A recursive approach for the determination of the nice sections of width three having a 4-crown stack as retract

Frank a Campo

Viersen, Germany
acampo.frank@gmail.com

Abstract

The characterization of the finite minimal automorphic posets of width three is still an open problem. Niederle has shown that this task can be reduced to the characterization of the nice sections of width three having a non-trivial tower of nice sections as retract. In our article, we characterize those of them which have a 4-crown stack as retract and we develop a recursive approach for their determination. We apply the approach on a subclass $\mathfrak{N}_2$ of nice sections of width three and determine all posets in $\mathfrak{N}_2$ with height up to six having a 4-crown stack as retract. For each integer $n \geq 2$, the class $\mathfrak{N}_2$ contains 2^{n-2} different isomorphism types of posets of height n.

Mathematics Subject Classification:
Primary: 06A07. Secondary: 06A06.

Key words: fixed point, fixed point property, retract, retraction, minimal automorphic, width three.

1 Introduction

We call a finite poset P *automorphic* if it has a fixed point free automorphism, and we call it *minimal automorphic* if additionally every proper retract of it has the fixed point property. Minimal automorphic posets have been present in the literature since the late eighties; an overview is provided by [10]. In particular, they have found interest as forbidden retracts because a poset does not have the fixed point property iff it has a minimal automorphic poset as retract [9, Th. 4.8].

However, the description or characterization of minimal automorphic posets is still an open field. (Unless otherwise stated, "poset" always means "finite poset" in this article.) Since the pioneering work of Rival [7, 8], we know that a poset of height one does not have the fixed point property iff it contains a crown, because a crown of minimal length will be a retract of it. Brualdi and Da Silva [1] generalized this result by showing that a poset does not have the fixed point property iff it has a "generalized crown" as retract. With respect to posets of small width, Fofanova and Rutkowski [4] showed that a poset of width two does not have the fixed point property iff it has a 4-crown stack as retract. (The result extends even to chain-complete infinite posets.) Niederle [5] characterized in 1989 posets of width three not having the fixed point property by having a non-trivial retract being a "tower of nice sections". Later on, in 2008, he extended his result to posets of width four [6].

Unfortunately, it is quite difficult to break down Niederle's abstract characterization of the fixed point property to concrete posets or classes of posets of width three. According to our knowledge the only real success in this field is a result of Farley [3] from 1997, who showed that the ranked nice sections of width three are exactly the 6-crown stacks, and that a 6-crown stack is minimal automorphic iff its height h is not a multiple of three. Otherwise, it has a 4-crown stack of height $\frac{2}{3}h$ as retract. Schröder [10, Lemma 2.6] generalized one direction of this result by showing that a $2n$-crown stack of height nm has a 4-crown stack of height $2m$ as retract. Moreover, in the case of $n \geq 3$, a $2n$-crown stack of height h is not minimal automorphic if h and n have a non-trivial common factor [10, Prop. 2.7].

Due to the results of Niederle [5], the investigation of the minimal automorphic posets of width three can be restricted to the investigation of the nice sections of width three. In our article, we characterize those of them which have a 4-crown stack as retract and we develop and apply a recursive approach for their determination.

After the preparatory Section 2, we deal with the classes $\mathfrak{N}$ of nice sections and $\mathfrak{C}$ of *crowned sections* in Section 3; $\mathfrak{N}$ is a proper subclass of $\mathfrak{C}$. We show in Theorem 3.4 that a poset $P \in \mathfrak{C}$ has a 4-crown stack as retract iff it can be decomposed by a *retractive split* in a certain way. This characterization gives raise to a recursive approach for the determination of the nice sections having a 4-crown stack as retract; it is described at the end of Section 3.

In order to test the approach, we introduce the class $\mathfrak{N}_2$ of nice sections with *horizon two* in Section 4. For each integer $n \geq 2$, this class contains 2^{n-2} isomorphism types of posets of height n; the isomorphism types are described in Theorem 4.2. We characterize in Theorem 4.5 the retractive splits of posets $P \in \mathfrak{N}_2$ fulfilling the requirements of Theorem 3.4, and using this result, we determine in Section 5 by means of the recursive approach all posets $P \in \mathfrak{N}_2$ of height up to six having a 4-crown stack as retract.

The characterization of the minimal automorphic posets of width three is regarded as a quite difficult task [9, p. 98], in parts even as “intractable” [3, p. 126]. Also the sparse sequence of publications dealing with the topic indicates the difficulties. Nevertheless, it gives some confidence that the nice section of width three used by Schröder [9, p. 99] for the illustration of the difficulties and considered as “nasty” by Farley [3, p. 136] is handled without a problem by the machinery developed in this article. It is the poset coded as 1011 in Section 5 and shown in Figure 7.

2 Preparation

In this section, we introduce our notation and recapitulate definitions of structures being in the focus of our investigation. For all other terms of order theory, the reader is referred to standard textbooks as [9]. For integers $i \leq j$, we write $[i, j]$ for the interval $\{i, \ldots, j\}$.

Let $P = (X, \leq)$ be a finite poset. (We deal with finite posets only in this paper.) For $Y \subseteq X$, the *induced sub-poset* $P|_Y$ of P is $(Y, \leq \cap (Y \times Y))$. To simplify notation, we identify a subset $Y \subseteq X$ with the poset $P|_Y$ induced by it.

For $y \in X$, we define the *down-set* and *up-set induced by* y as

$$\begin{aligned} \downarrow y &:= \{x \in X \mid x \leq y\}, \quad \mathring{\downarrow} y := (\downarrow y) \setminus \{y\}, \\ \uparrow y &:= \{x \in X \mid y \leq x\}, \quad \mathring{\uparrow} y := (\uparrow y) \setminus \{y\}. \end{aligned}$$

Given two points $x < y$ in P, the point x is called a *lower cover* of y and y an *upper cover* of x if $x \leq z \leq y$ implies $z \in \{x, y\}$ for all $z \in P$; this relation is denoted by $x \lessdot y$. A point in P is called *irreducible* if it has a single lower cover or a single upper cover.

Let $A, B \subseteq P$. We write $A < B$ if $a < b$ for all $a \in A$, $b \in B$. If $A = \{a\}$ is a singleton, we simply write $a < B$ and $B < a$. The notation for the relation $\leq$ is analogous.

The *length* of a chain is its cardinality minus 1, and the *height* of a poset is the maximal length of a chain contained in it. For a poset P, we denote its height by h_P. The *width* of a poset is the largest size of an antichain contained in it.

