FACTORIZATION OF POWER GCD MATRICES AND POWER LCM MATRICES ON CERTAIN GCD-CLOSED SETS

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ABSTRACT. For integers x and y, (x, y) and $[x, y]$ stand for the greatest common divisor and the least common multiple of x and y respectively. Denote by $|T|$ the number of elements of a finite set T. Let a, b and n be positive integers and let $S = \{x_1, \cdots, x_n\}$ be a set of n distinct positive integers. We denote by (S^a) (resp. $[S^a]$) the $n\times n$ matrix having the ath power of (x_i, x_j) (resp. $[x_i, x_j]$) as its (i, j)-entry. For any $x \in S$, define $G_S(x) := \{d \in S : d < x, d|x \text{ and } (d|y|x, y \in S) \Rightarrow y \in \{d, x\}\}$. In this paper, we show that if $a|b$ and S is gcd closed (namely, $(x_i, x_j) \in S$ for all integers i and j with $1 \le i, j \le n$) and $\max_{x\in S}\{|G_S(x)|\} = 3$ such that any elements $y_1, y_2 \in G_S(x)$ satisfy that $[y_1, y_2] = x$ and $(y_1, y_2) \in G_S(y_1) \cap G_S(y_2)$), then $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$ hold in the ring $M_n(\mathbb{Z})$. This extends the Chen-Hong-Zhao theorem gotten in 2022. This also partially confirms a conjecture of Hong raised in [S.F. Hong, Divisibility among power GCD matrices and power LCM matrices, *Bull. Aust. Math. Soc.*, doi:10.1017/S0004972725100361].

1. INTRODUCTION

Let (x, y) and $[x, y]$ denote the greatest common divisor and the least common multiple of the integers x and y, respectively. Let a, b and n be positive integers. Let $|T|$ stand for the cardinality of a finite set T of integers. Let f be an arithmetic function and let $S = \{x_1, \cdots, x_n\}$ be a set of n distinct positive integers. Let $(f(S))$ and $(f[S])$ stand for the $n \times n$ matrices having $f((x_i, x_j))$ and $(f[x_i, x_j])$ as its (i, j)-entry, respectively. A set S is called *factor closed* if the conditions $x \in S$ and $d|x$ together with $d > 0$ imply that $d \in S$. We say that S is *gcd closed* if $(x_i, x_j) \in S$ for all i and j with $1 \le i, j \le n$. Evidently, any factor closed set is gcd closed but not conversely. In 1875, Smith [36] published his famous theorem stating that if S is factor closed, then $\det(f(x_i, x_j)) = \prod_{k=1}^n (f * \mu)(x_k)$, where $f * \mu$ is the *Dirichlet convolution* of f and the Möbius function μ and is defined for any positive integer x by $(f*\mu)(x) := \sum_{d|x} f(d)\mu\big(\frac{x}{d}\big)$. Since then lots of generalizations of Smith's determinant and related results had published (see, for example, [2]- [34] and [37]- [45]). For any positive integer x, let ξ_a be the arithmetic function defined by $\xi_a(x) := x^a$. The $n \times n$ matrix $(\xi_a(x_i, x_j))$ (abbreviated by (S^a)) and $(\xi_a[x_i, x_j])$ (abbreviated by $[S^a]$) are called *ath power GCD matrix* on S and *ath power LCM matrix* on S, respectively. In 1993, Bourque and Ligh [7] extended the Beslin-Ligh result [4] and Smith's determinant by proving that if S is gcd closed,

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then $\det(S^a)=\prod_{k=1}^n \alpha_{\xi_a}(x_k)$, where

$$\alpha_{\xi_a}(x_k) := \sum_{\substack{d|x_k \\ d\nmid x_t, x_t<x_k}} (\xi_a * \mu)(d). \tag{1.1}$$

An integer $x \geq 1$ is called *square free* if x is not divisible by the square of any prime number. In 2018, Lin and Hong [34] generalized Smith's theorem [36] and the Hong-Hu-Lin theorem [14] by showing that for any integer t with $1 \leq t \leq n$, if S is factor closed, then

$$\det(f(x_i,x_j))_{\substack{1\leq i,j\leq n \\ i\neq t, j\neq t}} = \sum_{\substack{l=1 \\ x_t|x_l, \frac{x_l}{x_t}\text{ square free}}}^{n} \prod_{\substack{k=1 \\ k\neq l}}^{n} (f*\mu)(x_k).$$

Besides, the eigen structures of power GCD and power LCM matrices were explored by Wintner [39], Bourque and Ligh [6], Hong and Loewy [27, 28], Hong and Lee [25], Altinisik [1], Mattila and Haukkanen [35] and Kaarnioja [31].

Let $x, y \in S$ with $x<y$. If $x|y$ and the conditions $x|d|y$ and $d \in S$ imply that $d \in \{x, y\}$, then we say that x is a *greatest-type divisor* of y in S. For $x \in S$, $G_S(x)$ stands for the set of all greatest-type divisors of x in S. The concept of greatest-type divisor was introduced by Hong and played a key role in his solution [16] of the Bourque-Ligh conjecture [5]. Bourque and Ligh [8] showed that if S is factor closed and $a \geq 1$ is an integer, then the ath power GCD matrix (S^a) divides the ath power LCM matrix $[S^a]$ in the ring $M_n(\mathbb{Z})$ of $n\times n$ matrices over the integers. That is, there exists an $A \in M_n(\mathbb{Z})$ such that $[S^a]=(S^a)A$ or $[S^a]=A(S^a)$. Hong [18] showed that such a factorization is no longer true in general if S is gcd closed and $\max_{x\in S}\{|G_S(x)|\}=2$. Korkee and Haukkanen [32] and Chen et al. [9] extended the result of Bourque and Ligh and that of Hong. A characterization on the gcd-closed set S with $\max_{x\in S}\{|G_S(x)|\}=2$ (resp. $\max_{x\in S}\{|G_S(x)|\}=3$) such that $(S^a)|[S^a]$ holds in the ring $M_{|S|}(\mathbb{Z})$ was given in [11] (resp. [41]).

On the other hand, Hong [23] initially investigated the divisibility properties among power GCD matrices and among power LCM matrices. It was proved in [23] that $(S^a)|(S^b), (S^a)|[S^b]$ and $[S^a]|[S^b]$ hold in the ring $M_n(\mathbb{Z})$ if $a|b$ and S is a divisor chain (that is, $x_{\sigma(1)}|\cdots|x_{\sigma(n)}$ for a permutation σ of $\{1,\cdots,n\}$), and such factorizations are no longer true if $a\nmid b$ and $|S|\geq 2$. Evidently, a divisor chain is gcd closed but the converse is not true. Recently, Hong [24] established the same divisibility result when S is factor closed. As in [24], for any set S of positive integers and for any $x \in S$ with $|G_S(x)|\geq 2$, we say that the two distinct greatest-type divisors y_1 and y_2 of x in S *satisfy the condition* $\mathcal{G}$ if $[y_1,y_2]=x$ and $(y_1,y_2)\in G_S(y_1)\cap G_S(y_2)$. We say that x *satisfies the condition* $\mathcal{G}$ if any two distinct greatest-type divisors of x in S satisfy the condition $\mathcal{G}$. Moreover, we say that a set S of positive integers *satisfies the condition* $\mathcal{G}$ if any element x in S satisfies that either $|G_S(x)|\leq 1$, or $|G_S(x)|\geq 2$ and x satisfies the condition $\mathcal{G}$. In [24], Hong showed if $a|b$ and S is a factor closed set, then $(S^a)|(S^b), (S^a)|[S^b]$ and $[S^a]|[S^b]$ hold in the ring $M_n(\mathbb{Z})$. Meanwhile, Hong [24] proved that any factor-closed set is a gcd-closed set satisfying the condition $\mathcal{G}$. Furthermore, Hong proposed the following conjecture.

Conjecture 1.1. [24, Conjecture 3.4] *Let a and b be positive integers with $a|b$ and let S be a gcd-closed set satisfying the condition $\mathcal{G}$. Then in the ring $M_{|S|}(\mathbb{Z})$, we have $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$.*

By Zhu's theorem [42] and the Zhu-Li result [44], one knows that this conjecture is true when $a|b$ and S is a gcd-closed set with $\max_{x\in S}\{|G_S(x)|\}=1$. Also Conjecture 1.1 is proved by Wan and Zhu [38] when $a|b$ and S is a gcd-closed set satisfying the condition $\mathcal{G}$ and $\max_{x\in S}\{|G_S(x)|\}=2$. But Conjecture 1.1 is still kept open when S is a gcd-closed set satisfying the condition $\mathcal{G}$ and $\max_{x\in S}\{|G_S(x)|\}\geq 3$. One remarks that Wan and Zhu [38] showed the existences of gcd-closed sets S with $\max_{x\in S}\{|G_S(x)|\}=2$ and the condition $\mathcal{G}$ not being satisfied and infinitely many integers $b\geq 2$ such that $(S)\mid(S^b)$ (resp. $(S)\mid[S^b]$ and $[S]\mid[S^b]$).

In this paper, our main goal is to explore Conjecture 1.1 for the case $\max_{x\in S}\{|G_S(x)|\}=3$. Actually, we will show that Conjecture 1.1 is true if $a|b$ and S is a gcd-closed set with $\max_{x\in S}\{|G_S(x)|\}=3$ and the condition $\mathcal{G}$ being satisfied. That is, we have the following main results of this paper.

Theorem 1.2. *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and $\max_{x\in S}\{|G_S(x)|\}=3$ and let a and b be positive integers with $a|b$. Then the a-th power GCD matrix (S^a) divides each of the b-th power GCD matrix (S^b) and the b-th power LCM matrix $[S^b]$ in the ring $M_{|S|}(\mathbb{Z})$.*

Theorem 1.3. *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and $\max_{x\in S}\{|G_S(x)|\}=3$ and let a and b be positive integers with $a|b$. Then the a-th power LCM matrix $[S^a]$ divides the b-th power LCM matrix $[S^b]$ in the ring $M_{|S|}(\mathbb{Z})$.*

We define $S_\sigma:=\{x_{\sigma(1)},\cdots,x_{\sigma(n)}\}$ for any permutation σ on the set $\{1,\cdots,n\}$. It is easy to check that for any permutation σ on $\{1,\cdots,n\}$, a necessary and sufficient condition for $(S^a)|(S^b)$ to hold in the ring $M_n(\mathbb{Z})$ is $(S_\sigma^a)|(S_\sigma^b)$ being true in the ring $M_n(\mathbb{Z})$, and a necessary and sufficient condition for $(S^a)|[S^b]$ to hold in the ring $M_n(\mathbb{Z})$ is $(S_\sigma^a)|[S_\sigma^b]$ being true in the ring $M_n(\mathbb{Z})$. So, without loss of any generality, we always assume that the set $S=\{x_1,\cdots,x_n\}$ satisfies that $x_1<\cdots<x_n$.

This paper is organized as follows. We present in Section 2 several preliminary lemmas that are needed in the proofs of our main results. Subsequently, we give the proof of Theorem 1.2 in Section 3. Finally, Section 4 is devoted to the proof of Theorem 1.3.

