

Moments of the Zeros of Faber Polynomials of the Miller Basis

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Abstract

We study the zeros of modular forms in the Miller basis, a natural basis for the space of modular forms. We show that the zeros of their Faber polynomials have linear moments. By analyzing the moments we can extend the known range of the forms in the Miller basis for which at least one of the zeros is not on the arc - the circular part of the boundary of the fundamental domain. Additionally, for forms in the Miller basis of an index asymptotically linear in the weight such that all zeros are on the arc, we compute the limit distribution of the zeros, which depends on the asymptotic ratio of the index to the weight.

1 Introduction and Main Results

For $k \geq 0$ an even integer let M_k be the space of modular forms of weight k for the full modular group $SL_2(\mathbb{Z})$. Each $f \in M_k$ has the q expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_f(n) q^n$$

where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H} = \{\tau : \text{Im}\{\tau\} > 0\}$.

Writing $k = 12\ell + k'$, $k' \in \{0, 4, 6, 8, 10, 14\}$, the *Miller basis* of M_k consist of the unique elements $f_{k,m}$ of M_k with the q expansion

$$f_{k,m} = q^m + O(q^{\ell+1})$$

for $m = 0, \dots, \ell$.

Our goal is to understand the zeros of the forms in the Miller basis. An important method for studying the zeros was first used for the Eisenstein series in 1970 by F. Rankin and Swinnerton-Dyer [1]. They showed that the zeros in the fundamental domain of the Eisenstein series E_k all lie on the arc $\mathcal{A} = \{e^{i\theta} : \theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]\}$, and become uniformly distributed in $\mathcal{A}$ as $k \rightarrow \infty$. W. Duke and P. Jenkins [3] used a different approach and showed that the zeros of $f_{k,0} = 1 + O(q^{\ell+1})$ all lie on the arc $\mathcal{A}$ and become uniformly distributed. Their result was later extended by Raveh [5] who showed that all the zeros of $f_{k,m}$ are on the arc for $m < \frac{2}{9}\ell$ and showed the zeros are uniformly distributed for $m = o(\ell)$.

The main tool we will use to study the zeros are Faber polynomials. We write $F_{k,m}(t) = t^{\ell-m} + e_1 t^{\ell-m-1} + \dots + e_{\ell-m}$ for the Faber polynomial of $f_{k,m}$. The Faber polynomial is the polynomial that satisfies

$$f_{k,m} = \Delta^\ell \cdot F_{k,m}(j) \cdot E_{k'}$$

where Δ is the modular discriminant and j is the j -invariant.

Take $x_i = x_i(k, m)$ to be the zeros of the Faber polynomial, counted with multiplicity. Then $j^{-1}(x_i)$

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are the zeros of the modular form and j is a bijection from the fundamental domain to $\mathbb{C}$. This means that we can study the zeros of a modular form by studying the zeros of its Faber polynomial. Note that $j(\mathcal{A}) = [0, 1728]$, j maps the imaginary axis to $[1728, \infty)$ and the left boundary segment to the negative real numbers.

Using Faber polynomials, Rudnick [2] showed that if the degree of the Faber polynomial $\ell - m = D$ is fixed as $k \rightarrow \infty$ the zeros of $f_{k,m}$ lie on D vertical lines in the fundamental domain and are of approximate height $\log(k)$.

Our goal is to study the case where the degree of the polynomial is not bounded. It turns out the zeros do have a structure that can be analyzed with no assumptions on the degree.

Recall that we take $x_i = x_i(k, m)$ be the zeros of the Faber polynomial, counted with multiplicity.

Theorem 1. *There exists $A_n, B_n, C_n(k')$ such that for every $n \leq \ell - m$,*

$$\sum_{i=1}^{\ell-m} x_i^n = A_n \cdot k + B_n \cdot m + C_n(k')$$

where $A_n = \frac{1}{2\pi} \int_{\mathcal{A}} j^n(\theta) d\theta$, $-B_n$ is the coefficient of q^0 in j^n , $C_n(0) = C_n(4) = C_n(8) = 0$ and $C_n(6) = C_n(10) = C_n(14) = -\frac{1728^n}{2}$.

The proof of Theorem 1 uses only basic facts about Faber polynomials, like the fact that the leading coefficients of $F_{24\ell, 2m}$ are identical to the leading coefficients of $F_{12\ell, m}^2$. The value of A_n is a direct corollary of a result from W. Duke and P. Jenkins [3], which they proved by analytic methods. The value of B_n can be derived from a recent paper by R. Raveh [5] proven by similar methods. We also give a purely algebraic proof which has the added benefit of working in a more general setting.

This theorem gives a very structured constraint on the roots of the Faber polynomials and can be used in proving statements about the roots. The sums of the powers of the roots hold all the information about the coefficients of the polynomial (using Newton's identities we can go back and forth between them), but in some cases are much simpler to work with. Just from noticing that if all the zeros of the form are on the arc these sums are positive, we get the following

Corollary 1. *If $k' = 0, 4, 8$ and $\frac{30}{31}\ell + k' \frac{5}{62} < m \leq \ell - 1$, at least one of the zeros of $f_{k,m}$ is not on the arc.*

If $k' = 6, 10, 14$ and $\frac{30}{31}\ell + k' \frac{5}{62} - \frac{72}{31} < m \leq \ell - 1$, at least one of the zeros of $f_{k,m}$ is not on the arc.

Using Newton's identities, we obtain a new formula for the coefficients of the Faber polynomials. There are no previously known nonrecursive formulas for the coefficients.

Corollary 2. *The coefficient e_n of the Faber polynomial is given by*

$$e_n = \sum_{\substack{t_1+2t_2+\dots+nt_n=n \\ 0 \leq t_s}} \prod_{s=1}^n (-1)^{t_s} \frac{(A_s \cdot k + B_s \cdot m + C_s(k'))^{t_s}}{t_s! s^{t_s}}.$$

The above corollary is used in the computation of B_n .

The power sums also hold information on the distribution of the roots. As the n th moment of the distribution of the roots is

$$M_n(k, m) = \frac{1}{\ell - m} \sum_{i=1}^{\ell-m} x_i^n = \frac{A_n \cdot k + B_n \cdot m + C_n(k')}{\ell - m} = \frac{A_n(12 + \frac{k'}{\ell}) + B_n \cdot \frac{m}{\ell} + \frac{C_n(k')}{\ell}}{1 - \frac{m}{\ell}},$$

from bounds on the moments, as long as all the roots are on $\mathbb{R}$ (all the zeros of the modular form are on the boundary or on $i\mathbb{R}$), the moments uniquely define the distribution. Since in the limit the moments depend only on $\frac{m}{\ell}$, the limit distribution depends only on $\lim_{\ell \rightarrow \infty} \frac{m}{\ell}$.