For a poset P, the *level sets* $P(k)$, $k \in [0, h_P]$, are recursively defined by

$$\begin{aligned} P(0) &:= \min P, \\ P(k+1) &:= \min \left(P \setminus \cup_{i=0}^{k} P(i) \right) \quad \text{for all } 0 \leq k < h_P. \end{aligned}$$

Additionally, we write for all $k, \ell \in [0, h_P]$ with $k \leq \ell$,

$$\begin{aligned} P(k, \ell) &:= P(k) \cup P(\ell), \\ P(k \to \ell) &:= \cup_{i=k}^{\ell} P(i). \end{aligned}$$

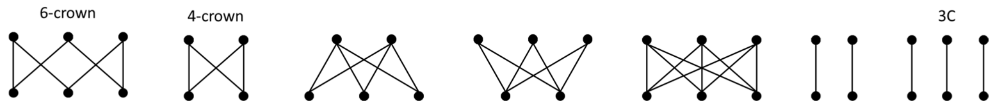

Figure 1: The level pairs $P(k,\ell)$, $k < \ell$, which are possible in an automorphic poset P of width at most three.

For $k_1 \leq k_2$ being level indices of P, we call the sub-poset $P(k_1 \to k_2)$ a *segment* of P, a *lower segment* if $k_1 = 0$, and an *upper segment* if $k_2 = h_P$.

For $n \in \mathbb{N} \setminus \{1\}$, a $2n$*-crown* is a poset with $2n$ points $x_{i,j}$, $i \in [0, n-1]$, $j \in \{0,1\}$, in which $x_{0,0} < x_{0,1} > x_{1,0} < \cdots > x_{n-1,0} < x_{n-1,1} > x_{0,0}$ are the only comparabilities between different points. We call a poset P of height ≥ 1 a $2n$*-crown stack* iff $P(k,k+1)$ is a $2n$-crown for all $k \in [0, h_P - 1]$ and all other comparabilites are induced by transitivity.

Given disjoint posets $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$, their *ordinal sum* is the poset $P \oplus Q := (X \cup Y, \leq_P \cup \leq_Q \cup (X \times Y))$. For P and Q being antichains of size 3, we simply write "33" as shortcut for the isomorphism type of $P \oplus Q$. Additionally, we write $3C$ for a poset consisting of three disjoint chains of height 1 without any additional comparabilities between different points. The shortcuts are illustrated in Figure 1.

For a mapping $f : X \to Y$, we denote by $f|_Z$ the pre-restriction of f to a subset Z of its domain and by $f|^Z$ its post-restriction to a subset Z of its codomain with $f[X] \subseteq Z$. An order homomorphism $r : P \to P$ is called a *retraction* of the poset P if r is idempotent, and an induced sub-poset R of P is called a *retract of* P if a retraction $r : P \to P$ exists with $r[X] = R$. For the sake of simplicity, we identify r with its post-restriction and write $r : P \to R$. A poset P has the fixed point property iff every retract of P has the fixed point property [9, Th. 4.8]. We need the following well-known result about the existence of *spanning retracts*:

Lemma 2.1 ([2, p. 232]). *Let P be be a finite poset and $r : P \to R$ a retraction. There exists a retraction $v : P \to V$ with $V \simeq R$,* $\min V \subseteq \min P$ *and* $\max V \subseteq \max P$.

A poset is called *automorphic* if it has a fixed point free automorphism, and it is called *minimal automorphic* if additionally every proper retract of it has the fixed point property. According to [9, Prop. 4.38], a poset P of width at most three is automorphic iff for all level indices $0 \leq k < \ell \leq h_P$, the level-pair $P(k,\ell)$ is one of the seven posets shown in Figure 1. The result goes back to [6, Th. 10].

3 Nice sections and crowned sections

3.1 Definition and basic properties

The following definition is due to Niederle [5]:

Definition 3.1. *A* section *is a two-element antichain or a poset P of heigth at least one with carrier* $X := \{ c_{k,j} \mid k \in [0, h_P], j \in \{0,1,2\} \}$ *for which*

- $C_j := c_{0,j} < \ldots < c_{h_P,j}$ *is a chain for all* $j \in \{0,1,2\}$*;*
- $\{c_{k,0}, c_{k,1}, c_{k,2}\}$ *is an antichain for all* $k \in [0, h_P]$*;*
- *The mapping* $c_{k,j} \mapsto c_{k,(j+1) \bmod 3}$ *for all* $k \in [0, h_P]$, $j \in \{0,1,2\}$ *is an automorphism of P;*
- $P(k, k+1) \not\simeq 33$ *for all level indices* $k \in [0, h_P - 1]$.

A section P is called nice *if P does not contain an irreducible point. The symbol* $\mathfrak{N}$ *denotes the class of nice sections. A* tower of sections *is an ordinal sum of sections.*

Simple examples of nice sections are the 6-crown stacks, whereas a 4-crown stack is a tower of nice sections, i.e., of 2-antichains.

For a section P of width 3, the first and second property imply $c_{0,j} \lessdot \ldots \lessdot c_{h_P,j}$ for all $j \in \{0,1,2\}$. We call the chains C_0, C_1, C_2 the *main chains* of P. Clearly, $P(k) = \{c_{k,0}, c_{k,1}, c_{k,2}\}$ for all $k \in [0, h_P]$. We define mappings $\lambda : P \to [0, h_P]$ and $\gamma : P \to \{0,1,2\}$ by setting $\lambda(c_{k,j}) := k$ and $\gamma(c_{k,j}) := j$, hence $\{p\} = P(\lambda(p)) \cap C_{\gamma(p)}$ for all $p \in P$. Two consecutive levels of P form a 6-crown or are of type $3C$. In what follows, the level-index function λ and the chain-index function γ always refer to the levels and main chains of P; for a retract R of P, we thus have $\{x\} = P(\lambda(x)) \cap C_{\gamma(x)}$ for all $x \in R$.

Niederle [5, p. 121 and p. 125] has shown that the decomposition of a tower of sections into sections is unique, and that a poset of width at most three has not the fixed point property iff it has a tower of nice sections as retract. We conclude that an automorphic poset P of width at most 3 not being a tower of nice sections is not minimal automorphic. Moreover, for $P = S_1 \oplus \cdots \oplus S_n$ being a tower of nice sections $S_1, \ldots S_n$, it is not hard to see that P is minimal automorphic iff S_i is minimal automorphic for every $i \in [1, n]$. The objects we have to deal with in investigating minimal automorphic posets of width three are thus the nice sections of width three.

In a nice section P not being a 2-antichain, the level-pairs $P(0,1)$ and $P(h_P - 1, h_P)$ must form 6-crowns. The only nice sections of height 0, 1, and 2 are thus the 2-antichains, the 6-crowns, and the 6-crown stacks of height 2. All three are

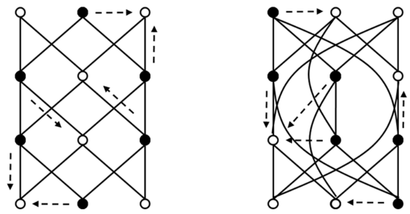

Figure 2: Two nice sections with 4-crown stacks as retracts (hollow dots). The retraction of the 6-crown stack on the left is from [3, Fig. 5.10].

minimal automorphic. Two nice sections with height 3 are shown in Figure 2. Both contain a 4-crown stack of height 2 as retract.

