4) = \frac{n}{3}$

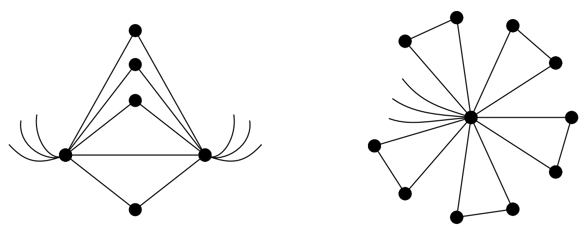
Turning to games, since P_3 is a subgraph of K_3, we will start looking at P_k for $k = 4$. In that case, Blocker can prevent Constructor from building a triangle.

Proposition 3.3. *We have*

$$g(n, K_3, P_4) = 0\,.$$

Proof. The only possibility for a K_3 to appear in a P_4-free graph is as an isolated triangle. Blocker can easily prevent such isolated triangles with the following strategy: whenever an isolated P_3 appears in $\mathcal{C}$, Blocker plays the edge that would complete a triangle, otherwise Blocker plays arbitrarily. Therefore $g(n, K_3, P_4) = 0$. $\square$

Now let us turn to P_6. This case is proved similarly to the case of P_5, which was proved by Patkós, Stojaković, and Vizier in [13] by looking at the possibilities of how components of a P_5-free graph could look like. For our case, these configurations are shown in Figure 3.1. Either there exists an edge which is in several triangles (case (a)) – Blocker will have to be careful about this situation and basically make sure that there are some vertices of the component that are not in any triangle to balance – or every edge is in at most one triangle (case (b)). In the latter case, only one vertex of the component can be in several triangles and the number of triangles is a bit less than half of the number of vertices of the component.



(a) (b)

FIGURE 3.1. How connected components with many triangles in a P_6-free graph can look like. The curved lines mean that there might or might not be such edges.

Proof of Theorem 1.2. We will start with a strategy for Constructor. Her strategy will be to create components of case (b), i.e. she will create components of many triangles intersecting in a single vertex. We will show that with this strategy for large enough n Constructor will claim at least $\frac{n}{2}(1 - \frac{2}{\ln(n)}) = (1 + o(1))\frac{n}{2}$ triangles. Her strategy is split into two stages.

Stage I: While there are more than $\frac{n}{\ln(n)}$ isolated vertices in $\mathcal{C}$, Constructor constructs big stars. She starts by picking an arbitrary vertex $v \in G$ and then afterwards she claims every available edge incident to v. Once there are no more edges available, Constructor picks an isolated vertex in $\mathcal{C}$ that has the lowest degree in $\mathcal{B}$ and repeats this process, until the number of isolated vertices in $\mathcal{C}$ is at most $\frac{n}{\ln(n)}$. Then she proceeds to Stage II.

Stage II: At the end of Stage I, Constructor's graph will consist of stars. During this stage, for each star Constructor creates a matching of the leaves by claiming arbitrary edges that extend her matching.

Once there are no more free edges available that would extend her matching, she stops playing.

These stages are illustrated in Figure 3.2.

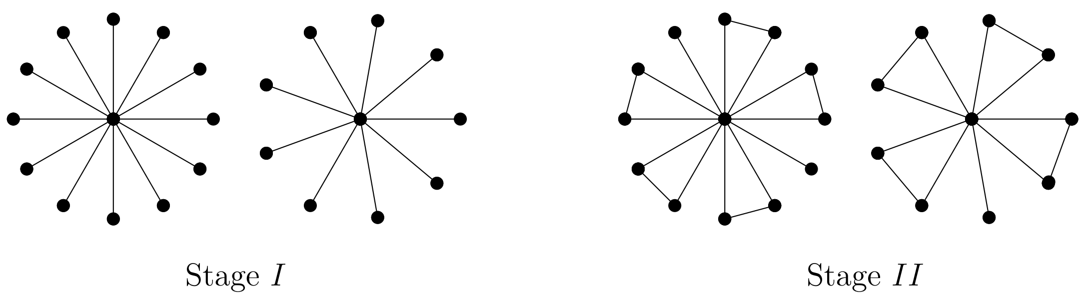


FIGURE 3.2. Constructor's strategy when forbidding P_6

Strategy discussion:

We will first look at Stage I and show that all of the stars built by Constructor will have almost linear size.

Claim 1. *If Constructor follows her strategy, then, for n large enough, at the end of the first stage, every star in $\mathcal{C}$ will have size at least $\frac{n}{3\ln(n)}$.*

Proof of the claim. Stage I lasts as most $(1 - \frac{1}{\ln(n)})n$ turns, therefore Constructor and Blocker each will have played at most $(1 - \frac{1}{\ln(n)})n$ edges at the end of Stage I. Thus, while there are more than $\frac{n}{\ln(n)}$ isolated vertices in $\mathcal{C}$, there is at least one vertex $v \in \mathcal{B} \backslash \mathcal{C}$ with $d_{\mathcal{B} \backslash \mathcal{C}}(v) < 2\ln(n)$. This means that all of the stars Constructor builds will have at least $\frac{1}{2}\left(\frac{n}{\ln(n)} - 2\ln(n)\right) \geqslant \frac{n}{3\ln(n)}$ leaves for n large enough. ∎

This claim also tells us, that there are at most $3\ln(n)$ star centers in Constructor's graph. Now, we argue that the matchings that Constructor will build during Stage II will cover most of the leaves. For each star, Constructor can match all of the leaves except at most $2\sqrt{n}$ of them. Indeed, if Constructor cannot add an edge between the k remaining leaves of a star, it means that Blocker has blocked all of the $\binom{k}{2}$ edges, but during the whole game Blocker claims less than $\frac{3n}{2}$ edges. We conclude that $\binom{k}{2} < \frac{3n}{2}$ has to hold, and for n large enough we have $k \leqslant 2\sqrt{n}$.

Finally, let us count the number of triangles in $\mathcal{C}$ at the end of the game. There are at most $\frac{n}{\ln(n)}$ isolated vertices and $\frac{(1-\frac{1}{\ln(n)})n \cdot 3\ln(n)}{n} \cdot 2\sqrt{n} < 3\ln(n) \cdot 2\sqrt{n}$ vertices that are leaves of stars but not in the matchings. Then there is one triangle every two vertices that are both leaves of a star and in the matching. This gives us at least $\frac{1}{2}(n - \frac{n}{\ln(n)} - 3\ln(n) - 3\ln(n) \cdot 2\sqrt{n}) \leqslant \frac{n}{2}(1 - \frac{2}{\ln(n)}) = (1+o(1))\frac{n}{2}$ triangles.

We now turn to the upper bound and provide a strategy for Blocker ensuring that Constructor does not create more than $\frac{n}{2}$ triangles.

If possible, Blocker always plays in the same connected component C in which Constructor just played, according to the following rules:

(1) If there exists an edge v_1v_2 which is in two triangles in C:
   - If Constructor plays vv_i ($i \in \{1, 2\}$) for some $v \in G$ and if vv_{3-i} is free, then Blocker claims vv_{3-i}.
   - Otherwise, if for some $v \in \mathcal{C}$ the edge vv_i ($i \in \{1, 2\}$) is in $\mathcal{C}$ and if vv_{3-i} is free, then Blocker claims vv_{3-i}.
   - If none of the two previous cases apply, Blocker plays arbitrarily.

   This corresponds to Figure 3.1.a.

(2) Otherwise, if Constructor completes the first C_4 of C and at least one diagonal is free, then Blocker plays this diagonal, prioritizing the diagonal with a vertex of largest Constructor degree out of the four vertices.

(3) Otherwise, if there is a free edge that could close a triangle in C, then Blocker takes it, prioritizing an edge adjacent to the edge Constructor just played.

(4) Otherwise Blocker plays an arbitrary edge.

We show that this strategy prevents Constructor from creating too many triangles. We will start with the following claim:

Claim 2. *At any time of the game, the number of triangles in a connected component C of $\mathcal{C}$ is at most $\frac{|C|}{2}$.*

Proof of the claim. We first deal with the small connected components. Let us assume, that Blocker plays according to the strategy in Lemma 2.3. Since Blocker always answers in the same connected component as the one Constructor just played an edge in (when possible), the following holds:

- If $|C| = 3$, then C contains no triangle.
- If $|C| = 4$, then Constructor has at most four edges, thus at most 1 triangle.

- If $|C| = 5$, then Constructor has at most six edges, thus at most two triangles (this can be shown by an easy case distinction).

Now assume that $|C| \geqslant 6$ and assume that there is an edge v_1v_2 that is in several triangles. The other vertices of C will be denoted by $u_1, u_2, \dots$ (see Figure 3.3). In order to have at least three triangles without any P_6, Constructor cannot have any edge between some u_i and u_j for $i \neq j$, thus every u_i is connected to v_1, v_2, or both. Let us look back at the time when the edge v_1v_2 was claimed by Constructor.

- Either it was the first edge of C played by Constructor. Then when Constructor played another edge in C (some v_1u_i ou v_2u_i), Blocker played (or had already played) v_2u_i or v_1u_i.
- Or there was already a C_4 in C, say $v_1u_1v_2u_2$ was the first one. Let us show that this is actually impossible. When this cycle was completed by Constructor, one of the following happened:
  - Either v_1 and v_2 both had degree 2. Then Blocker already had claimed one of the diagonals of the C_4, and claimed the other one, thus $v_1v_2 \in \mathcal{B}$.
  - Or v_1 or v_2 had degree at least 3. Then, because of Rule (2) in her strategy Blocker blocked v_1v_2.

  Thus it is impossible that Constructor completed a cycle in C before claiming v_1v_2.
- Or there was at least one edge v_1u_i or v_2u_i and no cycle. When Constructor played v_1v_2, then because of Rule (3) in her strategy Blocker played an edge of the form v_2u_i or v_1u_i.

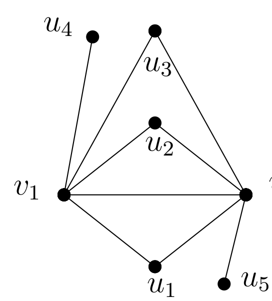


FIGURE 3.3. Notations for the proof of Theorem 1.2

In all cases, we made sure that when v_1v_2 becomes part of two triangles, some edge v_1u_i or v_2u_i is blocked. Then Blocker plays according to Rule (1). He will prevent at least half of the remaining vertices u_j from being in any triangle (that makes $\lfloor \frac{|C|-3}{2} \rfloor$ vertices). In the end there cannot be more than $\lfloor \frac{|C|}{2} \rfloor$ vertices u_j in a triangle and thus C contains at most $\lfloor \frac{|C|}{2} \rfloor$ triangles.

Note that if in some C no edge is in several triangles, then at most one vertex of C can be in several triangles (see Figure 3.1.b), otherwise there would be some P_6. Therefore, the number of triangles in such components cannot be more than $\frac{|C|-1}{2}$. ■

If we now consider all of Constructor's components at the end of the game, if Blocker follows his strategy the number of triangles in $\mathcal{C}$ is at most $\frac{n}{2}$, which gives us our desired result. □

3.3. **Forbidding a star.** We now investigate the case where we forbid Constructor from claiming a star S_k for small values of k. Note that for a graph being S_k-free is equivalent to being of maximum degree at most $k-1$.