The moments can also be used to determine the limit distribution of the zeros of the modular form on the arc.

Theorem 2. *For $m \sim c\ell$, $0 < c < 1$ for which all the zeros of $f_{k,m}$ are on the arc, as $\ell \rightarrow \infty$ the limit probability that a zero of the modular form on the arc is between $[\theta_1, \theta_2]$ is*

$$\frac{6}{\pi(1-c)}(\theta_2 - \theta_1) + \frac{2c}{1-c}(\cos(\theta_2) - \cos(\theta_1)).$$

Using a result of Raveh [5] we can take $c < \frac{2}{9}$.

Thus in our range, the limit distribution of the zeros depends on the asymptotic ratio of the index to the weight on the interval $[\frac{\pi}{2}, \frac{2\pi}{3}]$, unlike the uniform distribution in the work of Duke and Jenkins and of Raveh.

Numerical investigation of these polynomials revealed further remarkable properties. For example,

Conjecture 1. *We denote $m_0, m_1, \dots, m_{15} = 4, 3, 4, 5, 6, 7, 6, 7, 8, 9, 10, 9, 10, 11, 12, 13$. Take $\ell = 16s + r > 30$ with $0 \leq r < 16$. Let $m(k)$ be the minimal m for which $f_{k,m}$ has at least one zero not on the arc. Then $m(12\ell) = 10s + m_r$ and if $m \geq m(12\ell)$ at least one of the zeros is not on the arc.*

Conjecture 2. *The modular forms in the Miller basis have no zeros on the line $\text{Re}\{\tau\} = 0$. Equivalently, the Faber polynomials of these modular forms have no zeros in $[1728, \infty)$.*

These conjectures have been verified for $\ell \leq 300$.

Organization of the paper

Part 2 will introduce the needed background, including how to compute Faber polynomials.

In Part 3 we will prove Theorem 1, computing A_n , B_n and C_n and give a new formula for the coefficients of the Faber polynomials.

The computations of B_n will be algebraic and only based on the coefficients of Δ and the j invariant.

Part 4 will prove the upper bound that can be achieved using Theorem 1 on the value of m for which all the zeros of $f_{k,m}$ are on the arc.

Part 5 will discuss the limit distribution for the zeros of $f_{k,m}$ in the case of $m \sim c\ell$ using Theorem 1 and an additional proof using Raveh [5]. This result can be used for an additional proof of the formula for B_n .

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2 Background

Denote by $\mathbb{H} = \{\tau : \text{Im}(\tau) > 0\}$ the upper half-plane.

Let $k \geq 4$ be an even integer. A modular form of weight k is a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$, such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, and f is bounded as $\text{Im}(\tau) \rightarrow \infty$.

Equivalently, we can replace the functional equation in the previous definition with the equations

$$f(\tau) = f(\tau + 1), \quad f(-1/\tau) = \tau^k f(\tau)$$

as they generate the action of $SL_2(\mathbb{Z})$ on the upper half plane. We denote $f(i\infty) = \lim_{\text{Im}(\tau) \rightarrow \infty} f(\tau)$.

As $f(\tau) = f(\tau + 1)$, it has an expansion in terms of $q = e^{2\pi i\tau}$,

$$f(\tau) = \sum_{n=0}^{\infty} a(n)q^n.$$

For weight $k = 12\ell + k'$, the *Miller basis* of weight k is the sequence of modular forms of weight k , $\{f_{k,m}\}_{m=0}^{\ell}$ for which

$$f_{k,m} = q^m + O(q^{\ell+1}).$$

An important example of a sequence of modular forms are the Eisenstein series. For $k \geq 4$ even we denote

$$E_k(\tau) = \frac{1}{2} \sum_{\gcd(c,d)=1} \frac{1}{(c\tau + d)^k} = 1 + \gamma_k \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \in M_k$$

where $\sigma_s(n) = \sum_{d|n} d^s$ and $\gamma_k = \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \in \mathbb{Q}$.

Another important example is the modular discriminant $\Delta \in M_{12}$,

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 + \dots$$

We also define the space of modular functions as holomorphic functions on the upper half plane that are invariant under the action of $SL_2(\mathbb{Z})$ as defined for modular forms, where we allow a pole at the cusp $\text{Im}(\tau) \rightarrow \infty$. By the same argument they can also be written as a sum in q , but can have negative powers of q in the sum.

All modular functions are a polynomial in the j invariant

$$j = \frac{E_4^3}{\Delta} = \frac{1}{q} + 744 + 196884q + \dots$$

which is meromorphic at infinity.

Any modular form can be written as

$$f = \Delta^\ell E_{k'} \cdot F_f(j)$$

where $k = 12\ell + k'$, $k' \in \{0, 4, 6, 8, 10, 14\}$ and $\deg(F_f) = \ell - \text{ord}_\infty(f)$.

This gives us a way to study the zeros of a modular form by studying the zeros of a polynomial.

2.1 Computing the Faber Polynomial

Take $f \in M_k(SL_2(\mathbb{Z}))$ and $m := \text{ord}_\infty(f)$.

To compute the Faber polynomial of f , a modular form of weight k , we compare the Laurent series of $\frac{f}{\Delta^\ell E_{k'}}$ with a polynomial of j of degree $\ell - m$. We can also compare the q expansion of f and $\Delta^\ell E_{k'} F(j)$ with F of degree $\ell - m$.

For example, we will look at the Faber polynomial of $f_{12\ell, \ell-1} = q^{\ell-1} + O(q^{\ell+1})$, which satisfy

$$q^{\ell-1} + O(q^{\ell+1}) = \Delta^\ell F(j) = (q^\ell - 24\ell q^{\ell+1} + O(q^{\ell+2}))(e_0 j + e_1).$$

As $j = \frac{1}{q} + 744 + O(q)$ we can write $e_0 j + e_1 = \frac{e_0}{q} + (e_0 744 + e_1) + O(q)$ and

$$\left(q^\ell - 24\ell q^{\ell+1} + O(q^{\ell+2}) \right) \left(\frac{e_0}{q} + (e_0 744 + e_1) + O(q) \right) = e_0 q^{\ell-1} + (e_0 744 + e_1 - e_0 24\ell) q^\ell + O(q^{\ell+1})$$

and we must have $e_0 = 1$ and $(e_0 744 + e_1 - e_0 24\ell) = 0$, so

$$F_{12\ell, \ell-1}(t) = t - 24\ell - 744.$$

More generally, we can use the following algorithm -

Take $f \in M_k$ with $m = \text{ord}_\infty(f)$

$$f \Delta^{-\ell} E_{k'}^{-1} = F_f(j) = \sum_{n=0}^{\ell-m} e_n j^{\ell-m-n}.$$

For both sides the non-zero exponent of q with the smallest power is $q^{m-\ell}$. On the right-hand side it is the coefficient of $q^{m-\ell}$ that is only determined by e_0 .