2. Preliminary lemmas

In this section, we present some lemmas that are needed in the proofs of Theorems 1.2 and 1.3. We begin with a known result due to Bourque and Ligh [7].

Lemma 2.1. [7, Theorem 3] *If S is gcd closed and $(f(S))$ is nonsingular, then for any integers i and j with $1\leq i,j\leq n$, we have*

$$((f(S))^{-1})_{ij}:=\sum_{\substack{x_i|x_k\\x_j|x_k}}\frac{c_{ik}c_{jk}}{\alpha_f(x_k)}$$

with

$$\alpha_f(x_k):=\sum_{\substack{d|x_k\\d\nmid x_t,x_t<x_k}}(f*\mu)(d)$$

and

$$c_{ij}:=\sum_{\substack{dx_i|x_j\\dx_i\nmid x_t,x_t<x_j}}\mu(d). \tag{2.1}$$

Lemma 2.2. *If S is gcd closed, then the power GCD matrix (S^a) is nonsingular and for arbitrary integers i and j with $1 \le i, j \le n$, one has*

$$((S^a)^{-1})_{ij} := \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{c_{ik} c_{jk}}{\alpha_{\xi_a}(x_k)}$$

with c_{ij} being defined as in (2.1) *and $\alpha_{\xi_a}(x_k)$ being defined as in* (1.1).

Proof. By [7, Example 1 (ii)], one can deduce that the power GCD matrix (S^a) is positive definite, and so is nonsingular. Then Lemma 2.1 applied to $f = \xi_a$ gives us the expected result. Lemma 2.2 is proved. □

For any positive integer x, $\frac{1}{\xi_a}$ is the arithmetic function defined by $\frac{1}{\xi_a}(x) = \frac{1}{x^a}$.

Lemma 2.3. [20, Lemma 2.1] *If S is gcd closed, then*

$$\det[S^a] = \prod_{k=1}^{n} x_k^{2a} \alpha_{\frac{1}{\xi_a}}(x_k), \tag{2.2}$$

where

$$\alpha_{\frac{1}{\xi_a}}(x_k) = \sum_{\substack{d | x_k \\ d \nmid x_t, x_t < x_k}} \Big(\frac{1}{\xi_a} * \mu\Big)(d). \tag{2.3}$$

Lemma 2.4. [20] *Let S be a gcd-closed set and let $G_S(x_k) = \{y_{k1}, \cdots, y_{k,l_k}\}$ be the set of the greatest-type divisors of $x_k (1 \le k \le n)$ in S. Then*

$$\alpha_{\xi_a}(x_k) = x_k^a + \sum_{t=1}^{l_k} (-1)^t \sum_{1 \le i_1 < \cdots < i_t \le l_k} (x_k, y_{k,i_1}, \cdots, y_{k,i_t})^a$$

and

$$\alpha_{\frac{1}{\xi_a}}(x_k) = x_k^{-a} + \sum_{t=1}^{l_k} (-1)^t \sum_{1 \le i_1 < \cdots < i_t \le l_k} (x_k, y_{k,i_1}, \cdots, y_{k,i_t})^{-a}$$

with $\alpha_{\xi_a}(x_k)$ being determined as in (1.1) and $\alpha_{\frac{1}{\xi_a}}(x_k)$ being determined as in (2.3).

Lemma 2.5. [11, Lemma 2.3] *Let S be a gcd-closed set of $n \ge 2$ distinct positive integers and let c_{ij} be defined as in* (2.1). *Then*

$$c_{r1} = \begin{cases} 1 & if\ r = 1, \\ 0 & otherwise. \end{cases}$$

If $2 \le m \le n$ and $G_S(x_m) = \{x_{m_0}\}$, then

$$c_{rm} = \begin{cases} -1 & if\ r = m_0, \\ 1 & if\ r = m, \\ 0 & otherwise. \end{cases}$$

If $3 \le m \le n$ and $G_S(x_m) = \{x_{m_{01}},\ x_{m_{02}}\}$ and $x_{m_{03}} = (x_{m_{01}},\ x_{m_{02}})$, then

$$c_{rm} = \begin{cases} -1 & if\ r = m_{01}\ or\ r = m_{02}, \\ 1 & if\ r = m\ or\ m_{03}, \\ 0 & otherwise. \end{cases}$$

Lemma 2.6. *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$. Let $x_m \in S$ with $G_s(x_m) = \{x_{m_1}, x_{m_2}, x_{m_3}\}$, $x_{m_4} = (x_{m_1}, x_{m_2}, x_{m_3})$ and $x_{m_{ij}} = (x_{m_i}, x_{m_j})$ for $1 \le i < j \le 3$. Then*

$$c_{rm} = \begin{cases} 1, & r = m \text{ or } r = m_{ij} (1 \le i < j \le 3), \\ -1, & r = m_i (1 \le i \le 4), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. From the definition of c_{ij} as in (2.1), if $x_r \nmid x_m$, then we have $c_{rm} = 0$. In what follows we let $x_r \mid x_m$. First, let $r = m$. Then we can easily deduce that

$$c_{mm} = \sum_{\substack{dx_m | x_m \\ dx_m \nmid x_t, x_t < x_m}} \mu(d) = \mu(1) = 1.$$

Let $x_r = x_{m_i} (i = 1, 2, 3)$. We can compute and get that

$$c_{m_i m} = \sum_{\substack{dx_{m_i} | x_m \\ dx_{m_i} \nmid x_t, x_t < x_m}} \mu(d) = \sum_{\substack{d | \frac{x_m}{x_{m_i}} \\ d \nmid \frac{x_t}{x_{m_i}}, x_t < x_m}} \mu(d) = \sum_{d | \frac{x_m}{x_{m_i}}} \mu(d) - \sum_{d | \frac{x_{m_i}}{x_{m_i}}} \mu(d) = 0 - 1 = -1.$$

Let $x_r = x_{m_{ij}} (1 \le i < j \le 3)$. We have

$$\begin{aligned} c_{m_{ij} m} &= \sum_{\substack{dx_{m_{ij}} | x_m \\ dx_{m_{ij}} \nmid x_t, x_t < x_m}} \mu(d) \\ &= \sum_{\substack{d | \frac{x_m}{x_{m_{ij}}} \\ d \nmid \frac{x_t}{x_{m_{ij}}}, x_t < x_m}} \mu(d) \\ &= \sum_{d | \frac{x_m}{x_{m_{ij}}}} \mu(d) - \sum_{d | \frac{x_{m_i}}{x_{m_{ij}}}} \mu(d) - \sum_{d | \frac{x_{m_j}}{x_{m_{ij}}}} \mu(d) + \sum_{d | \frac{x_{m_{ij}}}{x_{m_{ij}}}} \mu(d) \\ &= 0 - 0 - 0 + 1 = 1. \end{aligned}$$

Let $x_r = x_{m_4}$. One has

$$\begin{aligned} c_{m_4 m} &= \sum_{\substack{dx_{m_4} | x_m \\ dx_{m_4} \nmid x_t, x_t < x_m}} \mu(d) \\ &= \sum_{\substack{d | \frac{x_m}{x_{m_4}} \\ d \nmid \frac{x_t}{x_{m_4}}, x_t < x_m}} \mu(d) \\ &= \sum_{d | \frac{x_m}{x_{m_4}}} \mu(d) - \sum_{d | \frac{x_{m_1}}{x_{m_4}}} \mu(d) - \sum_{d | \frac{x_{m_2}}{x_{m_4}}} \mu(d) - \sum_{d | \frac{x_{m_3}}{x_{m_4}}} \mu(d) \\ &\quad + \sum_{d | \frac{x_{m_{12}}}{x_{m_4}}} \mu(d) + \sum_{d | \frac{x_{m_{13}}}{x_{m_4}}} \mu(d) + \sum_{d | \frac{x_{m_{23}}}{x_{m_4}}} \mu(d) - \sum_{d | \frac{x_{m_4}}{x_{m_4}}} \mu(d) \\ &= 0 - 0 - 0 - 0 + 0 + 0 + 0 - 1 = -1. \end{aligned}$$

Now we treat with the case that $x_r | x_m$ with $r \ne m, m_i (i = 1, 2, 3), m_{ij} (1 \le i < j \le 3), m_4$. Since S is a gcd-closed set satisfying the condition $\mathcal{G}$, $x_m \in S$ and $|G_s(x_m)| = 3$, At this moment, without loss of generality, we only need to consider the following three cases:

Case 1. $x_r|x_{m_1}$, $x_r|x_{m_2}$ and $x_r|x_{m_3}$. This means $x_r|x_{m_4}$. Note that $x_r \neq x_{m_4}$. So $\frac{x_{m_4}}{x_r} \geq 2$. Thus

$$\begin{aligned}
c_{rm} &= \sum_{\substack{dx_r|x_m \\ dx_r\nmid x_t, x_t<x_m}} \mu(d) \\
&= \sum_{\substack{d|\frac{x_m}{x_r} \\ d\nmid \frac{x_t}{x_r}, x_t<x_m}} \mu(d) \\
&= \sum_{d|\frac{x_m}{x_r}} \mu(d) - \sum_{d|\frac{x_{m_1}}{x_r}} \mu(d) - \sum_{d|\frac{x_{m_2}}{x_r}} \mu(d) - \sum_{d|\frac{x_{m_3}}{x_r}} \mu(d) \\
&\quad + \sum_{d|\frac{x_{m_{12}}}{x_r}} \mu(d) + \sum_{d|\frac{x_{m_{13}}}{x_r}} \mu(d) + \sum_{d|\frac{x_{m_{23}}}{x_r}} \mu(d) - \sum_{d|\frac{x_{m_4}}{x_r}} \mu(d) \\
&= 0-0-0-0+0+0+0-0=0.
\end{aligned}$$

Case 2. $x_r|x_{m_1}$, $x_r|x_{m_2}$ and $x_r \nmid x_{m_3}$. We can deduce that $x_r|x_{m_{12}}$. Notice that $\frac{x_{m_{12}}}{x_r} \geq 2$. Therefore

$$\begin{aligned}
c_{rm} &= \sum_{\substack{dx_r|x_m \\ dx_r\nmid x_t, x_t<x_m}} \mu(d) \\
&= \sum_{\substack{d|\frac{x_m}{x_r} \\ d\nmid \frac{x_t}{x_r}, x_t<x_m}} \mu(d) \\
&= \sum_{d|\frac{x_m}{x_r}} \mu(d) - \sum_{d|\frac{x_{m_1}}{x_r}} \mu(d) - \sum_{d|\frac{x_{m_2}}{x_r}} \mu(d) + \sum_{d|\frac{x_{m_{12}}}{x_r}} \mu(d) \\
&= 0-0-0+0=0.
\end{aligned}$$

Case 3. $x_r|x_{m_1}$, $x_r \nmid x_{m_2}$ and $x_r \nmid x_{m_3}$. Notice that $\frac{x_{m_1}}{x_r} \geq 2$. Hence

$$c_{rm} = \sum_{\substack{dx_r|x_m \\ dx_r\nmid x_t, x_t<x_m}} \mu(d) = \sum_{\substack{d|\frac{x_m}{x_r} \\ d\nmid \frac{x_t}{x_r}, x_t<x_m}} \mu(d) = \sum_{d|\frac{x_m}{x_r}} \mu(d) - \sum_{d|\frac{x_{m_1}}{x_r}} \mu(d) = 0-0=0.$$

This finished the proof of Lemma 2.6. □

Lemma 2.7. [29] *Let S be gcd closed such that $\max_{x\in S}\{|G_S(x)|\} \leq 3$ and $|S|=n$. Let $\alpha_{\frac{1}{\xi_a}}(x_k)$ be defined as in (2.3). Then $\alpha_{\frac{1}{\xi_a}}(x_k) \neq 0$ for any integer k with $1 \leq k \leq n$.*

Lemma 2.8. *Let S be a gcd-closed set satisfying $\max_{x\in S}\{|G_S(x)|\} \leq 3$. Then the a-th power LCM matrix $[S^a]$ is nonsingular and for all integers i and j with $1 \leq i, j \leq n$, one has*

$$([S^a]^{-1})_{ij} := \frac{1}{x_i^a x_j^a} \sum_{\substack{x_i|x_k \\ x_j|x_k}} \frac{c_{ik}c_{jk}}{\alpha_{\frac{1}{\xi_a}}(x_k)}$$

with c_{ij} being defined as in (2.1) and $\alpha_{\frac{1}{\xi_a}}(x_k)$ being defined as in (2.3).