First, let us consider the maximum number of triangles a graph of a certain maximum degree can have. This was proven by Chase in [6]:

Proposition 3.4 (Theorem 1 in [6])**.** *For $n \geqslant 1$ and $k \geqslant 2$, the following holds:*

$$ex(n, K_3, S_k) = (1 + o(1))\frac{n}{k}\binom{k}{3} = (1 + o(1))\frac{nk^2}{6}\,.$$

The proof is done by induction on n and by exhibiting a vertex whose neighbourhood does not intersect too many triangles. The extremal graphs are once again disjoint unions of cliques of size k, that contain around $\frac{n}{k}\binom{k}{3}$ triangles.

Let us turn to the game version. The only way for Constructor to build a triangle while not exceeding maximum degree 2 in her graph (i.e. not claiming an S_3) is to build an isolated triangle, however Blocker can prevent this from happening, therefore $g(n, K_3, S_3) = 0$. Let us thus turn to the case of S_4 and prove Theorem 1.3 next.

3.3.1. *Forbidding S_4.* We will split this case into two parts, one focussing on a strategy for Constructor, and one for Blocker. Afterwards, Theorem 1.3 will trivially follow from both parts combined. We will start with Constructor's side to get the lower bound, and before we go into more details, we will briefly state the main ideas of the proof. We will show that Constructor can build a big chain of triangles connected by one edge (see Figure 3.4, induction). The vertices in this chain are (almost) all in a triangle, and we show that the number of vertices that are not in this chain is negligible. We will state this as a new proposition:

Proposition 3.5. *Constructor has a strategy to ensure that the following holds:*

$$g(n, K_3, S_4) \geqslant \frac{n}{3}(1 + o(1))\,.$$

Proof of Proposition 3.5. We provide a strategy for Constructor to build at least $\frac{n}{3} - o(n)$ triangles. This strategy consists of two stages.

Stage I: During this stage, Constructor creates many many isolated cherries. While there are at least $3\sqrt{n}$ isolated vertices in $\mathcal{C}$, Constructor picks 3 vertices v_1, v_2, v_3 that are isolated in $\mathcal{C}$ and are not connected in $\mathcal{B}$. She plays v_1v_2. If Blocker plays v_1v_3, then Constructor plays v_2v_3, otherwise she plays v_1v_3. Then v_1, v_2, v_3 form a new cherry.

Once there are less than $3\sqrt{n}$ isolated vertices in Constructor's graph, she proceeds to Stage II.

Stage II: During this stage, Constructor will recursively construct a component C_k, which will be a connected component of $\mathcal{C}$ satisfying the following properties:

- C_k contains exactly k triangles.
- C_k contains exactly $3k+1$ vertices.
- There is a vertex x_k in C_k of degree 1, connected to at most $\sqrt{n}+2$ cherries in $\mathcal{B}$.

Once she is not able to follow this strategy any more, she stops playing. Set k_0 to be equal to k at the final step.

We will briefly sketch how Constructor will play during Stage II. Assume, that so far Constructor could follow her strategy, thus there already exist C_k and x_k as described above. While there is a cherry that is connected to less than $\sqrt{n}$ other cherries in $\mathcal{B}$ and that is not connected to x_k in $\mathcal{B}$, Constructor picks this cherry to continue her chain. We will name its vertices v_1, v_2, v_3, where v_1 is the root of this cherry (meaning that $d_{\mathcal{C}}(v_1) = 2$). Constructor plays $x_k v_1$. If Blocker plays $x_k v_2$, then Constructor plays $x_k v_3$, otherwise she plays $x_k v_2$. Let this new connected component be C_{k+1} and x_{k+1} be v_2 or v_3, depending on what Constructor just played.

Remark. She starts Stage II with C_0 being an isolated vertex in $\mathcal{C}$ (namely x_0) and then she picks a cherry which is connected to at most $\sqrt{n}+2$ cherries in $\mathcal{B}$ and is not connected to x_0, then she follows the instructions as above. The resulting component C_1 is actually a triangle with a pendant leg, and its construction corresponds to the one in Lemma 2.2.

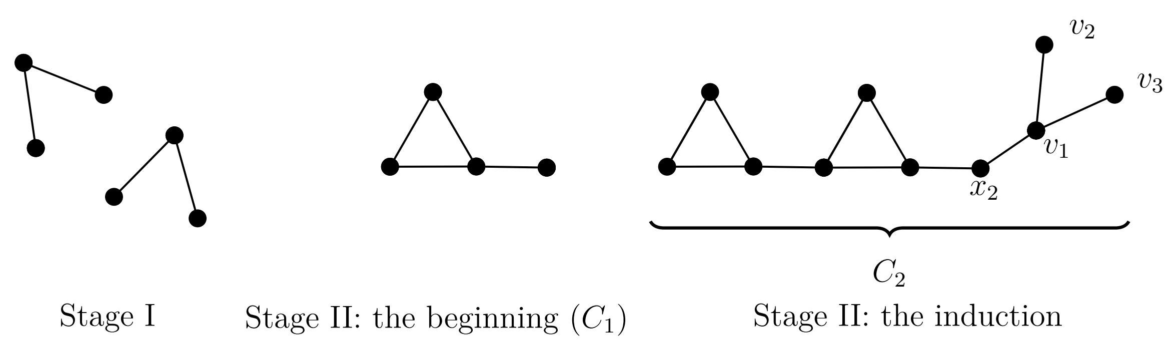


Figure 3.4. Constructor's strategy when S_4 is forbidden

This strategy is illustrated in Figure 3.4. We now show that Constructor can indeed play according to this strategy.

Strategy discussion:

We will start with a claim, which shows that Constructor can create many cherries.

Claim 3. *For n large enough, Constructor can follow Stage I.*

Proof of the claim. During Stage I, write L the set of isolated vertices in $\mathcal{C}$. Let $\ell := |L|$, and assume $\ell \geqslant 3\sqrt{n}$. The number of sets $\{v_1, v_2, v_3\}$ with all vertices isolated in $\mathcal{C}$ but somehow connected in $\mathcal{B}$ is at most

$$\sum_{S \in L^3} \sum_{e \in E(\mathcal{B})} 1_{\{e \in S^2\}} \leqslant \sum_{e \in E(\mathcal{B})} \ell \leqslant n\ell < \binom{\ell}{3}$$

for n large enough. Indeed an edge can be in at most ℓ sets of three vertices of L, and since in Stage I Constructor claims less than n edges, so does Blocker. Thus there exists a set of three vertices of L that are not connected in any way in $\mathcal{B}$. ■

Now we show that Constructor is able to find an isolated vertex to play the first move of Stage II afterwards.

Claim 4. *For n large enough, Constructor can construct C_0.*

Proof of the claim. At the end of Stage I, there are at least $3\sqrt{n} - 3$ vertices that are isolated in $\mathcal{C}$. Since Blocker has played less than n moves during Stage I, the number of vertices that are connected in $\mathcal{B}$ to at least $\sqrt{n} + 3$ cherries is at most $\frac{n}{\sqrt{n}+3} < 3\sqrt{n} - 3$ (for n large enough). Thus there is a vertex that is isolated in $\mathcal{C}$ and not connected to more than $\sqrt{n} + 2$ cherries in $\mathcal{B}$. Let it be x_0. ■

Lastly, let us show that Constructor can build a chain of triangles.

Claim 5. *For n large enough, Constructor can follow Stage II.*

Proof of the claim. By the previous claim, we can construct C_0. Now assume that, for some k, C_k is constructed (and satisfies the required properties). Assume that there is a cherry that is both connected to less than $\sqrt{n}$ other cherries in $\mathcal{B}$ and that is not connected to x_k in $\mathcal{B}$. The construction of C_{k+1} guarantees the first two properties. Since x_{k+1} was part of the selected cherry, it was not connected to more than $\sqrt{n}$ other cherries in $\mathcal{B}$ at that time. Constructor played two edges, thus x_{k+1} is connected to at most $\sqrt{n} + 2$ cherries after C_{k+1} is constructed. ■

We check that the strategy is valid, meaning that Constructor does not create an S_4.

Claim 6. *When Constructor follows this strategy, her graph $\mathcal{C}$ remains S_4-free.*

Proof of the claim. A cherry contains no vertex with degree more than 3, so $\mathcal{C}$ is S_4-free at the end of Stage I. When constructing C_{k+1}, Constructor first plays an edge between a vertex of degree one (x_k) and a vertex of degree 2 (v_1), and next an edge between a vertex of degree 2 (x_k) and a vertex of degree 1 (v_2 or v_3). Therefore all vertices remain of degree at most 3. ■

Finally, let us show that Constructor created at least $\frac{n}{3}(1 + o(1))$ triangles this way.

Claim 7. *Constructor can follow Stage II for $k_0 \geqslant \frac{n}{3}(1+o(1))$ steps.*

Proof of the claim. We will count the number of vertices that are not in a triangle at the end of the game. There are three possibilities for this to happen: either it was not put into a cherry during Stage I, or it was in a cherry but this cherry was not used by Constructor during Stage II or it is x_{k_0}.

Let ℓ_1 be the number of isolated vertices in $\mathcal{C}$ at the end of Stage I and ℓ_2 be the number of isolated cherries in $\mathcal{C}$ at the end of Stage II. We know that $\ell_1 < 3\sqrt{n}$. Let us estimate ℓ_2.

When Constructor ends Stage II, either all of the remaining cherries are connected to at least $\sqrt{n}$ cherries in $\mathcal{B}$, or all of the remaining cherries are connected to x_{k_0} in $\mathcal{B}$. In the first case, Blocker has claimed at least $\frac{\ell_2\sqrt{n}}{2}$ edges between cherries. Since Constructor has claimed less than $\frac{2}{3}(n-3\sqrt{n}+3) \leqslant n$ edges during Stage I, and less than n edges during Stage II, we know that $\frac{\ell_2\sqrt{n}}{2} \leqslant 2n$ and thus $\ell_2 \leqslant 4\sqrt{n}$. In the second case, since by assumption x_{k_0} is connected to at most $\sqrt{n}+2$ cherries in $\mathcal{B}$, $\ell_2 \leqslant \sqrt{n}+2$. In both cases, $\ell_2 \leqslant 4\sqrt{n}$, thus the number of vertices in C_{k_0} is at least $n-3\sqrt{n}-3\cdot 4\sqrt{n}-1 = n-15\sqrt{n}-1$. Hence $k_0 \geqslant \frac{n-15\sqrt{n}-1}{3} = \frac{n}{3}(1+o(1))$. ■

This claim concludes the proof of Proposition 3.5. □

To obtain the upper bound, we show that Blocker can prevent every vertex from being in several triangles, thus upper bounding the total number of triangles at $\frac{n}{3}$. The details are trickier, since Blocker's strategy has to tackle many cases. We will state this as a new proposition as well:

Proposition 3.6. *Blocker has a strategy to ensure that the following holds:*

$$g(n, K_3, S_4) \leqslant \frac{n}{3}(1+o(1)).$$

Proof of Proposition 3.6. We introduce a criterion that we call *priority*. Blocker will play according to Lemma 2.3, but we will use priority to help Blocker choose the exact edge he should play. In this context, we say that an edge is *free* if it has not been played by any of the two players and if its endvertices are in the same connected component of $\mathcal{C}$ and have degree 1 or 2. Let $e = uv$ be a free edge. Its priority number is:

(1) if $\mathcal{C} \cup \{e\}$ has 2 more triangles than $\mathcal{C}$.
(2) if (1) is not satisfied and $d_{\mathcal{C}}(u) = d_{\mathcal{C}}(v) = 1$.
(3) if (1) is not satisfied and exactly one vertex among u, v has degree 1 in $\mathcal{C}$.
(4) if (1) is not satisfied and $d_{\mathcal{C}}(u) = d_{\mathcal{C}}(v) = 2$.

