On Turán-type problems for Berge matchings

Xiamiao Zhao[*1], **Zixuan Yang**[†2,3], **Yichen Wang**[‡1], **Yuhang Bai**[§2,3], **Junpeng Zhou**[¶4,5]

[1]Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P.R. China.
[2]School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710129, Shaanxi, P.R. China.
[3]Xi'an-Budapest Joint Research Center for Combinatorics, Xi'an 710129, Shaanxi, P.R. China.
[4]Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China.
[5]Newtouch Center for Mathematics of Shanghai University, Shanghai 200444, P.R. China.

Abstract

For a graph F, an r-uniform hypergraph (r-graph for short) $\mathcal{H}$ is a Berge-F if there is a bijection $\phi : E(F) \to E(\mathcal{H})$ such that $e \subseteq \phi(e)$ for each $e \in E(F)$. Given a family $\mathcal{F}$ of r-graphs, an r-graph is $\mathcal{F}$-free if it does not contain any member in $\mathcal{F}$ as a subhypergraph. The Turán number of $\mathcal{F}$ is the maximum number of hyperedges in an $\mathcal{F}$-free r-graph on n vertices. Kang, Ni, and Shan [*Discrete Math. 345 (2022) 112901*] determined the exact value of the Turán number of Berge-M_{s+1} for all n when $r \leq s-1$ or $r \geq 2s+2$, where M_{s+1} denotes a matching of size $s+1$. In this paper, we settle the remaining case $s \leq r \leq 2s+1$. Moreover, we establish several exact and general results on the Turán numbers of Berge matchings together with a single r-graph, as well as of Berge matchings together with Berge bipartite graphs. Finally, we generalize the results on Turán problems for Berge hypergraphs proposed by Gerbner, Methuku, and Palmer [*Eur. J. Comb. 86 (2020) 103082*].

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1. Introduction

A *hypergraph* $\mathcal{H}$, denoted by $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, on a finite vertex set $V(\mathcal{H})$ is a family $E(\mathcal{H})$ of subsets of $V(\mathcal{H})$, called *hyperedges*. For an integer $r \geq 2$, a hypergraph $\mathcal{H}$ is called an *r-uniform hypergraph* (r-graph for short) if every hyperedge of $\mathcal{H}$ contains exactly r vertices. The size of $E(\mathcal{H})$ is denoted by $e(\mathcal{H})$. The *degree* of a vertex v in $\mathcal{H}$, denoted $d_{\mathcal{H}}(v)$, is the number of hyperedges of $\mathcal{H}$ that contain v. When there is no ambiguity regarding $\mathcal{H}$, we simply write $d(v)$ instead of $d_{\mathcal{H}}(v)$. Throughout this paper, we assume that all r-graphs are simple, i.e., no loops and no multiple edges (hyperedges).

[*]E-mail: zxm23@mails.tsinghua.edu.cn
[†]E-mail: yangzixuan@nwpu.edu.cn
[‡]E-mail: wangyich22@mails.tsinghua.edu.cn
[§]E-mail: yhbai@mail.nwpu.edu.cn
[¶]*Corresponding author.* E-mail: junpengzhou@shu.edu.cn

Given a family $\mathcal{F}$ of r-graphs, we say $\mathcal{H}$ is *$\mathcal{F}$-free* if $\mathcal{H}$ does not contain any member of $\mathcal{F}$ as a subhypergraph. The *Turán number* $\mathrm{ex}_r(n, \mathcal{F})$ of $\mathcal{F}$ is the maximum number of hyperedges in an $\mathcal{F}$-free r-graph on n vertices. When $r = 2$, we write $\mathrm{ex}(n, \mathcal{F})$ instead of $\mathrm{ex}_2(n, \mathcal{F})$.

Turán problems on graphs and hypergraphs are central topics in extremal combinatorics. Erdős and Gallai [3] determined the Turán number of paths and matchings. Let M_{s+1} denote a matching of size $s+1$, i.e., the graph consisting of $s+1$ independent edges, and K_{k+1} denote the complete graph on $k+1$ vertices. Recently, Alon and Frankl [1] initiated the study of Turán problems on graphs with bounded matching number. Specifically, they determined the exact value of $\mathrm{ex}(n, \{M_{s+1}, F\})$ for every n when F is a clique, or for n sufficiently large when F is an arbitrary color-critical graph.

Theorem 1.1 (Alon and Frankl [1]). *For $n \geq 2s+1$ and $k \geq 2$,*

$$\mathrm{ex}(n, \{M_{s+1}, K_{k+1}\}) = \max\left\{e(T(2s+1, k)), e(G(n, k, s))\right\},$$

where $T(n, k)$ is the Turán graph on n vertices, i.e., almost balanced complete k-partite graph, and $G(n, k, s)$ is the complete k-partite graph on n vertices with one part of order $n-s$ and each other part of order $\lfloor \frac{s}{k-1} \rfloor$ or $\lceil \frac{s}{k-1} \rceil$.

Later, Gerbner [4] generalized the results to the following theorem.

Theorem 1.2 (Gerbner [4]). *If $\chi(F) > 2$ and n is large enough, then $\mathrm{ex}(n, \{M_{s+1}, F\}) = \mathrm{ex}(s, \mathcal{D}(F)) + s(n-s)$, where $\mathcal{D}(F)$ is the family of graphs obtained by deleting an independent set from F.*

For two graphs H and H', let $\mathcal{N}(H', H)$ denote the number of copies of H' in H. Given a graph G and a finite family $\mathcal{F}$ of graphs, let $\mathrm{ex}(n, G, \mathcal{F})$ denote the maximum number of copies of G in an n-vertex $\mathcal{F}$-free graph. Ma and Hou [13] further extended the result of Alon and Frankl [1] as follows.

Theorem 1.3 (Ma and Hou [13]). *For $n \geq 2s+1$ and $k \geq r \geq 3$,*

$$\mathrm{ex}(n, K_r, \{M_{s+1}, K_{k+1}\}) = \max\left\{\mathcal{N}(K_r, T(2s+1, k)), \mathcal{N}(K_r, G(n, k, s))\right\}.$$

Hypergraph Turán problems are notoriously more difficult than graph versions. Two fundamental notions in hypergraph Turán problems are the expansion and the Begre hypergraph. For a given graph F, the *r-expansion* F^r of F is the r-graph obtained from F by adding $r-2$ distinct new vertices to each edge of F, such that the $(r-2)e(F)$ new vertices are distinct from each other and are not in $V(F)$. For the results on Turán problems for expansions, one can refer to Subsection 5.2.1 in [7]. Very recently, Gerbner, Tompkins and Zhou [8] considered the analogous Turán problems for hypergraphs with bounded matching number. In particular, they determined the asymptotics of $\mathrm{ex}_r(n, \{M^r_{s+1}, \mathcal{F}\})$ where $\mathcal{F}$ is an arbitrary r-graph. Furthermore, Zhou and Yuan [14] considered linear Turán problems on expansions of graphs with bounded matching number.

In 1989, Berge [2] introduced the notion of a *Berge cycle*. Győri, Katona and Lemons [9] defined the notion of *Berge paths* and generalized the Erdős–Gallai theorem to Berge paths in 2016. Later, Gerbner and Palmer [6] generalized the established notions of Berge cycle and Berge path to general graphs. For a graph F and an r-graph $\mathcal{F}$, we say $\mathcal{F}$ is a *Berge copy* of F if $V(F) \subseteq V(\mathcal{F})$ and there is a bijection $\phi : E(F) \to E(\mathcal{F})$ such that $e \subseteq \phi(e)$ for each

$e \in E(F)$. The graph F is called a *core* of $\mathcal{F}$. Observe that for a fixed graph F there are many hypergraphs that are a Berge-F. For convenience, we refer to this collection of hypergraphs as $\mathcal{B}F$. Turán problems on Berge hypergraphs have been extensively studied, yielding numerous related results. For a short survey on Turán problems on Berge hypergraphs, one can refer to Subsection 5.2.2 in [7]. In 2022, Khormali and Palmer [11] completely determined the Turán number of Berge matchings for sufficiently large n.

Theorem 1.4 (Khormali and Palmer [11]). *Fix integers $s \geq 1$ and $r \geq 2$. Then for n large enough,*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1}) = \begin{cases} s, & \text{if } r \geq 2s+1; \\ \binom{2s+1}{r}, & \text{if } s+1 < r < 2s+1; \\ n-s, & \text{if } r = s+1; \\ \binom{s}{r-1}(n-s) + \binom{s}{r}, & \text{if } r \leq s. \end{cases}$$

Note that if $n \leq 2s+1$, it is trivial that $\mathrm{ex}_r(n, \mathcal{B}M_{s+1}) = \binom{n}{r}$. For $n \geq 2s+2$ and $r \geq 3$, Kang, Ni and Shan [10] determined the exact value of $\mathrm{ex}_r(n, \mathcal{B}M_{s+1})$ in the case when $r \leq s-1$ or $r \geq 2s+2$. However, the case $s \leq r \leq 2s+1$ remains open.

Theorem 1.5 (Kang, Ni and Shan [10]). *Fix integers $s \geq 1$ and $r \geq 3$. For any $n \geq 2s+2$.*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1}) = \begin{cases} \max\{\binom{2s+1}{r}, \binom{s}{r-1}(n-s) + \binom{s}{r}\}, & \text{if } r \leq s-1; \\ s & \text{if } r \geq 2s+2. \end{cases}$$

The purpose of this paper is to settle the remaining cases of the Turán number of Berge matchings and to investigate Turán problems for families of Berge hypergraphs that include Berge matchings, providing both exact and general results.

2. Main results

Our first result is to determine the exact value of $\mathrm{ex}_r(n, \mathcal{B}M_{s+1})$ for all n in the case $s \leq r \leq 2s+1$ in Theorem 2.1, which will be proved in Section 3. For positive integers a and b, define $\binom{a}{b} = 0$ if $a < b$.

Theorem 2.1. *Let integers $s \geq 1$ and $r \geq 2$. For all $n \geq 2s+2$,*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1}) = \begin{cases} \max\left\{\binom{2s+1}{r}, \binom{s}{r-1}(n-s) + \binom{s}{r}\right\}, & \text{if } r \leq s+1; \\ \binom{2s+1}{r}, & \text{if } s+2 \leq r \leq 2s; \\ s, & \text{if } r \geq 2s+1. \end{cases}$$

Moreover, we provide a formula for the Turán number of Berge matching together with a single r-graph $\mathcal{F}$, namely $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \{\mathcal{F}\})$, and determine the asymptotics of the Turán number of Berge matchings together with Berge-G where G is a bipartite graph, namely $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}G)$. Finally, we extend two general lemmas for Turán problems on Berge hypergraphs from [5], and present some applications.

2.1. Berge matchings and a single r-graph

We first introduce some definitions from [8]. A *proper k-coloring* of $\mathcal{H}$ is a mapping from $V(\mathcal{H})$ to a set of k colors such that no hyperedge is monochromatic. The *chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ is the minimum colors needed for proper coloring $\mathcal{H}$.

If $\chi(\mathcal{H}) = 2$, then we take the colors of a proper 2-coloring of $\mathcal{H}$ to be red and blue. In the following, when we say *red-blue coloring*, we always mean a proper 2-coloring. Let $p(\mathcal{H})$ denote the minimum number of red vertices in a red-blue coloring of $\mathcal{H}$. We say that a red-blue coloring of $\mathcal{H}$ is *strong* if each hyperedge contains exactly one red vertex. We remark that the set of red vertices in a strong red-blue coloring is also called a *crosscut* in the literature. Let $q(\mathcal{H})$ denote the minimum number of red vertices in a strong red-blue coloring of $\mathcal{H}$, with $q(\mathcal{H}) = \infty$ if $\mathcal{H}$ does not have a strong red-blue coloring. Clearly, $p(\mathcal{H}) \le q(\mathcal{H})$.