Proof. Since $[x_i, x_j]^a = x_i^a x_j^a/(x_i, x_j)^a$, we have

$$[S^a] = \mathrm{diag}(x_1^a, \cdots, x_n^a) \cdot \Big(\frac{1}{\xi_a}(x_i, x_j)\Big) \cdot \mathrm{diag}(x_1^a, \cdots, x_n^a). \tag{2.4}$$

It then follows that

$$\det[S^a] = \det\big(\frac{1}{\xi_a}(x_i, x_j)\big) \cdot \prod_{k=1}^{n} x_k^{2a}. \tag{2.5}$$

Then from (2.2) and (2.5), we can derive that

$$\det\big(\frac{1}{\xi_a}(x_i, x_j)\big) = \prod_{k=1}^{n} \alpha_{\frac{1}{\xi_a}}(x_k).$$

Lemma 2.7 tells us that $\alpha_{\frac{1}{\xi_a}}(x_k) \neq 0$ for all integers k with $1 \le k \le n$. So the matrix $\big(\frac{1}{\xi_a}(x_i, x_j)\big)$ is nonsingular.

With Lemma 2.1 applied to $f = \frac{1}{\xi_a}$, one gets that

$$\Big(\Big(\frac{1}{\xi_a}(x_i, x_j)\Big)^{-1}\Big)_{ij} = \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{c_{ik}c_{jk}}{\alpha_{\frac{1}{\xi_a}}(x_k)}. \tag{2.6}$$

Therefore the required result follows immediately from (2.4) and (2.6). So Lemma 2.8 is proved. $\square$

Now define the set $A_S(z, x) := \{u \in S : z|u|x, u \neq z\}$.

Lemma 2.9. [41] *Let S be a gcd-closed set and let $x \in S$ satisfy $|G_S(x)| \ge 2$, $y \in G_S(x)$. Let $z \in S$ be such that $z|x, z \neq x$ and $z \nmid y$. If $A_S(z, x)$ satisfies the condition $\mathcal{G}$, then $[y, z] = x$.*

Lemma 2.10. *Let a and b be positive integers such that $a|b$. Let S be a gcd-closed set and $x, y, z \in S$ with $G_S(x) = \{y\}$.*

(i). [42, Lemma 2.5] *The integer $x^a - y^a$ divides each of $(x, z)^b - (y, z)^b$ and $[x, z]^b - [y, z]^b$.*

(ii). [44, Lemma 2.8] *If $r \in S$ and $r|x$, then $y^a[z, x]^b - x^a[z, y]^b$ is divisible by $r^a(y^a - x^a)$.*

Lemma 2.11. [38] *Let a and b be positive integers with $a|b$ and let S be a gcd-closed set. For $x, z \in S$ with $|G_S(x)| = 2$, let $G_S(x) = \{y_1, y_2\}$ and $y_3 := (y_1, y_2)$. Assume that the set $\{u \in S : (x, z)|u|x\}$ satisfies the condition $\mathcal{G}$. Then each of the following is true:*

(i). *The integer $x^a + y_3^a - y_1^a - y_2^a$ divides each of $(z, x)^b + (z, y_3)^b - (z, y_1)^b - (z, y_2)^b$ and $[z, x]^b + [z, y_3]^b - [z, y_1]^b - [z, y_2]^b$.*

(ii). *For any $r \in S$ with $r|x$, $x^a[z, y_3]^b + y_3^a[z, x]^b - y_1^a[z, y_2]^b - y_2^a[z, y_1]^b$ is divisible by $r^a(x^a + y_3^a - y_1^a - y_2^a)$.*

Lemma 2.12. *Let $a|b$ and let S be a gcd-closed set satisfying the condition $\mathcal{G}$. Let $x_m \in S$ with $G_S(x_m) = \{x_{m_1}, x_{m_2}, x_{m_3}\}$. Define $x_{m_4} := (x_{m_1}, x_{m_2}, x_{m_3})$ and $x_{m_{ij}} := (x_{m_i}, x_{m_j})$ for $1 \le i < j \le 3$.*

(i). *The integer $x_m^a - x_{m_1}^a - x_{m_2}^a - x_{m_3}^a + x_{m_{12}}^a + x_{m_{13}}^a + x_{m_{23}}^a - x_{m_4}^a$ divides $x_m^b - x_{m_1}^b - x_{m_2}^b - x_{m_3}^b + x_{m_{12}}^b + x_{m_{13}}^b + x_{m_{23}}^b - x_{m_4}^b$.*

(ii). *For any $x_t \in S$ with $x_t|x_m$, the fraction*

$$\frac{x_m^{b-a} - x_{m_1}^{b-a} - x_{m_2}^{b-a} - x_{m_3}^{b-a} + x_{m_{12}}^{b-a} + x_{m_{13}}^{b-a} + x_{m_{23}}^{b-a} - x_{m_4}^{b-a}}{x_t^a(x_m^{-a} - x_{m_1}^{-a} - x_{m_2}^{-a} - x_{m_3}^{-a} + x_{m_{12}}^{-a} + x_{m_{13}}^{-a} + x_{m_{23}}^{-a} - x_{m_4}^{-a})}$$

is an integer.

Proof. Since S satisfies the condition $\mathcal{G}$, we obtain that $[x_{m_i}, x_{m_j}] = x_m$ for $1 \leq i < j \leq 3$, $\{x_{m_{12}}, x_{m_{13}}\} \subseteq G_S(x_{m_1})$, $\{x_{m_{13}}, x_{m_{23}}\} \subseteq G_S(x_{m_3})$ and $\{x_{m_{12}}, x_{m_{23}}\} \subseteq G_S(x_{m_2})$. Further, we can get that $[x_{m_{12}}, x_{m_{13}}] = x_{m_1}$, $[x_{m_{13}}, x_{m_{23}}] = x_{m_3}$ and $[x_{m_{12}}, x_{m_{23}}] = x_{m_2}$. This implies that

$$\frac{x_m}{x_{m_2}} = \frac{x_{m_1}}{x_{m_{12}}} = \frac{x_{m_{13}}}{x_{m_4}} = \frac{x_{m_3}}{x_{m_{23}}}$$

and

$$\frac{x_m}{x_{m_1}} = \frac{x_{m_2}}{x_{m_{12}}} = \frac{x_{m_{23}}}{x_{m_4}}.$$

We can derive that

$$x_{m_{12}} = \frac{x_{m_1}x_{m_2}}{x_m}, \tag{2.7}$$

$$x_{m_{13}} = x_{m_4} \cdot \frac{x_m}{x_{m_2}}, \tag{2.8}$$

$$x_{m_{23}} = x_{m_4} \cdot \frac{x_m}{x_{m_1}}, \tag{2.9}$$

$$x_{m_3} = x_{m_{23}} \cdot \frac{x_m}{x_{m_2}} = x_{m_4} \cdot \frac{x_m}{x_{m_1}} \cdot \frac{x_m}{x_{m_2}}. \tag{2.10}$$

It readily follows from (2.7) to (2.10) that

$$\begin{aligned}
&x_m^a - x_{m_1}^a - x_{m_2}^a - x_{m_3}^a + x_{m_{12}}^a + x_{m_{13}}^a + x_{m_{23}}^a - x_{m_4}^a\\
=&x_m^a - x_{m_1}^a - x_{m_2}^a - \Big(\frac{x_{m_4}x_m^2}{x_{m_1}x_{m_2}}\Big)^a + \Big(\frac{x_{m_1}x_{m_2}}{x_m}\Big)^a + \Big(\frac{x_m x_{m_4}}{x_{m_2}}\Big)^a + \Big(\frac{x_m x_{m_4}}{x_{m_1}}\Big)^a - x_{m_4}^a\\
=&x_m^a - x_{m_1}^a - x_{m_2}^a + \Big(\frac{x_{m_1}x_{m_2}}{x_m}\Big)^a + x_{m_4}^a\Big(\Big(\frac{x_m}{x_{m_1}}\Big)^a + \Big(\frac{x_m}{x_{m_2}}\Big)^a - \Big(\frac{x_m^2}{x_{m_1}x_{m_2}}\Big)^a - 1\Big)\\
=&\frac{x_m^{2a} - x_m^a x_{m_1}^a - x_m^a x_{m_2}^a + x_{m_1}^a x_{m_2}^a}{x_m^a} - \frac{x_m^{2a} - x_m^a x_{m_1}^a - x_m^a x_{m_2}^a + x_{m_1}^a x_{m_2}^a}{x_{m_1}^a x_{m_2}^a} \cdot x_{m_4}^a\\
=&(x_m^{2a} - x_m^a x_{m_1}^a - x_m^a x_{m_2}^a + x_{m_1}^a x_{m_2}^a)\Big(\frac{1}{x_m^a} - \frac{x_{m_4}^a}{x_{m_1}^a x_{m_2}^a}\Big)\\
=&(x_m^a - x_{m_2}^a)(x_m^a - x_{m_1}^a)\frac{x_{m_1}^a x_{m_2}^a - x_m^a x_{m_4}^a}{x_m^a x_{m_1}^a x_{m_2}^a}\\
=&\Big(\Big(\frac{x_m}{x_{m_2}}\Big)^a - 1\Big)\Big(\Big(\frac{x_m}{x_{m_1}}\Big)^a - 1\Big)\Big(\Big(\frac{x_{m_1}x_{m_2}}{x_m}\Big)^a - x_{m_4}^a\Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{x_m^b - x_{m_1}^b - x_{m_2}^b - x_{m_3}^b + x_{m_{12}}^b + x_{m_{13}}^b + x_{m_{23}}^b - x_{m_4}^b}{x_m^a - x_{m_1}^a - x_{m_2}^a - x_{m_3}^a + x_{m_{12}}^a + x_{m_{13}}^a + x_{m_{23}}^a - x_{m_4}^a}\\
=&\frac{\big(\frac{x_m}{x_{m_2}}\big)^b - 1}{\big(\frac{x_m}{x_{m_2}}\big)^a - 1} \cdot \frac{\big(\frac{x_m}{x_{m_1}}\big)^b - 1}{\big(\frac{x_m}{x_{m_1}}\big)^a - 1} \cdot \frac{\big(\frac{x_{m_1}x_{m_2}}{x_m}\big)^b - x_{m_4}^b}{\big(\frac{x_{m_1}x_{m_2}}{x_m}\big)^a - x_{m_4}^a} \in \mathbb{Z}.
\end{aligned}$$

Part (i) is proved.