Our first goal is the characterization of nice sections of width three having a 4-crown stack as retract. However, our results will concern a larger class of sections containing the nice sections of width three as a proper subclass:

Definition 3.2. *We call a section P of width three a* crowned section *if $P(0,1)$ and $P(h_P-1,h_P)$ are 6-crowns. The symbol $\mathfrak{C}$ denotes the class of crowned sections.*

3.2 Four-crown stacks as retracts of crowned sections

Definition 3.3. *Let $P \in \mathfrak{C}$. We call a quadrupel (k,D,s,t) a* retractive down-split *of P if*

$$\begin{aligned} & k \in [0,h_P-1], \\ & D \subset P(k+1 \to h_P), \\ & s : P(k+1 \to h_P) \setminus D \to S \text{ is a retraction,} \\ \text{and} \quad & t : P(0 \to k) \to T \text{ is a retraction} \\ & \text{with } S \text{ and } T \text{ being 2-antichains or 4-crown stacks,} \end{aligned}$$

and we call a quadrupel (k,U,s,t) a retractive up-split *of P if*

$$\begin{aligned} & k \in [1,h_P], \\ & U \subset P(0 \to k-1), \\ & s : P(0 \to k-1) \setminus U \to S \text{ is a retraction,} \\ \text{and} \quad & t : P(k \to h_P) \to T \text{ is a retraction} \\ & \text{with } S \text{ and } T \text{ being 2-antichains or 4-crown stacks.} \end{aligned}$$

In order to avoid repetitions, we use the symbols S and T in what follows always as in this definition: for a retractive down-split (k,D,s,t) of $P \in \mathfrak{C}$, we always have $S = s[P(k+1 \to h_P) \setminus D]$ and $T = t[P(0 \to k)]$, and correspondingly for a retractive up-split. In a retractive down-split (k, D, s, t), the poset T is thus a retract of the lower segment of P ending with level k and D is a down-set, whereas in a retractive up-split (k, U, s, t), the poset T is a retract of the upper segment of P starting with level k and U is an up-set. The reader will observe that in a retractive down-split $(h_P - 1, D, s, t)$ and a retractive up-split $(1, U, s, t)$ we must have $\#D \leq 1$ and $\#U \leq 1$, respectively, because S contains at least two points.

Theorem 3.4. *A poset $P \in \mathfrak{C}$ has a 4-crown stack as retract iff there exists a retraction $r : P \to R$ of P onto a 4-crown stack R for which*

- *a retractive down-split (k, D, s, t) exists with*

$$\begin{aligned} R &= T \oplus S, \\ r|_{P(k+1\to h_P)\setminus D}^{P(k+1\to h_P)\setminus D} &= s, \\ r|_{P(0\to k)}^{P(0\to k)} &= t, \\ r[D] &\subseteq T, \end{aligned} \tag{1}$$

- *or a retractive up-split (k, U, s, t) exists with*

$$\begin{aligned} R &= S \oplus T, \\ r|_{P(0\to k-1)\setminus U}^{P(0\to k-1)\setminus U} &= s, \\ r|_{P(k\to h_P)}^{P(k\to h_P)} &= t, \\ r[U] &\subseteq T. \end{aligned} \tag{2}$$

The retraction of the poset on the left in Figure 2 yields for $k = 0$ a retractive down-split fulfilling (1) and for $k = 3$ a retractive up-split fulfilling (2). For the retraction of the poset on the right of the figure, the suitable level indices are $0, 2$ and $1, 3$, respectively. A retractive split of a crowned section P and a retraction of P onto a 4-crown stack are called *matching* if they are coupled by (1) or (2).

Only direction "$\Rightarrow$" of the theorem has to be proven. We agree on that for the rest of this section the poset P is always a crowned section and $r : P \to R$ is a retraction onto a 4-crown stack. We start the proof with two simple observations. Firstly, the down-sets $r^{-1}(a)$, $a \in R(0)$, are pairwise disjoint, hence

$$R(0) \cap P(0) \neq \emptyset. \tag{3}$$

Secondly,

Lemma 3.5. *Let $r[P(0 \to k)] = R(0 \to \ell) \subset P(0 \to k)$ with $\ell < h_R$. A matching down-split (k, D, s, t) is given by $t := r|_{P(0\to k)}^{P(0\to k)}$,*

$$D := \{x \in P(k+1 \to h_P) \mid r(x) \in R(0 \to \ell)\}, \quad s := r|_{P(k+1\to h_P)\setminus D}^{P(k+1\to h_P)\setminus D}.$$

In the next two propositions we see that two types of level sets of R always give raise to a matching retractive split of P: the level sets of R contained in single level sets of P, and the level sets of R with points in "maximal possible distance" in P. For examples, see Figure 2.

Corollary 3.6. *There exists a matching retractive split of P if there exists a level set of R contained in a single level set of P.*

Proof. Let ℓ and k be level indices of R and P, respectively, with $R(\ell) \subset P(k)$ and let $z \in P(k) \setminus R(\ell)$. Three cases are possible:

1. $\ell \in [1, h_R - 1]$: Then $k \in [1, h_P - 1]$. Depending on $\lambda(r(z)) \leq k$ or $\lambda(r(z)) \geq k$, Lemma 3.5 and its dual yield a matching retractive split.

2. $\ell = 0$: Then $k = 0$ due to (3). If $r(z) \notin R(0)$, define

$$t := r|_{P(1\to h_P)}^{P(1\to h_P)}, \quad U := \{z\}, \quad s := r|_{P(0)\setminus U}^{P(0)\setminus U}.$$

   For $x \in P(1 \to h_P)$, we have $R(0) < x$ or $z < x$, hence $r(x) \in P(1 \to h_P)$. Now apply the dual of Lemma 3.5. And in the case of $r(z) \in R(0)$, apply the lemma on

$$t := r|_{P(0)}^{P(0)}, \quad D := \{x \in P(1 \to h_P) \mid r(x) \in P(0)\}, \quad s := r|_{P(1\to h_P)\setminus D}^{P(1\to h_P)\setminus D}.$$

3. $\ell = h_R$: Dual to the previous case.

□

Lemma 3.7. *Let $R(\ell) = \{a, b\}$ with $\lambda(a) < \lambda(b)$. There exists a matching retractive split of P if*

1. *$\ell = 0$ and $P(0, \lambda(b) + 1) \simeq 33$;*

2. *$\ell = h_R$ and $P(\lambda(a) - 1, h_P) \simeq 33$;*

3. *$0 < \ell < h_R$ and $P(\lambda(a) - 1, \lambda(b)) \simeq 33 \simeq P(\lambda(a), \lambda(b) + 1)$.*

Proof. 1) We have $\lambda(a) = 0$ according to (3). Additionally, there exists a point $x \in P(0)$ with $r(x) = b$, hence $R(0) < r(y)$ for all $y \in P(\lambda(b)+1 \to h_P)$. Now apply the dual of Lemma 3.5.

2) Dual to the previous proposition.

3) Let $k := \lambda(a)$, $j := \lambda(b)$, and $x \in P(k) \setminus \{a\}$. If $r(x) \in \{b\} \cup R(\ell+1 \to h_R)$, then $P(k, j+1) \simeq 33$ yields $R(\ell) < r(z)$ for all $z \in P(j+1 \to h_P)$, and the dual of Lemma 3.5 yields a matching up-split $(j+1, U, s, t)$.