A *strongly independent set* of a hypergraph $\mathcal{H}$ is a subset of $V(\mathcal{H})$ that intersects every hyperedge in at most one vertex. Note that in a strong red-blue coloring, the set of red vertices is a strongly independent set. Our main result in this subsection is as follows.

Theorem 2.2. *Let integers $2 \le r \le s+1$ and n be sufficiently large. Suppose $\mathcal{F}$ is an r-graph with $\chi(\mathcal{F}) > 2$, or $\chi(\mathcal{F}) = 2$ and $q(\mathcal{F}) = \infty$. Then*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \{\mathcal{F}\}) = \mathrm{ex}_r(s, \mathcal{D}(\mathcal{F})) + \binom{s}{r-1}(n-s),$$

where $\mathcal{D}(\mathcal{F})$ is the family of r-graphs obtained by deleting a strongly independent set from $\mathcal{F}$.

Remark 2.3. *Note that in the above theorem we assume that $r \le s+1$. If $r \ge s+2$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \{\mathcal{F}\}) \le \mathrm{ex}_r(n, \mathcal{B}M_{s+1}) \le \binom{2s+1}{r}$ by Theorem 2.1. This implies that $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \{\mathcal{F}\})$ is a constant which does not depend on n when $r \ge s+2$.*

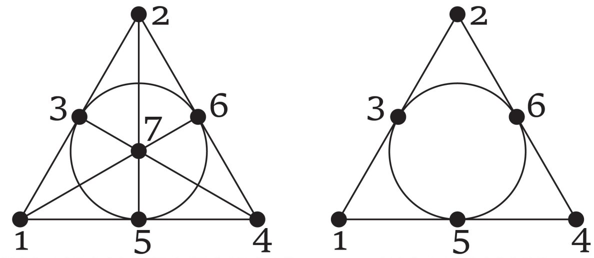
Theorem 2.2 can be viewed as a Berge-version extension of Theorem 1.2. We first provide two applications of Theorem 2.2. For $k > r$, let $\mathcal{K}_k^{(r)}$ denote the complete r-graph on k vertices. We color the vertices of $\mathcal{K}_k^{(r)}$. It is easy to see that the number of vertices of each color is at most $r-1$. Otherwise, a monochromatic hyperedge would exist. Then $\chi(\mathcal{K}_k^{(r)}) = \lceil \frac{k}{r-1} \rceil$. When $k > 2(r-1)$, we have $\chi(\mathcal{K}_k^{(r)}) \ge 3$. When $k \le 2(r-1)$, we claim that $q(\mathcal{K}_k^{(r)}) = \infty$. Indeed, if $q(\mathcal{K}_k^{(r)}) < \infty$, then $q(\mathcal{K}_k^{(r)}) = 1$ as any two vertices in $\mathcal{K}_k^{(r)}$ are contained in some hyperedge. However, after deleting one red vertex, there are still at least r blue vertices, which implies the existence of a monochromatic hyperedge, a contradiction. Thus, by Theorem 2.2, we derive the following corollary.

Corollary 2.4. *Let integers $2 \le r \le s+1$, $k \ge r+1$ and n be sufficiently large. Then*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \{\mathcal{K}_k^{(r)}\}) = \mathrm{ex}_r(s, \mathcal{K}_{k-1}^{(r)}) + \binom{s}{r-1}(n-s).$$

Observe that if $k > s+1$, then $\mathrm{ex}_r(s, \mathcal{K}_{k-1}^{(r)}) = \binom{s}{r}$ and $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \{\mathcal{K}_k^{(r)}\}) = \binom{s}{r} + \binom{s}{r-1}(n-s)$.

The other application is about the Fano plane. Let $\mathcal{F}_7$ denote the Fano plane. Since any two vertices in $\mathcal{F}_7$ are contained in some hyperedge, any single vertex in $\mathcal{F}_7$ forms a maximum strongly independent set. By deleting any vertex from $\mathcal{F}_7$, we obtain the 3-graph $\mathcal{F}_6$ with vertex set $\{1, 2, \ldots, 6\}$ and hyperedge set $\{123, 145, 246, 356\}$. See Figure 1 for an illustration of $\mathcal{F}_7$ and $\mathcal{F}_6$. Note that $\chi(\mathcal{F}_7) = 3$. Therefore, Theorem 2.2 implies the following result.



{123,145,246,257,167,347,356} {123,145,246,356}

$\mathcal{F}_7$ $\mathcal{F}_6 = \mathcal{D}(\mathcal{F}_7)$

Figure 1: The Fano plane $\mathcal{F}_7$ and the hypergraph $\mathcal{F}_6$.

Corollary 2.5. *Let integers $s \geq 2$ and n be sufficiently large. Then*

$$\mathrm{ex}_3(n, \mathcal{B}M_{s+1} \cup \{\mathcal{F}_7\}) = \mathrm{ex}_3(s, \mathcal{F}_6) + \binom{s}{2}(n-s).$$

The proof of Theorem 2.2 will be presented in Section 4.

2.2. Berge matchings and Berge bipartite graphs

Now let us consider forbidding Berge matchings together with Berge-G where G is a bipartite graph. We have the following theorem.

Theorem 2.6. *Let s and $r \geq 2$ be fixed integers. Let G be a bipartite graph with $p = p(G)$ and n sufficiently large.*

(i) *If $r \leq p \leq s$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}G) = \binom{p-1}{r-1}n + O(1)$.*

(ii) *If $p \leq s < r-1$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}G) \leq \max\left\{s, \binom{2s+1}{r}\right\}$.*

(iii) *If $p > s$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}G) = \mathrm{ex}_r(n, \mathcal{B}M_{s+1})$.*

In the above theorem, the case $p \leq r \leq s+1$ remains unresolved. For the complete bipartite graph $K_{p,q}$ with $p \leq q$, we obtain the asymptotics for $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q})$ in that case.

The 2-*shadow* $\partial\mathcal{H}$ of a hypergraph $\mathcal{H}$ is the graph consisting of all 2-sets contained in some hyperedge of $\mathcal{H}$. For $r \geq 3$, let $\mathcal{G}_s^{r-1}$ be an s-vertex $(r-1)$-graph with the maximum number of hyperedges such that $\partial\mathcal{G}_s^{r-1}$ is $K_{p,q}$-free. Note that when $r = p+q \leq s$, we have $e(\mathcal{G}_s^{r-1}) \geq 1$, since the 2-shadow of a single $(r-1)$-edge is $K_{p,q}$-free.

Theorem 2.7. *Fix integers s and $r \geq 2$. Let n be sufficiently large.*

(i) *If $p \leq r \leq \min\{s, p+q-1\}$, then*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) = (p-1)n + O(1).$$

(ii) *If $r = p+q \leq s$, then*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) = \min\{p-1, e(\mathcal{G}_s^{r-1})\} \cdot n + O(1).$$

(iii) *If $p+q < r \leq s$, then*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) \leq s + \binom{2s}{r} + \binom{2s}{r-1}(pq-1).$$

For the case $p \leq r = s+1$, we have the following result.

Proposition 2.8. *Let integers $r \geq 2$ and n be sufficiently large.*
(i) *If $p = r = s+1$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) = \mathrm{ex}_r(n, \mathcal{B}M_{s+1})$.*
(ii) *If $p < r = s+1 \leq p+q$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) = n + O(1)$.*
(iii) *If $p+q < r = s+1$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) \leq s + \binom{2s}{r} + \binom{2s}{r-1}(pq-1)$.*

The proofs of Theorems 2.6 and 2.7, and Proposition 2.8 will be presented in Section 5.

2.3. Finite families of Berge hypergraphs

In this subsection, we use a slightly different notation. In the Section 2.2, we consider the Turán problems on two Berge families. It is natural to ask what happens for more general families of Berge hypergraphs. Let $\mathcal{F}$ be a finite family of graphs and $\mathcal{BF} := \bigcup_{F \in \mathcal{F}} \mathcal{B}F$. Now we consider Turán problems on $\mathcal{BF}$.

Given a graph G, a *red-blue graph* $G_{\text{red-blue}}$ is formed by coloring the edges of G with two distinct colors, namely red and blue. For a red-blue graph $G_{\text{red-blue}}$, let G_{red} denote the subgraph spanned by the red edges of $G_{\text{red-blue}}$, and let G_{blue} denote the subgraph spanned by the blue edges of $G_{\text{red-blue}}$. Set $\varepsilon(r, G_{\text{red-blue}}) := e(G_{\text{red}}) + \mathcal{N}(K_r, G_{\text{blue}})$.

Inspired by the results of Gerbner, Methuku and Palmer in [5], we present the following more generalized results.

Lemma 2.9. *Let $\mathcal{F}$ be a finite family of graphs. Then*

$$\mathrm{ex}_r(n, \mathcal{BF}) \leq \max\Big\{\varepsilon(r, G_{\text{red-blue}}) \,\Big|\, G \text{ is an } n\text{-vertex } \mathcal{F}\text{-free graph}\Big\}.$$

Let

$$\mathcal{F}^- = \Big\{F^- \,\Big|\, F^- \text{ is a graph obtained by deleting a vertex from } F,\ F \in \mathcal{F}\Big\}.$$

By Lemma 2.9, we obtain a general upper bound for the Turán number of $\mathcal{BF}$.

Theorem 2.10. *Let $\mathcal{F}$ be a finite family of graphs and $f := f(n)$ such that $\mathrm{ex}(n, K_{r-1}, \mathcal{F}^-) \leq fn$ for every n, then*

$$\mathrm{ex}_r(n, \mathcal{BF}) \leq \max\Big\{\frac{2f}{r}, 1\Big\} \cdot \mathrm{ex}(n, \mathcal{F}).$$

Next, we present an application of Theorem 2.10. Let $\mathcal{F} = \big\{M_{s+1}, K_{k+1}\big\}$ and $\mathcal{F}^- = \big\{M_s \cup \{v\}, K_k\big\}$. For a matching M_s and a vertex $v \notin V(M_s)$, it is easy to see that if $n \geq 2s+1$, then a n-vertex graph is $M_s \cup \{v\}$-free if and only if it is M_s-free. Thus, $\mathrm{ex}(n, K_{r-1}, \{M_s \cup \{v\}, K_k\}) = \mathrm{ex}(n, K_{r-1}, \{M_s, K_k\})$ for $n \geq 2s+1$. By Theorems 1.1, 1.3 and 2.10, we obtain the following corollary.

Corollary 2.11. *Let $n \geq 2s+1$ and $k \geq r \geq 3$. Then*

$$\mathrm{ex}_r(n, \mathcal{B}\{M_{s+1}, K_{k+1}\}) \leq Z(n, r, s, k) \cdot \max\big\{e(T(2s+1, k)), e(G(n, k, s))\big\},$$

where $Z(n, r, s, k) := \max\Big\{\frac{2\mathcal{N}(K_{r-1}, T(2s-1, k-1))}{rn}, \frac{2\mathcal{N}(K_{r-1}, G(n, k-1, s-1))}{rn}, 1\Big\}$.

Note that if $n \leq 2s+1$, then $\mathrm{ex}_r(n, \mathcal{B}\{M_{s+1}, K_{k+1}\}) = \mathrm{ex}_r(n, \mathcal{B}K_{k+1})$. If $k \geq 2s+1$, then $M_{s+1} \subseteq K_{k+1}$. This implies that $\mathrm{ex}_r(n, \mathcal{B}\{M_{s+1}, K_{k+1}\}) = \mathrm{ex}_r(n, \mathcal{B}M_{s+1})$ for $k \geq 2s+1$.