On the other hand, by (2.7) to (2.10), we have

$$\begin{aligned}
&x_m^{-a} - x_{m_1}^{-a} - x_{m_2}^{-a} - x_{m_3}^{-a} + x_{m_{12}}^{-a} + x_{m_{13}}^{-a} + x_{m_{23}}^{-a} - x_{m_4}^{-a}\\
=&\frac{1}{x_m^a} - \frac{1}{x_{m_1}^a} - \frac{1}{x_{m_2}^a} - \Big(\frac{x_{m_1}x_{m_2}}{x_{m_4}x_m^2}\Big)^a + \Big(\frac{x_m}{x_{m_1}x_{m_2}}\Big)^a + \Big(\frac{x_{m_2}}{x_{m_4}x_m}\Big)^a + \Big(\frac{x_{m_1}}{x_{m_4}x_m}\Big)^a - \frac{1}{x_{m_4}^a}\\
=&\frac{1}{x_m^a} - \frac{1}{x_{m_1}^a} - \frac{1}{x_{m_2}^a} + \Big(\frac{x_m}{x_{m_1}x_{m_2}}\Big)^a - \Big(\frac{x_{m_1}^a x_{m_2}^a}{x_m^{2a}} - \frac{x_{m_2}^a}{x_m^a} - \frac{x_{m_1}^a}{x_m^a} + 1\Big) \cdot \frac{1}{x_{m_4}^a}
\end{aligned}$$

$$\begin{aligned}
=&\frac{x_{m_1}^a x_{m_2}^a - x_m^a x_{m_2}^a - x_m^a x_{m_1}^a + x_m^{2a}}{x_m^a x_{m_1}^a x_{m_2}^a} - \Big(\frac{x_{m_1}^a x_{m_2}^a - x_m^a x_{m_2}^a - x_m^a x_{m_1}^a + x_m^{2a}}{x_m^{2a}}\Big) \cdot \frac{1}{x_{m_4}^a}\\
=&(x_{m_1}^a x_{m_2}^a - x_m^a x_{m_2}^a - x_m^a x_{m_1}^a + x_m^{2a})\Big(\frac{1}{x_m^a x_{m_1}^a x_{m_2}^a} - \frac{1}{x_m^{2a} x_{m_4}^a}\Big)\\
=&(x_m^a - x_{m_2}^a)(x_m^a - x_{m_1}^a)\frac{x_m^a x_{m_4}^a - x_{m_1}^a x_{m_2}^a}{x_m^{2a} x_{m_1}^a x_{m_2}^a x_{m_4}^a}\\
=&\Big(\big(\frac{x_m}{x_{m_2}}\big)^a - 1\Big)\Big(\Big(\frac{x_m}{x_{m_1}}\Big)^a - 1\Big)\Big(x_{m_4}^a - \Big(\frac{x_{m_1}x_{m_2}}{x_m}\Big)^a\Big)\frac{1}{x_m^a x_{m_4}^a}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{x_m^{b-a} - x_{m_1}^{b-a} - x_{m_2}^{b-a} - x_{m_3}^{b-a} + x_{m_{12}}^{b-a} + x_{m_{13}}^{b-a} + x_{m_{23}}^{b-a} - x_{m_4}^{b-a}}{x_t^a(x_m^{-a} - x_{m_1}^{-a} - x_{m_2}^{-a} - x_{m_3}^{-a} + x_{m_{12}}^{-a} + x_{m_{13}}^{-a} + x_{m_{23}}^{-a} - x_{m_4}^{-a})}\\
=&-\frac{\big(\frac{x_m}{x_{m_2}}\big)^{b-a} - 1}{\big(\frac{x_m}{x_{m_2}}\big)^a - 1} \cdot \frac{\big(\frac{x_m}{x_{m_1}}\big)^{b-a} - 1}{\big(\frac{x_m}{x_{m_1}}\big)^a - 1} \cdot \frac{\big(\frac{x_{m_1}x_{m_2}}{x_m}\big)^{b-a} - x_{m_4}^{b-a}}{\big(\frac{x_{m_1}x_{m_2}}{x_m}\big)^a - x_{m_4}^a} \cdot \Big(\frac{x_m}{x_t}\Big)^a x_{m_4}^a \in \mathbb{Z}.
\end{aligned}$$

This finishes the proof of part (ii) and that of Lemma 2.12. $\square$

The following fact is taken from the proof of Theorem 1.3 in [41].

Lemma 2.13. *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ with $x_l, x_m \in S$. Let $G_S(x_m) := \{x_{m_1}, x_{m_2}, x_{m_3}\}, x_{m_{ij}} := (x_{m_i}, x_{m_j})$ for $1 \le i < j \le 3$ and $x_{m_4} := (x_{m_1}, x_{m_2}, x_{m_3})$. If $(x_l, x_m)|x_{m_1}$, $(x_l, x_m) \ne x_{m_1}$, $(x_l, x_m) \nmid x_{m_2}$ and $(x_l, x_m) \nmid x_{m_3}$, then*

$$(x_l, x_{m_3}) = (x_l, x_{m_{13}}), \tag{2.11}$$

$$[x_l, x_m] = [x_l, x_{m_2}], \tag{2.12}$$

$$[x_l, x_{m_1}] = [x_l, x_{m_{12}}], \tag{2.13}$$

$$[x_l, x_{m_3}] = [x_l, x_{m_{23}}], \tag{2.14}$$

$$[x_l, x_{m_{13}}] = [x_l, x_{m_4}]. \tag{2.15}$$

3. Proof of Theorem 1.2

In the present section, we use the lemmas presented in Section 2 to show Theorem 1.2. For arbitrary integers l and m with $1 \le l, m \le n$, the functions $f(l,m)$ and $g(l,m)$ are defined by

$$f(l,m) := \frac{1}{\alpha_{\xi_a}(x_m)} \sum_{x_r|x_m} c_{rm}(x_l, x_r)^b \tag{3.1}$$

and

$$g(l,m) := \frac{1}{\alpha_{\xi_a}(x_m)} \sum_{x_r|x_m} c_{rm}[x_l, x_r]^b, \tag{3.2}$$

respectively.

Lemma 3.1. *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and let $x_l, x_m \in S$ with $|G_S(x_m)| = 3$. Let a and b be positive integers with $a|b$. Then $f(l,m) \in \mathbb{Z}$ and $g(l,m) \in \mathbb{Z}$.*

Proof. Let $G_S(x_m) = \{x_{m_1}, x_{m_2}, x_{m_3}\}, x_{m_{ij}} = (x_{m_i}, x_{m_j})$ for $1 \le i < j \le 3, x_{m_4} = (x_{m_1}, x_{m_2}, x_{m_3})$. By Lemma 2.4, one has

$$\alpha_{\xi_a}(x_m) = x_m^a - x_{m_1}^a - x_{m_2}^a - x_{m_3}^a + x_{m_{12}}^a + x_{m_{13}}^a + x_{m_{23}}^a - x_{m_4}^a. \tag{3.3}$$

Since x_m satisfies the condition $\mathcal{G}$, then for any integers i and j with $1 \le i < j \le 3$, we have $[x_{m_i}, x_{m_j}] = x_m$. Now we claim that

$$x_{m_4} \neq x_{m_{ij}} \text{ for } 1 \le i < j \le 3.$$

Assume that $x_{m_4} = x_{m_{ij}}$. Without loss of generality, we may let $x_{m_{12}} = x_{m_4}$. It follows that

$$\begin{aligned} x_m =& [x_{m_1}, x_{m_2}, x_{m_3}] \\ =& \frac{x_{m_1} x_{m_2} x_{m_3} (x_{m_1}, x_{m_2}, x_{m_3})}{(x_{m_1}, x_{m_2})(x_{m_1}, x_{m_3})(x_{m_2}, x_{m_3})} \\ =& \frac{x_{m_1} x_{m_2} x_{m_3} x_{m_4}}{x_{m_{12}} x_{m_{13}} x_{m_{23}}} = \frac{x_{m_1} x_{m_2} x_{m_3}}{x_{m_{13}} x_{m_{23}}} \\ =& \frac{x_{m_2}}{x_{m_{23}}} [x_{m_1}, x_{m_3}] = \frac{x_{m_2} x_m}{x_{m_{23}}}. \end{aligned}$$

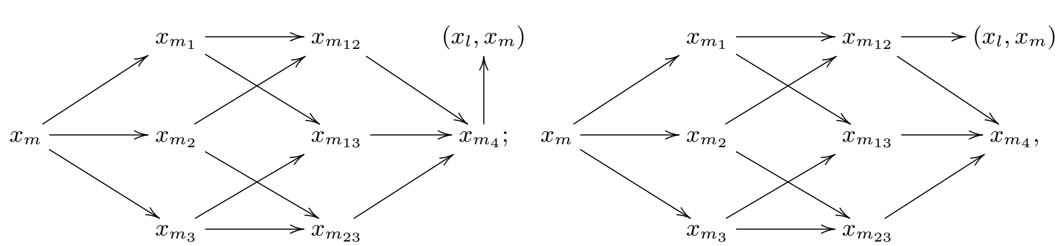
Thus $x_{m_2} = x_{m_{23}}$. This yields $x_{m_2} | x_{m_3}$. It contradicts with the facts that $G_S(x_m) = \{x_{m_1}, x_{m_2}, x_{m_3}\}$ and $x_{m_2} \neq x_{m_3}$. The claim is proved. From the claim and Lemma 2.6, we have

$$\begin{aligned} \sum_{x_r | x_m} c_{rm} (x_l, x_r)^b =& (x_l, x_m)^b - (x_l, x_{m_1})^b - (x_l, x_{m_2})^b - (x_l, x_{m_3})^b \\ & + (x_l, x_{m_{12}})^b + (x_l, x_{m_{13}})^b + (x_l, x_{m_{23}})^b - (x_l, x_{m_4})^b, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \sum_{x_r | x_m} c_{rm} [x_l, x_r]^b =& [x_l, x_m]^b - [x_l, x_{m_1}]^b - [x_l, x_{m_2}]^b - [x_l, x_{m_3}]^b \\ & + [x_l, x_{m_{12}}]^b + [x_l, x_{m_{13}}]^b + [x_l, x_{m_{23}}]^b - [x_l, x_{m_4}]^b. \end{aligned} \tag{3.5}$$

We only need to consider the following four cases corresponding to the four divisibility diagrams in Fig. 1.

Fig. 1. Divisibility relations among $x_m, x_l, x_{m_i} (1 \le i \le 4)$ and $x_{m_{ij}} (1 \le i, j \le 3)$.