We choose e_0 to be the coefficient of $q^{m-\ell}$ in the left-hand side.

Assume we chose the first s coefficients. Then we look at

$$f \Delta^{-\ell} E_{k'}^{-1} - \sum_{n=0}^{s-1} e_n j^{\ell-m-n} = \sum_{n=s}^{\ell-m} e_n j^{\ell-m-n}.$$

From our choices of $e_0, \dots, e_{s-1}$, the left-hand side is some constant times $q^{m+s-\ell} + O(q^{m+s-\ell+1})$. The right-hand side is $e_s q^{m+s-\ell} + O(q^{m+s-\ell+1})$ based on the q expansion of j . So we can choose e_s to be the coefficient of $q^{m+s-\ell}$ in the left-hand side.

Note that each e_s is a polynomial in the variables k, m whose coefficients depend on f, k' . Also, the $s+t$ coefficient of f does not affect e_s for $t > 0$.

2.2 Additional Notation

We will denote by $c_n(d)$ the coefficient of q^d in j^n and by $\tau_{-\ell}(n - \ell)$ the coefficient of $q^{n-\ell}$ in $\Delta^{-\ell}$.

3 Proof of the Linearity Theorem

3.1 Proof of Theorem 1

Let $f_{k,m} = q^m + O(q^{\ell+1})$ be the m -th modular form in the Miller basis of weight $k = 12\ell + k'$ and let $F_{k,m}$ be its Faber polynomial.

We will denote with $y_i = y_i(k, m)$ the zeros of the modular form $f_{k,m}$ and x_i the zeros of its Faber polynomial, both counted with multiplicity. Note that $\{j(y_i)\}_i$ contain the zeros of the Faber polynomial, with j of the zeros of the Eisenstein form of weight k' added, as $f_{k,m} = \Delta^\ell E_{k'} F_{k,m}(j)$.

We will denote $S_n(f_{k,m}) = \sum_{i=1}^{\ell-m} j(y_i)^n$.

Lemma 1. *$S_n(f_{k,m})$ can be written as a polynomial $P_{n,k'}(k, m)$ with integer coefficients.*

Proof. The coefficient of $t^{\ell-m-d}$ in the Faber polynomial can be written as a polynomial that depends on n, k' in the variables k, m .

Using Newton's identities we can write the sum of the n th powers of the zeros of the Faber polynomial $\sum_i x_i^n$ as a polynomial in the coefficients of the Faber polynomial. The difference $S_n(f_{k,m}) - \sum_i x_i^n$ is the sum of the n th powers of j of the zeros of $E_{k'}$, counted with multiplicity, which for a fixed n is a constant that depends only on k' . So over all, we get that $S_n(f_{k,m})$ can be written as a polynomial $P_{n,k'}(k, m)$. $\square$

Lemma 2. *Take $k = 12\ell$. For all $n \leq \ell - m$,*

$$P_{n,0}(k, m) = A_n \cdot k + B_n \cdot m.$$

Proof. Consider $f_{12\ell,m}$ and note that the first $\ell - m + 1$ coefficients of the q expansion of

$$f_{12\ell,m}^t = q^{tm} + O(q^{\ell+(t-1)m+1})$$

and

$$f_{12t\ell,tm} = q^{tm} + O(q^{t\ell+1})$$

are the same, for a fixed integer $t > 0$.

This means that the first $\ell - m + 1$ coefficients of their Faber polynomials are the same, and by Newton's identities they have the same sum of the n th powers of their roots, for $n \leq \ell - m$. As we are looking at $f_{12\ell,m}^t$, we have $t \cdot S_n(f_{k,m}) = S_n((f_{k,m}^t))$ and so for $n \leq \ell - m$,

$$t \cdot P_{n,0}(12\ell, m) = P_{n,0}(t \cdot 12\ell, tm).$$

This implies the polynomial is linear and homogeneous, and we can write

$$S_n(f_{k,m}) = \sum_{i=1}^{\ell-m} j^{-1}(y_i)^n = A_n \cdot k + B_n \cdot m$$

for some constants A_n, B_n . $\square$

We will now get the same result for all values of k' .

Lemma 3. *For all $n \leq \ell - m$,*

$$P_{n,k'}(k, m) = A_n \cdot k + B_n \cdot m.$$

Proof. Take $f_{12\ell+k',m}$ for some $k' \in \{4, 6, 8, 10, 14\}$. Note that the first $\ell - m + 1$ coefficients of the q expansion of

$$f_{k,m}^{12} = q^{12m} + O(q^{\ell+11m+1})$$

and

$$f_{12-k,12m} = q^{12m} + O(q^{k+1})$$

are the same.

As before, this means that the first $\ell - m + 1$ coefficients of their Faber polynomials are the same, and they have the same sum of the n th powers of their roots, for $n \leq \ell - m$. This implies that for $n \leq \ell - m$,

$$S_n((f_{k,m})^{12}) = S_n(f_{12 \cdot k, 12m}) = 12 \cdot S_n(f_{k,m}).$$

From the previous lemma,

$$S_n(f_{k,m}) = \frac{1}{12} S_n(f_{12 \cdot k, 12m}) = \frac{1}{12} (A_n \cdot 12k + B_n \cdot 12m) = A_n \cdot k + B_n \cdot m.$$

□

Lemma 4. For all $n \leq \ell - m$,

$$\sum_{i=1}^{\ell-m} x_i^n = A_n \cdot k + B_n \cdot m + C_n(k')$$

where $C_n(0) = C_n(4) = C_n(8) = 0$ and $C_n(6) = C_n(10) = C_n(14) = -\frac{1728^n}{2}$ and $A_n \in \mathbb{Z}$.

Proof. The difference $\sum_i x_i^n - S_n(f_{k,m})$ is the sum of the n th powers of j of the zeros of $E_{k'}$.

For $k' = 0$, $E_0 = 1$ has no zeros, so the difference is 0. For $k' \in \{4, 8\}$, $E_{k'}$ vanishes only at $e^{2\pi i/3}$ and as $j(e^{2\pi i/3}) = 0$ the difference is 0. As for $k' \in \{6, 10, 14\}$, $E_{k'}$ also has a zero at i of order $\frac{1}{2}$. Since $j(i) = 1728$ and the order of vanishing is $\frac{1}{2}$, the difference is $-\frac{1728^n}{2}$.

This means that

$$\sum_{i=1}^{\ell-m} x_i^n = A_n \cdot k + B_n \cdot m + C_n(k')$$

with the values of $C_n(k')$ as above.

As the coefficients of $f_{k,m}$ are integers, the coefficients of the Faber polynomial are integers. Newton's identities only use integer coefficients, so the coefficients of $P_{n,k'}(k, m)$ are integers so is A_n . □

So to finish proving Theorem 1 we only need to prove the formulas for the constants A_n, B_n .