The proof of Lemma 2.9 and Theorem 2.10 will be presented in Section 6.

3. Proof of Theorem 2.1

Given an r-graph $\mathcal{H}$, the *neighborhood* $N_{\mathcal{H}}(v)$ of a vertex v in $\mathcal{H}$ is the set of all vertices adjacent to v in $\mathcal{H}$. For a set $S \subseteq V(\mathcal{H})$, the *neighborhood* of S is defined as $N_{\mathcal{H}}(S) = \bigcup_{v \in S} N_{\mathcal{H}}(v) \setminus S$. Recall that $d_{\mathcal{H}}(v)$ denote the *degree* of a vertex v in $\mathcal{H}$. Let $\mathcal{H}[U]$ denote the subhypergraph of $\mathcal{H}$ induced by $U \subseteq V(\mathcal{H})$. We say that a Berge matching $\mathcal{M}$ in $\mathcal{H}$ is *maximum* if $\mathcal{H}$ contains no Berge matching with a core larger than that of $\mathcal{M}$. When referring to $\mathcal{H}$ contains a $\mathcal{B}M_{s+1}$, we mean $\mathcal{H}$ contains a Berge copy of M_{s+1}.

Now let us start with the proof of Theorem 2.1 that we restate here for convenience.

Theorem. *Let integers $s \geq 1$ and $r \geq 2$. For all $n \geq 2s+2$,*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1}) = \begin{cases} \max\left\{ \binom{2s+1}{r}, \binom{s}{r-1}(n-s) + \binom{s}{r} \right\}, & \text{if } r \leq s+1; \\ \binom{2s+1}{r}, & \text{if } s+2 \leq r \leq 2s; \\ s, & \text{if } r \geq 2s+1. \end{cases}$$

Note that for the cases $r \leq s-1$ and $r \geq 2s+2$ in Theorem 2.1, the result holds by Theorem 1.5, and the case $r = 2$ for Theorem 2.1 follows from the result of Erdős and Gallai [3]. Thus, in the following, we may assume $s \leq r \leq 2s+1$ and $r \geq 3$.

Proof. Let us first consider lower bound. For $r \leq s+1$, we consider the following two r-graphs: one is an r-graph formed by the union of a complete r-graph of order $2s+1$ and $n-2s-1$ isolated vertices (i.e., vertices of degree 0); the other is an n-vertex r-graph where there exists a vertex subset S of size s such that every hyperedge intersects S in at least $r-1$ vertices. For $s+2 \leq r \leq 2s$, the lower bound follows from the union of a complete r-graph of order $2s+1$ and $n-2s-1$ isolated vertices. For $r \geq 2s+1$, the lower bound follows from any r-graph with s hyperedges. It is easy to verify that these hypergraphs are $\mathcal{B}M_{s+1}$-free.

In the following, we prove the upper bound. Let $\mathcal{H}$ be a $\mathcal{B}M_{s+1}$-free r-graph on n vertices with maximum number of hyperedges. Note that adding a new hyperedge to $\mathcal{H}$ increases the size of the core of a maximum Berge matching by at most 1. Thus, by the maximality of $e(\mathcal{H})$, we may assume that $\mathcal{H}$ contains a $\mathcal{B}M_s$, denoted by $\mathcal{M}$. Let $\mathcal{E} = \{e_1, \ldots, e_s\}$ be the hyperedges of $\mathcal{M}$, with the core $\mathcal{C} = \{u_1, v_1, \ldots, u_s, v_s\}$ such that $\{u_i, v_i\} \subseteq e_i$ for each $i \in [s]$. For a subset $S \subseteq \mathcal{C}$, we write $\overline{S} = \{u_i \mid v_i \in S\} \cup \{v_i \mid u_i \in S\}$.

Let $\mathcal{H}'$ be the r-graph obtained from $\mathcal{H}$ by deleting s hyperedges of $\mathcal{E}$. Let $E_{\mathcal{H}'}(v)$ denote the set of hyperedges in $\mathcal{H}'$ that contain v. Observe that each hyperedge of $\mathcal{H}'$ intersects $\mathcal{C}$ in at least $r-1$ vertices. Indeed, if there exists a hypergraph $e \in E(\mathcal{H}')$ such that $e \setminus \mathcal{C} = \{u, v\}$, then $\mathcal{E} \cup \{e\}$ forms a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ with the core $\mathcal{C} \cup \{u, v\}$, a contradiction. This implies that $N_{\mathcal{H}'}(v) \subseteq \mathcal{C}$ for any $v \in V(\mathcal{H}) \setminus \mathcal{C}$.

Claim 1. *If $s \leq r \leq 2s$, then either $e(\mathcal{H}) \leq \binom{2s+1}{r}$, or there are at least two vertices $u, v \in V(\mathcal{H}) \setminus \mathcal{C}$ such that $d_{\mathcal{H}'}(u) \geq 1$ and $d_{\mathcal{H}'}(v) \geq 1$.*

Proof of Claim 1. It is sufficient to prove that if there is at most one vertex $u \in V(\mathcal{H}) \setminus \mathcal{C}$ with $d_{\mathcal{H}'}(u) \geq 1$, then $e(\mathcal{H}) \leq \binom{2s+1}{r}$. If $d_{\mathcal{H}'}(v) = 0$ for every $v \in V(\mathcal{H}) \setminus \mathcal{C}$, then $e(\mathcal{H}) \leq s + \binom{2s}{r} \leq \binom{2s+1}{r}$ as $r \leq 2s$.

Let $v \in V(\mathcal{H}) \setminus \mathcal{C}$ be the unique vertex with $d_{\mathcal{H}'}(v) \geq 1$. If $e_i \subseteq \mathcal{C} \cup \{v\}$ for every $i \in [s]$, then $e(\mathcal{H}) \leq \binom{2s+1}{r}$ and we are done. Now suppose that there exists some $i \in [s]$ such that

$e_i \nsubseteq \mathcal{C} \cup \{v\}$. Without loss of generality, suppose e_s contains a vertex $w \notin \mathcal{C} \cup \{v\}$. Note that $|\mathcal{C}| = 2s$. If $r = 2s$, then we have $u_s \in N_{\mathcal{H}'}(v)$ or $v_s \in N_{\mathcal{H}'}(v)$. Thus, there exists a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ by adding the hyperedge containing v and u_s (or v_s) to $\mathcal{E}$, and extending the core to $\mathcal{C} \cup \{v, w\}$, a contradiction. Now we asume that $s \le r < 2s$. We claim that $u_s \notin N_{\mathcal{H}'}(v)$ and $v_s \notin N_{\mathcal{H}'}(v)$. Indeed, if $u_s \in N_{\mathcal{H}'}(v)$ (or $v_s \in N_{\mathcal{H}'}(v)$), then as in the case $r = 2s$ we may find a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$, a contradiction.

Then $|E_{\mathcal{H}'}(v)| \le \binom{|N_{\mathcal{H}'}(v)|}{r-1}$. Note that $E(\mathcal{H}'[\mathcal{C}]) = E(\mathcal{H}') \setminus E_{\mathcal{H}'}(v)$. We now partition $E(\mathcal{H}'[\mathcal{C}])$ into the following three classes: $\mathcal{E}_1$ (consisting of hyperedges not containing $\overline{N_{\mathcal{H}'}(v)}$), $\mathcal{E}_2$ (consisting of hyperedges not containing $\{u_s, v_s\}$), and $\mathcal{E}_3$ (consisting of hyperedges containing both a vertex of $\{u_s, v_s\}$ and a vertex of $\overline{N_{\mathcal{H}'}(v)}$). We assert that $\mathcal{E}_3 = \emptyset$. Otherwise, assume that $e \in \mathcal{E}_3$ contains the vertex u_s and $u_i \in \overline{N_{\mathcal{H}'}(v)}$ (or $v_i \in \overline{N_{\mathcal{H}'}(v)}$) for some $i \in [s-1]$. By the definition of $\overline{N_{\mathcal{H}'}(v)}$, there is a hyperedge $e' \in E(\mathcal{H}')$ containing v and v_i (or u_i). Then $(\mathcal{E} \setminus \{e_i\}) \cup \{e', e\}$ forms a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ with the core $\mathcal{C} \cup \{v, w\}$, a contradiction. Note that $r - 1 \le |N_{\mathcal{H}'}(v)| \le 2s - 2$. Therefore, we have

$$e(\mathcal{H}) = s + |E_{\mathcal{H}'}(v)| + |\mathcal{E}_1| + |\mathcal{E}_2| \le s + \binom{|N_{\mathcal{H}'}(v)|}{r-1} + \binom{2s - |N_{\mathcal{H}'}(v)|}{r} + \binom{2s-2}{r} \le \binom{2s+1}{r}.$$

□

3.1. Proof of Theorem 2.1 for $r = s$

Claim 2. *If $d_{\mathcal{H}'}(v) \le 1$ for all $v \in V(\mathcal{H}) \setminus \mathcal{C}$, then $e(\mathcal{H}) \le \max\left\{\binom{2s+1}{s}, s(n-s)+1\right\}$.*

Proof of Claim 2. Note that $e(\mathcal{H}'[\mathcal{C}]) \le \binom{2s}{r} = \binom{2s}{s}$. Since $d_{\mathcal{H}'}(v) \le 1$ for all $v \in V(\mathcal{H}) \setminus \mathcal{C}$,

$$e(\mathcal{H}) = s + e(\mathcal{H}'[\mathcal{C}]) + \sum_{v \in V(\mathcal{H}) \setminus \mathcal{C}} d_{\mathcal{H}'}(v) \le s + \binom{2s}{s} + n - 2s = \binom{2s}{s} + n - s.$$

If $e(\mathcal{H}) > \max\left\{\binom{2s+1}{s}, s(n-s)+1\right\}$, then we have $\binom{2s}{s} + n - s \ge \binom{2s+1}{s} + 1$ and $\binom{2s}{s} + n - s \ge s(n-s)+2$. This implies that $\binom{2s}{s} \ge (s-1)(n-s)+2 \ge (s-1)\left[\binom{2s}{s-1}+1\right]+2 = (s-1)\binom{2s}{s-1}+s+1$, which is a contradiction when $s \ge 2$. □

Case 1. There exists $w \in V(\mathcal{H}) \setminus \mathcal{C}$ such that $\{u_i, v_i\} \subseteq N_{\mathcal{H}'}(w)$ for some $i \in [s]$.

We choose a such vertex w such that $d_{\mathcal{H}'}(w)$ achieves minimum. Without loss of generality, assume that $\{u_1, v_1\} \subseteq N_{\mathcal{H}'}(w)$. Then we have $\{u_1, v_1\} \subseteq e_w \in E_{\mathcal{H}'}(w)$, or $u_1 \in e_w^1$ and $v_1 \in e_w^2$, where $e_w^1, e_w^2 \in E_{\mathcal{H}'}(w)$ and $e_w^1 \ne e_w^2$. Let

$$Y = N_{\mathcal{H}'}\big(V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})\big).$$

Clearly, $Y \subseteq \mathcal{C}$. We claim that $u_1, v_1 \notin Y$. Otherwise, there exists a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ with the core $\mathcal{C} \cup \{w, z\}$, where $z \in V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$ and $u_1 \in N_{\mathcal{H}'}(z)$ (or $v_1 \in N_{\mathcal{H}'}(z)$).