Now assume $r(x) \in R(0 \to \ell-1) \cup \{a\}$ for all $x \in P(k) \setminus \{a\}$. Then $r(z) \in R(0 \to \ell-1) \cup \{a\}$ for all $z \in P(0 \to k-1)$, and $P(k-1, j) \simeq 33$ additionally yields $r(z) < b$, hence $r(z) \in P(0 \to k-1)$. According to Lemma 3.5, there exists a matching down-split $(k-1, D, s, t)$.

□

By means of Corollary 3.6, we easily prove direction "⇒" of Theorem 3.4. If $r' : P \to R'$ is a retraction of P onto a 4-crown stack, then Lemma 2.1 ensures the existence of a retraction $r : P \to R$ with $R \simeq R'$ being a spanning retract, hence $R(0) \subset P(0)$ and $R(h_R) \subset P(h_P)$. Now Corollary 3.6 yields the existence of a retractive split of P coupled with r by (1) or (2), respectively.

3.3 A recursive approach

Let $\mathfrak{S}$ be a subclass of $\mathfrak{C}$ closed under duality, e.g., the subclass of nice sections of width three. Theorem 3.4 states that $P \in \mathfrak{S}$ having a 4-crown stack as retract requires a proper lower segment $P(0 \to k)$ or a proper upper segment $P(k \to h_P)$ having a 2-antichain or a 4-crown as retract. This gives raise to a recursive approach for the determination of the sections in $\mathfrak{S}$ having a 4-crown stack as retract.

For a retractive down-split or up-split, we need a *t-base* V and an *s-base* W of a poset $P \in \mathfrak{S}$ of width 3, i.e.:

- V is a lower segment or an upper segment of P having a retraction t onto a 2-antichain or a 4-crown stack T.
- W is an upper segment of P having a retraction of $W \setminus D$ onto a 2-antichain or a 4-crown stack S for a down-set $D \subset W$, or W is a lower segment of P having a retraction of $W \setminus U$ onto a 2-antichain or a 4-crown stack S for an up-set $U \subset W$.
- V and W are coupled by $V \cap W = \emptyset$ and $P = V \cup W$.

Assume that we have listed all lower segments up to height n of sections contained in $\mathfrak{S}$ which have a 2-antichain or a 4-crown stack as retract. Given a poset

$P \in \mathfrak{S}$ of height $n+1$, we can quickly identify all candidates for t-bases in P by checking which of the lower segments in our list are lower segments $P(0 \to k)$ of P and which of their duals are upper segments $P(k \to n+1)$ of P. Of course, $P(0)$ and $P(n+1)$ are always candidates for a t-base. According to Theorem 3.4, the poset P has a 4-crown stack as retract iff one of the corresponding segments $P(k+1 \to n+1)$ or $P(0 \to k-1)$ provides an s-base fulfilling (1) or (2), respectively.

In order to test the approach we need a sub-class of $\mathfrak{C}$ combining plenty of interesting posets with good structural access. We choose the class of the nice sections with *horizon two*. We introduce them in Section 4, and in Section 5 we identify all of them with height up to six having a 4-crown stack as retract.

4 Nice sections with horizon two

4.1 Definition and isomorphism types

For a crowned section $P \in \mathfrak{C}$ with height $h_P \geq 2$, we have $P(0, h_P) \simeq 33$. There exists thus a smallest integer η with

$$P(k, k+\eta) \simeq 33 \quad \text{for all } k \in [0, h_P - \eta].$$

We call this integer the *horizon* of P. It is the distance in which all details become blurred. The sections in Figure 2 have horizon two.

For an integer $\eta \in \mathbb{N}$, the symbols $\mathfrak{C}_\eta$ and $\mathfrak{N}_\eta$ denote the classes of crowned sections and nice sections with horizon η and height ≥ 2. The set $\mathfrak{C}_1$ is empty, and the only crowned sections not contained in $\mathfrak{C}_\eta$ for any $\eta \geq 2$ are the 6-crowns. For the rest of this article we focus on $\eta = 2$, and for this horizon, we can skip the discrimination between nice sections and crowned sections:

Proposition 4.1. $\mathfrak{C}_2 = \mathfrak{N}_2$.

Proof. Let $P \in \mathfrak{C}_2$. We have to prove that P does not contain an irreducible point. Let $x \in P(k)$, $k \in [0, h_P - 1]$. If $P(k, k+1)$ is a 6-crown, x has two upper covers. Let thus $P(k, k+1) \simeq 3C$, hence $k \leq h_P - 2$ and $x < P(k+2)$. The layer $P(k+1)$ contains an upper cover of x, and depending on the type of $P(k+1, k+2)$, the layer $P(k+2)$ contains one or even two additional upper covers of x. The argument for lower covers is dual. □

The 6-crown stacks of height greater than one belong to $\mathfrak{N}_2$, and Farley [3, Prop. 4.1] showed that all 6-crown stacks with equal height are isomorphic. The following theorem generalizes this result:

Theorem 4.2. *The isomorphism type of a poset $P \in \mathfrak{N}_2$ is uniquely determined by the types of its level-pairs $P(k,k+1)$, $k \in [1,h_P-2]$. In particular, there exist 2^{n-2} isomorphism types of posets of height $n \geq 2$ in $\mathfrak{N}_2$. Furthermore, every permutation of $P(0)$ can be uniquely extended to an automorphism of P.*

Proof. For each integer $n \geq 2$ and each index set $K \subseteq \{0,\ldots,n-1\}$ with $0,n-1 \in K$, we can construct a poset $P \in \mathfrak{N}_2$ with $h_P = n$ and $P(k,k+1)$ being a 6-crown iff $k \in K$: We start with the union Q of three pairwise disjoint chains of length n, extend $Q(k,k+1)$ to an arbitrary 6-crown for every $k \in K$, and complete the partial order relation by adding $Q(k) \times Q(\ell)$ for all $k,\ell \in [0,n]$ with $k \leq \ell-2$. The resulting section is an element of $\mathfrak{C}_2$ and belongs thus to $\mathfrak{N}_2$ (Proposition 4.1).

Now let $P = (X,\leq_P)$ and $Q = (Y,\leq_Q)$ be elements of $\mathfrak{N}_2$ and let $\alpha : P(0) \to Q(0)$ be a bijection. We define on X and Y the relations

$$\begin{aligned} &\prec_P := \{(x,y) \in <_P \mid \lambda(y) - \lambda(x) = 1\} \\ \text{and} \quad &\prec_Q := \{(x,y) \in <_Q \mid \upsilon(y) - \upsilon(x) = 1\}, \end{aligned}$$

where υ is the level-function of Q. It is easily seen that we can extend α to an isomorphism $\beta : (X,\prec_P) \to (Y,\prec_Q)$ iff $h_P = h_Q$ and

$$\forall\, k \in [1,h_P-2]: \quad P(k,k+1) \text{ 6-crown} \;\Leftrightarrow\; Q(k,k+1) \text{ 6-crown},$$

and that in this case β is uniquely determined by α. Every isomorphism from $(X,\prec_P)$ to $(Y,\prec_Q)$ sends $P(k)$ onto $Q(k)$ for all $k \in [0,h_P]$, and because P and Q have horizon 2, the isomorphisms between $(X,\prec_P)$ and $(Y,\prec_Q)$ are exactly the isomorphisms between P and Q.