Remark. Define the normalized counting function of the zeros

$$\mu_{k,m} = \frac{1}{\ell - m} \sum_{i=1}^{\ell-m} \delta_{x_i(k,m)}.$$

This is a probability measure with n th moment

$$M_n(k, m) = \frac{1}{\ell - m} \sum_{i=1}^{\ell-m} x_i^n$$

which we just showed is equal to

$$M_n(k, m) = \frac{1}{\ell - m} (A_n \cdot k + B_n \cdot m + C_n(k')).$$

This means that results about the distribution can be written in the language of these sums.

Remark. Theorem 1 gives a strong structure to the zeros of Faber polynomials. We can see that in the limit as $k \rightarrow \infty$, the moments of the zeros only depend on the ratio $\frac{m}{\ell}$, as

$$M_n(k, m) = \frac{1}{\ell - m} (A_n \cdot k + B_n \cdot m + C_n(k')) = \frac{1}{1 - \frac{m}{\ell}} (12 \cdot A_n + B_n(k') \cdot \frac{m}{\ell} + \frac{C_n(k')}{\ell}).$$

As for $k' = 0, 4, 8$ we know $C_n(k') = 0$ and we don't even need to take a limit. So for $k' = 0, 4, 8$,

$$M_n(12\ell + k', m) = \frac{1}{1 - \frac{m}{\ell}} (12 \cdot A_n + B_n \cdot \frac{m}{\ell})$$

and the average of the powers of the roots is fixed if we multiply both ℓ and m by the same constant.

Corollary 3. We write $F_{k,m}(t) = t^{\ell-m} + e_1 t^{\ell-m-1} + \dots + e_{\ell-m}$ for the Faber polynomial.

Then, using Newton's identities

$$e_n = \sum_{t_1+2t_2+\dots+nt_n=n} \prod_{s=1}^n (-1)^{t_s} \frac{(A_s \cdot k + B_s \cdot m + C_s(k'))^{t_s}}{t_s! s^{t_s}}.$$

This gives us a non-recursive formula for the coefficients of the Faber polynomial.

3.2 Computing the Values of A_n

Proposition 1. For all n , $A_n = \frac{1}{2\pi} \int_{\mathcal{A}} j^n(\theta) d\theta$.

Proof. We will look at $f_{k,0}$ the first form in the Miller basis of weight k .

We can write

$$f_{k,0} = E_{k'} \Delta^\ell F_{k,0}(j).$$

$E_{k'}$ has finitely many zeros for all $k' \in \{0, 4, 6, 8, 10, 14\}$, Δ has a zero only at $i\infty$ and it is a simple zero, and j has a simple pole at $i\infty$. The zeros of Δ^ℓ and the poles of $F_{k,0}(j)$ cancel out as the Faber polynomial is of degree ℓ , and the form does not vanish at $i\infty$ (indeed, $f_{k,0}(\tau) = 1 + O(q^{\ell+1})$).

So the zeros of the form are the zeros of $E_{k'}$ (of bounded number) and $j^{-1}(x_i)$ where $x_i := x_i(k, 0)$ are the zeros of the Faber polynomial of $f_{k,0}$ counted with multiplicity.

From Duke and Jenkins [3] we know that the zeros of the modular form $y_i := j^{-1}(x_i)$ are uniformly distributed on the arc as $k \rightarrow \infty$.

$$A_n = \frac{1}{12} \lim_{k \rightarrow \infty} \frac{A_n \cdot k + B_n \cdot m + C_n(k')}{\ell} = \frac{1}{12} \lim_{k \rightarrow \infty} \frac{1}{\ell} \sum_{i=1}^{\ell} x_i^n = \frac{1}{12} \lim_{k \rightarrow \infty} \frac{1}{\ell} \sum_{i=1}^{\ell} (j(y_i))^n$$

as the zeros of $E_{k'}$ do not affect the limit.

This is a Riemann sum and from the uniform distribution of y_i in the limit we get

$$\frac{1}{12} \frac{6}{\pi} \int_{\mathcal{A}} j^n(\theta) d\theta = \frac{1}{2\pi} \int_{\mathcal{A}} j^n(\theta) d\theta.$$

□

3.3 Computing the Values of B_n

There are two ways to obtain the value of B_n . The first is an algebraic proof which we will present here and the second will be based on the limit distribution shown in Part 6.

Recall that we let $c_n(d)$ the coefficient of q^d in j^n and we will denote $c(d) := c_1(d)$.

Lemma 5.

$$c_n(0) = \sum_{d_1+\dots+nd_n=n} \frac{n!}{(n-\sum d_t)!} \prod_{t=1}^n \frac{(c(t-1))^{d_t}}{d_t!}$$

$$c_{\ell-m-i}(n+m-\ell) \equiv \sum_{d_1+\dots+(n-i)d_{n-i}=n-i} \left((-1)^{\sum d_t} \prod_{\alpha=0}^{\sum d_t-1} (i+m+\alpha) \right) \left(\prod_{t=1}^{n-i} \frac{(c(t-1))^{d_t}}{d_t!} \right) \pmod{\ell}$$

Proof. Let $c'_N(n)$ denote the coefficient of q^n in $(qj(q))^N$.

We write the q expansion

$$qj(q) = 1 + 744q + 196884q^2 + \dots = \sum_{t=0}^{\infty} c(t-1)q^t.$$

Then

$$c'_N(n) = \sum_{d_1+\dots+nd_n=n} \frac{N!}{(N-\sum_{1 \leq i} d_i)!} \prod_{t=1}^n \frac{(c(t-1))^{d_t}}{d_t!}.$$

By definition, $c_n(0) = c'_n(n)$ and this is

$$c_n(0) = c'_n(n) = \sum_{d_1+\dots+nd_n=n} \frac{n!}{(n-\sum_{1 \leq i} d_i)!} \prod_{t=1}^n \frac{(c(t-1))^{d_t}}{d_t!}.$$

In the same way, $c_{\ell-m-i}(n+m-\ell) = c'_{\ell-m-i}(n+m-\ell+\ell-m-i) = c'_{\ell-m-i}(n-i)$ and this is

$$c'_{\ell-m-i}(n-i) = \sum_{d_1+\dots+(n-i)d_{n-i}=n-i} \frac{(\ell-m-i)!}{(\ell-m-i-\sum d_t)!} \prod_{t=1}^{n-i} \frac{(c(t-1))^{d_t}}{d_t!}$$

$$\frac{(\ell-m-i)!}{(\ell-m-i-\sum d_t)!} = (\ell-m-i) \cdot (\ell-m-i-1) \cdot \dots \cdot (\ell-m-i-\sum d_t+1) \equiv (-1)^{\sum d_t} \prod_{\alpha=0}^{\sum d_t-1} (i+m+\alpha) \pmod{\ell}.$$

Putting it all together, we get

$$c'_{\ell-m-i}(n-i) \equiv \sum_{d_1+\dots+(n-i)d_{n-i}=n-i} \left((-1)^{\sum d_t} \prod_{\alpha=0}^{\sum d_t-1} (i+m+\alpha) \right) \left(\prod_{t=1}^{n-i} \frac{(c(t-1))^{d_t}}{d_t!} \right) \pmod{\ell}.$$

□

Proposition 2. For all n we have

$$B_n = -c_n(0)$$

where $c_n(0)$ is the coefficient of q^0 in j^n .