By Claim 1, we may suppose that there exists at least one vertex in $V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$ such that its degree in $\mathcal{H}'$ is at least 1. Otherwise, $e(\mathcal{H}) \le \binom{2s+1}{r}$ and we are done. Thus, $|Y| \ge r - 1 = s - 1$.

Case 1.1. $|Y| = s - 1$.

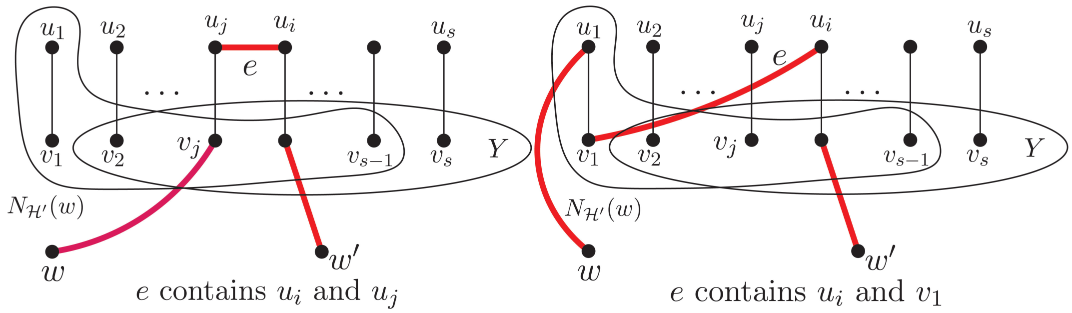


Figure 2: The illustration of $\mathcal{B}M_{s+1}$ by the thick red lines. The left figure shows when a hyperedge $e \in \mathcal{E}_3$ contains u_i, u_j (the figure shows the case $2 \le i, j \le s-1$, the case when $i \in \{1, s\}$ holds similarly). The right figure shows when a hyperedge $e \in \mathcal{E}_3$ contains u_i, v_1 ($2 \le i \le s$). The hyperedge connecting w' comes from the definition of Y.

It is easy to see that $d_{\mathcal{H}'}(v) \le 1$ for any $v \in V(\mathcal{H}) \setminus \{\mathcal{C} \cup w\}$. By Claim 2, we may suppose $d_{\mathcal{H}'}(w) \ge 2$. Otherwise, $e(\mathcal{H}) \le \max\left\{\binom{2s+1}{s}, s(n-s)+1\right\}$ and we are done. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the set of hyperedges in $\mathcal{H}'$ containing a vertex in $V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$, and the vertex w, respectively. Note that $\mathcal{E}_2 = E_{\mathcal{H}'}(w)$. Let $\Omega = \{v_1, \ldots, v_s\}$ and $\mathcal{E}_3 = E(\mathcal{H}'[\mathcal{C}]) \backslash E(\mathcal{H}'[\Omega])$. Then

$$E(\mathcal{H}) = \mathcal{E} \cup E(\mathcal{H}'[\Omega]) \cup \bigcup_{i=1}^{3} \mathcal{E}_i.$$

Clearly, $e(\mathcal{H}'[\Omega]) \le 1$. Note that $d_{\mathcal{H}'}(v) \le 1$ for any $v \in V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$. So $|\mathcal{E}_1| \le n - 2s - 1$.

Recall that $u_1, v_1 \notin Y$. We claim that $\{u_i, v_i\} \nsubseteq Y$ for all $2 \le i \le s$. Otherwise, there exists some j such that $\{u_j, v_j\} \subseteq Y$. Then $V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$ contains a vertex v such that $\{u_j, v_j\} \subseteq N_{\mathcal{H}'}(v)$ and $d_{\mathcal{H}'}(v) \le 1$, which contradicts the minimality of $d_{\mathcal{H}'}(w)$. Without loss of generality, we assume that $Y = \{v_2, \ldots, v_s\}$. We claim that $N_{\mathcal{H}'}(w) \subseteq \Omega \cup \{u_1\}$. Indeed, if $u_i \in N_{\mathcal{H}'}(w)$ for some $2 \le i \le s$, then there exists a $\mathcal{B}M_{s+1}$, obtained by adding to $\mathcal{E} \setminus \{e_i\}$ the hyperedge containing wu_i and hyperedge containing v_iw' for some $w' \in V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$ by the definition of Y. So $|\mathcal{E}_2| \le \binom{s+1}{s-1} = \binom{s+1}{2}$.

Since $d_{\mathcal{H}'}(w) \ge 2$, we have $|N_{\mathcal{H}'}(w)| \ge r = s$. Recall that $\{u_1, v_1\} \subseteq N_{\mathcal{H}'}(w)$. Without loss of generality, we may assume that $\{u_1, v_1, \ldots, v_{s-1}\} \subseteq N_{\mathcal{H}'}(w)$. For any hyperedge $e \in E(\mathcal{H}'[\mathcal{C}])$ containing u_1, e does not contain any u_j for $2 \le j \le s$. Otherwise, by the definition of Y, we may find a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ whose core has edge set $\{wv_1, u_1u_j, v_jw', u_2v_2, \ldots, u_sv_s\} \backslash \{u_jv_j\}$. For any hyperedge $e \in E(\mathcal{H}'[\mathcal{C}])$ containing u_i with $2 \le i \le s$, e does not contain any other u_j ($1 \le j \le s$ and $j \ne i$) or v_1. Otherwise, by the definition of Y, we may find a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ (see Figure 2). It means that $|\mathcal{E}_3| \le \binom{s}{r-1} + s - 1 = 2s - 1$.

Therefore, we have

$$\begin{aligned} e(\mathcal{H}) \le |\mathcal{E}| + e(\mathcal{H}'[\Omega]) + \sum_{i=1}^{3} |\mathcal{E}_i| &\le s + 1 + (n - 2s - 1) + \binom{s+1}{2} + 2s - 1 \\ &= n + s - 1 + \binom{s+1}{2} \\ &\le \max\left\{\binom{2s+1}{s}, s(n-s)+1\right\}, \end{aligned}$$

where the last inequality can be proved by a method similar to that of Claim 2 when $s \geq 3$.

Case 1.2. $|Y| \geq s$.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the set of hyperedges in $\mathcal{H}'$ containing the vertex w, and a vertex in $V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$, respectively. Let $\mathcal{E}_3 = E(\mathcal{H}'[\mathcal{C}])$. Then

$$E(\mathcal{H}) = \mathcal{E} \cup \bigcup_{i=1}^{3} \mathcal{E}_i.$$

Recall that $\{u_1, v_1\} \subseteq N_{\mathcal{H}'}(w)$. We claim that $e_1 \subseteq \mathcal{C} \cup \{w\}$. Otherwise, we may find a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ with the core $\mathcal{C} \cup \{w, z\}$, where $z \in e_1 \setminus (\mathcal{C} \cup \{w\})$. So we can choose a set X of s vertices from $(e_1 \cup e_w) \cap \mathcal{C}$ or $(e_1 \cup e_w^1 \cup e_w^2) \cap \mathcal{C}$ such that $\{u_1, v_1\} \subseteq X$.

Recall that $u_1, v_1 \notin Y$. It follows that $u_1, v_1 \notin \overline{Y}$ and $|\overline{X} \cap \overline{Y}| \leq s - 2$. We claim that $\overline{X} \cap \overline{Y} = \emptyset$. Otherwise, assume that $u_j \in \overline{X} \cap \overline{Y}$ (or $v_j \in \overline{X} \cap \overline{Y}$) for some $2 \leq j \leq s$. Then for the hyperedge $e_j \in \mathcal{E}$, apart from u_j (or v_j), it does not contain vertices of $V(\mathcal{H}) \setminus \mathcal{C}$, vertices in $\overline{X}$, and vertices in $\overline{Y}$. Otherwise, we may obtain a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ (see Figure 3 for illustration), a contradiction. Thus, we have

$$s - 1 = |e_j| - 1 \leq |\mathcal{C} \setminus (\overline{X} \cup \overline{Y})| = 2s - |X| - |Y| + |\overline{X} \cap \overline{Y}| \leq s - |Y| + s - 2.$$

It yields $|Y| \leq s - 1$, which contradicts $|Y| \geq s$. Thus, we have $\overline{X} \cap \overline{Y} = \emptyset$.

Combining $|X| = |\overline{X}| = s$, $|Y| = |\overline{Y}| \geq s$, $|\overline{X} \cup \overline{Y}| \leq |\mathcal{C}| = 2s$, and $|\overline{X} \cap \overline{Y}| = 0$, we have $|Y| = |\overline{Y}| = s$ and $|X \cap Y| = 0$. We claim that $X \cap \overline{Y} = \emptyset$. Otherwise, there exists a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$. So we have $X = \overline{X}$ and $Y = \overline{Y}$. Thus, all hyperedges of $\mathcal{E}_1$ are contained in $\{w\} \cup X$ (otherwise, there exists a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ since $Y = \overline{Y}$). Then $|\mathcal{E}_1| \leq \binom{|X|}{s-1} = \binom{s}{s-1}$. Furthermore, if $e \in \mathcal{E}_3$ contains a vertex from X, then $e = X$, and if $e \in \mathcal{E}_3$ contains a vertex from Y, then $e = Y$ (otherwise, there exists a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$). Then $|\mathcal{E}_3| \leq 2$. By the definition of Y, we have $|\mathcal{E}_2| \leq \binom{|Y|}{s-1}(n - 2s - 1) = \binom{s}{s-1}(n - 2s - 1)$. Therefore,

$$\begin{aligned}
e(\mathcal{H}) &= |\mathcal{E}| + \sum_{i=1}^{3} |\mathcal{E}_i| \\
&\leq s + \binom{s}{s-1} + \binom{s}{s-1}(n - 2s - 1) + 2 \\
&= s(n - s) + 1 - (s^2 - s - 1) \leq s(n - s) + 1.
\end{aligned}$$

Case 2. For any $v \in V(\mathcal{H}) \setminus \mathcal{C}$, $\{u_i, v_i\} \nsubseteq N_{\mathcal{H}'}(v)$ for each $i \in [s]$.

By Claim 2, we may suppose $d_{\mathcal{H}'}(w) \geq 2$ for some $w \in V(\mathcal{H}) \setminus \mathcal{C}$. Then we have that $N_{\mathcal{H}'}(w) \subseteq \mathcal{C}$ and $|N_{\mathcal{H}'}(w)| \geq r = s$. Note that $\{u_i, v_i\} \nsubseteq N_{\mathcal{H}'}(w)$ for every $i \in [s]$. Hence, without loss of generality, we may assume that $N_{\mathcal{H}'}(w) = \Omega$.

By Claim 1, we may suppose that there exists at least one vertex in $V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$ such that its degree in $\mathcal{H}'$ is at least 1. Otherwise, $e(\mathcal{H}) \leq \binom{2s+1}{r}$ and we are done. It follows that $N_{\mathcal{H}'}(w') \subseteq \Omega$ for any $w' \in V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$. Otherwise, there exists a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ with the core $\mathcal{C} \cup \{w, w'\}$.

Case 2.1. There exists a set $W \subseteq V(\mathcal{H}) \setminus \mathcal{C}$ such that each v_i $(i \in [s])$ is contained in at least two members of the family $\{N_{\mathcal{H}'}(v) \mid v \in W\}$.