Case 1 Case 2

$$\begin{tikzcd}
(x_l, x_m) & x_{m_1} \arrow[l] \arrow[r] \arrow[dr] & x_{m_{12}} \arrow[dr] & \\
x_m \arrow[ur] \arrow[r] \arrow[dr] & x_{m_2} \arrow[ur] \arrow[dr] & x_{m_{13}} \arrow[r] & x_{m_4}, \\
& x_{m_3} \arrow[ur] \arrow[r] & x_{m_{23}} \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
x_l \arrow[d] & x_{m_1} \arrow[r] \arrow[dr] & x_{m_{12}} \arrow[dr] & \\
x_m \arrow[ur] \arrow[r] \arrow[dr] & x_{m_2} \arrow[ur] \arrow[dr] & x_{m_{13}} \arrow[r] & x_{m_4}, \\
& x_{m_3} \arrow[ur] \arrow[r] & x_{m_{23}} \arrow[ur] &
\end{tikzcd}$$

Case 3 Case 4

Case 1. $(x_l, x_m)|x_{m_4}$. In this case, for any i and j with $1 \le i < j \le 3$, we have

$$(x_l, x_m)|(x_l, x_{m_4})|(x_l, x_{m_{ij}})|(x_l, x_{m_i})|(x_l, x_m).$$

So

$$(x_l, x_m) = (x_l, x_{m_4}) = (x_l, x_{m_{ij}}) = (x_l, x_{m_i}). \tag{3.6}$$

The identities (3.4) and (3.6) yield that $\sum\limits_{x_r|x_m} c_{rm}(x_l, x_r)^b = 0$. Hence by (3.1), we have $f(l, m) = 0$.

On the other hand, from (3.5) and (3.6), one has

$$\begin{aligned}
&\sum_{x_r|x_m} c_{rm}[x_l, x_r]^b \\
=&\frac{x_l^b x_m^b}{(x_l, x_m)^b} - \frac{x_l^b x_{m_1}^b}{(x_l, x_{m_1})^b} - \frac{x_l^b x_{m_2}^b}{(x_l, x_{m_2})^b} - \frac{x_l^b x_{m_3}^b}{(x_l, x_{m_3})^b} \\
&+ \frac{x_l^b x_{m_{12}}^b}{(x_l, x_{m_{12}})^b} + \frac{x_l^b x_{m_{13}}^b}{(x_l, x_{m_{13}})^b} + \frac{x_l^b x_{m_{23}}^b}{(x_l, x_{m_{23}})^b} - \frac{x_l^b x_{m_4}^b}{(x_l, x_{m_4})^b} \\
=&\frac{x_l^b}{(x_l, x_m)^b}(x_m^b - x_{m_1}^b - x_{m_2}^b - x_{m_3}^b + x_{m_{12}}^b + x_{m_{13}}^b + x_{m_{23}}^b - x_{m_4}^b). \qquad (3.7)
\end{aligned}$$

It follows from (3.2), (3.3), (3.5), (3.7) and Lemma 2.12 (i) that

$$g(l, m) = \frac{x_l^b}{(x_l, x_m)^b} \frac{x_m^b - x_{m_1}^b - x_{m_2}^b - x_{m_3}^b + x_{m_{12}}^b + x_{m_{13}}^b + x_{m_{23}}^b - x_{m_4}^b}{x_m^a - x_{m_1}^a - x_{m_2}^a - x_{m_3}^a + x_{m_{12}}^a + x_{m_{13}}^a + x_{m_{23}}^a - x_{m_4}^a} \in \mathbb{Z}$$

as required. Case 1 is proved.

Case 2. $(x_l, x_m)|x_{m_{12}}$, $(x_l, x_m) \nmid x_{m_{13}}$ and $(x_l, x_m) \nmid x_{m_{23}}$. Clearly, for any $i \in \{1, 2\}$, we have $(x_l, x_m)|(x_l, x_{m_{12}})|(x_l, x_{m_i})|(x_l, x_m)$. Thus

$$(x_l, x_m) = (x_l, x_{m_{12}}) = (x_l, x_{m_i})(i \in \{1, 2\}). \tag{3.8}$$

Meanwhile, we have $(x_l, x_m) \nmid x_{m_3}$ and $(x_l, x_m) \nmid x_{m_4}$. Since $A_S\big((x_l, x_m), x_{m_1}\big)(\subseteq S)$ satisfies the condition $\mathcal{G}$, $x_{m_{13}} \in G_S(x_{m_1})$ and $(x_l, x_m) \nmid x_{m_{13}}$, by Lemma 2.9 one has $[(x_l, x_m), x_{m_{13}}] = x_{m_1}$. So

$$[x_l, x_{m_1}] = [x_l, [(x_l, x_m), x_{m_{13}}]] = [x_l, x_{m_{13}}]. \tag{3.9}$$

Similarly, we can prove that

$$[x_l, x_m] = [x_l, x_{m_3}], \tag{3.10}$$

$$[x_l, x_{m_2}] = [x_l, x_{m_{23}}], \tag{3.11}$$

$$[x_l, x_{m_{12}}] = [x_l, x_{m_4}]. \tag{3.12}$$

By (3.4) and (3.8) to (3.12), we have

$$\begin{aligned}
&\sum_{x_r|x_m} c_{rm}(x_l,x_r)^b\\
=&(x_l,x_m)^b-(x_l,x_{m_1})^b-(x_l,x_{m_2})^b-(x_l,x_{m_3})^b\\
&+(x_l,x_{m_{12}})^b+(x_l,x_{m_{13}})^b+(x_l,x_{m_{23}})^b-(x_l,x_{m_4})^b\\
=&-(x_l,x_{m_3})^b+(x_l,x_{m_{13}})^b+(x_l,x_{m_{23}})^b-(x_l,x_{m_4})^b\\
=&-\frac{x_l^bx_{m_3}^b}{[x_l,x_{m_3}]^b}+\frac{x_l^bx_{m_{13}}^b}{[x_l,x_{m_{13}}]^b}+\frac{x_l^bx_{m_{23}}^b}{[x_l,x_{m_{23}}]^b}-\frac{x_l^bx_{m_4}^b}{[x_l,x_{m_4}]^b}\\
=&-\frac{x_l^bx_{m_3}^b}{[x_l,x_m]^b}+\frac{x_l^bx_{m_{13}}^b}{[x_l,x_{m_1}]^b}+\frac{x_l^bx_{m_{23}}^b}{[x_l,x_{m_2}]^b}-\frac{x_l^bx_{m_4}^b}{[x_l,x_{m_{12}}]^b}\\
=&-\frac{x_l^bx_{m_3}^b(x_l,x_m)^b}{x_l^bx_m^b}+\frac{x_l^bx_{m_{13}}^b(x_l,x_{m_1})^b}{x_l^bx_{m_1}^b}+\frac{x_l^bx_{m_{23}}^b(x_l,x_{m_2})^b}{x_l^bx_{m_2}^b}-\frac{x_l^bx_{m_4}^b(x_l,x_{m_{12}})^b}{x_l^bx_{m_{12}}^b}\\
=&(x_l,x_m)^b\Big(-\big(\frac{x_{m_3}}{x_m}\big)^b+\big(\frac{x_{m_{13}}}{x_{m_1}}\big)^b+\big(\frac{x_{m_{23}}}{x_{m_2}}\big)^b-\big(\frac{x_{m_4}}{x_{m_{12}}}\big)^b\Big). \qquad (3.13)
\end{aligned}$$

Notice that both of x_m and x_{m_2} satisfy the condition $\mathcal{G}$, $\{x_{m_1},x_{m_3}\}\subseteq G_S(x_m)$ and $\{x_{m_{12}},x_{m_{23}}\}\subseteq G_S(x_{m_2})$. Then we have $[x_{m_1},x_{m_3}]=x_m$ and $[x_{m_{12}},x_{m_{23}}]=x_{m_2}$. So

$$\frac{x_{m_3}}{x_m}=\frac{x_{m_{13}}}{x_{m_1}} \qquad (3.14)$$

and

$$\frac{x_{m_{23}}}{x_{m_2}}=\frac{x_{m_4}}{x_{m_{12}}}. \qquad (3.15)$$

The identities (3.13) to (3.15) tell us that

$$\sum_{x_r|x_m} c_{rm}(x_l,x_r)^b=0. \qquad (3.16)$$

It follows from (3.1) and (3.16) that $f(l,m)=0$.

On the other hand, it follows from (3.5), and (3.9) to (3.12) that

$$\begin{aligned}
\sum_{x_r|x_m} c_{rm}[x_l,x_r]^b=&[x_l,x_m]^b-[x_l,x_{m_1}]^b-[x_l,x_{m_2}]^b-[x_l,x_{m_3}]^b\\
&+[x_l,x_{m_{12}}]^b+[x_l,x_{m_{13}}]^b+[x_l,x_{m_{23}}]^b-[x_l,x_{m_4}]^b\\
=&0. \qquad (3.17)
\end{aligned}$$

So by (3.2) and (3.17), we have $g(l,m)=0$. Case 2 is proved.

Case 3. $(x_l,x_m)|x_{m_1}$, $(x_l,x_m)\nmid x_{m_2}$ and $(x_l,x_m)\nmid x_{m_3}$. Since both of x_{m_1} and x_{m_3} satisfy the condition $\mathcal{G}$, $\{x_{m_{12}},x_{m_{13}}\}\subseteq G_S(x_{m_1})$ and $\{x_{m_{13}},x_{m_{23}}\}\subseteq G_S(x_{m_3})$, we have $[x_{m_{12}},x_{m_{13}}]=x_{m_1}$ and $[x_{m_{13}},x_{m_{23}}]=x_{m_3}$. So

$$\frac{x_{m_1}}{x_{m_{12}}}=\frac{x_{m_{13}}}{x_{m_4}}=\frac{x_{m_3}}{x_{m_{23}}}. \qquad (3.18)$$

Consider the following two subcases:

Subcases 3-1. $(x_l,x_m)\neq x_{m_1}$. Obviously, $(x_l,x_m)|(x_l,x_{m_1})|(x_l,x_m)$. Thus

$$(x_l,x_m)=(x_l,x_{m_1}). \qquad (3.19)$$

Moreover, it follows from (2.12) and (2.13) that

$$(x_l, x_{m_2}) = \frac{x_l x_{m_2}}{[x_l, x_{m_2}]} = \frac{x_l x_{m_2}}{[x_l, x_m]} = \frac{x_l x_{m_2}(x_l, x_m)}{x_l x_m} = \frac{x_{m_2}(x_l, x_m)}{x_m} \tag{3.20}$$

and

$$(x_l, x_{m_{12}}) = \frac{x_l x_{m_{12}}}{[x_l, x_{m_{12}}]} = \frac{x_l x_{m_{12}}}{[x_l, x_{m_1}]} = \frac{x_l x_{m_{12}}(x_l, x_{m_1})}{x_l x_{m_1}} = \frac{x_{m_{12}}(x_l, x_{m_1})}{x_{m_1}}. \tag{3.21}$$

Note that $\dfrac{x_{m_2}}{x_m} = \dfrac{x_{m_{12}}}{x_{m_1}}$. By (3.19) to (3.21), one has

$$(x_l, x_{m_2}) = (x_l, x_{m_{12}}). \tag{3.22}$$

On the other hand, from (3.19), we have

$$(x_l, x_{m_4}) = (x_l, x_{m_1}, x_{m_2}, x_{m_3}) = (x_l, x_m, x_{m_2}, x_{m_3}) = (x_l, x_m, x_{m_{23}}) = (x_l, x_{m_{23}}). \tag{3.23}$$

It follows from (3.4), (2.11), (3.19), (3.22) and (3.23) that $\sum_{x_r|x_m} c_{rm}(x_l, x_r)^b = 0$. Thus $f(l, m) = 0$. By (3.5), and (2.12) to (2.15), we have $\sum_{x_r|x_m} c_{rm}[x_l, x_r]^b = 0$ and so $g(l, m) = 0$.