□

Due to this result, we can always draw the Hasse-diagram of a poset $P \in \mathfrak{N}_2$ in a standardized way: all level-pairs $P(k,k+1)$ being 6-crowns are drawn with a "void" in the center as in Figure 2.

We denote the classes of segments, lower segments, and upper segments of nice sections with horizon 2 by $\mathfrak{N}_2^{Seg}, \mathfrak{N}_2^{LS}$, and $\mathfrak{N}_2^{US}$, respectively. Theorem 4.2 extends to these classes: Two segments of equal height n are isomorphic iff both are 3-antichains (the case $n = 0$) or all of their consecutive level-pairs are isomorphic (the case $n \geq 1$), and also the extension of a level permutation to an automorphism is always possible.

4.2 Properties of retracts

We start our investigation with three simple observations; in particular implication (4) will play an important role in several proofs because it yields via Lemma 3.5 a matching retractive up-split:

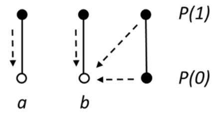

Figure 3: The construction in the proof of Corollary 4.4.

Corollary 4.3. *Let $P \in \mathfrak{N}_2^{Seg}$ and let $r : P \to R$ be a retraction.*

1) For all $\ell \in [0, h_R]$ there exists an index $k \in [0, h_P - 1]$ with $R(\ell) \subset P(k, k+1)$. In particular, $R(0) \subset P(0,1)$ if $\#R(0) \geq 2$.

2) Let $\ell \in [0, h_R]$ with $\#R(\ell) = 2$ and $0 < \rho := \max\{\lambda(x) \mid x \in R(\ell)\} < h_P$. Then

$$\begin{aligned} & \forall\, a \in R(\ell) : \; r^{-1}(a) \cap P(\rho - 1) \neq \emptyset \\ \textit{implies} \quad & r[P(\rho+1 \to h_P)] = R(\ell+1 \to h_R) \subseteq P(\rho+1 \to h_P). \end{aligned} \tag{4}$$

Proof. The first proposition follows from the definition of the horizon and (3). For the second one, observe that the assumption implies $a \leq r(y)$ for all $a \in R(\ell)$ and all $y \in P(\rho+1 \to h_P)$, hence $R(\ell) < r(y)$.

□

Corollary 4.4. *Let $P \in \mathfrak{N}_2^{Seg}$ with $h_P \geq 2$ and $P(0,1) \simeq 3C$. P has a retract R with $R(0,1)$ being a 4-crown iff $P(2 \to h_P)$ has a retract W with $W(0)$ being a 2-antichain.*

Proof. "$\Rightarrow$": Let $R(0) = \{a, b\}$ with $\lambda(a) \leq \lambda(b)$. Due to the first part of Corollary 4.3, we have $R(0) \subset P(0,1)$. In the case of $\lambda(b) = 1$, the equation $r[P(2 \to h_P)] = R(1 \to h_P)$ follows with implication (4). In the case of $\lambda(b) = 0$, the relation $R(0) < P(2 \to h_P)$ enforces $r[P(2 \to h_P)] \subseteq R(1 \to h_P)$, and $P(0,1) \simeq 3C$ yields $R(1) \subset P(2 \to h_P)$.

"$\Leftarrow$": If $w : P(2 \to h_P) \to W$ is a retraction with $W(0)$ being a 2-antichain, select two points $a, b \in P(0)$ and define a retraction $r : P \to W \cup \{a, b\}$ by (cf. Figure 3)

$$r(x) := \begin{cases} w(x), & \text{if } x \in P(2 \to h_P), \\ a, & \text{if } x \in \{a, c_{1,\gamma(a)}\}, \\ b, & \text{if } x \in P(0,1) \setminus \{a, c_{1,\gamma(a)}\}. \end{cases}$$

□

4.3 Retractive up- and down-splits

Let $P \in \mathfrak{N}_2$ and let $r : P \to R$ be a retraction onto a 4-crown stack R. Due to the horizon of P, every level set of R is either contained in a single level set of P or consists of points belonging to consecutive level sets of P. Now Corollary 3.6 and Lemma 3.7 tell us that *every* level-set of R gives raise to matching retractive split of P. On the other hand, for a given retractive split of P it is easy to check if there exists a matching retraction of P onto a 4-crown stack:

Theorem 4.5. *Let $P \in \mathfrak{N}_2$. For a retractive down-split (k,D,s,t) of P, the existence of a retraction $r : P \to R$ onto a 4-crown stack R coupled with (k,D,s,t) by* (1) *is equivalent to*

$$\begin{gathered} T(h_T) <_P S(0), \\ \forall\, d \in D\, \exists\, v \in T(h_T) :\ p \not<_P d \text{ for all } p \in t^{-1}(v). \end{gathered} \tag{5}$$

which implies $D \subseteq P(k+1)$, and for a retractive up-split (k,U,s,t) of P, there exists a retraction $r : P \to R$ onto a 4-crown stack R coupled with (k,U,s,t) by (2) *iff*

$$\begin{gathered} S(h_S) <_P T(0), \\ \forall\, u \in U\, \exists\, v \in T(0) :\ u \not<_P p \text{ for all } p \in t^{-1}(v), \end{gathered} \tag{6}$$

which implies $U \subseteq P(k-1)$.

Proof. Let $r : P \to R$ be a retraction onto a 4-crown stack R coupled with a retractive up-split (k,U,s,t) by (2). Due to $r[P(k \to h_P)] = T$, the second part of Condition (6) is necessary for $r[U] \subseteq T$. The addendum follows because of $P(0 \to k-2) <_P P(k \to h_P) \supset T(0)$.

Now assume that (6) holds for (k,U,s,t) and let $u \in U \subseteq P(k-1)$. There exists a point $\tau(u) \in T(0)$ with $u \not<_P p$ for all $p \in t^{-1}(\tau(u))$. We define $\rho(u)$ as the single point contained in $T(0) \setminus \{\tau(u)\}$. Now we define a mapping $r : P \to P$ by setting for $x \in P$

$$r(x) := \begin{cases} s(x), & \text{if } x \in P(0 \to k-1) \setminus U, \\ \rho(x), & \text{if } x \in U, \\ t(x), & \text{if } x \in P(k \to h_P), \end{cases}$$

The mapping r is clearly idempotent. In order to show that it is order-preserving, let $x <_P y$. Only three cases have to be discussed:

- $x \in P(0 \to k-1) \setminus U$ and $y \in U \cup P(k \to h_P)$ yields $r(x) \in S <_{S \oplus T} T \ni r(y)$.