Proof. Take $k' = 0$ and $m > 0$. For a general ℓ, m , we will compute the coefficients of the Faber polynomial modulo ℓ, m^2 .

As e_n is the n th coefficient in the Faber polynomial, we know it is the coefficient of $q^{n+m-\ell}$ in

$$q^m \Delta^{-\ell} - \sum_{i=0}^{n-1} e_i j^{\ell-m-i}.$$

So if $\tau_{-\ell}(n-\ell)$ is the coefficient of $q^{n-\ell}$ in $\Delta^{-\ell}$ and $c_{\ell-m-i}(n+m-\ell)$ is the coefficient of $q^{n+m-\ell}$ in $j^{\ell-m-i}$, the formula is

$$e_n = \tau_{-\ell}(n-\ell) - \sum_{i=0}^{n-1} e_i c_{\ell-m-i}(n+m-\ell),$$

as we multiply Δ^ℓ by q^m .

We are working mod ℓ , and so as

$$\Delta^{-1} = \sum_{i=1}^{\infty} \tau_{-1}(i) q^i$$

when we look at $\Delta^{-\ell}$ mod ℓ , we have

$$\Delta^{-\ell} \equiv \sum_{i=1}^{\infty} (\tau_{-1})^\ell(i) q^{i \cdot \ell} \equiv q^{-\ell} + O(1) \pmod{\ell}.$$

We only care about the negative powers of q in $\Delta^{-\ell}$. This gives us the value of $e_0 = 1$.

So for $1 < n < \ell - m$,

$$e_n \equiv - \sum_{i=0}^{n-1} e_i c_{\ell-m-i}(n+m-\ell) \pmod{\ell}$$

with the initial condition of $e_0 = 1$.

Another way to get the coefficients of the Faber polynomials are using Newton's identities on the sums of powers of the roots. In our case, $\sum_{i=1}^{\ell-m} x_i^n \equiv B_n \cdot m \pmod{\ell}$, and so we get that

$$e_n \equiv \sum_{d_1 + \dots + n d_n = n} \prod_{s=1}^n \frac{(-m \cdot B_s)^{d_s}}{d_s! s^{d_s}} \pmod{\ell}.$$

Combining the two equation together,

$$\sum_{d_1 + \dots + n d_n = n} \prod_{s=1}^n \frac{(-m \cdot B_s)^{d_s}}{d_s! s^{d_s}} \equiv - \sum_{i=0}^{n-1} e_i c_{\ell-m-i}(n+m-\ell) \pmod{\ell}.$$

Let's look at this mod m^2 . The only choice of $d_1, \dots, d_n$ where we get something that is not zero mod m^2 is for $d_1 = \dots d_{n-1} = 0$ and $d_n = 1$. Our equivalence is now

$$-m \frac{B_n}{n} \equiv - \sum_{i=0}^{n-1} e_i c_{\ell-m-i}(n+m-\ell) \pmod{\ell, m^2}.$$

As $m|e_i$ for $0 < i$, and we only care about $e_n \pmod{m^2}$, we can replace $c_{\ell-m-i}(n+m-\ell)$ with it's residue mod m for $0 < i$ and it's residue mod m^2 for $i = 0$.

From Lemma 5 , for $0 < i$,

$$\begin{aligned}
c_{\ell-m-i}(n+m-\ell) &\equiv \sum_{d_1+\dots+(n-i)d_{n-i}=n-i} \left((-1)^{\sum d_t} \prod_{\alpha=0}^{\sum d_t-1} (i+m+\alpha) \right) \left(\prod_{t=1}^{n-i} \frac{(c(t-1))^{d_t}}{d_t!} \right) \equiv \\
&\sum_{d_1+\dots+(n-i)d_{n-i}=n-i} \left((-1)^{\sum d_t} \prod_{\alpha=0}^{\sum d_t-1} (i+\alpha) \right) \left(\prod_{t=1}^{n-i} \frac{(c(t-1))^{d_t}}{d_t!} \right) \equiv \\
&\sum_{d_1+\dots+(n-i)d_{n-i}=n-i} \left((-1)^{\sum d_t} \frac{(i+\sum d_t-1)!}{(i-1)!} \right) \left(\prod_{t=1}^{n-i} \frac{(c(t-1))^{d_t}}{d_t!} \right) \pmod{\ell, m}
\end{aligned}$$

and for $i = 0$ this is

$$\begin{aligned}
c_{\ell-m}(n+m-\ell) &\equiv \sum_{d_1+\dots+(n)d_n=n} \left((-1)^{\sum d_t} \prod_{\alpha=0}^{\sum d_t-1} (m+\alpha) \right) \left(\prod_{t=1}^{n-i} \frac{(c(t-1))^{d_t}}{d_t!} \right) \equiv \\
&\sum_{d_1+\dots+(n)d_n=n} \left((-1)^{\sum d_t} m \cdot \prod_{\alpha=1}^{\sum d_t-1} \alpha \right) \left(\prod_{t=1}^{n-i} \frac{(c(t-1))^{d_t}}{d_t!} \right) \equiv \\
&m \cdot \sum_{d_1+\dots+(n)d_n=n} \left((-1)^{\sum d_t} (\sum d_t-1)! \right) \left(\prod_{t=1}^{n-i} \frac{(c(t-1))^{d_t}}{d_t!} \right) \pmod{\ell, m^2}.
\end{aligned}$$

Over all, we get that

$$m \frac{B_n}{n} \equiv m \cdot \sum_{i=0}^{n-1} \left(\sum_{d_1+\dots+(n-i)d_{n-i}=n-i} c_i(0) \left((-1)^{\sum d_t} \frac{(i+\sum d_t-1)!}{(i)!} \right) \left(\prod_{t=1}^{n-i} \frac{(c(t-1))^{d_t}}{d_t!} \right) \right) \pmod{\ell, m^2}$$

expanding $c_i(0)$ as a sum gives us

$$\begin{aligned}
m \cdot \sum_{i=0}^{n-1} \sum_{d_1+\dots+(n-i)d_{n-i}=n-i} \sum_{g_1+\dots+ig_i=i} \frac{i!}{(i-\sum g_s)!} \prod_{s=1}^n \frac{(c(t-1))^{g_s}}{g_s!} \left((-1)^{\sum d_t} \frac{(i+\sum d_t-1)!}{(i)!} \right) \left(\prod_{t=1}^{n-i} \frac{(c(t-1))^{d_t}}{d_t!} \right) = \\
m \cdot \sum_{i=0}^{n-1} \sum_{d_1+\dots+(n-i)d_{n-i}=n-i} \sum_{g_1+\dots+ig_i=i} (-1)^{\sum d_t} \frac{(i+\sum d_t-1)!}{(i-\sum g_s)!} \left(\prod_{1 \leq t \leq n} \frac{(c(t-1))^{g_t+d_t}}{g_t!d_t!} \right) \pmod{\ell, m^2}
\end{aligned}$$

where $d_t = 0$ for $t > n-i$ and $D_s = 0$ for $s > i$.