Subcases 3-2. $(x_l, x_m) = x_{m_1}$. Then $x_{m_1}|x_l$ and so $x_{m_{12}}|x_l$, $x_{m_{13}}|x_l$ and $x_{m_4}|x_l$. It is clear that

$$(x_l, x_{m_1}) = x_{m_1}, \tag{3.24}$$
$$(x_l, x_{m_{12}}) = x_{m_{12}}, \tag{3.25}$$
$$(x_l, x_{m_{13}}) = x_{m_{13}}, \tag{3.26}$$
$$(x_l, x_{m_4}) = x_{m_4} \tag{3.27}$$

and

$$[x_l, x_{m_1}] = [x_l, x_{m_{12}}] = [x_l, x_{m_{13}}] = [x_l, x_{m_4}] = x_l. \tag{3.28}$$

Moreover, we can obtain that

$$(x_l, x_{m_3}) = (x_l, x_m, x_{m_3}) = (x_{m_1}, x_{m_3}) = x_{m_{13}}, \tag{3.29}$$
$$(x_l, x_{m_{23}}) = (x_l, x_m, x_{m_{23}}) = (x_{m_1}, x_{m_{23}}) = x_{m_4}, \tag{3.30}$$
$$(x_l, x_{m_2}) = (x_l, x_m, x_{m_2}) = (x_{m_1}, x_{m_2}) = x_{m_{12}}. \tag{3.31}$$

So from (3.24) to (3.27) and from (3.29) to (3.31), we obtain that

$$(x_l, x_m) = (x_l, x_{m_1}),$$
$$(x_l, x_{m_{12}}) = (x_l, x_{m_2}),$$
$$(x_l, x_{m_{13}}) = (x_l, x_{m_3}),$$

and

$$(x_l, x_{m_4}) = (x_l, x_{m_{23}}).$$

These together with (3.4) imply that

$$\sum_{x_r|x_m} c_{rm}(x_l, x_r)^b = 0$$

and so $f(l, m) = 0$.

Since x_m satisfies the condition $\mathcal{G}$ and $G_S(x_m) = \{x_{m_1}, x_{m_2}, x_{m_3}\}$. one has $[x_{m_1}, x_{m_2}] = x_m$. This together with $x_{m_1}|x_l$ yields that

$$[x_l, x_m] = [x_l, [x_{m_1}, x_{m_2}]] = [x_l, x_{m_2}]. \tag{3.32}$$

Meanwhile, by (3.18), (3.29) and (3.30) we have

$$[x_l, x_{m_3}] = \frac{x_l x_{m_3}}{(x_l, x_{m_3})} = \frac{x_l x_{m_3}}{x_{m_{13}}} = \frac{x_l x_{m_{23}}}{x_{m_4}} = \frac{x_l x_{m_{23}}}{(x_l, x_{m_{23}})} = [x_l, x_{m_{23}}]. \tag{3.33}$$

It then follows from (3.5), (3.28), (3.32) and (3.33) that

$$\sum_{x_r|x_m} c_{rm}[x_l, x_r]^b = 0.$$

This yields $g(l, m) = 0$ as desired. Case 3 is proved.

Case 4. $x_m|x_l$. It is easy to check that $(x_l, x_m) = x_m$, and for any integers r, i and j with $1 \le r \le 4$ and $1 \le i < j \le 3$, one has $(x_l, x_{m_r}) = x_{m_r}$, $(x_l, x_{m_{ij}}) = x_{m_{ij}}$ and $[x_l, x_m] = [x_l, x_{m_r}] = [x_l, x_{m_{ij}}] = x_l$. So by (3.4) and (3.5), one derives that

$$\sum_{x_r|x_m} c_{rm}(x_l, x_r)^b = x_m^b - x_{m_1}^b - x_{m_2}^b - x_{m_3}^b + x_{m_{12}}^b + x_{m_{13}}^b + x_{m_{23}}^b - x_{m_4}^b \tag{3.34}$$

and

$$\sum_{x_r|x_m} c_{rm}[x_l, x_r]^b = 0. \tag{3.35}$$

Thus from Lemma 2.12 (i), (3.1), (3.3) and (3.34), it deduces that

$$f(l, m) = \frac{x_m^b - x_{m_1}^b - x_{m_2}^b - x_{m_3}^b + x_{m_{12}}^b + x_{m_{13}}^b + x_{m_{23}}^b - x_{m_4}^b}{x_m^a - x_{m_1}^a - x_{m_2}^a - x_{m_3}^a + x_{m_{12}}^a + x_{m_{13}}^a + x_{m_{23}}^a - x_{m_4}^a} \in \mathbb{Z}.$$

By (3.2) and (3.35), we have $g(l, m) = 0$. Case 4 is proved and so is Lemma 3.1. □

Proof of Theorem 1.2. Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and $\max_{x\in S} |G_S(x)| = 3$. For any integers l and j with $1 \le l, j \le n$, from Lemma 2.2, we have

$$\begin{aligned}
\big((S^b)(S^a)^{-1}\big)_{lj} &= \sum_{r=1}^{n} (x_l, x_r)^b \sum_{\substack{x_r|x_m \\ x_j|x_m}} \frac{c_{rm}c_{jm}}{\alpha_{\xi_a}(x_m)} \\
&= \sum_{x_j|x_m} c_{jm} \sum_{x_r|x_m} \frac{c_{rm}(x_l, x_r)^b}{\alpha_{\xi_a}(x_m)} \\
&= \sum_{x_j|x_m} c_{jm} \frac{1}{\alpha_{\xi_a}(x_m)} \sum_{x_r|x_m} c_{rm}(x_l, x_r)^b \\
&= \sum_{x_j|x_m} c_{jm} f(l, m)
\end{aligned}$$

and

$$\begin{aligned}
\big([S^b](S^a)^{-1}\big)_{lj} &= \sum_{r=1}^{n} [x_l, x_r]^b \sum_{\substack{x_r|x_m \\ x_j|x_m}} \frac{c_{rm}c_{jm}}{\alpha_{\xi_a}(x_m)} \\
&= \sum_{x_j|x_m} c_{jm} \sum_{x_r|x_m} \frac{c_{rm}[x_l, x_r]^b}{\alpha_{\xi_a}(x_m)} \\
&= \sum_{x_j|x_m} c_{jm} \frac{1}{\alpha_{\xi_a}(x_m)} \sum_{x_r|x_m} c_{rm}[x_l, x_r]^b
\end{aligned}$$

$$= \sum_{x_j|x_m} c_{jm} g(l,m),$$

where $f(l,m)$ and $g(l,m)$ are defined as in (3.1) and (3.2), respectively. By the definition of c_{jm}, we know that $c_{jm} \in \mathbb{Z}$. So in order to show $\big((S^b)(S^a)^{-1}\big)_{lj} \in \mathbb{Z}$ and $\big([S^b](S^a)^{-1}\big)_{lj} \in \mathbb{Z}$, we only need to prove that $f(l,m) \in \mathbb{Z}$ and $g(l,m) \in \mathbb{Z}$ for all $1 \le l,m \le n$. Consider the following three cases:

Case 1. $|G_S(x_m)| = 1$. Let $G_S(x_m) = \{x_{m_0}\}$. By Lemmas 2.4, 2.5 and 2.10 (i), we can arrive at that

$$f(l,m) = \frac{(x_l,x_m)^b - (x_l,x_{m_0})^b}{x_m^a - x_{m_0}^a} \in \mathbb{Z}$$

and

$$g(l,m) = \frac{[x_l,x_m]^b - [x_l,x_{m_0}]^b}{x_m^a - x_{m_0}^a} \in \mathbb{Z}.$$

Case 2. $|G_S(x_m)| = 2$. Let $G_S(x_m) = \{x_{m_{01}}, x_{m_{02}}\}$ and $(x_{m_{01}}, x_{m_{02}}) = x_{m_{03}}$. It follows from Lemmas 2.4, 2.5 and 2.11 (i) that

$$f(l,m) = \frac{(x_l,x_m)^b - (x_l,x_{m_{01}})^b - (x_l,x_{m_{02}})^b + (x_l,x_{m_{03}})^b}{x_m^a - x_{m_{01}}^a - x_{m_{02}}^a + x_{m_{03}}^a} \in \mathbb{Z}$$

and

$$g(l,m) = \frac{[x_l,x_m]^b - [x_l,x_{m_{01}}]^b - [x_l,x_{m_{02}}]^b + [x_l,x_{m_{03}}]^b}{x_m^a - x_{m_{01}}^a - x_{m_{02}}^a + x_{m_{03}}^a} \in \mathbb{Z}.$$

Case 3. $|G_S(x_m)| = 3$. Then from Lemma 3.1, we know that both of $f(l,m)$ and $g(l,m)$ are integers.

This completes the proof of Theorem 1.2. $\square$

4. Proof of Theorem 1.3

In this section, with help of the lemmas presented in Section 2, we show Theorem 1.3. For this purpose, for arbitrary integers l, m and s with $1 \le l,m,s \le n$ and $x_s|x_m$, we define the function $h(l,m)$ as follows:

$$h(l,m) := \frac{1}{x_s^a \alpha_{\frac{1}{\xi_a}}(x_m)} \sum_{x_r|x_m} \frac{c_{rm}[x_l,x_r]^b}{x_r^a}. \tag{4.1}$$

We have the following result.

Lemma 4.1. *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and let $x_l, x_m \in S$ with $|G_S(x_m)| = 3$. Let a and b be positive integers with $a|b$. Then $h(l,m) \in \mathbb{Z}$.*

Proof. Let

$$\begin{aligned} &G_S(x_m) = \{x_{m_1}, x_{m_2}, x_{m_3}\},\\ &x_{m_{ij}} = (x_{m_i}, x_{m_j}) \text{ for } 1 \le i < j \le 3,\\ &x_{m_4} = (x_{m_1}, x_{m_2}, x_{m_3}).\end{aligned}$$

By Lemma 2.4, one has

$$\alpha_{\frac{1}{\xi_a}}(x_m) = x_m^{-a} - x_{m_1}^{-a} - x_{m_2}^{-a} - x_{m_3}^{-a} + x_{m_{12}}^{-a} + x_{m_{13}}^{-a} + x_{m_{23}}^{-a} - x_{m_4}^{-a}. \tag{4.2}$$

From Lemma 2.6, we have

$$\sum_{x_r|x_m}\frac{c_{rm}[x_l,x_r]^b}{x_r^a}=\frac{[x_l,x_m]^b}{x_m^a}-\frac{[x_l,x_{m_1}]^b}{x_{m_1}^a}-\frac{[x_l,x_{m_2}]^b}{x_{m_2}^a}-\frac{[x_l,x_{m_3}]^b}{x_{m_3}^a}$$
$$+\frac{[x_l,x_{m_{12}}]^b}{x_{m_{12}}^a}+\frac{[x_l,x_{m_{13}}]^b}{x_{m_{13}}^a}+\frac{[x_l,x_{m_{23}}]^b}{x_{m_{23}}^a}-\frac{[x_l,x_{m_4}]^b}{x_{m_4}^a}. \tag{4.3}$$

We only need to consider the following four cases as the proof of Lemma 3.1.