Assume Constructor plays the edge e. If there is a free edge that might complete a triangle with e, then Blocker plays it. If there are several such free edges, then he plays the one with the smallest priority number (breaking the ties arbitrarily). If Constructor just completed a triangle t with e and if there is no free edge that might complete another triangle with e, then

Blocker looks for a free edge that might complete a triangle adjacent to t. If there is one, then Blocker blocks it (if there are several such edges, he uses the priority order). Otherwise he plays arbitrarily.

Claim 8. *If Blocker follows this strategy, then at any time during the game, at most one edge has priority number (1).*

Proof of the claim. Since such an edge is immediately claimed by Blocker when it appears, it is enough to verify that two edges of priority (1) cannot appear simultaneously. Edges of priority (1) are diagonals of C_4. If an edge is in two C_4, then none of the diagonals can be played by Constructor (because at least one of the endvertices has degree 3). Therefore the only way for two edges of priority (1) to appear simultaneously is if Constructor completes an isolated C_4 with no diagonal blocked. However, since Blocker has played in the same connected component as Constructor, he necessarily has claimed one of the diagonals before this C_4 was completed. Hence the claim is proven. ■

We now show that each edge in $\mathcal{C}$ cannot be in several triangles. It is enough to show that there is no *diamond* in $\mathcal{C}$ (i.e. a C_4 with one diagonal, see Figure 3.5 for a drawing) whenever Blocker follows the strategy described above.

Claim 9. *If Blocker follows this strategy, then no diamond can occur in $\mathcal{C}$.*

Proof of the claim. Consider the four vertices v_1, v_2, v_3, v_4 and the five edges $v_1v_2, v_2v_3, v_3v_4, v_1v_4, v_1v_3$, making a diamond (see Figure 3.5). We want to show that these edges cannot all be claimed by Constructor. If this were to happen, then vertices v_2 and v_4 might have degrees 2 or 3. We will consider these different cases, one of which is illustrated in Figure 3.5.

- **Case 1:** $d_{\mathcal{C}}(v_2) = d_{\mathcal{C}}(v_4) = 2$.
  Since the connected component of v_1, v_2, v_3, v_4 is of size four, Blocker has always answered in it whenever Constructor has played in it. Therefore, by Lemma 2.3, the five edges cannot all be claimed by Constructor during the game.
- **Case 2:** $d_{\mathcal{C}}(v_2) = 3$, $d_{\mathcal{C}}(v_4) = 2$.
  - If Constructor manages to play all the edges but v_1v_3, then Blocker will block this edge because it will be the only one with priority number (1) (by the first claim).
  - If Constructor manages to play all the edges but v_1v_4, then let e be the last edge played by Constructor among $v_1v_2, v_2v_3, v_3v_4, v_1v_3$.
    - If $e = v_1v_3$ or v_3v_4, then after this move the only free edge that might complete a triangle with e is v_1v_4, so Blocker claims it. This case is illustrated in Figure 3.5.

    * If $e = v_1v_2$, then we look back at the time when the last edge between v_1v_3 and v_3v_4 was played by Constructor. If v_2 had degree 2, then v_1v_4 was the only edge with priority (2), so Blocker has already claimed it. Otherwise, observing the first case, Blocker would have played two edges among v_1v_2, v_2v_4, v_1v_4. Since Constructor played v_1v_2 later, Blocker necessarily played v_1v_4.
    * If $e = v_2v_3$, then as before we look back at the time when the last edge between v_1v_3 and v_3v_4 was played by Constructor. In order to prevent a triangle, Blocker has already played v_1v_4 or v_2v_3. Since Constructor played v_2v_3 later, Blocker necessarily played v_1v_4 at that time.
  - If Constructor manages to play all the edges but v_1v_2, then let e be the last edge played by Constructor among $v_2v_3, v_3v_4, v_1v_4, v_1v_3$.
    * e cannot be v_2v_3 because Constructor cannot build an isolated triangle.
    * If $e = v_1v_3$, then when Constructor played it Blocker necessarily played v_1v_2.
    * If $e = v_1v_4$, then v_2 had degree 1 when all other three edges were claimed (otherwise Blocker would have played v_1v_4 because this would have been the only edge with priority (1)). But this is covered by Case 1.
    * If $e = v_3v_4$, then when the last edge between v_1v_3 and v_1v_4 was played, Blocker necessarily played v_1v_2 or v_3v_4 (and it cannot be v_3v_4 because this was played by Constructor later).
- **Case 3:** $d_{\mathcal{C}}(v_2) = d_{\mathcal{C}}(v_4) = 3$.
  - If Constructor manages to play all the edges but v_1v_3, then again Blocker will block it because of the first claim.
  - Otherwise, by symmetry, assume that Constructor manages to play all the edges but v_1v_4 and let e be the last edge played by Constructor among $v_1v_2, v_2v_3, v_3v_4, v_1v_3$.
    * If $e = v_1v_3$ or v_3v_4, then after this move the only free edge that might complete a triangle with e is v_1v_4, thus Blocker claims it. Indeed v_3 already has degree 3.
    * If $e = v_1v_2$, then we look back at the time when the last edge between v_1v_3, v_2v_3, and v_3v_4 was played by Constructor. If at that time both v_2 and v_4 had degree one, then at the end of his turn Blocker would have claimed two edges among v_1v_2, v_1v_4, and v_2v_4. If it is not the case then because of the priority order Blocker claimed v_1v_2 or v_1v_4, which means that he claimed v_1v_4 because we assumed that the other one was played by Constructor later.
    * If $e = v_2v_3$, then let v_5 be the third vertex adjacent to v_2 (at the end of the process) and look back at when the last edge between v_1v_2 and v_1v_3 was played by Constructor. Blocker then either claimed v_1v_4, v_2v_3, or v_1v_5. In the latter case, when Constructor plays v_2v_3 Blocker will answer according to the part of

his strategy when there is no free edge that might complete a triangle with v_2v_3 (because v_3 will have degree 3), and therefore he will play v_1v_4.

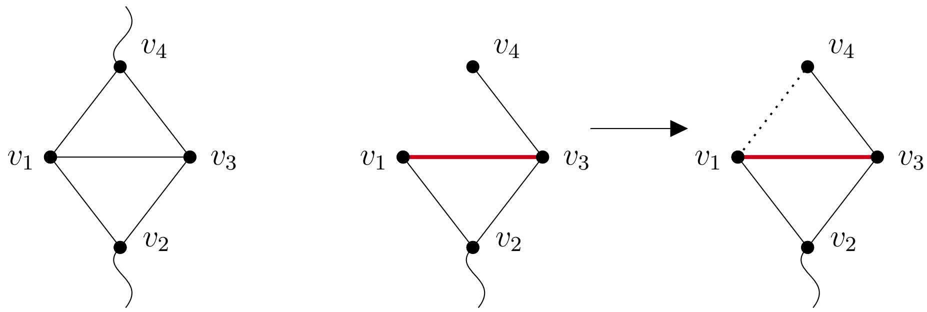


FIGURE 3.5. Support for the proof of Proposition 3.6: the diamond, and one of the many cases. The curved lines on the first diamond represent edges that might or might not be there. Constructor's last move is in bold red, and Blocker's answer is in dotted line.

In all cases, Blocker can prevent a diamond from appearing in $\mathcal{C}$. ■

Since Constructor's graph will not contain any diamond, each edge in $\mathcal{C}$ can be in at most one triangle. Since each vertex has degree at most 3 in $\mathcal{C}$, it cannot be in two triangles. This implies that the number of triangles in $\mathcal{C}$ is at most $\frac{n}{3}$. □

3.3.2. *Forbidding S_5.* In this section we will prove Theorem 1.4. In contrary to the previous case of forbidding S_4, here our bounds are not tight. As before we split the proof into two parts and the proofs of two propositions, from which Theorem 1.4 will then trivially follow. Let us start with the lower bound and a strategy for Constructor.

In order to obtain the lower bound, we will reuse the big chain of triangles created via the method presented in the proof of Theorem 1.3 and show that when we allow Constructor to have vertices of degree 4, she can roughly double the number of triangles by connecting vertices of this chain in a certain way. We will state the result as a new proposition:

Proposition 3.7. *Constructor has a strategy to ensure that the following holds:*

$$g(n, K_3, S_5) \geqslant \frac{2n}{3}(1 + o(1)).$$

Proof of Proposition 3.7. We provide a strategy for Constructor to build at least $\frac{2n}{3} - o(n)$ triangles in two stages. We illustrate the second stage in Figure 3.6.

Stage I: During this stage, Constructor applies the strategy presented in the proof of Proposition 3.5 to obtain a chain of $k_0 = \frac{n}{3} - o(n)$ triangles on $3k_0 + 1$ vertices. Once she is done constructing this chain, we number these triangles in an arbitrary order. Then, for the i^{th} triangle, let s_i be the vertex of degree 2 in $\mathcal{C}$ and let l_i, r_i be its neighbours. Afterwards, Constructor proceeds with Stage II.

Stage II: During this stage, Constructor will pair the triangles created in Stage I and connect those in a certain way. While there exist $i \neq j$ such that the sets $\{s_i, l_i, r_i\}$ and $\{s_j, l_j, r_j\}$ are not connected in $\mathcal{B}$, Constructor will play the following three moves and then repeat this process.

She starts by claiming $s_i s_j$. Then pair the edges $s_i l_j$ and $s_i r_j$, and also pair the edges $s_j l_i$ and $s_j r_i$. If Blocker plays a paired edge, then Constructor plays the other one. Otherwise she plays one of these four edges arbitrarily. Constructor repeats this move once (playing adjacent to s_j if she played adjacent to s_i before). After these two moves, Constructor will have created two more triangles in her graph while maintaining degree at most 4.

Once there are no more free edges as required available any more, Constructor stops playing.

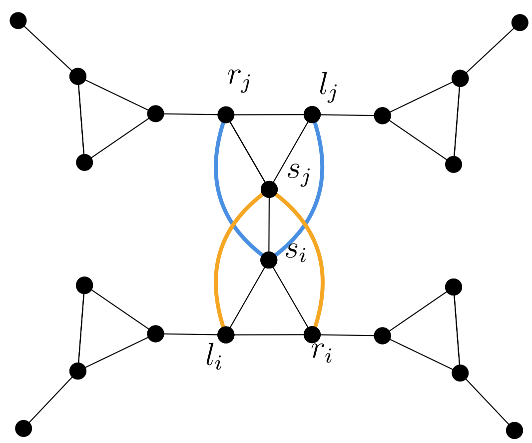


FIGURE 3.6. The second phase of Constructor's strategy in the game $K_3 - S_5$. The blue edges are paired together, and so are the orange ones.

Strategy discussion:

The proof of Proposition 3.5 ensures that Constructor can play according to this strategy for Stage I, and there is nothing to check for Stage II. We now show that this strategy gives the right number of triangles.