Over all, as B_n is a constant and does not depend on ℓ, m ,

$$B_n = n \sum_{i=0}^{n-1} \sum_{d_1+\dots+(n-i)d_{n-i}=n-i} \sum_{g_1+\dots+ig_i=i} (-1)^{\sum d_t} \frac{(i+\sum d_t-1)!}{(i-\sum g_s)!} \left(\prod_{1 \leq t \leq n} \frac{(c(t-1))^{g_t+d_t}}{g_t!d_t!} \right).$$

If we take $i = n$ in the right sum, we get $c_n(0)$, so it means

$$B_n = -c_n(0) + n \sum_{i=0}^n \sum_{d_1+\dots+(n-i)d_{n-i}=n-i} \sum_{g_1+\dots+ig_i=i} (-1)^{\sum d_t} \frac{(i+\sum d_t-1)!}{(i-\sum g_t)!} \left(\prod_{1 \leq t \leq n} \frac{(c(t-1))^{g_t+d_t}}{g_t!d_t!} \right).$$

The sum can be expressed as

$$\begin{aligned}
& \sum_{\sum_{t=1}^n t(d_t + g_t) = n} (-1)^{\sum d_t} \frac{(\sum_t t g_t + \sum d_t - 1)!}{(\sum (t-1) g_t)!} \left(\prod_{1 \leq t \leq n} \frac{(c(t-1))^{g_t + d_t}}{g_t! d_t!} \right) = \\
& \sum_{\sum_{t=1}^n t b_t = n} \sum_{d_t + g_t = b_t} (-1)^{\sum d_t} \frac{(\sum_t t g_t + \sum d_t - 1)!}{(\sum (t-1) g_t)!} \left(\prod_{1 \leq t \leq n} \frac{(c(t-1))^{g_t + d_t}}{g_t! d_t!} \right) = \\
& \sum_{\sum_{t=1}^n t b_t = n} \sum_{0 \leq g_t \leq b_t} (-1)^{\sum b_t - g_t} \frac{(\sum_t (t-1) g_t + \sum b_t - 1)!}{(\sum (t-1) g_t)!} \left(\prod_{1 \leq t \leq n} \frac{(c(t-1))^{b_t}}{g_t! (b_t - g_t)!} \right) = \\
& \sum_{\sum_{t=1}^n t b_t = n} \sum_{\substack{0 \leq g_t \leq b_t \\ 1 < t}} (-1)^{b_1 + \sum_{1 < t} b_t - g_t} \frac{(\sum_t (t-1) g_t + \sum b_t - 1)!}{(\sum (t-1) g_t)!} \\
& \left(\prod_{2 \leq t \leq n} \frac{(c(t-1))^{b_t}}{g_t! (b_t - g_t)!} \right) c(0)^{b_1} \sum_{0 \leq g_1 \leq b_1} \prod_{1 \leq t \leq n} \frac{(-1)^{g_0}}{g_0! (b_0 - g_0)!}.
\end{aligned}$$

We can now see that the sum vanishes as the inner sum cancels out, which leaves us with $B_n = -c_n(0)$. $\square$

This finishes the proof of Theorem 1.

4 Upper Bound

We will now prove an upper bound for the value of m for which all of the zeros of $f_{k,m}$ are on the arc.

Lemma 6. *For all n ,*

$$B_n = - \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin(\theta) j^n(e^{i\theta}) d\theta.$$

Proof. As $c_n(0)$ is the coefficient of q^0 in j^n , we can write

$$-B_n = c_n(0) = \text{Res}_{q=0}(q^{-1} j^n(q)) = \frac{1}{2\pi i} \int_{|t|=r} (q^{-1} j^n(q)) dq$$

for some $0 < r < 1$.

After a change of variables $q = e^{2\pi i \tau}$ with $A > 1$, we have

$$-B_n = \int_{-\frac{1}{2} + Ai}^{\frac{1}{2} + Ai} j^n(\tau) d\tau.$$

Let γ denote the following contour where $\gamma_t : [t-1, t] \rightarrow \mathbb{C}$ for $t = 1, \dots, 4$,

$$\begin{aligned}
\gamma_1(\tau) &= e^{i\frac{\pi}{3}(1+\tau)} & \gamma_2(\tau) &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i + (\tau-1)(A - \frac{\sqrt{3}}{2})i \\
\gamma_3(\tau) &= -\frac{1}{2} + (\tau-2) + Ai & \text{and} & \quad \gamma_4(\tau) = \frac{1}{2} + Ai + (\tau-3)(\frac{\sqrt{3}}{2} - A)i.
\end{aligned}$$

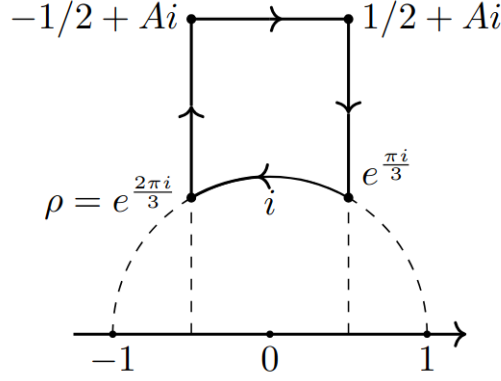


Figure 1: The contour of integration

As j is a modular function, we have

$$\int_{\gamma_2} j^n = - \int_{\gamma_4} j^n,$$

which means that as $-B_n = \int_{\gamma_1} j^n(\tau) d\tau$,

$$B_n = \int_{\gamma_3} j^n(\tau) d\tau.$$

We will do a change of variables of $\tau = e^{i\theta}$ so $d\tau = ie^{i\theta} d\theta$, so

$$B_n = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} ie^{i\theta} j^n(e^{i\theta}) d\theta = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} i(\cos(\theta) + i\sin(\theta)) j^n(e^{i\theta}) d\theta.$$