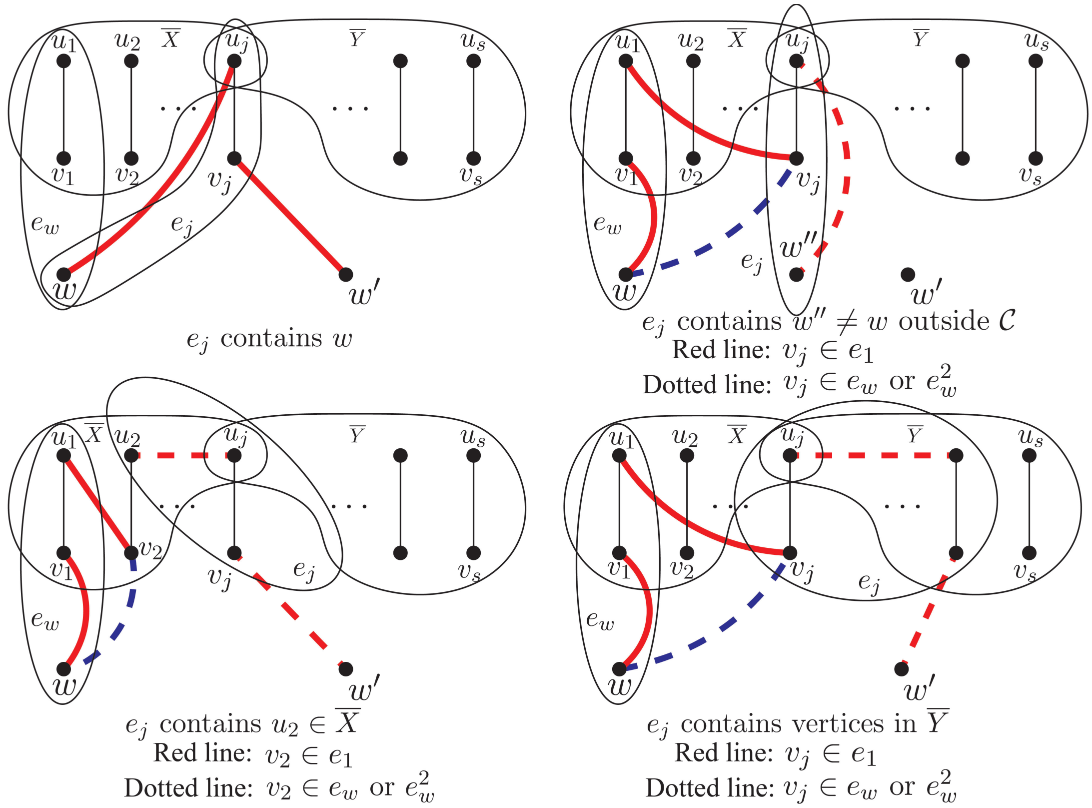


Figure 3: The four cases: e_j contains w, $w'' \neq w$, a vertex in $\overline{X}$, or a vertex in $\overline{Y}$. In each case, we can find a $\mathcal{B}M_{s+1}$ using either red lines or dotted lines. In the picture, we only list some typical cases. For example, when e_j contains $u_2 \in \overline{X}$, we only consider $u_2 \in e_j \cap \overline{X}$. However, other cases hold similarly.

It follows that for each $e \in E(\mathcal{H})$, the hyperedge e can not contain both u_i and u_j for any $i \neq j$. Otherwise, we would get a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ by connecting $u_i u_j$ with the edge, and connecting u_i, u_j to two different elements in W by definition. Then every hyperedge $e \in E(\mathcal{H})$ intersects Ω in at least $s-1$ vertices. Therefore, we have

$$e(\mathcal{H}) \leq 1 + \binom{s}{s-1}(n-s) = 1 + s(n-s).$$

Case 2.2. There does not exist $W \subseteq V(\mathcal{H}) \setminus \mathcal{C}$ such that each v_i ($i \in [s]$) is contained in at least two members of the family $\{N_{\mathcal{H}'}(v) \mid v \in W\}$.

Note that $N_{\mathcal{H}'}(w) = \Omega$ and $N_{\mathcal{H}'}(w') \subseteq \Omega$ for any $w' \in V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$. It follows that for every $w' \in V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$, we have $d_{\mathcal{H}'}(w') \leq 1$. Otherwise, there exists a set $W = \{w, w'\}$ that satisfies Case 2.1, a contradiction. So, by Claim 1, there exists one vertex $w' \in V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$ such that $d_{\mathcal{H}'}(w') = 1$. Similarly, if $w'_1, w'_2 \in V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$ satisfies $d_{\mathcal{H}'}(w'_1) = d_{\mathcal{H}'}(w'_2) = 1$, then $N_{\mathcal{H}'}(w'_1) = N_{\mathcal{H}'}(w'_2)$. Without loss of generality, assume that $N_{\mathcal{H}'}(w') = \Omega \setminus \{v_1\}$ for all $w' \in V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$ with $d_{\mathcal{H}'}(w') = 1$. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the set of hyperedges in $\mathcal{H}'$ containing the vertex w, and a vertex in $V(\mathcal{H}) \setminus (\mathcal{C} \cup \{w\})$, respectively. Let $\mathcal{E}_3 = E(\mathcal{H}'[\mathcal{C}])$. Clearly, we have $|\mathcal{E}_1| \leq \binom{s}{s-1} = s$, $|\mathcal{E}_2| \leq n - 2s - 1$.

Now we consider the hyperedges in $\mathcal{E}_3$. If $e \in \mathcal{E}_3$ contains u_i $(1 \le i \le s)$, then it must lie in $\{u_i\} \cup \Omega$, at most $\binom{s}{s-1} = s$ such hyperedges. Otherwise, we would get a $\mathcal{B}M_{s+1}$ in $\mathcal{H}$ by connecting $u_i u_j$ with the edge, and connecting u_i, u_j to w, w'. With only one hyperedge on Ω, we have $|\mathcal{E}_3| \le s^2 + 1$.

This implies that

$$\begin{aligned} e(\mathcal{H}) &\le |\mathcal{E}| + |\mathcal{E}_1| + |\mathcal{E}_2| + |\mathcal{E}_3| \\ &\le s + s + n - 2s - 1 + s^2 + 1 \\ &= n + s^2 \le \max\left\{\binom{2s+1}{s}, 1 + s(n-s)\right\}, \end{aligned}$$

where the last inequality can be proved by a method similar to that of Claim 2 when $s \ge 2$.

3.2. Proof of Theorem 2.1 for $s + 1 \le r \le 2s + 1$

Case (i): $r = s + 1$.

By Claim 1, if $e(\mathcal{H}) \le \binom{2s+1}{r}$, then the result holds. So we assume that w_1, w_2 are two vertices such that $d_{\mathcal{H}'}(w_i) \ge 1$ for $i = 1, 2$. Then $|N_{\mathcal{H}'}(w_i)| \ge r - 1 = s$. And $\overline{N_{\mathcal{H}'}(w_1)} \cap N_{\mathcal{H}'}(w_2) = \emptyset$, otherwise we can find a Berge copy of M_{s+1}. Thus, $|N_{\mathcal{H}'}(w_1)| = |N_{\mathcal{H}'}(w_2)| = s$.

For the case $N_{\mathcal{H}'}(w_1) \cap N_{\mathcal{H}'}(w_2) = \emptyset$, we write $N_{\mathcal{H}'}(w_1) = V_1$ and $N_{\mathcal{H}'}(w_2) = V_2$. Then for every $x \in V_1$ and $y \in V_2$, there is no hyperedge $e \in E(\mathcal{H}) \setminus \mathcal{E}$ with $\{x, y\} \subseteq e$, otherwise we can find a $\mathcal{B}M_{s+1}$. Therefore, $E(\mathcal{H}'[\mathcal{C}]) =$ and for every vertex $v \in V(\mathcal{H}) \setminus \mathcal{C}$, either $N_{\mathcal{H}'}(v) \subseteq V_1$ or $N_{\mathcal{H}'}(v) \subseteq V_2$. Thus, $e(\mathcal{H}) \le s + n - 2s = n - s$.

For the case $N_{\mathcal{H}'}(w_1) \cap N_{\mathcal{H}'}(w_2) \neq \emptyset$, there exists u_j (or v_j) in $\overline{N_{\mathcal{H}'}(w_1)} \cap \overline{N_{\mathcal{H}'}(w_2)}$. Now we consider the hyperedge e_j in $\mathcal{E}$. Apart from u_j (or v_j), e_j does not contain vertices in $V(\mathcal{H}) \setminus \mathcal{C}$, $\overline{N_{\mathcal{H}'}(w_1)}$, and $\overline{N_{\mathcal{H}'}(w_2)}$ by a similar argument as Figure 3. Then we have

$$s = |e_j| - 1 \le |\mathcal{C} \setminus (\overline{N_{\mathcal{H}'}(w_1)} \cup \overline{N_{\mathcal{H}'}(w_2)})| \le |\overline{N_{\mathcal{H}'}(w_1)} \cap \overline{N_{\mathcal{H}'}(w_2)}|.$$

It derives that $\overline{N_{\mathcal{H}'}(w_1)} = \overline{N_{\mathcal{H}'}(w_2)}$ and thus $N_{\mathcal{H}'}(w_1) = N_{\mathcal{H}'}(w_2)$. This only happens when $N_{\mathcal{H}'}(w_1)$ contains exactly one vertex from $\{u_i, v_i\}$. Without loss of generality, we may assume $N_{\mathcal{H}'}(w_1) = N_{\mathcal{H}'}(w_2) = \Omega = \{v_1, \ldots, v_s\}$. In this case, for every $w \in V(\mathcal{H}) \setminus \mathcal{C}$, either $N_{\mathcal{H}'}(w) = \Omega$ or $d_{\mathcal{H}'}(w) = 0$. Also, every hyperedge containing u_i (including e_i) contains Ω. Thus, we have

$$e(\mathcal{H}) \le e(\mathcal{H}[\Omega]) + \sum_{u \in V(\mathcal{H}) \setminus \Omega} d_{\mathcal{H}'}(u) \le \binom{s}{s+1} + (n - s) = n - s.$$

Case (ii): $s + 2 \le r \le 2s$.

Suppose that the result does not hold. By Claim 1, we may assume $w_1, w_2 \in V(\mathcal{H}) \setminus \mathcal{C}$ with $d_{\mathcal{H}'}(w_i) \ge 1$ for $i = 1, 2$. Then $N_{\mathcal{H}'}(w_i) \ge r - 1 \ge s + 1$. It implies $\overline{N_{\mathcal{H}'}(w_1)} \cap N_{\mathcal{H}'}(w_2) \neq \emptyset$. Similar as above, we can find a $\mathcal{B}M_{s+1}$, a contradiction.

Case (iii): $r = 2s + 1$.

Suppose that the result does not hold, i.e., $e(\mathcal{H}) \ge s+1$. Then there is at least one hyperedge $e \in E(\mathcal{H}) \setminus \mathcal{E}$. Since $r = 2s + 1$, e must be of the form $\{w\} \cup \mathcal{C}$ for a $w \in V(\mathcal{H}) \setminus \mathcal{C}$. Then e_1 contains a vertex outside $\{w\} \cup \mathcal{C}$, and thus we may find a $\mathcal{B}M_{s+1}$, a contradiction.

$\square$

4. Proof of Theorems 2.2

Let $\mathcal{H}$ be an r-graph and $U \subseteq V(\mathcal{H})$ be a nonempty subset. Let $\mathcal{H}[U]$ denote the subhypergraph of $\mathcal{H}$ induced by U. Given a hypergraph $\mathcal{H}$ and a vertex $v \in V(\mathcal{H})$, the link hypergraph $\mathcal{L}_v$ is defined as

$$\mathcal{L}_v = \{e \backslash \{v\} \mid e \in E(\mathcal{H}), v \in e\}.$$

Let us start with the proof of Theorem 2.2 that we restate here for convenience.