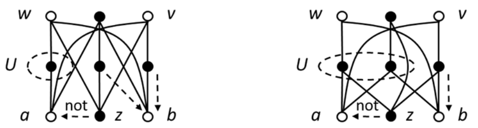

Figure 4: Illustrations for the proof of Corollary 4.6. The three levels show $P(k-1 \to k+1)$ for choices $b \in P(k-1)$, $w \in P(k+1)$ (which are not mandatory). The point z is not mapped to a by the retraction s of $P(0 \to k-1)$.

- $x \in U$, $y \in P(0 \to k-1)$ is not possible due to $U \subseteq P(k-1)$.
- $x \in U$ and $y \in P(k \to h_P)$: $x \not< p$ for all $p \in t^{-1}(\tau(x))$ delivers $t(y) \in T \setminus \{\tau(x)\}$. Because $\rho(x)$ is the only minimal point of this poset, $r(x) = \rho(x) \leq t(y) = r(y)$.

□

Corollary 4.6. *Let $k \in [1, h_P - 1]$ and assume that we have retractions $s : P(0 \to k-1) \to S$ and $t : P(k+1 \to h_P) \to T$ with S and T being 2-antichains or 4-crown stacks with*

$$s^{-1}(a) = \{a\} \textit{ for a point } a \in S(h_S),$$
$$\textit{and} \quad t^{-1}(v) = \{v\} \textit{ for a point } v \in T(0).$$

The 4-crown stack $S \oplus T$ is a retract of P if $P(k-1 \to k+1)$ is not a 6-crown stack.

Proof. Let $S(h_S) = \{a, b\}$ and $T(0) = \{v, w\}$. The assumptions imply $a \in P(k-1)$ and $v \in P(k+1)$. Assume $P(k, k+1) \simeq 3C$. According to Lemma 2.1, we can assume $S(h_S) \subset P(k-1)$ without loss of generality. Let $P(k-1) = \{a, b, z\}$.

Let $U \subset P(k)$ be the set of upper covers of a in $P(k)$. Applying Theorem 4.2 on $P(0 \to k)$, we see that without loss of generality we can assume that none of the points in U is below v (Figure 4).

No point of $P(k) \setminus U$ is above a and the point b is the only maximal point of $S \setminus \{a\}$. Due to $s^{-1}(a) = \{a\}$, the mapping

$$s' : P(0 \to k) \setminus U \to S,$$
$$x \mapsto \begin{cases} s(x), & \text{if } x \in P(0 \to k-1), \\ b, & \text{if } x \in P(k) \setminus U \end{cases}$$

1		n									
11		n	10		y						
111		y	101		y	110		n	100		n
1111	0,3	n	1101	0	n	1011	0,2,3	n	1001	0,2	y
1110		n	1100		n	1010		y	1000		y
11111	0,3	n	11101	0,3	y	11011	0	n	10111	0,2,3	y
11001	0	y	10101	0,2,3,4	y	10011	0,2,4	y	10001	0,2,4	y
11110		y	11100		y	11010		n	10110		y
11000		n	10100		y	10010		n	10000		n
111111	0,3	y	111101	0,3,5	y	111011	0,3,5	n	110111	0	n
101111	0,2,3,5	y	111001	0,3,5	n	110101	0	n	101101	0,2,3,5	y
110011	0,5	n	101011	0,2,3,4,5	n	100111	0,2,4,5	n	110001	0	n
101001	0,2,3,4,5	y	100101	0,2,4	y	100011	0,2,4,5	n	100001	0,2,4	y
111110		n	111100		n	111010		n	110110		n
101110		n	111000		n	110100		n	101100		n
110010		n	101010		y	100110		y	110000		n
101000		y	100100		y	100010		y	100000		y

Table 1: The lower segments from $\mathfrak{N}_2^{LS}$ with height from one to six. The letters "y" and "n" indicate if a segment has 4-crown stack as retract or not. Additional explanation in text.

is a retraction, and due to $t^{-1}(v) = \{v\}$, the quadrupel $(k+1, U, s', t)$ is a retractive up-split fulfilling (6).

In the case of $P(k-1,k) \simeq 3C$, apply the dual construction.

□

5 Application

With the recursive approach described in Section 3.3, we will now identify all posets $P \in \mathfrak{N}_2^{LS}$ with height up to six having a 4-crown stack as retract. For some of them, e.g. the 6-crown stacks, we already know the result. We include them in our investigation because our purpose is to test our approach with as many nice sections as possible.

We encode the lower segments not being 3-antichains by the sequence of their level-pair types (cf. Theorem 4.2). 1 indicates a 6-crown, 0 a type $3C$. The lower segment 111 is thus the 6-crown stack of height 3, whereas 110 indicates the lower segment starting with two 6-crowns followed by a single $3C$.

The lower segments with height from one to six are listed in Table 1. The letters "y" and "n" indicate the result of our investigation whether the segment has a 4-crown stack as retract or not. For the posets 1, 10, 11, 111, and 101, we refer to common knowledge. The lower segment 10 even has a retraction r to a 4-crown with $r^{-1}(v) = \{v\}$ for a point v from the top level of the 4-crown, as shown in Figure 5a.

For the posets with a final 0, we can decide by applying the dual of Corollary

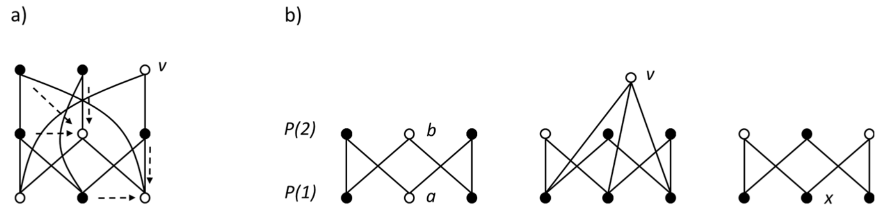

Figure 5: a) The lower segment 10 and a retraction r onto a 4-crown with $r^{-1}(v) = \{v\}$ for a point v from the top level of the 4-crown. b) Illustrations for the case discrimination in the proof of Criterion 5.1.5. The hollow dots belong to $S(0)$.

4.4 referring to previous results in Table 1. As an example, the posets 110 and 100 do not have a 4-crown stack as retract, because the 6-crown $P(0,1)$ does not have a suitable retract. And 1010 and 1000 have a 4-crown stack as retract because 10 is marked with "y".

The bulk of the work concerns thus the lower segments P with a 6-crown as final level-pair. They all are elements of $\mathfrak{N}_2$. For them, the integers in Table 1 indicate the levels $k \in [0, h_P - 1]$ for which $P(0 \to k)$ can be a t-base according to the results for lower segments with less height. For 1111 it is 0 and 3: 0, because the antichain $P(0)$ is always a candidate for a t-base, and 3, because the poset $1111(0 \to 3) = 111$ is marked with "y" in the previous row. Because 1 and 11 are marked with "n" , the level indices 1 and 2 are missing in the list for 1111.