As for $\alpha \leq \frac{\pi}{6}$, $\cos(\frac{\pi}{2} + \alpha) = -\cos(\frac{\pi}{2} - \alpha)$ and $j(\frac{\pi}{2} + \alpha) = j(\frac{\pi}{2} - \alpha)$, the imaginary part of the integral vanishes and we are left with

$$B_n = - \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin(\theta) j^n(e^{i\theta}) d\theta.$$

□

Lemma 7.

$$\lim_{n \rightarrow \infty} \frac{A_n}{-B_n} = \frac{1}{4\pi}.$$

Proof. As j is a modular function we can extend the interval of integration,

$$A_n = \frac{1}{2\pi} \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} j^n(e^{i\theta}) d\theta = \frac{1}{4\pi} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} j^n(e^{i\theta}) d\theta$$

and we have seen that

$$B_n = - \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin(\theta) j^n(e^{i\theta}) d\theta.$$

Take $\varepsilon > 0$ and some $\delta > 0$ such that $\sin(\frac{\pi}{2} - \delta) > 1 - \varepsilon$.

For an upper bound

$$-B_n = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin(\theta) j^n(e^{i\theta}) d\theta \leq \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} j^n(e^{i\theta}) d\theta = 4\pi A_n.$$

For a lower bound,

$$\frac{A_n}{-B_n} = \frac{1}{4\pi} \frac{\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} j^n(e^{i\theta}) d\theta}{\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin(\theta) j^n(e^{i\theta}) d\theta} = \frac{1}{4\pi} \frac{\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} (\frac{j}{1728})^n(e^{i\theta}) d\theta}{\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin(\theta) (\frac{j}{1728})^n(e^{i\theta}) d\theta}.$$

The function $\frac{j}{1728} : [\frac{\pi}{3}, \frac{\pi}{2}] \rightarrow [0, 1]$ is non-decreasing and continuous and so there exists $n \gg 1$ large enough such that $\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} (\frac{j}{1728})^n(e^{i\theta}) d\theta \geq (1 - \varepsilon) \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\frac{j}{1728})^n(e^{i\theta}) d\theta$.

For such n ,

$$\frac{1}{4\pi} \frac{\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\frac{j}{1728})^n(e^{i\theta}) d\theta}{\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(\theta) (\frac{j}{1728})^n(e^{i\theta}) d\theta} \geq \frac{1}{4\pi} \frac{\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\frac{j}{1728})^n(e^{i\theta}) d\theta}{(1 - \varepsilon) \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} (\frac{j}{1728})^n(e^{i\theta}) d\theta} \geq \frac{1}{4\pi} \frac{\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\frac{j}{1728})^n(e^{i\theta}) d\theta}{(1 - \varepsilon)^2 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\frac{j}{1728})^n(e^{i\theta}) d\theta} = \frac{1}{4\pi} \cdot \frac{1}{(1 - \varepsilon)^2},$$

and so this goes to $\frac{1}{4\pi}$ as $n \rightarrow \infty$. $\square$

Proposition 3. *If $\frac{A_n}{-B_n}k + \frac{C_n(k')}{-B_n} < m \leq \ell - n$, at least one of the zeros is not on the arc.*

Proof. We know that for $m \leq \ell - n$, the sum of the powers of the zeros is

$$\sum_{i=1}^{\ell-m} x_i(k, m) = A_n \cdot k + B_n \cdot m + C_n(k').$$

If the sum of the zeros is negative, at least one of the zeros of the polynomial is not in $[0, 1728]$, and the corresponding zero of the form is not on the arc. Note that the sum is negative if and only if $\frac{A_n}{-B_n}k + \frac{C_n(k')}{-B_n} < m$. If n is even, we also know two of the zeros are complex.

Over all, this means that if $\frac{A_n}{-B_n}k + \frac{C_n(k')}{-B_n} < m \leq \ell - n$ at least one of the zeros is not on the arc. $\square$

If we take $n = 1$ we get

Corollary 4. *If $k' = 0, 4, 8$ and $\frac{30}{31}\ell + k' \frac{5}{62} < m \leq \ell - 1$, at least one of the zeros of $f_{k,m}$ is not on the arc.*

If $k' = 6, 10, 14$ and $\frac{30}{31}\ell + k' \frac{5}{62} - \frac{72}{31} < m \leq \ell - 1$, at least one of the zeros of $f_{k,m}$ is not on the arc.

More generally, if we take $m = \ell - n$,

Corollary 5. *Fix $n > 0$. For all ℓ such that $\ell > \frac{A_n k' + C_n(k') + n B_n}{-B_n - 12 A_n}$, at least one of the zeros of $f_{k, \ell - n}$ is not on the arc.*

Proof. Take $m = \ell - n$. So we need $\frac{A_n}{-B_n}k + \frac{C_n(k')}{-B_n} < m = \ell - n \leq \ell - n$. This is equivalent to

$$\frac{A_n}{-B_n}(12\ell + k') + \frac{C_n(k')}{-B_n} < \ell - n$$

and

$$\frac{A_n}{-B_n}k' + \frac{C_n(k')}{-B_n} + n < \ell - 12 \frac{A_n}{-B_n}\ell = \left(1 - 12 \frac{A_n}{-B_n}\right)\ell.$$

Which is equivalent to

$$\ell > \frac{\frac{A_n}{-B_n}k' + \frac{C_n(k')}{-B_n} + n}{1 - 12\frac{A_n}{-B_n}} = \frac{A_n k' + C_n(k') + nB_n}{-B_n - 12A_n}.$$

□

Corollary 6. *Assume $c\ell < m$ for some $\frac{3}{\pi} < c < 1$. Then for ℓ large enough, at least one of the zeros of $f_{k,m}$ is not on the arc.*

Proof. If $\ell - m$ is bounded as $\ell \rightarrow \infty$, we can use Corollary 4. So we will assume $\ell - m \rightarrow \infty$ as $\ell \rightarrow \infty$.

We will take $n \rightarrow \infty$ with $n \leq \ell - m$ (which we can, as $\ell - m \rightarrow \infty$) which means that asymptotically, the right inequality is satisfied, in

$$\frac{A_n}{-B_n}k + \frac{C_n(k')}{-B_n} < m \leq \ell - n.$$

As $C_n(k')$ is non-positive so is the ratio $\frac{C_n(k')}{-B_n}$, and it is enough to show that for ℓ large enough,

$$\frac{A_n}{-B_n}k < c\ell,$$

which is

$$\frac{A_n}{-B_n}k' < \left(c - 12\frac{A_n}{-B_n}\right)\ell.$$

The LHS is bounded, so if $c > 12\frac{A_n}{-B_n}$ for all n , for ℓ large enough at least one of the zeros is not on the arc. As $n \rightarrow \infty$, $12\frac{A_n}{-B_n} \rightarrow \frac{3}{\pi}$ and we get an asymptotic condition on where all the zeros can still be on the arc. □

5 Limit Distribution and Proof of Theorem 2

As we are interested in the distribution of the zeros of the Faber polynomials we think of the sums in the following way. We denote $\mu_{k,m} = \frac{1}{\ell-m} \sum_{i=1}^{\ell-m} \delta_{x_i}$, and so the moments of the measure are

$$M_n(k, m) = \int x^n d\mu_{k,m} = \frac{1}{\ell-m} \sum_{i=1}^{\ell-m} x_i^n,$$

which is the average of the powers of the zeros. The limit of $M_n(k, m)$ as $k \rightarrow \infty$ will give us the n -th moment of the limit distribution of the zeros.