Theorem. *Let integers $3 \le r \le s+1$ and n be sufficiently large. Suppose $\mathcal{F}$ is an r-graph with $\chi(\mathcal{F}) > 2$, or $\chi(\mathcal{F}) = 2$ and $q(\mathcal{F}) = \infty$. Then*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \{\mathcal{F}\}) = \mathrm{ex}_r(s, \mathcal{D}(\mathcal{F})) + \binom{s}{r-1}(n-s),$$

where $\mathcal{D}(\mathcal{F})$ is the family of r-graphs obtained by deleting a strongly independent set from $\mathcal{F}$.

Proof. Observe that for any strongly independent set S of $\mathcal{F}$, the r-graph $\mathcal{F} - S$ is not empty. Otherwise, $\chi(\mathcal{F}) = 2$ and $q(\mathcal{F}) < \infty$, a contradiction. This implies that $\mathcal{D}(\mathcal{F})$ does not contain the empty graph. Thus there exists an s-vertex $\mathcal{D}(\mathcal{F})$-free r-graph $\mathcal{H}_0$ with $\mathrm{ex}_r(s, \mathcal{D}(\mathcal{F}))$ hyperedges. Now let us add $n-s$ new vertices and add all hyperedges containing one new vertex and $r-1$ vertices in $\mathcal{H}_0$. The resulting r-graph is $\mathcal{B}M_{s+1}$-free since $|V(\mathcal{H}_0)| < s+1$, and $\mathcal{F}$-free since $\mathcal{H}_0$ is $\mathcal{D}(\mathcal{F})$-free and the $n-s$ vertices outside $\mathcal{H}_0$ form a strongly independent set. The construction gives the lower bound.

Now we consider the upper bound. Let $\mathcal{H}$ be an n-vertex $\mathcal{B}M_{s+1} \cup \{\mathcal{F}\}$-free r-graph with $e(\mathcal{H}) > \mathrm{ex}_r(s, \mathcal{D}(\mathcal{F})) + \binom{s}{r-1}(n-s)$. Let $V(\mathcal{H}) = \{v_1, v_2, \ldots, v_n\}$ and $d_{\mathcal{H}}(v_1) \ge \cdots \ge d_{\mathcal{H}}(v_n)$.

Claim 3. $d_{\mathcal{H}}(v_{s+1}) \le s + \binom{2s}{r-1}$.

Proof of Claim 3. Suppose otherwise that $d_{\mathcal{H}}(v_{s+1}) > s + \binom{2s}{r-1}$. We may greedily find a $\mathcal{B}M_{s+1}$ in the following way. First, we pick a hyperedge e_1 containing v_1 and a vertex v_1' distinct from $v_1, v_2, \ldots, v_{s+1}$, and take v_1 and v_1' as the core of e_1. In step i $(2 \le i \le s+1)$, we pick a hyperedge e_i containing v_i and a vertex v_i' distinct from $v_1, v_2, \ldots, v_{s+1}, v_1', \ldots, v_{i-1}'$ which is different from all previously selected hyperedges, and take v_i and v_i' as the core of e_i. This is possible since the number of the previously selected hyperedges is at most s, and there are at most $\binom{s+i-1}{r-1} \le \binom{2s}{r-1}$ hyperedges containing v_i and $r-1$ vertices from $\{v_1, \ldots, v_{s+1}, v_1', \ldots, v_{i-1}'\} \backslash v_i$. Then $e_1, \ldots, e_{s+1}$ forms a $\mathcal{B}M_{s+1}$ whose core is $\{v_1v_1', \ldots, v_{s+1}v_{s+1}'\}$, a contradiction. $\square$

Let $\mathcal{M}_t$ be a maximum Berge matching in $\mathcal{H}$, and let A be the core of $\mathcal{M}_t$. Then $|A| = 2t \le 2s$. Let $\{e_i\}_{1 \le i \le t}$ be the hyperedges of $\mathcal{M}_t$. Every hyperedge in $E(\mathcal{H})$ other than $\{e_i\}_{1 \le i \le t}$ intersects A in at least $r-1$ vertices, otherwise a larger Berge matching would appear. Assume $A = \{a_1, a_2, \ldots, a_{2t}\}$. This implies that the number of hyperedges in $\mathcal{H}$ is at most

$$t + \frac{1}{r-1}\sum_{i=1}^{2t} d_{\mathcal{H}}(a_i) \le t + \frac{1}{r-1}\sum_{i=1}^{2s} d_{\mathcal{H}}(v_i) \le s + \frac{1}{r-1}\sum_{i=1}^{s} d_{\mathcal{H}}(v_i) + \frac{s}{r-1}\left[s + \binom{2s}{r-1}\right]. \quad (4.1)$$

The upper bound in this inequality follows from Claim 3.

Claim 4. $d_{\mathcal{H}}(v_i) \le \binom{s-1}{r-2}(n-1) + \binom{s-1}{r-1}$.

Proof of Claim 4. Let $d := d(s,r)$ be a large enough fixed constant. If $d_{\mathcal{H}}(v_i) \leq d$, then $d_{\mathcal{H}}(v_i) \leq \binom{s-1}{r-2}(n-1) + \binom{s-1}{r-1}$ as $\binom{s-1}{r-2} \geq 1$ and n is sufficiently large. If $d_{\mathcal{H}}(v_i) > d$, then we claim that the $(r-1)$-graph $\mathcal{L}_{v_i}$ is $(r-1)$-uniform $\mathcal{B}M_s$-free. Suppose otherwise that $\mathcal{L}_{v_i}$ contains a $\mathcal{B}M_s$, denoted by $\mathcal{M}_s$. Let $E(\mathcal{M}_s) = \{e'_1, \ldots, e'_s\}$. Since n is sufficiently large, we have $e(\mathcal{L}_{v_i}) = d_{\mathcal{H}}(v_i) > d > s + \binom{2s}{r-1}$. Then there exists an $(r-1)$-edge $e' \in E(\mathcal{L}_{v_i})$ with $e' \neq e'_1, \ldots, e'_s$ that contains a vertex v' outside the core of $\mathcal{M}_s$. Then $e'_1 \cup \{v_i\}, \ldots, e'_s \cup \{v_i\}, e' \cup \{v_i\}$ forms a r-uniform $\mathcal{B}M_{s+1}$ whose core is the core of $\mathcal{M}_s$ together with the vertices v_i and v', a contradiction. By Theorem 1.4, we have $e(\mathcal{L}_{v_i}) = d_{\mathcal{H}}(v_i) \leq \binom{s-1}{r-2}(n-1) + \binom{s-1}{r-1}$. $\square$

Claim 5. *$d_{\mathcal{H}}(v_s) > \binom{s-1}{r-2}n - c$ for some $c = c(s,r) > 0$.*

Proof of Claim 5. Suppose to the contrary that $d_{\mathcal{H}}(v_s) \leq \binom{s-1}{r-2}n - c$ for any c. By Claim 4, we have

$$
\begin{aligned}
e(\mathcal{H}) &\leq \frac{1}{r-1}\sum_{i=1}^{s} d_{\mathcal{H}}(v_i) + \frac{s}{r-1}\left[s + \binom{2s}{r-1}\right] + s \\
&\leq \frac{s-1}{r-1}\left[\binom{s-1}{r-2}(n-1) + \binom{s-1}{r-1}\right] + \binom{s-1}{r-2}\frac{n}{r-1} - \frac{c}{r-1} + \frac{s}{r-1}\left[s + \binom{2s}{r-1}\right] + s \\
&= \binom{s}{r-1}(n-s) + C - \frac{c}{r-1},
\end{aligned}
$$

where C is a constant depending only on r and s. When c is large, we have $e(\mathcal{H}) \leq \binom{s}{r-1}(n-s)$, a contradiction. $\square$

Let $U = \{v_1, \ldots, v_s\}$. If there exists a hyperedge e intersecting $V(\mathcal{H})\backslash U$ in at least two vertices, then we take $u, u' \in e\backslash U$ as the core of e. Since $d_{\mathcal{H}}(v_1) \geq \cdots \geq d_{\mathcal{H}}(v_s) > \binom{s-1}{r-2}n - c$ by Claim 5, we may greedily find a $\mathcal{B}M_{s+1}$ in a way analogous to the proof of Claim 3. Thus, $\mathcal{H}$ contains only two types of hyperedges: those containing exactly $r-1$ vertices from U, and those containing exactly r vertices from U. The number of hyperedges of the first type is at most $\binom{s}{r-1}(n-s)$, and the number of hyperedges of the second type is equal to $e(\mathcal{H}[U]) \leq \binom{s}{r}$.

If $\mathcal{H}[U]$ is $\mathcal{D}(\mathcal{F})$-free, then $e(\mathcal{H}) \leq \mathrm{ex}_r(s, \mathcal{D}(\mathcal{F})) + \binom{s}{r-1}(n-s)$ and we are done. Assume now that $\mathcal{H}[U]$ contains an r-graph $\mathcal{F}_0 \in \mathcal{D}(\mathcal{F})$. Then for each $(|V(\mathcal{F})| - |V(\mathcal{F}_0)|)$-set X outside U, this copy of $\mathcal{F}_0$ could be extended to a copy of $\mathcal{F}$ in $V(\mathcal{F}_0) \cup X$ using only hyperedges that contains exactly $r-1$ vertices from $V(\mathcal{F}_0)$ and one vertex from X, by the definition of $\mathcal{D}(\mathcal{F})$. This means that at least one such hyperedge is missing from $\mathcal{H}$. Let us partition the $n-s$ vertices outside U to $\lfloor (n-s)/(|V(\mathcal{F})| - |V(\mathcal{F}_0)|)\rfloor$ sets of size $|V(\mathcal{F})| - |V(\mathcal{F}_0)|$ and a smaller set, arbitrarily. Then we have a distinct missing hyperedge for each of the sets of size $|V(\mathcal{F})| - |V(\mathcal{F}_0)|$, thus there are at most $\binom{s}{r-1}(n-s) - \lfloor (n-s)/(|V(\mathcal{F})| - |V(\mathcal{F}_0)|)\rfloor$ hyperedges containing exactly $r-1$ vertices from U. Therefore, the number of hyperedges of H is at most $\binom{s}{r} + \binom{s}{r-1}(n-s) - \lfloor (n-s)/(|V(\mathcal{F})| - |V(\mathcal{F}_0)|)\rfloor$. Since n is sufficiently large, we have $\binom{s}{r} - \lfloor (n-s)/(|V(F)| - |V(\mathcal{F}_0)|)\rfloor \leq \mathrm{ex}_r(s, \mathcal{D}(\mathcal{F})) + \binom{s}{r-1}(n-s)$. This completes the proof. $\square$

5. Proofs of Theorems 2.6, 2.7 and 2.8

Let S_ℓ be a star with ℓ edges. The non-leaf vertex is called the *center* of S_ℓ. Khormali and Palmer [11] proved the following lemma establishing the degree conditions for the existence of a $\mathcal{B}S_\ell$.

Lemma 5.1 (Khormali and Palmer [11])**.**
(i) Fix integers $\ell > r \geq 2$ and let $\mathcal{H}$ be an r-graph. If x is a vertex of degree $d(x) > \binom{\ell-1}{r-1}$ in $\mathcal{H}$, then $\mathcal{H}$ contains a $\mathcal{B}S_\ell$ with center x.
(ii) Fix integers $r \geq 2$ and $\ell \leq r$ and let $\mathcal{H}$ be an r-graph. If x is a vertex of degree $d(x) > \ell - 1$ in $\mathcal{H}$, then $\mathcal{H}$ contains a $\mathcal{B}S_\ell$ with center x.