The following criteria will be useful in our investigation:

Criteria 5.1. *Let $P \in \mathfrak{N}_2$ with $h_P \geq 3$.*

1. *For every retractive down-split $(h_P - 1, D, s, t)$ we have $t[P(0 \to h_P - 3)] = T(0 \to h_T - 1)$. In particular, the choice of $P(h_P)$ as an s-base implies that the poset $P(0 \to h_P - 3)$ has a 2-antichain or a 4-crown tower as retract.*

2. *For the choice of $P(h_P - 1, h_P)$ as s-base, the set $D \subseteq P(h_P - 1)$ must contain at least two points.*

3. *The level-pair $P(h_P - 1, h_P)$ is not a candidate for an s-base if the level-pair $P(h_P - 2, h_P - 1)$ is a 6-crown.*

4. *$P(1 \to h_P)$ can serve as an s-base only together with a subset $D \subset P(1)$ containing at most a single point.*

5. *If $P(1,2)$ is a 6-crown, then $P(1 \to h_P)$ being an s-base implies that $P(3 \to h_P)$ has a 2-antichain or a 4-crown stack as retract.*

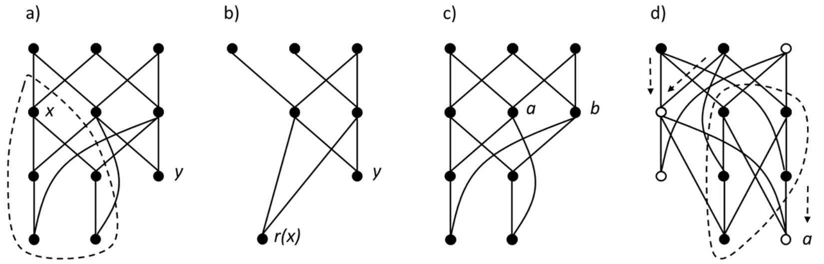

Figure 6: The first three diagrams are illustrations for the proof of Lemma 5.2. Additionally the rightmost diagram shows for $P = 001$ a retraction of $P \setminus D$ to a 4-crown with a singleton $D \subset P(0)$.

Proof. 1. The first condition in (5) yields $T(h_T) \not\subset P(h_P - 1)$. Now the dual of Implication (4) yields $t[P(0 \to h_P - 3)] = T(0 \to h_T - 1)$.

2. The poset $P(h_P - 1, h_P) \setminus D$ has to be disconnected.

3. Is implied by the second criterion, because if $P(h_P - 2, h_P - 1)$ is a 6-crown, the second condition in (5) can be fulfilled for at most a single point $d \in P(h_P - 1)$.

4. Trivial.

5. We discriminate three cases (cf. Figure 5b). If $S(0)$ contains a point $a \in P(1)$, then it has to contain a point $b \in P(2)$, too, because of the first condition in (5). At least one of the points in $P(1) \setminus \{a\}$ has to be mapped to b (fourth criterion), thus $s[P(3 \to h_P)] = S(1 \to h_S)$ due to implication (4).

Now assume that $S(0)$ does not contain a point of $P(1)$. If $S(0)$ contains a point $v \in P(3 \to h_P)$, then at least two points in $P(1)$ are below $S(0)$ in contradiction to the fourth criterion. And in the case of $S(0) \subset P(2)$, exactly one point $x \in P(1)$ is below $S(0)$ and has to be sent to $P(0)$. From the two remaining points in $P(1)$ each is under a single different point of $S(0)$. Due to the fourth criterion, they have to be mapped onto $S(0)$, and $s[P(3 \to h_P)] = S(1 \to h_S)$ follows again with implication (4).

□

The rightmost poset in Figure 7 confirms that we can neither drop the condition "$P(h_P - 2, h_P - 1)$ is a 6-crown" in Criterion 5.1.3 nor the condition "$P(1,2)$ is a 6-crown" in Criterion 5.1.5.

Lemma 5.2. *Let $P = 011$. If $r : P \setminus D \to R$ is a retraction onto a 4-crown stack or a 2-antichain for a down-set D, then D contains at least two points of $P(0)$.*

Proof. Assume that a retraction exists for $D = \{d\}$ with $d \in P(0)$. Figure 6a shows $P \setminus \{d\}$. Because r cannot be extended to P, both upper covers x and y of

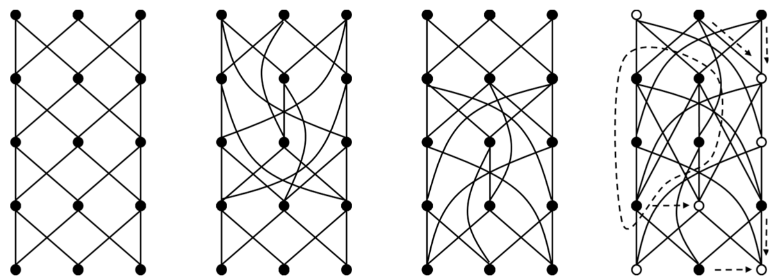

Figure 7: The posets 1111, 1101, 1011, and 1001. As already mentioned at the end of the introduction, the poset 1011 is also treated in [9, p. 99] and mentioned in [3, p. 136] as a very complicated case. For the poset 1001, a retraction onto a 4-crown tower is indicated resulting from a retractive down-split $(2, D, s, t)$ based on the retraction t of $P(0 \to 2) = 10$ shown in Figure 5a.

d have to be mapped to different points of $R(0)$. The encircled set $\downarrow x$ has thus to be mapped to a single point in $R(0)$. The resulting poset is shown in Figure 6b. Clearly, it does not have a 4-crown stack or a 2-antichain as retract.

Now assume $D = \{d, y\}$ with $d \in P(0)$ and $d \lessdot y$. The poset $P \setminus D$ is shown in Figure 6c. If a retraction as described in the lemma exists, it cannot be extended to $P \setminus \{d\}$. The upper covers a and b of y are thus mapped to different points of $R(0)$. But that is impossible because of $P(0) \setminus \{d\} < \{a, b\}$.

□

Now we can start with the posets of height four. They are shown in Figure 7.

- P = 1111: Criterion 5.1.1 prevents $P(4)$ from being a candidate for an s-base because the 6-crown $P(0,1)$ does not have a suitable retract. Furthermore, selecting $P(1 \to 4)$ as s-base does not work due to Criterion 5.1.5, because $P(3,4) \simeq 1$ is marked with "n" in Table 1. 1111 does thus have no retractive down-split fulfilling (5), and because the poset is self-dual, we conclude that it does not have a 4-crown stack as retract.

- $P = 1101$: Again, $P(1 \to 4)$ fails as an s-base due to Criterion 5.1.5. 1101 thus does not have a retractive down-split fulfilling (5).

- $P = 1011$: $P(1 \to 4)$, $P(3,4)$, or $P(4)$ being a successful s-base is prevented by Criterion 5.1.4 together with Lemma 5.2, Criterion 5.1.3, and Criterion 5.1.1, respectively. Also 1011 does thus not have a retractive down-split fulfilling (5). Therefore, neither 1101 nor 1011 has a 4-crown stack as retract.