Claim 10. *During Stage II, all but at most $o(n)$ triangles are paired.*

Proof of the claim. Denote by r the number of triangles that are in the chain C_{k_0} but that are not paired to any other triangle during Stage II. At the end of the game, all of these r triangles are pairwise connected in $\mathcal{B}$. During the whole game, Constructor has played less than $2n$ edges, which means, that Blocker also played less than $2n$ edges. Therefore we get $\frac{r(r-1)}{2} < 2n$, and solving this in r yields $r \leqslant 2\sqrt{n}$ (for n large enough). ■

All the triangles that were paired added one more triangle in $\mathcal{C}$ (actually each pair of two triangles added two more triangles). Since the number of triangles at the end of Stage I was $\frac{n}{3} - o(n)$, the number of triangles created by Constructor during Stage II is $\left(\frac{n}{3} - o(n)\right) - o(n)$. Finally, the number of triangles in $\mathcal{C}$ at the end of the game is $\frac{2n}{3} - o(n)$. □

Now let us turn our attention to the upper bound and a strategy for Blocker. To obtain the upper bound of Theorem 1.4, we show that Blocker can prevent any K_4 from appearing in Constructor's graph. By looking at all possibilities for a vertex to be in many (i.e. more than three) triangles, we show that the maximum number of triangles is bounded by n. We also state this result as a new proposition:

Proposition 3.8. *Blocker has a strategy to ensure that the following holds:*

$$g(n, K_3, S_5) \leqslant n(1 + o(1))\,.$$

Proof of Proposition 3.8. We now investigate the upper bound. We first provide a strategy for Blocker ensuring that Constructor does not build a K_4, and we will deduce that via this strategy Blocker prevents Constructor from building more than n triangles. Let us state and prove the following lemma:

Lemma 3.1. *For all $n \geqslant 0$, Blocker has a strategy such that the following holds:*

$$g(n, K_4, S_5) = 0\,.$$

Proof. The strategy we provide for Blocker is very similar to the one presented in the proof of Proposition 3.6, except that there are more cases to handle. Whenever possible, Blocker will play according to Lemma 2.3 and adjacent to Constructor's last move (with some additional priority rules), otherwise Blocker will play arbitrarily.

Similarly as in the proof of Proposition 3.6, an edge is said to be *free* if it has not been played by any of the two players and if its endvertices are in the same connected component of $\mathcal{C}$ and have degree 1, 2, or 3. Let $e = uv$ be a free edge. Its priority number is:

(0) if $\mathcal{C} \cup \{e\}$ has three more triangles than $\mathcal{C}$, and additionally there is no edge between the remaining vertices of these triangles.
(1) if (0) is not satisfied but if $\mathcal{C} \cup \{e\}$ has three more triangles than $\mathcal{C}$.
(2) if (0) and (1) are not satisfied and if $\mathcal{C} \cup \{e\}$ has two more triangles than $\mathcal{C}$.
(3) if (0), (1), and (2) are not satisfied and $\mathcal{C} \cup \{e\}$ has one more edge that is in three triangles than $\mathcal{C}$.
(4) if (0), (1), (2), and (3) are not satisfied and we have $d_{\mathcal{C}}(u) \leqslant 2$ as well as $d_{\mathcal{C}}(u) \leqslant 2$.
(5) otherwise.

Assume Constructor plays the edge e. If there is a free edge that might complete a triangle containing e, then Blocker plays it. If there are several such free edges, then he plays the one with the smallest priority number (breaking ties arbitrarily). Otherwise Blocker plays arbitrarily.

The only way for Constructor to create a K_4 is to create a *dangerous diamond* that will not be blocked by Blocker. We call a dangerous diamond a C_4 in $\mathcal{C}$ with one diagonal in $\mathcal{C}$ and the other diagonal being free. Since edges of priority (0) or (1) are diagonals of dangerous diamonds, Blocker claims such edges as soon as they appears. Assume that at some point during the game all dangerous diamonds have been blocked (thus so far there is no K_4). Let us show that two dangerous diamonds cannot occur in one Constructor's move. This will prove the lemma, because Blocker will be able to block all dangerous diamonds immediately when they appear and thus prevent any K_4.

There are two possibilities for two dangerous diamonds to appear at the same time: either these diamonds share exactly one common side edge, or they share a common diagonal. We present these two possibilities in Figure 3.7, the black edges have to be claimed by Constructor, the yellow ones need to be free for the two diamonds to be dangerous. We also write $\varnothing$ next to vertices that have degree less than 4 in $\mathcal{C}$ but cannot have degree 4 in order for the diamonds to be dangerous.

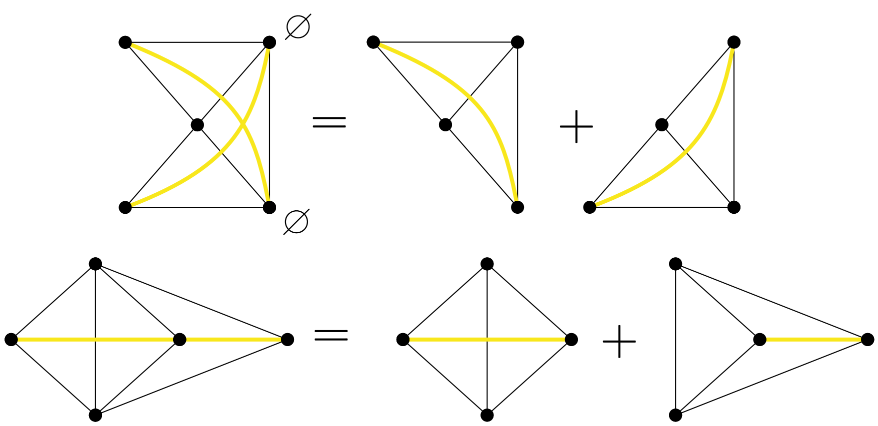


FIGURE 3.7. The two possibilities for having two dangerous diamonds simultaneously. In black, Constructor's edges. In yellow, edges that need to be free.

We examine these two cases and show that Constructor cannot have all the black edges while the yellow ones remain free. To to this, we will further distinguish between black edges whether they have been played by Constructor during her last turn (red) or at a previous point in time (blue). An illustration can be found in Figure 3.8.

Now we present an overview of the cases that can happen during the course of the game and how Blocker's strategy handles all these cases.

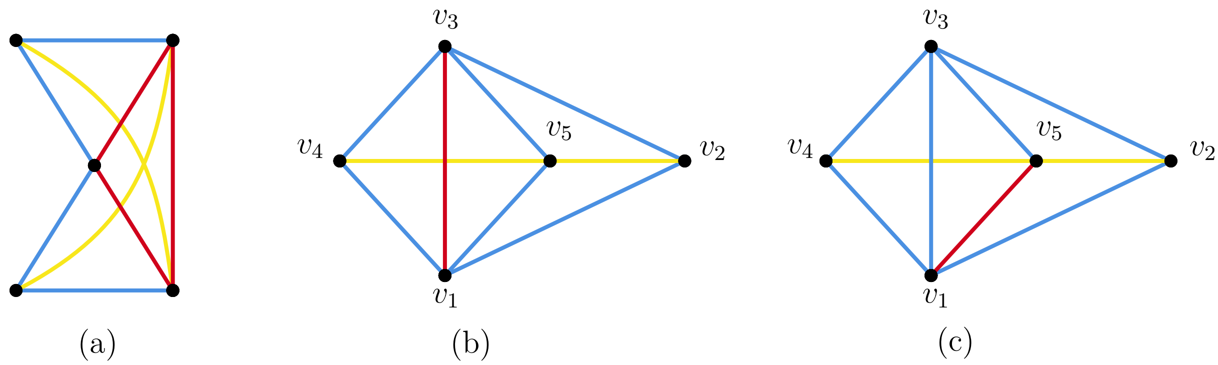


FIGURE 3.8. Support for the proof of Lemma 3.1. In blue and red, Constructor's edges, the last one being red. In yellow, edges that need to be free when Constructor plays her last edge.

- **Case 1:** There is exactly one common side edge.

  This configuration is presented in Figure 3.8.a. The last edge claimed by Constructor needs to be a red one, so that both dangerous diamonds appear at the same time, and the two yellow diagonals need to be free. When the second edge among the red ones was added, Blocker necessarily claimed a yellow, blue, or red edge since there was no other adjacent free edge.
- **Case 2:** There is a common diagonal.

  The two dangerous diamonds are labelled $v_1v_4v_3v_5$ and $v_1v_2v_3v_5$, the diagonals that need to be free being v_4v_5 and v_2v_5 (in yellow in Figures 3.8b and 3.8c).
  - If Constructor manages to play every edge but the common diagonal v_1v_3 (Figure 3.8.b), then this edge has priority (0) or (1). If it has priority (0), then it is the only one (note that if two edges of priority (0) would appear at the same time, it would mean that Blocker would not have always played adjacent to Constructor's moves when he could have), hence Blocker plays it and Constructor cannot claim this red edge.

    If this edge has priority (1), then either v_2v_4, v_2v_5, or v_4v_5 was already blocked. There can be at most one other edge with priority (0) or (1) in this connected component. Since v_2, v_4, v_5 can be connected to at most one other vertex outside of v_1, v_2, v_3, v_4, v_5, the edge with priority (0) or (1) can only be one of the following:
    * v_4v_5: This corresponds to v_4 and v_5 sharing a third common neighbour. But if Blocker claims this one, then not all of the yellow edges will be free.
    * v_2v_4: In this case we said that an edge among v_2v_5 and v_4v_5 (a yellow edge) was already blocked.

In each case, Constructor cannot obtain a graph as in Figure 3.8.b, either because she will not be able to claim the red edge, or because some yellow edge will have already been claimed by Blocker.

– If Constructor manages to play every edge but say v_1v_5 (Figure 3.8.c), then v_1 and v_3 cannot have any other adjacent edges in $\mathcal{C}$, while v_2, v_4, v_5 can each have at most one. Note that v_1v_3 has to be the first edge claimed by Constructor among all of the black edges in Figure 3.7, otherwise Blocker would have blocked one of the others. Let e be the last edge played by Constructor among $v_1v_2, v_1v_4, v_2v_3, v_3v_4, v_3v_5$. Now, we distinguish between the following cases for her choice of e:
  * v_1v_4: When Constructor played the last edge among v_3v_4 and v_3v_5, then clearly no edge of priority at most (3) can have been created, thus Blocker blocked either one of the edges v_1v_4, v_1v_5, v_2v_5, or v_4v_5 (which are yellow, red, or blue edges) or Blocker blocked v_2v_4. In the latter case, when Constructor played e, Blocker blocked v_1v_5 because there was no free edge of priority (0), (1), or (2), and this was the only one of priority (3) (blocking it would prevent v_1v_3 from being in three triangles). So either Constructor cannot claim v_1v_5 (red), or a yellow edge is blocked.
  * v_3v_4: Without loss of generality, assume v_1 reached degree 3 in $\mathcal{C}$ before v_3 did. At that time, Blocker played an edge of priority (4), thus she claimed v_2v_3 or v_3v_4 (which is not possible because we know for sure that Constructor claimed those later), or v_2v_4. In the latter case, when Constructor played e, Blocker blocked v_4v_5 because there was no free edge of priority (0), (1), (2), or (3), and this was the only one of priority (4).
  * v_3v_5: If this was the last edge Constructor claimed, then just before that the vertices v_1, v_2, v_3, v_4 formed a dangerous diamond, and we assumed that at that time Blocker could block all dangerous diamonds, so that is what he did by claiming v_1v_5.
  * We handle all the remaining cases by symmetry.