Now let us start with the proof of Theorem 2.6 that we restate here for convenience.

Theorem. *Fix integers s and $r \geq 2$. Let G be a bipartite graph with $p = p(G)$ and n be sufficiently large.*
(i) If $r \leq p \leq s$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}G) = \binom{p-1}{r-1}n + O(1)$.
(ii) If $p \leq s < r - 1$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}G) \leq \max\left\{s, \binom{2s+1}{r}\right\}$.
(iii) If $p > s$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}G) = \mathrm{ex}_r(n, \mathcal{B}M_{s+1})$.

Proof. (i) For the lower bound, we consider the following r-graph: take a set A of order $p-1$ and a set B of order $n - p + 1$, and add each hyperedge that contains exactly $r - 1$ vertices of A. Clearly, this resulting r-graph is $\mathcal{B}M_p$-free as $|A| < p$, and thus $\mathcal{B}M_{s+1} \cup \mathcal{B}G$-free as $p \leq s$ and $\mathcal{B}G$ contains a $\mathcal{B}M_p$.

We now continue with the upper bound. Let $\mathcal{H}$ be an n-vertex $\mathcal{B}M_{s+1} \cup \mathcal{B}G$-free r-graph with maximum number of hyperedges. Let $\mathcal{M}$ be a largest Berge matching in $\mathcal{H}$, and let U be the core of $\mathcal{M}$. Then $|U| \leq 2s$ as $\mathcal{H}$ is $\mathcal{B}M_{s+1}$-free. Observe that each hyperedge not belonging to $\mathcal{M}$ intersects U in at least $r-1$ vertices. Otherwise, a larger Berge matching would appear. Now we delete these at most s hyperedges from $\mathcal{M}$. Denote by $\mathcal{H}'$ this resulting r-graph. Thus, $\mathcal{H}'$ contains only two types of hyperedges: those containing exactly $r-1$ vertices from U, and those containing exactly r vertices from U. The number of hyperedges of the first type is at most $\binom{|U|}{r-1}(n - |U|)$, and the number of hyperedges of the second type is at most $\binom{|U|}{r}$.

Let $X = \{v \in V(\mathcal{H})\backslash U \,|\, d_{\mathcal{H}'}(v) > \binom{p-1}{r-1}\}$ and $Y = V(\mathcal{H})\backslash(U \cup X)$. By Lemma 5.1 **(i)**, for any $v \in X$, there exists a $\mathcal{B}S_p$ in $\mathcal{H}'$ with center v such that every vertex other than v belongs to U. In particular, the core of this $\mathcal{B}S_p$ is contained in U, except for the vertex v. Note that there are at most $\binom{|U|}{p}$ p-sets in U. If $|X| \geq \binom{|U|}{p}(|V(G)| - p)$, then by pigeonhole principle, there exists a p-set $U' \subseteq U$ that forms the p leaf vertices in the cores of at least $|V(G)| - p$ Berge copies of S_p. This implies that $\mathcal{H}$ contains a $\mathcal{B}K_{p,|G|-p}$, and therefore a $\mathcal{B}G$, a contradiction. Thus, $|X| < \binom{|U|}{p}(|V(G)| - p)$. Therefore, the number of hyperedges of $\mathcal{H}$ is at most

$$s + \binom{|U|}{r} + \binom{|U|}{r-1}|X| + \binom{p-1}{r-1}|Y| \leq \binom{p-1}{r-1}n + O(1).$$

(ii) If $r - 1 > s$, then by Theorem 1.4,

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}G) \leq \mathrm{ex}_r(n, \mathcal{B}M_{s+1}) \leq \max\left\{s, \binom{2s+1}{r}\right\}.$$

(iii) By the definition of p, G contains a matching M_p. Then any Berge copy of G contains a Berge copy of M_p. Since $p \geq s+1$, we obtain $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}G) = \mathrm{ex}_r(n, \mathcal{B}M_{s+1})$. □

Recall that for $r \geq 3$, let $\mathcal{G}_s^{r-1}$ be an s-vertex $(r-1)$-graph with the maximum number of hyperedges such that $\partial\mathcal{G}_s^{r-1}$ is $K_{p,q}$-free.

Let us continue with the proof of Theorem 2.7 that we restate here for convenience.

Theorem. *Fix integers s and $r \geq 2$. Let n be sufficiently large.*

(i) *If $p \leq r \leq \min\{s, p+q-1\}$, then*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) = (p-1)n + O(1).$$

(ii) *If $r = p+q \leq s$, then*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) = \min\{p-1, e(\mathcal{G}_s^{r-1})\} \cdot n + O(1).$$

(iii) *If $p+q < r \leq s$, then*

$$\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) \leq s + \binom{2s}{r} + \binom{2s}{r-1}(pq-1).$$

Proof. We first prove the lower bounds in **(i)** and **(ii)**. For the lower bound in **(i)**, we consider the following r-graph. Let A be a fixed r-set with vertices $\{a_1, a_2, \ldots, a_r\}$. Let $A_i = A \backslash \{a_i\}$ for $i = 1, 2, \ldots, p-1$. For each of the remaining $n-r$ vertices, add all hyperedges that contain A_i and the vertex, $i = 1, 2, \ldots, p-1$. Clearly, this resulting r-graph is $\mathcal{B}M_{s+1}$-free as $r \leq s$. Observe that in a $\mathcal{B}K_{p,q}$, there are at least $p+q$ vertices whose degree is at least p. However, in this resulting r-graph, the number of vertices of degree at least p is at most $r \leq p+q-1$. This implies that this resulting r-graph is $\mathcal{B}K_{p,q}$-free.

For the lower bound in **(ii)**, we consider the following r-graph. If $e(\mathcal{G}_s^{r-1}) \leq p-1$, let $\mathcal{A} = E(\mathcal{G}_s^{r-1})$. Otherwise, let $\mathcal{A}$ be a collection consisting of exactly $p-1$ hyperedges of $\mathcal{G}_s^{r-1}$. For each of the remaining $n-s$ vertices and each $A \in E(\mathcal{A})$, add the hyperedge containing the vertex and A. Note that $|\mathcal{A}| = \min\{p-1, e(\mathcal{G}_s^{r-1})\}$ and the resulting r-graph has exactly $(n-s)|\mathcal{A}|$ hyperedges. Clearly, this resulting r-graph is $\mathcal{B}M_{s+1}$-free as $\mathcal{G}_s^{r-1}$ has only s vertices. Observe that only the vertices in $V(\mathcal{G}_s^{r-1})$ have degree greater than p. Therefore, the core of any $\mathcal{B}K_{p,q}$ is contained only in $\mathcal{G}_s^{r-1}$. However, by the definition of $\mathcal{G}_s^{r-1}$, $\partial\mathcal{G}_s^{r-1}$ is $K_{p,q}$-free. This implies that this resulting r-graph is $\mathcal{B}K_{p,q}$-free.

We now prove the upper bounds in **(i)**, **(ii)** and **(iii)** together. Let $\mathcal{H}$ be an n-vertex $\mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}$-free r-graph. Similar to the proof of Theorem 2.6, let $\mathcal{M}$ be a largest Berge matching in $\mathcal{H}$, and let U be the core of $\mathcal{M}$. Now we delete at most s hyperedges from $\mathcal{M}$ and denote by $\mathcal{H}'$ this resulting r-graph. Then $\mathcal{H}'$ contains only two types of hyperedges: those containing exactly $r-1$ vertices from U, and those containing exactly r vertices from U.

Case 1. $p \leq r \leq p+q$.

Let $X = \{v \in V(\mathcal{H}) \backslash U \mid d_{\mathcal{H}'}(v) > p-1\}$ and $Y = V(\mathcal{H}) \backslash (U \cup X)$. By Lemma 5.1 **(ii)**, for any $v \in X$, there exists a $\mathcal{B}S_p$ in $\mathcal{H}'$ with center v such that every vertex other than v belongs to U. In particular, the core of this $\mathcal{B}S_p$ is contained in U, except for the vertex v. Note that there are at most $\binom{|U|}{p}$ p-sets in U. If $|X| \geq \binom{|U|}{p}q$, then by pigeonhole principle, there exists a p-set $U' \subseteq U$ that forms the p leaf vertices in the cores of at least q Berge copies of S_p. This implies that $\mathcal{H}$ contains a $\mathcal{B}K_{p,q}$, a contradiction. It follows that $|X| < \binom{|U|}{p}q$. Therefore, the number of hyperedges of $\mathcal{H}$ is at most

$$s + \binom{|U|}{r} + \binom{|U|}{r-1}|X| + (p-1)|Y| \leq (p-1)n + O(1).$$

Case 2. $r = p+q \leq s$ and $e(\mathcal{G}_s^{r-1}) < p-1$.

Let $d := d(s,r,p,q)$ be a large enough fixed constant. For every $(r-1)$-set S in U, set $E_{\mathcal{H}'}(S) = \{e \in E(\mathcal{H}') \,|\, e \cap U = S\}$ and $X = \Big\{S \subseteq U \,\Big|\, |E_{\mathcal{H}'}(S)| > d\Big\}$. We claim that $|\bigcup_{S\in X} S| \leq s$. Otherwise, we may greedily construct a $\mathcal{B}M_{s+1}$ such that the maximum independent set of its core is contained in $\bigcup_{S\in X} S$, a contradiction. For convenience, let $|\bigcup_{S\in X} S| := x \leq s$. Now let $\tilde{\mathcal{G}}$ denote the $(r-1)$-graph with vertex set $\bigcup_{S\in X} S$ and hyperedge set X. We claim that $\partial\tilde{\mathcal{G}}$ is $K_{p,q}$-free. Otherwise, there exists a copy of $K_{p,q}$ in $\partial\tilde{\mathcal{G}}$, denoted by K. Since each edge of K is contained in some $(r-1)$-set from X, by the definition of X, we may greedily construct a $\mathcal{B}K_{p,q}$ with K as its core, a contradiction. Thus, we have $|X| = e(\tilde{\mathcal{G}}) \leq e(\mathcal{G}_x^{r-1}) \leq e(\mathcal{G}_s^{r-1}) < p-1$. Note that $|E_{\mathcal{H}'}(S)| \leq n - |U|$ for any $S \subseteq U$. Therefore, the number of hyperedges of $\mathcal{H}$ is at most

$$s + \binom{|U|}{r} + e(\mathcal{G}_s^{r-1}) \cdot (n - |U|) + \left(\binom{|U|}{r-1} - e(\mathcal{G}_s^{r-1})\right) d = e(\mathcal{G}_s^{r-1}) \cdot n + O(1).$$

Case 3. $p + q < r \leq s$.

We claim that any $(r-1)$-set $S \subseteq U$ is contained in at most $pq-1$ hyperedges. Indeed, If there exists such an $(r-1)$-set $S \subseteq U$, then we may greedily construct a $\mathcal{B}K_{p,q}$ such that its core is contained in S. Thus, the number of hyperedges in $\mathcal{H}'$ that contain exactly $r-1$ vertices from U is at most $\binom{|U|}{r-1}(pq-1)$. Therefore, the number of hyperedges of $\mathcal{H}$ is at most

$$s + \binom{|U|}{r} + \binom{|U|}{r-1}(pq-1) \leq s + \binom{2s}{r} + \binom{2s}{r-1}(pq-1).$$

This completes the proof. □

We now begin with the proof of Theorem Theorem 2.8 that we restate here for convenience.