Case 1. $(x_l,x_m)|x_{m_4}$. From (3.6) and (4.3), one has

$$\begin{aligned}&\sum_{x_r|x_m}\frac{c_{rm}[x_l,x_r]^b}{x_r^a}\\
=&\frac{x_l^bx_m^{b-a}}{(x_l,x_m)^b}-\frac{x_l^bx_{m_1}^{b-a}}{(x_l,x_{m_1})^b}-\frac{x_l^bx_{m_2}^{b-a}}{(x_l,x_{m_2})^b}-\frac{x_l^bx_{m_3}^{b-a}}{(x_l,x_{m_3})^b}\\
&+\frac{x_l^bx_{m_{12}}^{b-a}}{(x_l,x_{m_{12}})^b}+\frac{x_l^bx_{m_{13}}^{b-a}}{(x_l,x_{m_{13}})^b}+\frac{x_l^bx_{m_{23}}^{b-a}}{(x_l,x_{m_{23}})^b}-\frac{x_l^bx_{m_4}^{b-a}}{(x_l,x_{m_4})^b}\\
=&\frac{x_l^b}{(x_l,x_m)^b}(x_m^{b-a}-x_{m_1}^{b-a}-x_{m_2}^{b-a}-x_{m_3}^{b-a}+x_{m_{12}}^{b-a}+x_{m_{13}}^{b-a}+x_{m_{23}}^{b-a}-x_{m_4}^{b-a}).\end{aligned}\tag{4.4}$$

It follows from (4.1), (4.2), (4.4) and Lemma 2.12 (ii) that

$$h(l,m)=\frac{x_l^b}{(x_l,x_m)^b}\frac{x_m^{b-a}-x_{m_1}^{b-a}-x_{m_2}^{b-a}-x_{m_3}^{b-a}+x_{m_{12}}^{b-a}+x_{m_{13}}^{b-a}+x_{m_{23}}^{b-a}-x_{m_4}^{b-a}}{x_s^a(x_m^{-a}-x_{m_1}^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{12}}^{-a}+x_{m_{13}}^{-a}+x_{m_{23}}^{-a}-x_{m_4}^{-a})}\in\mathbb{Z}$$

as expected.

Case 2. $(x_l,x_m)|x_{m_{12}}$, $(x_l,x_m)\nmid x_{m_{13}}$ and $(x_l,x_m)\nmid x_{m_{23}}$. By (4.3) and (3.8) to (3.12), we have

$$\begin{aligned}&\sum_{x_r|x_m}\frac{c_{rm}[x_l,x_r]^b}{x_r^a}\\
=&\frac{[x_l,x_m]^b}{x_m^a}-\frac{[x_l,x_{m_1}]^b}{x_{m_1}^a}-\frac{[x_l,x_{m_2}]^b}{x_{m_2}^a}-\frac{[x_l,x_m]^b}{x_{m_3}^a}\\
&+\frac{[x_l,x_{m_{12}}]^b}{x_{m_{12}}^a}+\frac{[x_l,x_{m_1}]^b}{x_{m_{13}}^a}+\frac{[x_l,x_{m_2}]^b}{x_{m_{23}}^a}-\frac{[x_l,x_{m_{12}}]^b}{x_{m_4}^a}\\
=&[x_l,x_m]^b(\frac{1}{x_m^a}-\frac{1}{x_{m_3}^a})+[x_l,x_{m_1}]^b(\frac{1}{x_{m_{13}}^a}-\frac{1}{x_{m_1}^a})+[x_l,x_{m_2}]^b(\frac{1}{x_{m_{23}}^a}-\frac{1}{x_{m_2}^a})\\
&+[x_l,x_{m_{12}}]^b(\frac{1}{x_{m_{12}}^a}-\frac{1}{x_{m_4}^a})\\
=&\frac{x_l^bx_m^b}{(x_l,x_m)^b}(\frac{1}{x_m^a}-\frac{1}{x_{m_3}^a})+\frac{x_l^bx_{m_1}^b}{(x_l,x_{m_1})^b}(\frac{1}{x_{m_{13}}^a}-\frac{1}{x_{m_1}^a})+\frac{x_l^bx_{m_2}^b}{(x_l,x_{m_2})^b}(\frac{1}{x_{m_{23}}^a}-\frac{1}{x_{m_2}^a})\\
&+\frac{x_l^bx_{m_{12}}^b}{(x_l,x_{m_{12}})^b}(\frac{1}{x_{m_{12}}^a}-\frac{1}{x_{m_4}^a})\\
=&\frac{x_l^b}{(x_l,x_m)^b}(x_m^{b-a}-x_{m_1}^{b-a}-x_{m_2}^{b-a}+x_{m_{12}}^{b-a}-\frac{x_m^b}{x_{m_3}^a}+\frac{x_{m_1}^b}{x_{m_{13}}^a}+\frac{x_{m_2}^b}{x_{m_{23}}^a}-\frac{x_{m_{12}}^b}{x_{m_4}^a}).\end{aligned}\tag{4.5}$$

Since S satisfies the condition $\mathcal{G}$, we have $[x_{m_i},x_{m_j}]=x_m$ for $1\le i<j\le 3$, $\{x_{m_{12}},x_{m_{13}}\}\subseteq G_S(x_{m_1})$ and $\{x_{m_{13}},x_{m_{23}}\}\subseteq G_S(x_{m_3})$. This implies that

$$\frac{x_{m_2}}{x_{m_{23}}}=\frac{x_m}{x_{m_3}}=\frac{x_{m_1}}{x_{m_{13}}}=\frac{x_{m_{12}}}{x_{m_4}}:=\lambda \tag{4.6}$$

and

$$\frac{x_m}{x_{m_1}} = \frac{x_{m_2}}{x_{m_{12}}} = \frac{x_{m_3}}{x_{m_{13}}} = \frac{x_{m_{23}}}{x_{m_4}} := \delta. \tag{4.7}$$

From (4.5) to (4.7), we obtain that

$$\begin{aligned}
\sum_{x_r|x_m} \frac{c_{rm}[x_l,x_r]^b}{x_r^a} =& \frac{x_l^b}{(x_l,x_m)^b}\big(x_m^{b-a} - x_{m_1}^{b-a} - x_{m_2}^{b-a} + x_{m_{12}}^{b-a} - \frac{x_m^b}{x_{m_3}^a} + \frac{x_{m_1}^b}{x_{m_{13}}^a} + \frac{x_{m_2}^b}{x_{m_{23}}^a} - \frac{x_{m_{12}}^b}{x_{m_4}^a}\big)\\
=& \frac{x_l^b}{(x_l,x_m)^b}\big(x_m^{b-a} - x_{m_1}^{b-a} - x_{m_2}^{b-a} + x_{m_{12}}^{b-a} - \lambda^a x_m^{b-a} + \lambda^a x_{m_1}^{b-a} + \lambda^a x_{m_2}^{b-a} - \lambda^a x_{m_{12}}^{b-a}\big)\\
=& \frac{x_l^b}{(x_l,x_m)^b}(1-\lambda^a)\big(x_m^{b-a} - x_{m_1}^{b-a} - x_{m_2}^{b-a} + x_{m_{12}}^{b-a}\big)\\
=& \frac{x_l^b}{(x_l,x_m)^b}(1-\lambda^a)\big(x_m^{b-a} - \big(\frac{x_m}{\delta}\big)^{b-a} - x_{m_2}^{b-a} + \big(\frac{x_{m_2}}{\delta}\big)^{b-a}\big)\\
=& \frac{x_l^b}{(x_l,x_m)^b}(1-\lambda^a)(\delta^{b-a}-1)\Big(\big(\frac{x_m}{\delta}\big)^{b-a} - \big(\frac{x_{m_2}}{\delta}\big)^{b-a}\Big).
\end{aligned} \tag{4.8}$$

By (4.2), (4.6) and (4.7), one has

$$\alpha_{\frac{1}{\xi_a}}(x_m) = (1-\lambda^a)\delta^a(\delta^a-1)\frac{\big(\frac{x_m}{\delta}\big)^a - \big(\frac{x_{m_2}}{\delta}\big)^a}{x_m^a x_{m_2}^a}. \tag{4.9}$$

It follows from (4.8) and (4.9) that

$$h(l,m) = \frac{x_m^a}{x_s^a}\frac{x_{m_2}^a}{\delta^a}\frac{x_l^b}{(x_l,x_m)^b}\frac{\big(\frac{x_m}{\delta}\big)^{b-a} - \big(\frac{x_{m_2}}{\delta}\big)^{b-a}}{\big(\frac{x_m}{\delta}\big)^a - \big(\frac{x_{m_2}}{\delta}\big)^a}\frac{\delta^{b-a}-1}{\delta^a-1} \in \mathbb{Z}$$

as desired.

CASE 3. $(x_l,x_m)|x_{m_1}$, $(x_l,x_m)\nmid x_{m_2}$ and $(x_l,x_m)\nmid x_{m_3}$. We should consider the following two cases:

SUBCASES 3-1. $(x_l,x_m)\neq x_{m_1}$. Obviously, $(x_l,x_m)|(x_l,x_{m_1})|(x_l,x_m)$. Thus

$$(x_l,x_m) = (x_l,x_{m_1}). \tag{4.10}$$

Notice that $(x_l,x_m)\nmid x_{m_2}$, $(x_l,x_m)\nmid x_{m_3}$, $(x_l,x_m)\nmid x_{m_{12}}$, $(x_l,x_m)\nmid x_{m_{13}}$, $\{x_{m_2},x_{m_3}\}\subseteq G_S(x_m)$ and $\{x_{m_{12}},x_{m_{13}}\}\subseteq G_S(x_{m_1})$. Since $A_S((x_l,x_m),x_m)(\subseteq S)$ satisfies the condition $\mathcal{G}$, from Lemma 2.9, we have

$$[(x_l,x_m),x_{m_2}] = x_m = [(x_l,x_m),x_{m_3}] \tag{4.11}$$

and

$$[(x_l,x_m),x_{m_{12}}] = x_{m_1} = [(x_l,x_m),x_{m_{13}}]. \tag{4.12}$$

So from (4.11) and (4.12), we get that

$$[x_l,x_{m_2}] = [x_l,(x_l,x_m),x_{m_2}] = [x_l,x_m] = [x_l,(x_l,x_m),x_{m_3}] = [x_l,x_{m_3}] \tag{4.13}$$

and

$$[x_l,x_{m_{12}}] = [x_l,(x_l,x_m),x_{m_{12}}] = [x_l,x_{m_1}] = [x_l,(x_l,x_m),x_{m_{13}}] = [x_l,x_{m_{13}}]. \tag{4.14}$$