After examining all cases, we showed that against Blocker's strategy, Constructor cannot create two dangerous diamonds simultaneously. Thus during the whole game every dangerous diamond is neutralized by Blocker and therefore no K_4 appears in $\mathcal{C}$. □

To prove Proposition 3.8, we will apply Blocker's strategy described in Lemma 3.1, and show that when there does not appear a K_4 in $\mathcal{C}$ the number of triangles Constructor claims is at most n (which means that every vertex will be on average in at most 3 triangles). We argue that there cannot be many vertices which are in more than 3 triangles, more precisely that for every such vertex there are some vertices that are in less than 3 triangles.

Naturally, we first investigate the possibilities for a vertex to be in four triangles, in a graph that has maximal degree at most 4 and no K_4. The only way for a vertex v to be in four triangles is to be the center of a wheel of size 4, that is, to have a cycle $v_1v_2v_3v_4$ in $\mathcal{C}$ such that $vv_1, \dots, vv_4 \in \mathcal{C}$. Since the degree of every v_i in $\mathcal{C}$ is at most 4, it can only be in a triangle if this triangle contains at least another vertex among v, v_1, v_2, v_3, v_4. By symmetry, five cases might occur and are illustrated in Figure 3.9:

- A: No vertex v_i is in another triangle.
- B: There is exactly one additional triangle with some other vertex ($v_1v_2u_1$).
- C: There are exactly two additional triangles with two other vertices ($v_1v_2u_1$ and $v_3v_4u_2$).
- D: There are exactly two additional triangles with one other vertex ($v_1v_2u_1$ and $v_1v_4u_1$).
- E: There are exactly three additional triangles with one other vertex ($v_1v_2u_1$, $v_1v_4u_1$, and $v_2v_3u_1$). We can immediately exclude this structure for the following reason. If this structure were to happen, then, since all vertices are of degree 4, this would be a complete connected component. Constructor would have twelve edges in a connected component of six vertices, which is in contradiction to Lemma 2.3.

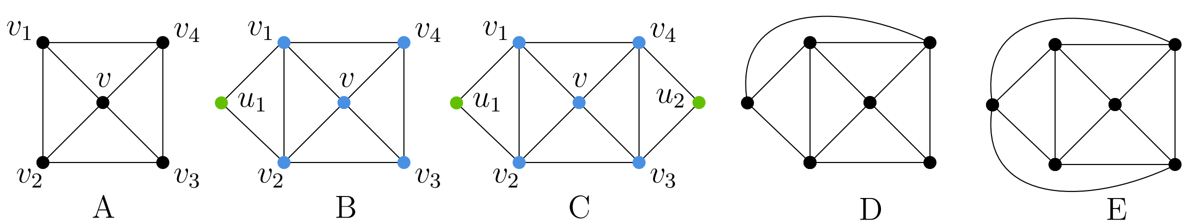


FIGURE 3.9. The structures of types A,B,C,D,E that appear in the proof of Proposition 3.8

We will denote by $S_{\mathrm{A}}, \dots, S_{\mathrm{D}}$ the sets of all structures of type A,. . . ,D in Constructor's graph at the end of the game. Note that the vertices that are in a structure of type A or D cannot be in another structure. When we consider structures of type B or C, the vertices $v, v_1, \dots, v_4$ are in a unique such structure, while the vertices u_1 and u_2 are in at most two structures of such types. Therefore, we can partition the set of all vertices into the following disjoint sets:

- V_{A}: the set of vertices that are in a structure of type A.
- V_{D}: the set of vertices that are in a structure of type D.
- V_α: the set of vertices that are in a structure of type B or C, and labeled as v, v_1, v_2, v_3, or v_4 (colored in blue in Figure 3.9).
- V_β: the set of vertices that are in at least one structure of type B or C, labeled as u_1 or u_2 (colored in green in Figure 3.9).

- V_{U}: the vertices that are in no structure of type A,B,C, or D.

For any vertex v, denote by $t(v)$ the number of triangles containing v. The total number of triangles in Constructor's graph is

$$\frac{1}{3}\sum_{v\in V} t(v) = \frac{1}{3}\left(\sum_{v\in V_{\mathrm{A}}} t(v) + \sum_{v\in V_{\mathrm{D}}} t(v) + \sum_{v\in V_{\alpha}} t(v) + \sum_{v\in V_{\beta}} t(v) + \sum_{v\in V_{\mathrm{U}}} t(v)\right).$$

Let us upper bound these sums.

- By assumption, all the vertices that are in four triangles are in a structure of type A,B,C, or D, thus $\sum_{v\in V_{\mathrm{U}}} t(v) \leqslant 3|V_{\mathrm{U}}|$.
- The vertices that are in a structure of type A or D are in a unique such structure, thus we can partition the sets V_{A} and V_{D} according to the structures the vertices belong to. Moreover, these vertices cannot be in a triangle with vertices that are not in this structure (simply by assumption on $v, v_1, \ldots, v_4$ and because u_1 is already of degree 3). Therefore,

$$\sum_{v\in V_{\mathrm{A}}} t(v) = \sum_{s\in S_{\mathrm{A}}}\sum_{v\in s} t(v) = \sum_{s\in S_{\mathrm{A}}} 12 = \frac{12}{5}|V_{\mathrm{A}}|$$

and

$$\sum_{v\in V_{\mathrm{D}}} t(v) = \sum_{s\in S_{\mathrm{D}}}\sum_{v\in s} t(v) = \sum_{s\in S_{\mathrm{D}}} 18 = \frac{18}{6}|V_{\mathrm{D}}|.$$

- The vertices in V_β that are in a structure s can be in at most one triangle outside of s, either in another structure of type B or C, or with some other vertices of the graph. Therefore,

$$\sum_{v\in V_\alpha\cup V_\beta} t(v) \leqslant \sum_{s\in S_{\mathrm{B}}\cup S_{\mathrm{C}}}\ \sum_{v\in s\cap V_\alpha} t(v) + 2|V_\beta| = 14|S_{\mathrm{B}}| + 16|S_{\mathrm{C}}| + 2|V_\beta|.$$

Since the vertices in V_α are in a unique structure, $|V_\alpha| = 5|S_{\mathrm{B}}| + 5|S_{\mathrm{C}}|$. Moreover, since every structure of type C contains two vertices of V_β and since a vertex of V_β is in at most two structures, it holds that $|S_{\mathrm{C}}| \leqslant |V_\beta|$. Therefore,

$$\sum_{v\in V_\alpha\cup V_\beta} t(v) \leqslant 3|V_\alpha| + |V_\beta| + 2|V_\beta|.$$

Putting it all together, we get that the following:

$$\frac{1}{3}\sum_{v\in V} t(v) \leqslant \frac{1}{3}\left(\frac{12}{5}|V_{\mathrm{A}}| + \frac{18}{6}|V_{\mathrm{D}}| + 3|V_\alpha| + 3|V_\beta| + 3|V_{\mathrm{U}}|\right) \leqslant \frac{1}{3}\cdot 3|V| = n.$$

Therefore, the number of triangles in Constructor's graph is at most n. $\square$

3.3.3. *Forbidding larger stars.* The strategies for Blocker for $k = 4$ and $k = 5$ are extremely tricky because we basically explore all of the possible cases. We were not able to find a more general, nicer approach, not even for these small k. However, we present an idea for a lower bound when k goes to infinity but does not grow too fast in comparison to n.

The idea behind this result is that Constructor can create many stars of size $k(n)$ and then maximizes the number of triangles between the leaves of each star, playing independent copies of the game where we do not forbid anything (see Section 3.1). We illustrate this in Figure 3.10. Even though Blocker can claim some edges while Constructor creates the stars, the number of triangles Blocker blocks becomes negligible.

This bound does not seem tight to us, because there might be many triangles between different stars that we have not accounted for in the lower bound. Moreover, it does not give us any information about what happens when k is fixed.

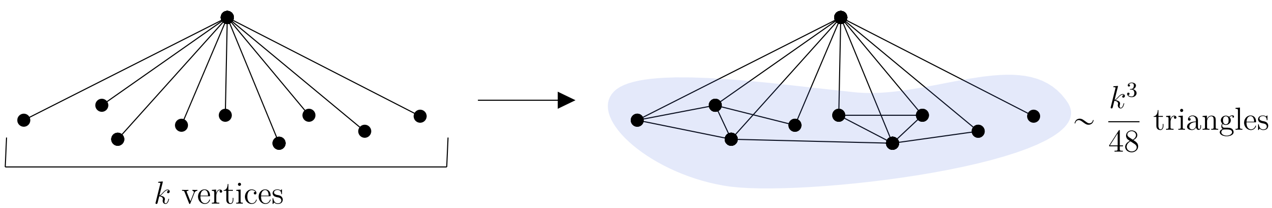


FIGURE 3.10. Constructor's strategy for Theorem 1.5 (one of the stars)

Proof of Theorem 1.5. We will provide a strategy for Constructor. She will start by creating a a star of size k. Then, whenever Blocker plays between leaves of this star, so does Constructor, according to Erdős-Selfridge's strategy. Whenever it is not the case and while there are at least $3k\sqrt{n}$ isolated vertices in $\mathcal{C}$, Constructor picks a set of $2k + 1$ such vertices that are also isolated in $\mathcal{B}$ and creates another star of size k.

We could also look at her strategy as playing on many stars by themselves, where every star that is created by Constructor becomes a game independent from what happens in the rest of the graph, since Constructor always answers in a star whenever Blocker claims an edge between its leaves.

Let us state the following claim, which tells us that Constructor can find large sets of independent vertices for a long time.

Claim 11. *While there are at least $3k\sqrt{n}$ isolated vertices in $\mathcal{C}$, Constructor can find a set of $2k + 1$ vertices that are isolated in $\mathcal{C}$ and not connected in $\mathcal{B}$.*

Proof of the claim. Similarly as in previous proofs, let L be the set of isolated vertices in $\mathcal{C}$ and $\ell := |L|$. Assume $l \geqslant 3k\sqrt{n}$. Then the number of sets of size $2k + 1$ of isolated vertices in $\mathcal{C}$, with at least one Blocker edge, is at most

$$\sum_{S \in L^{2k+1}} \sum_{e \in E(\mathcal{B})} 1_{\{|S \cap e| = 2\}} \leqslant \sum_{e \in E(\mathcal{B}) \text{ not inside of a star}} \binom{\ell}{2k-1}.$$

Since Constructor will use less than n moves to construct her stars, the number of Blocker edges that are not between leaves of a star is also less than n. Thus the previous sum is less than

$$n\binom{\ell}{2k-1} < \binom{\ell}{2k+1}$$

for n large enough. ■

Inside of every star, Constructor tries to maximize the number of triangles between the leaves of this star. Once Constructor finishes a new star, Blocker might already have claimed (at most) k edges among the leaves, that would appear in $k(k-1)$ triangles. Thanks to Theorem 2.1, we know that Blocker can play in such a way, that Constructor does not claim more than $\frac{k^3}{48}(1+o(1)) - O(k^2)$ triangles per star. Therefore, the total number of triangles Constructor creates is at least $\frac{n-3k\sqrt{n}}{k+1}\left(\frac{k^3}{48} - O(k^2)\right)$. Since $k(n) = o(\sqrt{n})$, this becomes $\frac{nk^2}{48}(1+o(1))$. □

We can also wonder what happens when k is greater than $\sqrt{n}$. Constructor could secretly pick $k+1$ vertices and play the game where nothing is forbidden only in the smaller induced subgraph K_{k+1}. Thanks to Theorem 1.1, Constructor can create at least $\frac{k^3}{48}$ triangles. In particular, if k is linear in n, then this tells us that $g(n, K_3, S_{k+1})$ is of order n^3. More precisely, if $k = cn$ with $c \in [0,1]$, then $g(n, K_3, S_{k+1}) \geqslant \frac{c^3n^3}{48}(1+o(1))$. However, we do not know the order of $g(n, K_3, S_{k+1})$ when k is of order between $\sqrt{n}$ and n, especially we wonder if it is of the same order as $ex(n, K_3, S_{k+1})$ – that is nk^2. Another interesting question is when $k = \frac{n}{2}$. We observed via some simulations that if both players follow the Erdős-Selfridge's strategy that gave the result when there is no constraint (see Theorem 1.1 and Theorem 2.1), then Constructor's graph had maximal degree $\frac{n}{2}$. However we do not know how Blocker would play with the knowledge that Constructor cannot add any edge to a vertex of high degree. We wonder if the result is the same as with no degree constraint or if the constant factor is different.