Proposition. *Let integers $r \geq 2$ and n be sufficiently large.*

(i) *If $p = r = s+1$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) = \mathrm{ex}_r(n, \mathcal{B}M_{s+1})$.*

(ii) *If $p < r = s+1 \leq p+q$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) = n + O(1)$.*

(iii) *If $p+q < r = s+1$, then $\mathrm{ex}_r(n, \mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}) \leq s + \binom{2s}{r} + \binom{2s}{r-1}(pq-1)$.*

Here we make a slight adjustment to the proof of the upper bound in Theorem 2.7.

Proof. Obviously, **(i)** follows directly from Theorem 2.6 **(iii)**. For the lower bound in **(ii)**, we consider the following r-graph. For a fixed $(r-1)$-set, add hyperedges by joining it with every vertex from the remaining $n-r+1$ vertices. Clearly, this resulting r-graph is $\mathcal{B}M_{s+1}$-free as $r-1 < s+1$. Observe that in a $\mathcal{B}K_{p,q}$, there are at least $p+q$ vertices whose degree is at least p. However, in this resulting r-graph, the number of vertices of degree at least p is at most $r-1$. This implies that this resulting r-graph is $\mathcal{B}K_{p,q}$-free as $r-1 < p+q$.

We now prove the upper bounds in **(ii)** and **(iii)** together. Let $\mathcal{H}$ be an n-vertex $\mathcal{B}M_{s+1} \cup \mathcal{B}K_{p,q}$-free r-graph. Similar to the proof of Theorem 2.6, let $\mathcal{M}$ be a largest Berge matching in $\mathcal{H}$, and let U be the core of $\mathcal{M}$. Now we delete at most s hyperedges from $\mathcal{M}$ and denote by $\mathcal{H}'$ this resulting r-graph. Then $\mathcal{H}'$ contains only two types of hyperedges: those containing exactly $r-1$ vertices from U, and those containing exactly r vertices from U.

Suppose $p < r = s+1 \leq p+q$. Let $d := d(s,r,p,q)$ be a large enough fixed constant. For every $(r-1)$-set S in U, set $E_{\mathcal{H}'}(S) = \{e \in E(\mathcal{H}') \,|\, e \cap U = S\}$ and $X = \Big\{S \subseteq U \,\Big|\, |E_{\mathcal{H}'}(S)| > d\Big\}$.

We claim that $|X| \leq 1$. Otherwise, there are two $(r-1)$-sets S_1 and S_2 such that $|E_{\mathcal{H}'}(S_1)| > d$ and $|E_{\mathcal{H}'}(S_2)| > d$. Since $|S_1 \cup S_2| \geq r = s+1$, we may greedily construct a $\mathcal{B}M_{s+1}$ such that the maximum independent set of its core is contained in $S_1 \cup S_2$, a contradiction. Note that $|E_{\mathcal{H}'}(S)| \leq n - |U|$ for any $S \subseteq U$. Therefore, the number of hyperedges of $\mathcal{H}$ is at most

$$s + \binom{|U|}{r} + (n - |U|) + \left(\binom{|U|}{r-1} - 1\right) d = n + O(1).$$

If $p + q < r = s + 1$, an argument analogous to that of Theorem 2.7 **(iii)** yields $e(\mathcal{H}) \leq s + \binom{2s}{r} + \binom{2s}{r-1}(pq-1)$. This completes the proof. □

6. Proofs of Lemma 2.9 and Theorem 2.10

We first introduce Gallai-Edmonds Structure Theorem on bipartite graphs. For a graph G, we denote by D the set of all vertices in G that are not covered by at least one maximum matching of G. Let $A := N_G(D)$ and $C = V(G) \setminus (D \cup A)$. The partition (D, A, C) is called the *Gallai-Edmonds decomposition* or *G-E decomposition* shortly. Here we use the following result (see Theorem 3.2.4 in [12]) to prove Lemma 2.9, which extends the result of in [5].

Theorem 6.1 (Lovász and Plummer [12])**.** *For a bipartite graph $X = (X_1, X_2)$ and $i = 1, 2$, let $A_i = A \cap X_i$, $C_i = C \cap X_i$ and $D_i = D \cap X_i$, where A, C, and D are the G-E decomposition for X. Then*

(i) *$D_1 \cup D_2$ is an independent set of X.*

(ii) *The subgraph $X[C_1 \cup C_2]$ has a perfect matching.*

(iii) *$N_X(D_1) = A_2$ and $N_X(D_2) = A_1$.*

(iv) *For any maximum matching M of X, it contains a perfect matching of $X[C_1 \cup C_2]$, and contains all vertices of A_1 (resp. A_2) with neighbors in D_2 (resp. D_1).*

Let $\mathcal{H}$ be an r-graph. Define an *auxiliary bipartite graph* X with parts X_1 and X_2, where X_1 is the hyperedges of $\mathcal{H}$ and X_2 is the collection of all pairs of vertices contained in some hyperedges of $\mathcal{H}$. A vertex $x \in X_1$ is joined to $y \in X_2$ if and only if the hyperedge x contains the pair y.

Now let us start with the proof of Lemma 2.9 that we restate here for convenience.

Lemma. *Let $\mathcal{F}$ be a finite family of graphs. Then*

$$\mathrm{ex}_r(n, \mathcal{BF}) \leq \max\left\{\varepsilon(r, G_{\text{red-blue}}) \,\middle|\, G \text{ is an } n\text{-vertex } \mathcal{F}\text{-free graph}\right\}.$$

Proof. Let $\mathcal{H}$ be a Berge-$\mathcal{F}$-free r-graph of order n, and let $X = (X_1, X_2)$ be an auxiliary bipartite graph defined as above. For $i = 1, 2$, we let $A_i = A \cap X_i$, $C_i = C \cap X_i$ and $D_i = D \cap X_i$, where A, C, and D are the G-E decomposition of X. Let M be a maximum matching in X. By Theorem 6.1 **(iv)**, there exist vertices $D_1' \subseteq D_1$ and $D_2' \subseteq D_2$ respectively that are uncovered by M. The vertices in $X_2 \setminus D_2'$ correspond to a collection of pairs on $V(\mathcal{H})$. Let G^* be the graph on $V(\mathcal{H})$ with edges $X_2 \setminus D_2'$. Clearly, $e(G^*) = |X_2 \setminus D_2'|$ and $|V(G^*)| \leq n$.

We assert that G^* is $\mathcal{F}$-free. If G^* contains some $F \in \mathcal{F}$, then by the definition of X, we see that for each pair $\{v_1 v_2\} \in E(F) \subseteq X_2 \setminus D_2'$, there exists a distinct hyperedge $e \in X_1 \setminus D_1'$ in $\mathcal{H}$ such that $\{e, \{v_1, v_2\}\} \in M$, which implies that $\mathcal{H}$ contains a Berge copy of F, a contradiction. Thus the assertion holds.

We next find a red-blue coloring of G^* that satisfies the statement of the lemma. If $D_1' = \emptyset$, then $|D_1| = |A_2|$ by Theorem 6.1 **(iii)** and **(iv)**. We color all edges of G^* red. By Theorem 6.1 **(ii)** and **(iv)**, one can see that

$$e(\mathcal{H}) = |X_1| = |C_1| + |A_1| + |D_1| = |C_2| + |D_2 \setminus D_2'| + |A_2| = |X_2 \setminus D_2'| = e(G^*).$$

So we may assume that $D_1' \neq \emptyset$. Let $G^*_{\text{red-blue}}$ be the red-blue graph obtained by coloring the edges of G^* corresponding to vertices in $C_2 \cup (D_2 \setminus D_2')$ red, and those corresponding to vertices in A_2 blue. By Theorem 6.1 **(ii)**, **(iii)**, and **(iv)**, we have $|C_1 \cup A_1| = |C_2 \cup (D_2 \setminus D_2')| = e(G^*_{\text{red}})$, and $N_X(D_1) = A_2$. Note that each vertex in D_1 is adjacent to exactly $\binom{r}{2}$ vertices in A_2. So each vertex in D_1 corresponds to a blue K_r in $G^*_{\text{red-blue}}$, which implies that $|D_1| \leq \mathcal{N}(K_r, G^*_{\text{blue}})$. Therefore,

$$e(\mathcal{H}) = |X_1| = |C_1| + |A_1| + |D_1| \leq e(G^*_{\text{red}}) + \mathcal{N}(K_r, G^*_{\text{blue}}).$$

□

We now continue with the proof of Theorem 2.10 that we restate here for convenience.

Theorem. *Let $\mathcal{F}$ be a finite family of graphs and $f := f(n)$ such that* $\mathrm{ex}(n', K_{r-1}, \mathcal{F}^-) \leq fn$ *for every n, then*

$$\mathrm{ex}_r(n, \mathcal{BF}) \leq \max\left\{\frac{2f}{r}, 1\right\} \cdot \mathrm{ex}(n, \mathcal{F}).$$

We will follow the strategy of the proof in [5] by Gerbner, Methuku and Palmer.

Proof. Let $G_{\text{red-blue}}$ be an n-vertex $\mathcal{F}$-free red-blue graph. By Lemma 2.9, it is sufficient to prove that

$$\varepsilon(r, G_{\text{red-blue}}) \leq \max\left\{\frac{2f}{r}, 1\right\} \cdot \mathrm{ex}(n, \mathcal{F}).$$

Write $e(G_{\text{blue}}) = m$, then

$$e(G_{\text{red}}) = e(G_{\text{red-blue}}) - m \leq \mathrm{ex}(n, \mathcal{F}) - m. \tag{6.1}$$

Since the underlying graph G is $\mathcal{F}$-free, the neighborhood of every vertex $v \in V(G_{\text{blue}})$ in G_{blue} is $\mathcal{F}^-$-free. Note that an $\mathcal{F}^-$-free graph on $|N_{G_{\text{blue}}}(v)|$ vertices contains at most

$$\mathrm{ex}(|N_{G_{\text{blue}}}(v)|, K_{r-1}, \mathcal{F}^-) \leq f|N_{G_{\text{blue}}}(v)| = fd_{G_{\text{blue}}}(v)$$

copies of K_{r-1}. This implies that v is contained in at most $fd_{G_{\text{blue}}}(v)$ copies of K_r in G_{blue}. Summing the number of copies of K_r containing each vertex in $V(G_{\text{blue}})$, we count each K_r exactly r times. So by $\sum_{v \in V(G_{blue})} d(v) = 2m$, one can see that

$$\begin{aligned}
\mathcal{N}(K_r, G_{\text{blue}}) &\leq \frac{1}{r} \sum_{v \in V(G_{\text{blue}})} \mathrm{ex}(|N_{G_{\text{blue}}}(v)|, K_{r-1}, \mathcal{F}^-) \\
&\leq \frac{1}{r} \sum_{v \in V(G_{\text{blue}})} fd_{G_{\text{blue}}}(v) \\
&= \frac{2fm}{r}. \qquad (6.2)
\end{aligned}$$

Thus, by (6.1) and (6.2), we have

$$\begin{aligned}
\varepsilon(r, G_{\text{red-blue}}) &\leq \mathrm{ex}(n, \mathcal{F}) - m + \frac{2fm}{r} \\
&\leq \max\left\{\frac{2f}{r}, 1\right\} \cdot \big(\mathrm{ex}(n, \mathcal{F}) - m\big) + \max\left\{\frac{2f}{r}, 1\right\} \cdot m \\
&= \max\left\{\frac{2f}{r}, 1\right\} \cdot \mathrm{ex}(n, \mathcal{F}).
\end{aligned}$$

This completes the proof. □

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Declaration of interest

The authors declare no known conflicts of interest.

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