- $P = 1001$: A retraction to a 4-crown stack resulting from a retractive down-split $(2, D, s, t)$ is indicated in Figure 7 on the right.

We continue with the posets of height five:

- $P = 11111$: Due to the Criteria 5.1.5 and 5.1.3, the upper segments $P(1 \to 5)$ and $P(4,5)$ cannot work as s-bases. The self-dual poset has thus no 4-crown stack as retract.

- $P = 11101$: see $P = 10111$.

- $P = 11011$: Criterion 5.1.5 prevents $P(1 \to 5)$ from being a successful s-base. Because 11011 is self-dual, the poset cannot have a 4-crown stack as retract.

- $P = 10111$: Figure 2 shows that the 6-crown stack $Q : P(2 \to 5) = 111$ has a retraction $t : Q \to T$ with T being a 4-crown stack and $t^{-1}(a) = \{a\}$ for a point $a \in T(0)$. With $s : P(0) \to P(0)$ being a mapping with $s[P(0)]$ being a 2-antichain, apply Corollary 4.6 with $k = 1$.

- $P = 11001$: see 10011.

- $P = 10101, 10001$: Let $s : P(0 \to 2) \to S$ and $t : P(3 \to 5) \to T$ retractions onto 4-crowns as shown in Figure 5a and its dual. Without loss of generality (Theorem 4.2) we assume that the single point in $P(2) \cap S$ is below the single point in $P(3) \cap T$. Now $(3, \emptyset, s, t)$ is a retractive down-split fulfilling (5).

- $P = 10011$: Let $Q := 1001$. The retraction t of Q indicated in Figure 7 on the right and a self-mapping s of $P(5)$ onto a 2-antichain can be combined to a retractive down-split $(4, \emptyset, s, t)$ of P fulfilling (5).

For some of the posets of height six, a quick positive decision is possible:

- $P = 111111, 111101, 101111, 101101$: Combine the retractions in Figure 2.

- $P = 101001, 100001$: Let P be one of these posets. With $Q := P(3 \to 6) = 001$, select a point $d \in Q(0)$ and let s be the retraction of $Q \setminus \{d\}$ onto a 4-crown S shown in Figure 6 on the right with $a \in S \cap Q(0)$ as in the figure. Furthermore, let t be the retraction of $P(0 \to 2) = 10$ onto a 4-crown T as shown in Figure 5a with $v \in T \cap P(2)$ as in the figure. Applying Theorem 4.2 on $P(0 \to 2)$, we can assume $v \lessdot a$ and $v \not< d$ and get a retractive down-split $(2, \{d\}, s, t)$ fulfilling (5).

- $P = 100101$: dual to 101001.

The remaining posets of height six are 111011, 111001, 101011, 110001 and their duals and the self-dual poset 110011. They do not have a 4-crown stack as retract:

Lemma 5.3. *None of the posets* 111011*,* 111001*,* 101011*,* 110001*, and* 110011 *has a 4-crown stack as retract.*

Proof. Firstly, let $P \in \mathfrak{N}_2$ be any nice section of height six and R a retract of P being a 4-crown stack. Every level set of R has to be contained in a single level set or in two consecutive level sets of P, and we conclude that at least one of the posets $P(0 \to 3) \cap R$ and $P(3 \to 6) \cap R$ is a 4-crown stack.

Now let P be any of the five posets and r a retraction onto a 4-crown stack R. With $S := P(3 \to 6) \cap R$, we show in a first step that S cannot be a 4-crown stack.

$P(0 \to 3) = 111, 110$: S being a 4-crown stack yields a point $x \in P(3)$ with $v := r(x) \in P(0 \to 2) = 11$. Let $\{v, w\}$ be the level set of R containing v. $x \in P(3)$ yields $r[P(0,1)] \leq v$, hence $w \notin P(0,1)$. But then, $r(z) < \{v, w\}$ holds for at least five of the points $z \in P(0,1)$, and $P(0,1)$ cannot be mapped onto a 2-antichain.

$P = 101011$: S being a 4-crown stack requires due to Lemma 5.2 two points $x, y \in P(3)$ to be mapped into $P(0 \to 2) = 10$. The points $r(x)$ and $r(y)$ belong thus to $R(0,1)$. Three cases are possible, each of it leading to a contradiction:

- $\{r(x), r(y)\}$ is a level set of R: Let $z \in P(2)$ be the common lower cover of x and y. Due to $P(0) < z$, the point $r(z)$ is the only minimal point of R.
- $r(x) < r(y)$: Then $r(x) \in R(0)$, and $P(0) < x$ yields that $r(x)$ is the only minimal point of R.
- $v := r(x) = r(y)$: Let $\{v, w\}$ be the level set of R containing v. Due to $r[P(0 \to 2)] \leq v$, the point w must belong to $P(3 \to h_P)$, and due to $\lambda(v) \leq 2$, we must have $w \in P(3)$ and $v \in P(2)$. But then $w \in S \not\ni v$.

S is thus not a 4-crown stack. This requires a level set $R(\ell) = \{a, b\}$ with $a \in P(2)$ and $b \in P(3)$. Clearly, $\ell = 1$. Due to $P(1) < b$, no point of $P(1)$ can be mapped to a. And if a point of $P(1)$ is mapped to b, implication (4) yields $R(2 \to h_R)$ being a retract of $P(4 \to 6) = 11$ which is impossible. But if no point of $P(1)$ can be mapped to $R(1)$, then $r[P(0,1)] = R(0)$ which is impossible, too. □

References

[1] Brualdi, R. A., Dias da Silva, J. A.: A retract characterization of posets with the fixed-point property. *Discr. Math.* **169** (1997), 199–203.

[2] Duffus, D., Poguntke, W., and Rival, I.: Retracts and the fixed point problem for finite partially ordered sets. *Canad. Math. Bull.* **23** (1980), 231–236.

[3] Farley, J. D.: The fixed point property for posets of small width. *Order* **14** (1997), 125–143.

[4] Fofanova, T. S., Rutkowski, A.: The fixed point property in ordered sets of width two, *Order* **4** (1987), 101—106.

[5] Niederle, J.: A characterization of finite posets of the width at most three with the fixed point property. *Czech. Math. J.* **39** (1989) 120–126.

[6] Niederle, J.: Forbidden retracts for finite ordered sets of width at most four. *Discrete Mathematics* **308** (2008) 1774–1778.

[7] Rival, I.: A fixed point theorem for partially ordered sets. *J. Comb. Th. (A)* **21** (1976), 309–318.

[8] Rival, I.: The retract construction. In Rival, I. (Ed.), *Ordered Sets. Proceedings of the NATO Advanced Study Institute held at Banff, Canada, August 28 to September 12, 1981*. Dordrecht, D.-Reidel (1982), 97–122.

[9] Schröder, B.: *Ordered Sets. An Introduction with Connections from Combinatorics to Topology.* Birkhäuser (2016).

[10] Schröder, B. S. W.: Minimal automorphic superpositions of crowns. Arab. J. Math. (2022), https://doi.org/10.1007/s40065-022-00399-5.