Theorem 3. *For $m \sim c\ell$, $0 < c < \frac{2}{9}$, as $\ell \rightarrow \infty$ the limit probability that a zero of the modular form on the arc is between $[\theta_1, \theta_2]$ is*

$$\frac{6}{\pi(1-c)}(\theta_2 - \theta_1) + \frac{2c}{1-c}(\cos(\theta_2) - \cos(\theta_1)).$$

We will give two proofs.

First proof. Raveh [5] showed that for $h(\theta) = \frac{k}{2}\theta + 2\pi m \cos(\theta)$, there is a unique zero of the modular form $f_{k,m}$ between every two values of h which are integer multiples of π . For $c < \frac{2}{9}$, h' is non negative and is increasing in $[\frac{\pi}{2}, \frac{2\pi}{3}]$.

The number of zeros of $f(e^{i\theta})$ for $\theta_1 \leq \theta \leq \theta_2$ is the number of integer values of $\frac{h}{\pi}$ in this interval with error

bounded by 2, which is $\lfloor (h(\theta_2) - h(\theta_1))/\pi \rfloor = \frac{h(\theta_2) - h(\theta_1) + O(1)}{\pi}$.

For such c , all the zeros of the modular form are on the arc, so there are $\ell - m$ such zeros.

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \frac{h(\theta_2) - h(\theta_1) + O(1)}{\pi(\ell - m)} &= \lim_{\ell \rightarrow \infty} \frac{(6\ell + \frac{k'}{2})\theta_2 + 2\pi c\ell \cos(\theta_2) - (6\ell + \frac{k'}{2})\theta_1 + 2\pi c\ell \cos(\theta_1) + O(1)}{\pi(1 - c)\ell} = \\ &= \frac{6}{\pi(1 - c)}(\theta_2 - \theta_1) + \frac{2c}{1 - c}(\cos(\theta_2) - \cos(\theta_1)). \end{aligned}$$

□

Remark. The second proof uses the moments to compute the distribution. The distribution computed from the moments can only be determined for all such $0 \leq c$ for which all the zeros of the modular form are on the arc.

As long as all the zeros of the Faber polynomial are in $[0, 1728]$, we can use the fact that the Hausdorff moment problem has a unique solution, if one exists. This means that as long as $0 \leq c$ is small enough such that all the zeros of the Faber polynomial are in $[0, 1728]$, and as long as f_c is monotone increasing, f_c is the density function of the zeros of the Faber polynomial.

For f_c to be monotone increasing it is enough if $c \leq \frac{3}{\pi}$, as for such c , the probability a zero is in $[\theta_1, \theta_2]$ is

$$\begin{aligned} \frac{6}{\pi(1 - c)}(\theta_2 - \theta_1) + \frac{2c}{1 - c}(\cos(\theta_2) - \cos(\theta_1)) &= \frac{2}{1 - c}(\frac{3}{\pi}(\theta_2 - \theta_1) + c(\cos(\theta_2) - \cos(\theta_1))) \geq \\ &= \frac{6}{(1 - c)\pi}(\theta_2 - \theta_1 + \cos(\theta_2) - \cos(\theta_1)). \end{aligned}$$

As the derivative of $x + \cos(x) = 1 - \sin(x) \geq 0$, the probability is indeed non negative and f_c is a density function.

Second proof. Let f_c denote the limit density function of the zeros of $F_{k,m}$, the Faber polynomial of $f_{k,m}$, as $m \sim c\ell$ and $k \rightarrow \infty$.

As $0 < c < \frac{2}{9}$, all the zeros of $F_{k,m}$ are in $[0, 1728]$. Using the linearity of the sums of the powers of the zeros, we showed that

$$m_n = \frac{12}{1 - c}A_n + \frac{c}{1 - c}B_n$$

and f_c should satisfy

$$m_n = \int_{-\infty}^{\infty} x^n f_c(x) dx.$$

From the Hausdorff moment problem, there exists a unique density function with these moments. We will look at

$$f_c(x) := \mathbb{1}_{[0, 1728]} \frac{\frac{72}{\pi} - 2c \cdot \sin(j^{-1}(x))}{(1 - c)j'(j^{-1}(x))}.$$

Note that

$$\int_0^{1728} x^n f(x) dx = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} j^n(\theta) f(j(\theta)) j'(\theta) d\theta = \frac{72}{\pi(1 - c)} \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} j^n(\theta) d\theta - \frac{2c}{1 - c} \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sin(\theta) j^n(\theta) d\theta.$$

Notice that

$$\frac{72}{\pi(1 - c)} \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} j^n(\theta) d\theta = \frac{12}{1 - c} A_n$$

and

$$-\frac{2c}{1-c} \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sin(\theta) j^n(\theta) d\theta = \frac{c}{1-c} B_n.$$

These shows that this choice of a density function gives the correct moments and is the density function of the limit distribution of the zeros.

Take $\theta_1, \theta_2 \in [\frac{\pi}{2}, \frac{2\pi}{3}]$. The limit probability that a zero is in $\{e^{i\theta} : \theta_1 \leq \theta \leq \theta_2\}$ is

$$\begin{aligned} \int_{\theta_1}^{\theta_2} f_c(x) dx &= \int_{j(\theta_1)}^{j(\theta_2)} f_c(j(x)) j'(x) dx \\ &= \int_{j(\theta_1)}^{j(\theta_2)} \mathbb{1}_{[\frac{\pi}{2}, \frac{2\pi}{3}]} \cdot \frac{\frac{72}{\pi} - 2c \cdot \sin(x)}{(1-c)j'(x)} j'(x) dx \\ &= \int_{j(\theta_1)}^{j(\theta_2)} \frac{\frac{72}{\pi} - 2c \cdot \sin(x)}{(1-c)} dx \\ &= \frac{72}{\pi(1-c)} (\theta_2 - \theta_1) + \frac{2c}{1-c} (\cos(\theta_2) - \cos(\theta_1)) \end{aligned} \tag{1}$$

This gives us another proof for the value of B_n . By starting from the distribution and using it to compute the moments, if we know A_n we can get B_n . $\square$

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