4. Results on the planar and embedded Constructor-Blocker game

In this section, we will focus on our versions of game where we focus on planarity and present our proofs of Theorem 1.6 and Theorem 1.7.

4.1. **The Planar Constructor-Blocker game.** We will start with the planar Constructor-Blocker game and the proof of Theorem 1.6. Before we proceed to prove our result, let us first state a trivial upper bound to $g_{\mathrm{PCB}}(n, K_3)$.

Proposition 4.1 ([9]). *For $n \geqslant 3$, the maximum number of triangles in a planar graph is $3n-8$.*

The extremal graphs matching this number are called *Apollonian networks* (see Figure 4.1) and can be constructed inductively. Start with one triangle, then in each step add a new vertex to some face and connect it to the three vertices of this face.

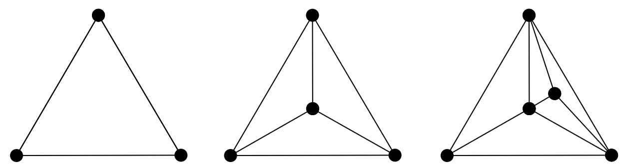


FIGURE 4.1. The Apollonian networks on three, four, and five vertices

Since Constructor can match this bound from below, we will only consider a strategy for her. Before we go into more details, let us briefly give a sketch of Constructor's strategy. She will try to build a graph which is very close to an Apollonian network. At the beginning of the game, Constructor will take a few turns to build a fan of size 6. Then, as in the inductive construction of the Apollonian networks, Constructor will be able to create three triangles with every new vertex: She will connect one new vertex to the center of the fan, and then (since she does not need to tell Blocker on which face this vertex will end up) she has many possibilities to create new triangles, and Blocker cannot block all of them.

Proof of Theorem 1.6. In the following, we will denote the center of an F_k by v and the k other vertices that form a path by $v_1, \ldots, v_k$. We call a vertex red if it is the center of a wheel of size at least 4 or of a fan of size at least 6. We call a vertex orange if this is not true, but it is the center of a W_3. Orange vertices might turn red during the game, and the red vertices will remain red forever. Before we state Constructor's strategy, we will prove two useful claims.

Claim 12. *Constructor can build an F_6 using 12 moves and 8 vertices.*

Proof of the claim. Following the strategy of Lemma 2.2, Constructor can build a triangle with a pendant leg within four moves (note that this does not break any planarity constraints), and afterwards she arbitrarily denotes these vertices as v_1, v_2, and v_3. Then (after Blocker's turn) she picks a vertex u that is isolated from this triangle, both in $\mathcal{C}$ and $\mathcal{B}$, and claims the edge uv_1. Now, Blocker cannot block both uv_2 and uv_3, so Constructor can claim one of those edges after Blocker's next turn. Constructor successively adds four vertices this way and creates a fan with center v_1 and leaves (after relabelling them) $w_1, \ldots, w_6$ such that $w_1w_2, \ldots, w_5w_6 \in \mathcal{C}$. She can create such a fan within a total of $4 + 4 \cdot 2 = 12$ moves. ∎

We will say that a vertex u is *safe* from a W_4 (resp. a W_5 or an F_6) if it is not connected to any of the five (resp. six or seven) vertices of this structure in $\mathcal{B}$. The following claim will be crucial for Constructor in the creation of three new triangles for every vertex she adds to her graph.

Claim 13. ***(Crucial claim)*** *If u is safe from a W_4, a W_5, or an F_6, then, in three moves, Constructor can build three new triangles with v, and v becomes an orange vertex.*

Proof of the claim. The following strategy is illustrated in Figure 4.2.

- **Case 1:** If v is the center of a W_4 (with leaves $v_1, \dots, v_4$) and u is not connected to $v, v_1, \dots, v_4$ in $\mathcal{B}$, then Constructor plays uv. By symmetry, assume that Blocker plays uv_4 (or Blocker does not connect any of the v_i to u). Constructor then plays uv_2. Now, Blocker cannot block both uv_1 and uv_3, thus Constructor can build a K_4 centered at u. This strategy can easily be adapted to the case of a W_5.
- **Case 1:** If v is the center of an F_6 (with leaves $v_1, \dots, v_6$) and u is not connected to $v, v_1, \dots, v_6$ in $\mathcal{B}$, then Constructor plays uv. By symmetry, assume that Blocker plays uv_i for some $i \geqslant 4$ (or that Blocker does not connect any of the v_i to u). Constructor then plays uv_2. Now, Blocker cannot block both uv_1 and uv_3, thus Constructor can build a K_4 centered at u. Furthermore u becomes the center of a W_3 (with leaves among $v, v_1, \dots, v_6$).

■

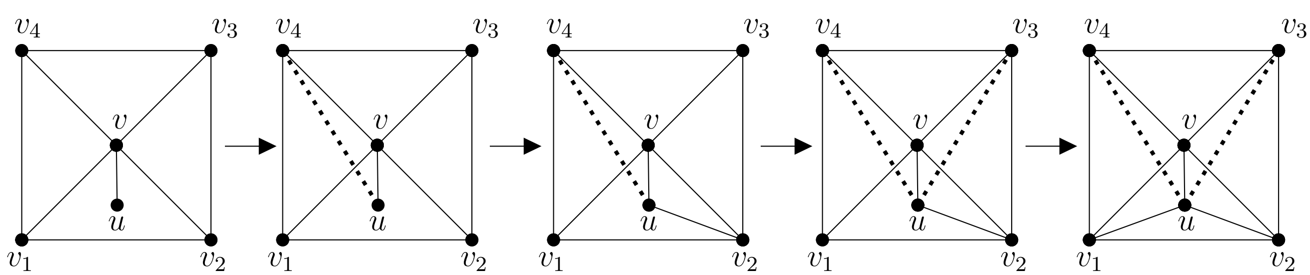


FIGURE 4.2. Constructor's strategy to create three triangles with a new vertex u and a W_4. Constructor's moves are in plain lines, Blocker's moves are in dotted lines.

With these claims at hand, we can now describe the global strategy for Constructor in three stages.

Stage I: Constructor creates an F_6 using at most twelve moves and eight vertices, then she proceeds to Stage II.

Stage II: During this stage, Constructor builds up a set $\mathcal{S}$ of structures (for example fans of size 6) that will used later to create further triangles. She repeats the following process $\frac{n}{105}$ times:

(1) First she picks a red vertex v that is not the center of a structure in $\mathcal{S}$. Let $S \in \mathcal{S}$ be an associated W_4, W_5, or F_6.

Remark. The first vertex she picks will be the center of the F_6 created in Stage I. Afterwards, new red vertices will appear during this stage.

(2) Constructor applies successively the crucial claim 26 times, with vertices that are isolated in $\mathcal{C}$ and safe from S. By the pigeonhole principle, there will be at least one face of S that contains at least six of these vertices. These vertices, together with v, will form a new F_6, which we then add to $\mathcal{S}$. Note that these six vertices were in no structure of $\mathcal{S}$ before. They will be either orange or red, but at least one of them will be red and can take over the role of v during the next iteration of Stage II.

Once Constructor is done with Stage II, she proceeds to Stage III.

Stage III: While there are more than $6 \cdot 105 + 1$ isolated vertices in $\mathcal{C}$, Constructor picks an isolated vertex u and a structure $S \in \mathcal{S}$ such that u is safe from S. She then creates three triangles according to the strategy in the crucial claim. When there are not enough isolated vertices available any more, she stops playing.

Strategy discussion:

Let us show, that Constructor can actually follow her strategy and during this process creates at least $3n - o(n)$ triangles. She can follow Stage I because of Claim 12. For Stage II, we will show that there are enough vertices available for Constructor as desired.

The total number of edges that Constructor (and therefore also Blocker) plays during Stage I and Stage II combined is upper bounded by $12 + 3 \cdot 26 \cdot \frac{n}{105} = 12 + \frac{78n}{105}$. Thus, if we pick a fixed $S \in \mathcal{S}$, at most $12 + \frac{78n}{105}$ isolated vertices are not safe from S, and we can find enough safe vertices to repeat the process during Stage II.

For Stage III, note that a vertex can be in at most two elements of $\mathcal{S}$. Indeed, it can be used only once as a center of a fan, and only once as a leaf of a fan (because the leaves of fans are vertices that were not in an element of $\mathcal{S}$ before). Thus one Blocker edge can make one vertex not safe from at most two elements of $\mathcal{S}$. During the whole game, Constructor – and therefore Blocker as well – will play less than $3n$ edges. Therefore, at any time during Stage III, the number of pairs (u, S) which are unavailable for Constructor is upper bounded by

$$\begin{aligned}
|\{(u,S) \mid u \text{ isolated in } \mathcal{C}, S \in \mathcal{S}, u \text{ not safe from } S\}| &= \sum_{S\in\mathcal{S}} \sum_{u \text{ isolated in } \mathcal{C}} \sum_{uv\in E(\mathcal{B})} 1_{\{v\in S\}} \\
&= \sum_{uv\in E(\mathcal{B})} \sum_{S\in\mathcal{S}} 1_{\{u \text{ isolated in } \mathcal{C}\}} 1_{\{v\in S\}} \\
&\leqslant 2 \cdot |E(\mathcal{B})| \\
&\leqslant 6n\,.
\end{aligned}$$

However, during Stage III, the following holds true:

$$|\{(u,S) \mid u \text{ isolated in } \mathcal{C}, S \in \mathcal{S}\}| \geqslant \frac{n}{105} \cdot (6 \times 105 + 1) = 6n + \frac{n}{105} > 6n\,.$$

Thus, Constructor can always find a pair (u,S) with u isolated in $\mathcal{C}$ and safe from S, and construct three new triangles with u.

Since Constructor can follow her strategy, at the end of the game she used $n - o(n)$ vertices which each created 3 triangles, thus the number of triangles is asymptotically equivalent to $3n$. □

4.2. **The Embedded Constructor-Blocker game.** This section is devoted to the proof of Theorem 1.7. Since here our bounds do not match, we split the proof into two parts, corresponding to the lower bound (Proposition 4.2) and the upper bound (Proposition 4.3). As before, we first present the main ideas for both player's strategies to achieve their respective bounds, as well as a result for the non-game version to compare our results to.