Hence by (2.14), (2.15) (4.13) and (4.14), we have

$$[x_l,x_{m_2}] = [x_l,x_m] = [x_l,x_{m_3}] = [x_l,x_{m_{23}}], \tag{4.15}$$

$$[x_l,x_{m_{12}}] = [x_l,x_{m_1}] = [x_l,x_{m_{13}}] = [x_l,x_{m_4}]. \tag{4.16}$$

It then follows from (4.3), (4.7) , (4.10), (4.15) and (4.16) that

$$\begin{aligned}
&\sum_{x_r|x_m}\frac{c_{rm}[x_l,x_r]^b}{x_r^a}\\
=&\frac{[x_l,x_m]^b}{x_m^a}-\frac{[x_l,x_{m_1}]^b}{x_{m_1}^a}-\frac{[x_l,x_m]^b}{x_{m_2}^a}-\frac{[x_l,x_m]^b}{x_{m_3}^a}\\
&+\frac{[x_l,x_{m_1}]^b}{x_{m_{12}}^a}+\frac{[x_l,x_{m_1}]^b}{x_{m_{13}}^a}+\frac{[x_l,x_m]^b}{x_{m_{23}}^a}-\frac{[x_l,x_{m_1}]^b}{x_{m_4}^a}\\
=&[x_l,x_m]^b\big(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a}\big)+[x_l,x_{m_1}]^b\big(-x_{m_1}^{-a}+x_{m_{12}}^{-a}+x_{m_{13}}^{-a}-x_{m_4}^{-a}\big)\\
=&\frac{x_l^bx_m^b}{(x_l,x_m)^b}\big(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a}\big)+\frac{x_l^bx_{m_1}^b}{(x_l,x_{m_1})^b}\big(-x_{m_1}^{-a}+x_{m_{12}}^{-a}+x_{m_{13}}^{-a}-x_{m_4}^{-a}\big)\\
=&\frac{x_l^b}{(x_l,x_m)^b}\big(x_m^b(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a})+x_{m_1}^b(-x_{m_1}^{-a}+x_{m_{12}}^{-a}+x_{m_{13}}^{-a}-x_{m_4}^{-a})\big)\\
=&\frac{x_l^b}{(x_l,x_m)^b}\big(x_m^b(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a})+x_{m_1}^b\delta^a(-x_m^{-a}+x_{m_2}^{-a}+x_{m_3}^{-a}-x_{m_{23}}^{-a})\big)\\
=&\frac{x_l^b}{(x_l,x_m)^b}(x_m^b-x_{m_1}^b\delta^a)(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a})\\
=&\frac{x_m^ax_l^b}{(x_l,x_m)^b}(x_m^{b-a}-x_{m_1}^{b-a})(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a}).
\end{aligned}\tag{4.17}$$

Using (4.2) and (4.7) gives us that

$$\alpha_{\frac{1}{\xi_a}}(x_m)=x_{m_1}^{-a}(x_{m_1}^a-x_m^a)(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a}).\tag{4.18}$$

It follows from (4.1), (4.17) and (4.18) that

$$h(l,m)=x_{m_1}^a\big(\frac{x_m}{x_s}\big)^a\big(\frac{x_l}{(x_l,x_m)}\big)^b\frac{x_m^{b-a}-x_{m_1}^{b-a}}{x_{m_1}^a-x_m^a}\in\mathbb{Z}$$

as asserted.

Subcases 3-2. $(x_l,x_m)=x_{m_1}$. Then $x_{m_1}|x_l$ and so $x_{m_{12}}|x_l$, $x_{m_{13}}|x_l$ and $x_{m_4}|x_l$. Since x_m satisfies the condition $\mathcal{G}$, we have $[x_{m_1},x_{m_3}]=x_m$. It follows that

$$\frac{x_m}{x_{m_1}}=\frac{x_{m_3}}{x_{m_{13}}}.$$

Thus from (3.29), we get that

$$[x_l,x_m]=\frac{x_lx_m}{(x_l,x_m)}=\frac{x_lx_m}{x_{m_1}}=\frac{x_lx_{m_3}}{x_{m_{13}}}=\frac{x_lx_{m_3}}{(x_l,x_{m_3})}=[x_l,x_{m_3}].\tag{4.19}$$

Therefore by (3.32), (3.33) and (4.19), one has

$$[x_l,x_{m_2}]=[x_l,x_m]=[x_l,x_{m_3}]=[x_l,x_{m_{23}}].\tag{4.20}$$

One can deduce from (3.28), (4.3), (4.7) and (4.20) that

$$\begin{aligned}
&\sum_{x_r|x_m}\frac{c_{rm}[x_l,x_r]^b}{x_r^a}\\
=&\frac{[x_l,x_m]^b}{x_m^a}-\frac{x_l^b}{x_{m_1}^a}-\frac{[x_l,x_m]^b}{x_{m_2}^a}-\frac{[x_l,x_m]^b}{x_{m_3}^a}+\frac{x_l^b}{x_{m_{12}}^a}+\frac{x_l^b}{x_{m_{13}}^a}+\frac{[x_l,x_m]^b}{x_{m_{23}^a}}-\frac{x_l^b}{x_{m_4}^a}\\
=&[x_l,x_m]^b(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a})+x_l^b(x_{m_{12}}^{-a}+x_{m_{13}}^{-a}-x_{m_1}^{-a}-x_{m_4}^{-a})
\end{aligned}$$

$$
\begin{aligned}
=&\frac{x_l^b x_m^b}{(x_l,x_m)^b}\big(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a}\big)+x_l^b\big(x_{m_{12}}^{-a}+x_{m_{13}}^{-a}-x_{m_1}^{-a}-x_{m_4}^{-a}\big)\\
=&\frac{x_l^b x_m^b}{x_{m_1}^b}(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a})+x_l^b\big(x_{m_{12}}^{-a}+x_{m_{13}}^{-a}-x_{m_1}^{-a}-x_{m_4}^{-a}\big)\\
=&\frac{x_l^b x_m^b}{x_{m_1}^b}(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a})-x_l^b\delta^a(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a})\\
=&x_l^b\cdot\frac{x_m^b-x_m^a x_{m_1}^{b-a}}{x_{m_1}^b}(x_m^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{23}}^{-a}). \qquad (4.21)
\end{aligned}
$$

Combining (4.18) and (4.21) gives us that

$$
h(l,m)=x_{m_1}^a\Big(\frac{x_m}{x_s}\Big)^a\Big(\frac{x_l}{(x_l,x_m)}\Big)^b\frac{x_m^{b-a}-x_{m_1}^{b-a}}{x_{m_1}^a-x_m^a}\in\mathbb{Z}
$$

as claimed.

Case 4. $x_m|x_l$. It is easy to check that $(x_l,x_m)=x_m$, and for any integers r,i and j with $1\le r\le 4$ and $1\le i<j\le 3$, one has $(x_l,x_{m_r})=x_{m_r}$, $(x_l,x_{m_{ij}})=x_{m_{ij}}$ and

$$
[x_l,x_m]=[x_l,x_{m_r}]=[x_l,x_{m_{ij}}]=x_l.
$$

So by (4.3), one derives that

$$
\sum_{x_r|x_m}\frac{c_{rm}[x_l,x_r]^b}{x_r^a}=x_l^b(x_m^{-a}-x_{m_1}^{-a}-x_{m_2}^{-a}-x_{m_3}^{-a}+x_{m_{12}}^{-a}+x_{m_{13}}^{-a}+x_{m_{23}}^{-a}-x_{m_4}^{-a}). \quad (4.22)
$$

Notice that $x_s|x_m$ and $x_m|x_l$. Thus $x_s|x_l$. From (4.1), (4.2) and (4.22), it deduces that

$$
h(l,m)=\frac{x_l^b}{x_s^a}=x_l^{b-a}\Big(\frac{x_l}{x_s}\Big)^a\in\mathbb{Z}
$$

as desired. This finishes the proof of Lemma 4.1. □

Finally, we present the proof of Theorem 1.3.

Proof of Theorem 1.3. Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and $\max_{x\in S}|G_S(x)|=3$. For arbitrary integers l and s with $1\le l,s\le n$, from Lemma 2.8, we have

$$
\begin{aligned}
\big([S^b][S^a]^{-1}\big)_{ls}=&\sum_{r=1}^n[x_l,x_r]^b\frac{1}{x_r^a x_s^a}\sum_{\substack{x_r|x_m\\x_s|x_m}}\frac{c_{rm}c_{sm}}{\alpha_{\frac{1}{\xi_a}}(x_m)}\\
=&\sum_{x_s|x_m}c_{sm}\frac{1}{x_s^a\alpha_{\frac{1}{\xi_a}}(x_m)}\sum_{x_r|x_m}\frac{c_{rm}[x_l,x_r]^b}{x_r^a}\\
=&\sum_{x_s|x_m}c_{sm}h(l,m),
\end{aligned}
$$

where $h(l,m)$ is defined as in (4.1). To show the required result, it suffices to prove that for any positive integers m such that $x_s|x_m$, one has $h(l,m)\in\mathbb{Z}$ which will be done as follows. Consider the following three cases:

Case 1. $|G_S(x_m)|=1$. Let $G_S(x_m)=\{x_{m_0}\}$. By Lemmas 2.4, 2.5 and 2.10 (ii), we can arrive at that

$$
h(l,m)=\frac{x_{m_0}^a[x_l,x_m]^b-x_m^a[x_l,x_{m_0}]^b}{x_s^a(x_{m_0}^a-x_m^a)}\in\mathbb{Z}.
$$

Case 2. $|G_S(x_m)| = 2$. Let $G_S(x_m) = \{x_{m_{01}}, x_{m_{02}}\}$ and $(x_{m_{01}}, x_{m_{02}}) = x_{m_{03}}$. It follows from Lemmas 2.4, 2.5 and 2.11 (ii) that

$$\begin{aligned}h(l,m) =& \frac{\frac{[x_l,x_m]^b}{x_m^a} - \frac{[x_l,x_{m_{01}}]^b}{x_{m_{01}}^a} - \frac{[x_l,x_{m_2}]^b}{x_{m_{02}}^a} + \frac{[x_l,x_{m_{03}}]^b}{x_{m_{03}}^a}}{x_s^a\big(\frac{1}{x_m^a} - \frac{1}{x_{m_{01}}^a} - \frac{1}{x_{m_{02}}^a} + \frac{1}{x_{m_{03}}^a}\big)}\\ =& \frac{x_m^a[x_l,x_{m_{03}}]^b + x_{m_{03}}^a[x_l,x_m]^b - x_{m_{02}}^a[x_l,x_{m_{01}}]^b - x_{m_{01}}^a[x_l,x_{m_{02}}]^b}{x_s^a(x_m^a + x_{m_{03}}^a - x_{m_{01}}^a - x_{m_{02}}^a)} \in \mathbb{Z}.\end{aligned}$$

Case 3. $|G_S(x_m)| = 3$. Then by Lemma 4.1, we know that $h(l,m)$ is an integer.

This concludes the proof of Theorem 1.3. □

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