To get this aforementioned comparison for our result, let us briefly look at the maximum number of triangles a planar graph embedded with vertex set $\mathbb{U}_n$ (the n^{th} roots of unity) can have. Such a graph is an outerplanar graph, that is a planar graph where all vertices are on the same face, and this question reduces to the maximum number of triangles in an outerplanar graph with n vertices. For such graphs it is known that these can have at most $n-2$ triangles.

Now let us talk about strategies. Constructor will play a strategy in two phases. First, she claims edges between vertices that are (almost) consecutive and thus she creates a polygon of size between $\frac{n}{2}$ and n. Then she builds many triangles inside of this polygon, basically by taking edges between vertices that are close to each other.

Considering a strategy for Blocker, he can claim around $\frac{n}{3}$ distinct outer edges (i.e. edges between two consecutive vertices). At the end of the game, we can contract these edges (i.e. merge their endvertices) without losing any triangle of $\mathcal{C}$. With this operation we end up with a graph on at most $\frac{2n}{3}$ vertices, thus the number of triangles is at most $\frac{2n}{3} - 2$.

In both strategies, the outer edges seem crucial. Our intuition is the following: Whenever Constructor plays some edge e, all the edges crossing e become useless since Constructor cannot play them any more. Thus Blocker does not have to block all these edges and can thus focus on somewhere else. The further away the endvertices of the edge e are, the more edges it crosses and thus the more it helps Blocker in his job of blocking many triangles. In the same vein, Constructor wants the endvertices to be close to each other to not destroy too many potential triangles.

Now, let us present a strategy for Constructor for the lower bound. Therefore, let us state Constructor's side as a new proposition:

Proposition 4.2. *Constructor has a strategy to ensure that the following holds:*

$$g_{\mathrm{ECB}}(n, K_3) \geqslant (1 + o(1))\frac{n}{2}\,.$$

Proof of Proposition 4.2. We provide a strategy for Constructor in two stages:

Stage I: During the first m turns of the game, Constructor creates a polygon $\mathcal{P}$ of size m (where $\frac{n}{2} \leqslant m \leqslant n$). Details of how she achieves this can be found in the strategy discussion. Afterwards, Constructor proceeds to Stage II.

Stage II: Constructor creates a graph that is close to a triangulation of $\mathcal{P}$, i.e. she creates many triangles inside of $\mathcal{P}$. Details can be found in the strategy discussion. Once there are no more edges available, Constructor stops playing.

Strategy discussion:

We first focus on Stage I, when Constructor wants to build a large polygon with not so many Blocker edges inside of it. Let us state and prove the following claim:

Claim 14. *For any $n \geqslant 1$, Constructor can claim all (maybe except for one) edges of a polygon $\mathcal{P}$, that has $m \in \left[\frac{n}{2}, n\right]$ edges, and such that Blocker will have claimed at most $2m - n$ edges inside this polygon.*

Proof of the claim. We give the precise strategy for Constructor to achieve this goal. First, we order the vertices $v_1, \ldots, v_n$ clockwise. Then Constructor starts by playing v_1v_2, and then initialises the sets $V_C := \{v_1, v_2\}$ and $V_B := \varnothing$.

Now, whenever Blocker plays v_iv_j with $i < j$, if $v_j \notin V_B \cup V_C$, then we add it to V_B, otherwise we do not change the sets.

Next, assume it is Constructor's turn and let $i := \max\{j, v_j \in V_C\}$ and $k := \min\{j > i, v_j \notin V_B\}$. Constructor then plays v_iv_k and we add v_k to V_C. If at some point $\{j > i, v_j \notin V_B\} = \varnothing$, then Constructor plays v_iv_1 (if this edge was not taken by Blocker). For Stage II, we will consider that this edge is claimed by Blocker, if it is not the case then the number of triangles completed by Constructor might decrease by one. The polygon $\mathcal{P}$ is made of the vertices of V_C that are connected in increasing order. An example is given in Figure 4.3.

Now let us analyze the number of Blocker edges inside of $\mathcal{P}$ with regards to the size of $\mathcal{P}$. Denote by m the number of edges – and therefore of vertices – in $\mathcal{P}$. Since $V(C) \cup V(B) = \mathbb{U}_n$ and $|V_B| \leqslant |V_C|$, we immediately have $m \geqslant \frac{n}{2}$. Moreover, the number of Blocker edges between vertices of V_C is at most $m - (n - m) = 2m - n$. Indeed each player played m edges and from

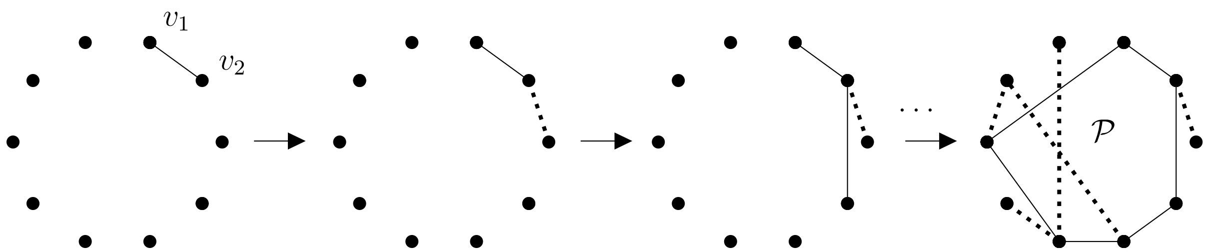


FIGURE 4.3. An example of Constructor creating her polygon, in the early stages of the process and at the end. Blocker's edges are displayed as dotted lines.

these Blocker used $n-m$ moves to put all of the $n-m$ vertices into V_B (thus he played edges that are outside of $\mathcal{P}$). ■

We now explain how Constructor can (partially) triangulate $\mathcal{P}$ during Stage II. We forget about the vertices in V_B and Blocker's edges that were adjacent to those vertices. We relabel the vertices of our polygon $\mathcal{P}_0 := \mathcal{P}$ as $v_1(0) < v_2(0) < \cdots < v_m(0)$. Once both players will have played t further moves, we will consider another polygon $\mathcal{P}_t$ (see below) of size s_t and denote its vertices by $v_1(t) < \cdots < v_{s_t}(t)$. We will also denote by l_t the number of Blocker's edges between vertices of $\mathcal{P}_t$, and by T_t the number of triangles in $\mathcal{C}$ at this time (when each player has played $m+t$ moves in total). Recall that $l_0 \leqslant 2m-n$.

We will call a *k-ear* of $\mathcal{P}_t$ an edge between $v_i(t)$ and $v_{i+k+1}(t)$ for some i. The strategy for Constructor when she has to play her $(t+1)^{th}$ move is the following:

Assume that there is at least one Blocker edge and one available edge in $\mathcal{P}_t$. Let $k_t = k$ be the minimal integer such that all of the 1-ears, 2-ears,..., $(k-1)$-ears of $\mathcal{P}_t$ are blocked but not all of the k-ears. Two cases arise and will be illustrated in Figure 4.4:

(a) If there is a Blocker's edge in $\mathcal{P}_t$ that is not a 1-ear,...,$(k-1)$-ear, then there is an available k-ear that crosses such an edge, which Constructor can claim as her next move. Then she keeps playing in $\mathcal{P}_{t+1}$, the polygon with $s_{t+1} = s_t - k$ vertices that she just created.
(b) Otherwise, Constructor plays any $(k+1)$-ear, say $v_1(t)v_{k+2}(t)$, and now consider the polygon $\mathcal{P}_{t+1}$ with $s_{t+1} = s_t - k - 1$ vertices that she just created. Whenever Blocker plays inside of $\mathcal{P}_{t+1}$, Constructor answers according to her strategy, and whenever Blocker claims $v_1(t)v_{k+1}(t)$ or $v_2v_{k+2}(t)$ Constructor claims the other one of those edges, thus creating one triangle. We can therefore assume that the latter immediately happens without loss of generality.

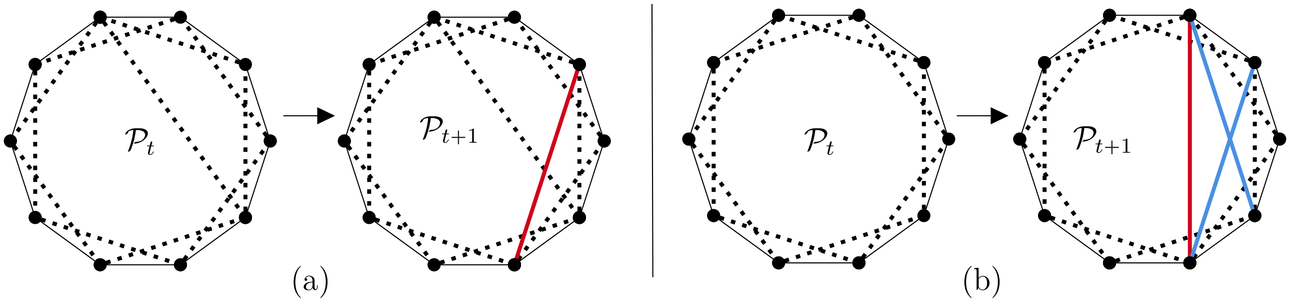


FIGURE 4.4. Constructor's strategy to triangulate her polygon. In both examples, $k = 2$, Blocker's edges are depicted in dotted lines, and the red edge is Constructor's next move. In case (b), Constructor takes a $(k+1)$-ear, and pairs the blue edges, thus she will be able to create a triangle with one of them.

Constructor stops this strategy at t_f when either all of the edges inside $\mathcal{P}_t$ are claimed by Blocker or all of the edges inside $\mathcal{P}_t$ are free. In the latter case, the following claim shows that Constructor then has a strategy to perfectly triangulate $\mathcal{P}_t$.

Claim 15. *If at time t_f there is no Blocker edge in $\mathcal{P}_{t_f}$, then Constructor can perfectly triangulate $\mathcal{P}_{t_f}$ and thus create $s_{t_f} - 2$ more triangles.*

Proof of the claim. We proceed by induction on s_{t_f}. If $s_{t_f} = 4$, then, even if there is one Blocker edge, Constructor is sure to completely triangulate her polygon. If $s_{t_f} \geqslant 5$, then Constructor can claim any 1-ear and build one triangle. Blocker will be able to play only one edge in $\mathcal{P}_{t+1}$ and Constructor will then claim a 1-ear crossing this edge, and so on. ■

Let us compute the number of triangles T_{t_f} next.

Claim 16. *During the game, if Constructor follows her strategy, the number of triangles in her graph is at least*

$$T_{t_f} \geqslant m - s_{t_f} - \frac{l_0 - l_{t_f}}{2}\,.$$

Proof of the claim. Let us see how T_t, s_t, and l_t evolve depending on the cases:

(a) Let us say that the ear Constructor plays is $v_i(t)v_{i+k+1}(t)$. Then for all $1 \leqslant j \leqslant k-1$, the $k+j+1$ edges $v_{i-j}(t)v_{i+1}(t), \dots, v_{i+k}(t)v_{i+k+j+1}(t)$ plus the additional existing Blocker edge were in $\mathcal{P}_t$ but not in $\mathcal{P}_{t+1}$. In one move Blocker can claim one more edge in $\mathcal{P}_{t+1}$. Thus $l_t - l_{t+1} \geqslant \sum_{j=1}^{k}(k+j-1) + 1 - 1 = \frac{(k-1)(3k+2)}{2}$. Therefore, in this case we get the following:

$$\begin{aligned} T_{t+1} &= T_t + 1_{\{k=1\}}\,, \\ s_{t+1} &= s_t - k\,, \\ l_{t+1} &\leqslant l_t - \tfrac{(k-1)(3k+2)}{2}\,. \end{aligned}$$

(b) Note that since in this situation there is at least one Blocker edge and $k \geqslant 2$. For all $1 \leqslant j \leqslant k-1$, the $k+j+2$ edges $v_{s_t-j+1}(t)v_2(t), \ldots, v_{k+1}(t)v_{k+2+j}(t)$ were in $\mathcal{P}_t$ but not in $\mathcal{P}_{t+1}$. In one move Blocker can claim one more edge in $\mathcal{P}_{t+1}$. Therefore, in this case we get the following:

$$\begin{aligned} T_{t+1} &= T_t + 1\,, \\ s_{t+1} &= s_t - k - 1\,, \\ l_{t+1} &\leqslant l_t - \tfrac{(k-1)(3k+4)}{2} + 1 \leqslant l_t - \tfrac{(k-1)(3k+2)}{2}\,. \end{aligned}$$

Thus, using telescopic sums we get the following:

$$\begin{aligned} l_0 - l_{t_f} &= \sum_{t=0}^{t_f-1} (l_t - l_{t+1}) \\ &\geqslant \sum_{t=0}^{t_f-1} \frac{(k_t-1)(3k_t+2)}{2} = \sum_{t=0}^{t_f-1} 1_{\{k_t \geqslant 2\}} \frac{(k_t-1)(3k_t+2)}{2} \\ &\geqslant \sum_{t=0}^{t_f-1} 1_{\{k_t \geqslant 2\}} \frac{4k_t}{2} \end{aligned}$$

and

$$\begin{aligned} m - s_{t_f} - T_{t_f} &= \sum_{t=0}^{t_f-1} (s_t - s_{t+1} + T_t - T_{t+1}) = \sum_{t=0}^{t_f-1} (k_t - 1_{\{k_t=1\}}) = \sum_{t=0}^{t_f-1} k_t 1_{\{k_t \geqslant 2\}} \\ &\leqslant \frac{l_0 - l_{t_f}}{2}\,, \end{aligned}$$

thus we have

$$T_{t_f} \geqslant m - s_{t_f} - \frac{l_0 - l_{t_f}}{2}\,.$$

■

Finally let us bound the number of triangles in $\mathcal{C}$ at the end of the game.

- If at time t_f all the edges inside of $\mathcal{P}_{t_f}$ are claimed by Blocker, then

$$l_{t_f} = \frac{s_{t_f}(s_{t_f}-1)}{2} - s_{t_f} = \frac{s_{t_f}(s_{t_f}-3)}{2}$$

  and

$$T_{t_f} \geqslant m - \frac{l_0}{2} + s_{t_f} \cdot \frac{s_{t_f}-7}{4} \geqslant m - 4 - \frac{l_0}{2}\,.$$

- Otherwise, all the edges inside of $\mathcal{P}_{t_f}$ are free, and we know from Claim 15 that Constructor can perfectly triangulate $\mathcal{P}_{t_f}$. Therefore, in this case, the number of triangles in $\mathcal{C}$ at the end of the game is at least $T_{t_f} + s_{t_f} - 2 \geqslant m - \frac{l_0}{2}$.

In both cases, using $l_0 \leqslant 2m-n$, we get that the number of triangles in $\mathcal{C}$ at the end of the game is at least $\frac{n}{2}-4$ (actually $\frac{n}{2}-5$ if we consider the case where the polygon $\mathcal{P}$ was not actually closed). □

Let us switch to Blocker's side now. The upper bound we provide does not match our lower bound, but still gives a non-trivial upper bound on the number of triangles Constructor can create. We can state this as a new proposition:

Proposition 4.3. *Blocker has a strategy to ensure that the following holds:*

$$g_{\mathrm{ECB}}(n, K_3) \leqslant (1+o(1))\frac{2n}{3}\,.$$

Proof of Proposition 4.3. The strategy we present for Blocker consists of taking as many non-adjacent outer edges as possible. We first count the number of such edges Blocker can claim.

Claim 17. *Blocker can claim at least $\lfloor \frac{n}{3} \rfloor$ non-adjacent outer edges.*

Proof of the claim. We give a strategy for Blocker. Assume that Constructor's first move is an outer edge (or else can pretend that she claimed an outer edge). Order the outer edges $\{1,\dots,n\}$ clockwise, the one claimed by Constructor being the n^{th} one. Blocker then claims the $(n-1)^{th}$ edge. Now group the edges from 1 to $3\cdot\lfloor\frac{n-2}{3}\rfloor$ in groups of three, and assign a type to each edge, clockwise $a, b, c, a, b, c, \dots$

- Whenever Constructor plays an edge of type a (resp. b), Blocker plays the adjacent edge of type b (resp. a).
- If Constructor plays an edge of type c or an inner edge, then Blocker plays an arbitrary edge of type a or b that is not adjacent to an edge he has already claimed. If at some point he would claim a particular edge of type a or b that he has already claimed, he plays an arbitrary edge of type a or b as well.
- When Blocker cannot play according to this strategy, he plays arbitrarily.

An example of this strategy is given in Figure 4.5. At the end of the game, Blocker will have an edge of type a or b in all blocks of three consecutive edges, thus he will have claimed $\lfloor\frac{n-2}{3}\rfloor + 1 \geqslant \lfloor\frac{n}{3}\rfloor$ non-adjacent outer edges. ■

The next claim will now give us an upper bound on the number of triangles Constructor can still create when many non-adjacent outer edges are claimed by Blocker.

Claim 18. *If Blocker has claimed m non-adjacent outer edges, then Constructor can create at most $n-m-2$ triangles.*

Proof of the claim. Let $e_1,\dots,e_m$ be the non-adjacent outer edges claimed by Blocker. The idea is to contract these edges to obtain a bound on the number of triangles in $\mathcal{C}$. We consider

Constructor's graph $\mathcal{C}$ at the end of the game. Let $\hat{\mathcal{C}}$ be the graph obtained from $\mathcal{C}$ by contracting every edge e_i.

Every triangle in $\mathcal{C}$ corresponds to a triangle in $\hat{\mathcal{C}}$, because it cannot have an edge e_i as an edge. Thus the number of triangles in $\mathcal{C}$ is upper bounded by the number of triangles in $\hat{\mathcal{C}}$, which is at most $|V(\hat{\mathcal{C}})| - 2$, hence $n - m - 2$. Note that we use that the edges are not adjacent. Indeed if two edges v_1v_2 and v_2v_3 were adjacent, then contracting these edges would mean contracting v_1v_3 as well, but this edge might be claimed by $\mathcal{C}$ and inside a triangle. ■

If we now combine both claims we get that

$$g(n, K_3) \leqslant n - \lfloor \frac{n}{3} \rfloor - 2 \leqslant \frac{2n}{3} \,.$$

□

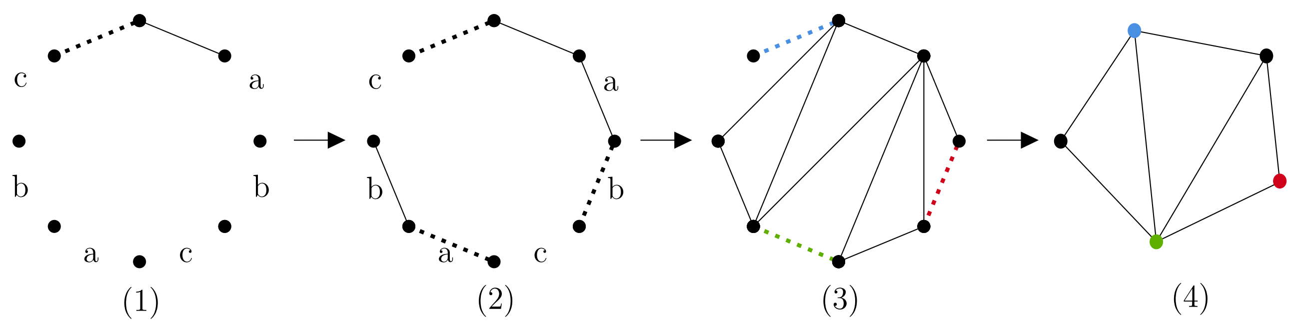


FIGURE 4.5. An example for Blocker's strategy for ECB. Constructor's edges are in solid lines and Blocker's edges are in dotted lines. (1) Labels of the outer edges after one move for each player; (2) Example of the graph later on; (3) Example of the graph at the end of the game; (4) Contraction of the Blocker edges.

5. Concluding remarks

In this paper, we studied the Constructor-Blocker game where Constructor wants to maximize the number of triangles in her graph while Blocker wants to minimize it. Figure 5.1 summarizes new and existing results, our contributions being in bold font.

Despite some similarities in the methods and strategies for the different forbidden subgraphs, we were not able to provide very generic statements that would handle many cases of H and F. This would be really interesting and provide a deeper understanding of this problem. More precisely, we would be very interested in finding some strategies leading to the asymptotics of $g(n, K_3, S_k)$ and $g(n, K_3, P_k)$ for a general k. We would also like to close the gaps between the bounds for $g(n, K_3, S_5)$ and $g_{\mathrm{ECB}}(n, K_3)$.

Many other questions remain open. For instance, it is common in positional games to introduce a bias. Instead of the two players each claiming one edge during their turns, we can allow

F	$ex(n, K_3, F)$	$g(n, K_3, F)$	ref. for $g(n, K_3, F)$
$\varnothing$	$\frac{n^3}{6}$	$\boldsymbol{\frac{n^3}{48}}$	Thm 1.1
P_5	n	$\frac{n}{4}$	[13]
P_6	$2n$	$\boldsymbol{\frac{n}{2}}$	Thm 1.2
S_4	n	$\boldsymbol{\frac{n}{3}}$	Thm 1.3
S_5	$2n$	$\boldsymbol{\frac{2n}{3} \leqslant \cdot \leqslant n}$	Thm 1.4
C_4	$\Theta(n^{3/2})$	$\Theta\left(n^{3/2}e^{-c\sqrt{\log(n)}}\right) \leqslant \cdot \leqslant O(n^{3/2})$	[5]
F, $\chi(F) = s > 3$	$\frac{(s-2)(s-3)}{6(s-1)^2}n^3$	$\frac{(s-2)(s-3)}{48(s-1)^2}n^3$	[5]

game	max. number of triangles	$g_{\text{game}}(n, K_3)$	ref.
PCB	$3n$	$\boldsymbol{3n}$	Thm 1.6
ECB	n	$\boldsymbol{\frac{n}{2} \leqslant \cdot \leqslant \frac{2n}{3}}$	Thm 1.7

Figure 5.1. Recap table of asymptotic results

Constructor to claim a edges and Blocker to play b edges. This bias could change the score of the game, even if $a = b$ a priori. It would be interesting to see how the score of the game evolves with a and b.

Finally, even without improving the bounds, we would like to have nicer strategies, especially for Blocker. In most of the proofs, we describe the strategies for Blocker very locally, with a case study, but we think there could be easier strategies for Blocker giving the same results.

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